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Discrete-time network-based control under scheduling and actuator constraints

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SUMMARY

This paper is concerned with the solution bounds for discrete-time networked control systems via delay-dependent Lyapunov–Krasovskii methods. Solution bounds are widely used for systems with input saturation caused by actuator saturation or by the quantizers with saturation. The time-delay approach has been developed recently for the stabilization of continuous-time networked control systems under the round-robin protocol and under a weighted try-once-discard protocol, respectively. Actuator saturation has not been taken into account. In the present paper, for the first time, the time-delay approach is extended to the stability analysis of the discrete-time networked control systems under both scheduling protocols and actuators saturation. The communication delays are allowed to be larger than the sampling intervals. A novel Lyapunov-based method is presented for finding the domain of attraction. Polytopic uncertainties in the system model can be easily included in our analysis. The efficiency of the time-delay approach is illustrated on the example of a cart–pendulum system.

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KEY WORDS: networked control systems; time-delay approach; scheduling; input saturation; Lyapunov–Krasovskii method

1. INTRODUCTION

Network control systems (NCSs) are spatially distributed systems in which the communication between sensors, actuators, and controllers occurs through a communication network [1]. In many such systems, only one node is allowed to use the communication channel at once. In the present paper, we focus on the stability analysis of discrete-time NCSs with communication constraints and actuator constraints. The scheduling of sensor information toward the controller is ruled by the round-robin (RR) protocol and by a weighted try-once-discard (TOD) protocol, respectively. A linear (probably, uncertain) system with distributed sensors is considered. Three recent approaches for NCSs are based on discrete-time systems [2, 3], impulsive/hybrid systems [4, 5], and time-delay systems [6–8].

The time-delay approach has been developed for the stabilization of continuous-time NCSs under the RR protocol in [9] and under a weighted TOD protocol in [10], respectively. The closed-loop system is modeled as a switched system with multiple and ordered time-varying delays under RR protocol or as a hybrid system with time-varying delays in the dynamics and in the reset equations under TOD protocol. Differently from the existing hybrid and discrete-time approaches on the stabilization of NCS with scheduling protocols, the time-delay approach allows treating the case

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of large communication delays. Actuator saturation was not taken into account in [10] and [9]. Recently, the stabilization of sampled-data systems under variable samplings and actuator saturation was studied in [11], where scheduling protocols and delays were not included.

As shown in [12], when one deals with the solution bounds of time-delay systems via Lyapunov–Krasovskii method, the first time-interval of the delay length needs a special analysis. Solution bounds are widely used for systems with input saturation caused by actuator saturation or by the quantizers with saturation. This first time-interval does not influence on the stability and the exponential decay rate analysis. The analysis of the first time-interval of the delay length is important for nonlinear systems, for example, for finding the domain of attraction.

In the present paper, the time-delay approach is extended to the stability analysis of discrete-time NCSs with actuator constraints under the RR [9] or under a weighted TOD [10] scheduling. Following [12], we present a direct Lyapunov approach for finding the domain of attraction under both scheduling protocols. The conditions are given in terms of LMIs. Polytopic uncertainties in the system model can be easily included in the analysis. The efficiency of the presented approach is illustrated by a cart–pendulum system.

Notation: Throughout the paper, the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. For any matrix $A \in \mathbb{R}^{n\times n}$ and vector $x \in \mathbb{R}^n$, the notations A_j and x_j denote, respectively, the j^{th} line of matrix A and the j^{th} component of vector x. $\mathbb{Z}^+$, $\mathbb{N}$, and $\mathbb{R}^+$ denote the set of non-negative integers, positive integers, and non-negative real numbers, respectively. Given $\bar{u} = [\bar{u}_1, \ldots, \bar{u}_{n_u}]^T$, $0 < \bar{u}_i$, $i = 1, \ldots, n_u$, for any $u = [u_1, \ldots, u_{n_u}]^T$, we denote by $sat(u)$ the vector with coordinates $sign(u_i) min\,(|u_i|,\ \bar{u}_i)$. $MATI$ denotes the maximum allowable transmission interval.

2. STABILIZATION OF DISCRETE-TIME NCSS WITH ACTUATOR SATURATION UNDER RR SCHEDULING

2.1. Problem formulation and a switched system model

Consider the system architecture in Figure 1 with plant

$$x(t+1) = Ax(t) + Bu(t), \quad t \in \mathbb{Z}^+, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input, and A and B are (probably, uncertain) system matrices with appropriate dimensions. The initial condition is given by $x(0) = x_0$. We suppose that the control input is subject to the following amplitude constraints

$$|u_i(t)| \leqslant \bar{u}_i,\ 0 < \bar{u}_i,\ i = 1, \ldots, n_u,\ t \in \mathbb{Z}^+. \tag{2}$$

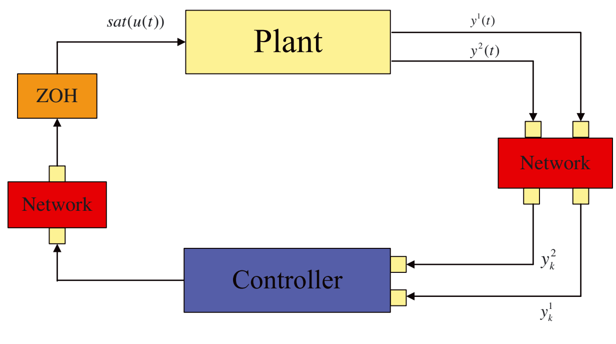

Figure 1. NCSs with actuator saturation under RR scheduling.

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The NCS has several nodes (distributed sensors, a controller node, and an actuator node), which are connected via networks. For the sake of simplicity, we consider two sensor nodes $y^i(t) = C^i x(t), i = 1,2$ and we denote $C = \begin{bmatrix} C^1 \\ C^2 \end{bmatrix}, y(t) = \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \in \mathbb{R}^{n_y}$. The results can be easily extended to any finite number of sensors. We let s_k denote the unbounded and monotonously increasing sequence of sampling instants, that is,

$$0 = s_0 < s_1 < \cdots < s_k < \cdots, \quad k \in \mathbb{Z}^+, \lim_{k\to\infty} s_k = \infty, \quad s_{k+1} - s_k \leqslant MATI, \tag{3}$$

where $\{s_0, s_1, s_2, \ldots\}$ is a subsequence of $\{0, 1, 2, \ldots\}$. At each sampling instant s_k, one of the outputs $y^i(t) \in \mathbb{R}^{n_i}\,(n_1 + n_2 = n_y)$ is sampled and transmitted via the network. First, we consider the RR scheduling protocol for the choice of the active output node: the outputs are transmitted one after another, that is, $y^i(t) = C^i x(t), t \in \mathbb{Z}^+$ is transmitted only at the sampling instant $t = s_{2p+i-1},\ p \in \mathbb{Z}^+, i = 1,2$. After each transmission and reception, the values in $y^i(t)$ are updated with the newly received values, whereas the values of $y^j(t)$ for $j \neq i$ remain the same, as no additional information is received. This leads to the constrained data exchange expressed as

$$y_k^i = \begin{cases} y^i(s_k) = C^i x(s_k), & k = 2p + i - 1, \\ y_{k-1}^i, & k \neq 2p + i - 1, \end{cases} \quad p \in \mathbb{Z}^+.$$

It is assumed that no packet dropouts and no packet disorders will happen during the data transmission over the network. The transmission of the information over the two networks (between the sensor and the actuator) is subject to a variable delay $\eta_k = \eta_k^{sc} + \eta_k^{ca} \in \mathbb{Z}^+$, where η_k^{sc} and η_k^{ca} are the network-induced delays from the sensor to the controller and from the controller to the actuator, respectively. Then $t_k = s_k + \eta_k$ is the updating time instant of the zero-order hold device.

As in [9] and [13], we allow the delays to be non-small (larger than the sampling intervals) provided that the old sample cannot get to the destination (to the controller or to the actuator) after the current one. Assume that the network-induced delay η_k and the time span between the updating and the current sampling instants are bounded

$$t_{k+1} - 1 - t_k + \eta_k \leqslant \tau_M,\ 0 \leqslant \eta_m \leqslant \eta_k \leqslant \eta_M,\ k \in \mathbb{Z}^+, \tag{4}$$

where τ_M, η_m, and η_M are known non-negative integers. Then,

$$\begin{aligned} (t_{k+1} - 1) - s_k &= s_{k+1} - s_k + \eta_{k+1} - 1 \leqslant MATI + \eta_M - 1 = \tau_M, \\ (t_{k+1} - 1) - s_{k-1} &= s_{k+1} - s_{k-1} + \eta_{k+1} - 1 \leqslant 2MATI + \eta_M - 1 = 2\tau_M - \eta_M + 1 \triangleq \bar{\tau}_M, \\ t_{k+1} - t_k &\leqslant \tau_M - \eta_m + 1. \end{aligned} \tag{5}$$

In [9], a time-delay approach was developed for the stability and L_2-gain analysis of continuous-time NCSs with RR scheduling. Actuator saturation was not taken into account. In this section, we consider the stability analysis of discrete-time NCSs with actuator saturation under RR scheduling protocol. Because of the control bounds defined in (2), the effective control signal to be applied to the system (1) is given by

$$u(t) = sat\left(K^1 y_k^1 + K^2 y_k^2\right),\ t \in [t_k, t_{k+1} - 1],\ t \in \mathbb{N},\ k \in \mathbb{N},$$

where $K = [K^1\ K^2]$, $K^1 \in \mathbb{R}^{n_u \times n_1}$, $K^2 \in \mathbb{R}^{n_u \times n_2}$ such that $A + BKC$ is Schur.

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We define the polyhedron

$$\mathcal{L}\left(K^j, \bar{u}\right) = \left\{x \in \mathbb{R}^n \ : \left|(K^j C^j)_i x\right| \leqslant \frac{1}{2}\bar{u}_i,\ i = 1, \ldots, n_u\right\},\ j = 1, 2.$$

If the control is such that $x(t) \in \mathcal{L}\left(K^1, \bar{u}\right) \cap \mathcal{L}\left(K^2, \bar{u}\right)$, then $\left|(K^1C^1)_i x + (K^2C^2)_i x\right| \leqslant \bar{u}_i$. Following [9], the closed-loop system with RR scheduling is modeled as a switched system

$$\begin{aligned} x(t+1) &= Ax(t) + A_1 x(t_{k-1} - \eta_{k-1}) + A_2 x(t_k - \eta_k),\ t \in [t_k, t_{k+1} - 1], \\ x(t+1) &= Ax(t) + A_1 x(t_{k+1} - \eta_{k+1}) + A_2 x(t_k - \eta_k),\ t \in [t_{k+1}, t_{k+2} - 1], \end{aligned} \tag{6}$$

where $k = 2p - 1,\ p \in \mathbb{N},\ A_i = BK^iC^i,\ i = 1, 2.$

For $t \in [t_k, t_{k+1} - 1]$, we can represent $t_k - \eta_k = t - \tau_1(t),\ t_{k-1} - \eta_{k-1} = t - \tau_2(t)$, where

$$\begin{aligned} &\tau_1(t) = t - t_k + \eta_k < \tau_2(t) = t - t_{k-1} + \eta_{k-1}, \\ &\tau_1(t) \in [\eta_m, \tau_M], \quad \tau_2(t) \in [\eta_m, \quad \bar{\tau}_M], \quad t \in [t_k, t_{k+1} - 1]. \end{aligned}$$

Therefore, (6) for $t \in [t_k, t_{k+1} - 1]$ can be considered as a system with two time-varying interval delays, where $\tau_1(t) < \tau_2(t)$. Similarly, for $t \in [t_{k+1}, t_{k+2} - 1]$, (6) is a system with two time-varying delays, one of which is less than another.

2.2. *Solution bounds*

Applying the following discrete-time Lyapunov–Krasovskii functional (LKF) to system (6) with time-varying delay from the maximum delay interval $[\eta_m, \bar{\tau}_M]$

$$\begin{aligned} V_{RR}(t) &= x^T(t)Px(t) + \sum_{s=t-\eta_m}^{t-1} \lambda^{t-s-1} x^T(s) S_0 x(s) \\ &\quad + \eta_m \sum_{j=-\eta_m}^{-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_0 \eta(s) + \sum_{s=t-\bar{\tau}_M}^{t-\eta_m-1} \lambda^{t-s-1} x^T(s) S_1 x(s) \\ &\quad + (\bar{\tau}_M - \eta_m) \sum_{j=-\bar{\tau}_M}^{-\eta_m-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_1 \eta(s),\ \eta(t) = x(t+1) - x(t), \\ &P > 0, \quad S_i > 0, \quad R_i > 0, \quad i = 0, 1, \quad 0 < \lambda < 1, \quad t \geqslant 0, \end{aligned} \tag{7}$$

where following [12], we define (for simplicity)

$$x(t) = x_0, \quad t \leqslant 0. \tag{8}$$

We find then conditions that guarantee

$$V_{RR}(t+1) - \lambda V_{RR}(t) \leqslant 0, \quad t = t_1, t_1 + 1, \ldots, \tag{9}$$

which will imply

$$V_{RR}(t) \leqslant \lambda^{t-t_1} V_{RR}(t_1),\ t = t_1, t_1 + 1, \ldots.$$

In order to derive a bound on $V_{RR}(t_1)$ in terms of x_0 in a simple way, on the controller side, we need to wait for all (both) latest transmitted measurements $y_1(s_0),\ y_2(s_1)$ and then send them together to the actuator side. Therefore, the first updating time $t_0 = t_1$ corresponds to the updating time instant when the first data are received by the actuator, which means that $u(t) = 0, t \in [0, t_1 - 1]$. Then for $t \in [0, t_1 - 1]$, (1) is given by

$$x(t+1) = Ax(t), \quad t = 0, 1, \ldots, t_1 - 1, \quad t \in \mathbb{Z}^+. \tag{10}$$

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Denote by $x(t, x_0)$, the state trajectory of (6), (10) with the initial condition $x_0 \in \mathbb{R}^n$. The domain of attraction of the closed-loop system (6), (10) is the set

$$\mathcal{A} = \left\{ x_0 \in \mathbb{R}^n : \lim_{t\to\infty} x(t, x_0) = 0 \right\}.$$

Given K^1, K^2 and positive integers $0 \leqslant \eta_m \leqslant \eta_M < \tau_M$, our objective is to get an estimate $\mathcal{X}_\beta \subset \mathcal{A}$ (as large as we can get) on the domain of attraction, for which exponential stability of the closed-loop system is ensured, where

$$\mathcal{X}_\beta = \left\{ x_0 \in \mathbb{R}^n : x_0^T P x_0 \leqslant \beta^{-1} \right\}, \tag{11}$$

and where $\beta > 0$ is a scalar, $P > 0$ is an $n \times n$ matrix.

By extending the direct Lyapunov approach suggested in [12] for time-delay system to the switched system given by (6), (10), we obtain

Lemma 1

Consider LKF $V_{RR}(t)$ given by (7) and denote $V_0(t) = x^T(t)Px(t)$. Under (8), let there exist $0 < \lambda < 1$ and $\mu > 1$ such that the following inequalities

$$V_0(t+1) - \mu V_0(t) \leqslant 0, \; t = 0, 1, \ldots, t_1 - 1, \tag{12a}$$

$$V_{RR}(t+1) - \lambda V_{RR}(t) - (\mu - 1)V_0(t) \leqslant 0, \; t = 0, 1, \ldots, t_1 - 1. \tag{12b}$$

hold along (10). Then the solutions of (6), (10) at time t_1 satisfy

$$V_{RR}(t_1) \leqslant x_0^T \left[\lambda^{\eta_m} \Xi_{RR} + (\mu^{\tau_M+1} - 1)P \right] x_0, \tag{13}$$

where

$$\Xi_{RR} = P + \eta_m S_0 + \lambda^{\eta_m} (\bar{\tau}_M - \eta_m) S_1. \tag{14}$$

Proof

From (12a), $V_0(t) \leqslant \mu^t V_0(0)$ for $t = 0, 1, \ldots, t_1$. Under the constant initial condition (8) and $V_{RR}(t)$ of (7), we have for $t = 0$

$$V_{RR}(0) = x_0^T P x_0 + \sum\nolimits_{s=-\eta_m}^{-1} \lambda^{-s-1} x^T(s) S_0 x(s) + \sum\nolimits_{s=-\bar{\tau}_M}^{-\eta_m-1} \lambda^{-s-1} x_0^T S_1 x_0.$$

Hence, $V_{RR}(0) \leqslant x_0^T \Xi_{RR} x_0$. Noting that $\eta_m < t_1 \leqslant \tau_M + 1$, (12b) yields (13) because

$$V_{RR}(t_1) \leqslant \lambda^{t_1} V_{RR}(0) + (\mu^{t_1} - 1) x_0^T P x_0 \leqslant x_0^T \left[\lambda^{t_1} \Xi_{RR} + (\mu^{t_1} - 1)P \right] x_0.$$

□

Lemma 1 implies the following result:

Theorem 1

Given scalars $0 < \lambda \leqslant 1, \beta > 0, \mu > 1, \sigma > 0$, positive integers $0 \leqslant \eta_m \leqslant \eta_M < \tau_M$, and K^1, K^2, let there exist scalars $n \times n$ matrices $P > 0$, $S_\vartheta > 0$, $R_\vartheta > 0 (\vartheta = 0, 1)$, G_1^i, G_2^i, G_3^i $(i = 1, 2)$ such that $\eta_m S_0 + \lambda^{\eta_m}(\bar{\tau}_M - \eta_m)S_1 \leqslant \sigma P$ and the following matrix inequalities are feasible:

$$\Omega_i = \begin{bmatrix} R_1 & G_1^i & G_2^i \\ * & R_1 & G_3^i \\ * & * & R_1 \end{bmatrix} \geqslant 0, \tag{15}$$

DOI: 10.1002/rnc

$$\begin{bmatrix} -\mu P & A^T P \\ * & -P \end{bmatrix} < 0, \tag{16}$$

$$\begin{bmatrix} P\rho_{RR}^{-1} & (K^i C^i)_j^T \\ * & \frac{1}{4}\beta \bar{u}_j^2 \end{bmatrix} \geqslant 0, \ j = 1, \dots, n_u, \tag{17}$$

$$F_0^T P F_0 + \Sigma + F_{01}^T H F_{01} - \lambda^{\eta_m} F_{12}^T R_0 F_{12} - \lambda^{\bar{\tau}_M} F^T \Omega_i F < 0, \tag{18}$$

$$\bar{F}_0^T P \bar{F}_0 + \bar{\Sigma} + \bar{F}_{01}^T H \bar{F}_{01} - \lambda^{\eta_m} F_{12}^T R_0 F_{12} - \lambda^{\bar{\tau}_M} F^T \Omega_1 F < 0, \ i = 1, 2, \tag{19}$$

where

$$\begin{aligned}
F_0 &= [A \ 0 \ A_{3-i} \ A_i \ 0], \ F_{01} = [A - I \ 0 \ A_{3-i} \ A_i \ 0], \ i = 1, 2, \\
F_{12} &= [I \ -I \ 0 \ 0 \ 0], \ \bar{F}_0 = [A \ 0 \ 0 \ 0 \ 0], \ \bar{F}_{01} = [A - I \ 0 \ 0 \ 0 \ 0], \\
F &= \begin{bmatrix} 0 & I & -I & 0 & 0 \\ 0 & 0 & I & -I & 0 \\ 0 & 0 & 0 & I & -I \end{bmatrix}, \\
\Sigma &= \text{diag}\left\{S_0 - \lambda P, -\lambda^{\eta_m}(S_0 - S_1), 0, 0, -\lambda^{\bar{\tau}_M} S_1\right\}, \\
\bar{\Sigma} &= \text{diag}\left\{S_0 - \lambda P + (1-\mu)P, -\lambda^{\eta_m}(S_0 - S_1), 0, 0, -\lambda^{\bar{\tau}_M} S_1\right\}, \\
\rho_{RR} &= \lambda^{\eta_m}(1+\sigma) + \mu^{\tau_M + 1} - 1, \\
H &= \eta_m^2 R_0 + (\bar{\tau}_M - \eta_m)^2 R_1.
\end{aligned} \tag{20}$$

Then, for all initial conditions x_0 belonging to $\mathcal{X}_\beta$, the closed-loop system (6), (10) is exponentially stable.

Proof

Suppose that $x(t) \in \mathcal{L}\left(K^1, \bar{u}\right) \cap \mathcal{L}\left(K^2, \bar{u}\right)$. Following [9], we take advantage of the ordered delays and use convex analysis of [14]. We show that (16) and (15), (19) guarantee, respectively, (12a) and (12b) for $t = 0, \dots, t_1 - 1$; (15) and (18) guarantee (9) for $t = t_1, t_1 + 1, \dots$.

Noting that (9), (13), and $\eta_m S_0 + \lambda^{\eta_m}(\bar{\tau}_M - \eta_m) S_1 \leqslant \sigma P$, we arrive at

$$\begin{aligned}
x^T(t) P x(t) \leqslant V_{RR}(t) &\leqslant \lambda^{t - t_1} V_{RR}(t_1) \\
&\leqslant x_0^T [\lambda^{\eta_m} \Xi_{RR} + (\mu^{\tau_M + 1} - 1) P] x_0 \\
&\leqslant x_0^T [\lambda^{\eta_m}(P + \sigma P) + (\mu^{\tau_M + 1} - 1) P] x_0 \\
&= \rho_{RR} x_0^T P x_0, \ \ t = t_1, t_1 + 1, \dots .
\end{aligned}$$

So for all $x(t) : x^T(t) P x(t) \leqslant \rho_{RR} \beta^{-1} \Rightarrow x^T(t)(K^i C^i)_j^T (K^i C^i)_j x(t) \leqslant \frac{1}{4}\bar{u}_j^2$, if

$$x^T(t)(K^i C^i)_j^T (K^i C^i)_j x(t) \leqslant \frac{1}{4}\beta \rho_{RR}^{-1} x^T(t) P x(t) \bar{u}_j^2, \ j = 1, \dots, n_u, \ i = 1, 2.$$

The latter inequality is guaranteed if $\frac{1}{4}\beta\rho_{RR}^{-1} P \bar{u}_j^2 - (K^i C^i)_j^T (K^i C^i)_j \geqslant 0$, and, thus, by Schur complements if (17) is feasible. Hence, the trajectories of system (6), (10) converge to the origin exponentially, provided that $x_0 \in \mathcal{X}_\beta$. □

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3. DISCRETE-TIME NCSS WITH ACTUATOR SATURATION UNDER A WEIGHTED TOD PROTOCOL

3.1. Problem formulation and a hybrid time-delay model

In [10], a weighted TOD protocol was analyzed for the stabilization of continuous-time NCSs in the framework of time-delay approach. Actuator saturation was not taken into account. In this section, we consider discrete-time NCSs with actuator saturation under the weighted TOD protocol.

Consider (1) with two sensor nodes $y^i(t) = C^i x(t), i = 1,2$ under the saturated control input (2). Consider a sequence of sampling instants (3). At each sampling instant s_k, one of the outputs $y^i(t) \in \mathbb{R}^{n_i}(n_1+n_2 = n_y)$ is sampled and transmitted via network. Denote by $\hat{y}(s_k) = \begin{bmatrix} \hat{y}^1(s_k) \\ \hat{y}^2(s_k) \end{bmatrix} \in \mathbb{R}^{n_y}$ the output information submitted to the scheduling protocol. At each sampling instant s_k, one of $\hat{y}^i(s_k)$ values is updated with the recent output $y^i(s_k)$.

It is assumed that no packet dropouts and no packet disorders will happen during the data transmission over the network. The transmission of the information over the two networks (between the sensor and the actuator) is subject to a variable delay η_k. Then $t_k = s_k + \eta_k$ is the updating time instant. As in the previous section, we allow the delays to be non-small (larger than the sampling intervals) provided that the old sample cannot get to the destination (to the controller or to the actuator) after the current one. Assume that the network-induced delay η_k and the time span between the updating and the current sampling instants satisfy (4).

The choice of the active output node is ruled by a weighted TOD protocol. Following [10], consider the error between the system output $y(s_k)$ and the last available information $\hat{y}(s_{k-1})$

$$\begin{aligned} e(t) &= col\{e_1(t), e_2(t)\} \equiv \hat{y}(s_{k-1}) - y(s_k),\ t \in [t_k, t_{k+1}-1], \\ & t \in \mathbb{Z}^+, k \in \mathbb{Z}^+,\ \hat{y}(s_{-1}) \triangleq 0,\ e(t) \in \mathbb{R}^{n_y}. \end{aligned}$$

Let $Q_i > 0, i = 1,2$ be some weighting matrices (they will be found from matrix inequalities in Lemma 2). The node that has the largest error, $\left|\sqrt{Q}_i e_i(t)\right|^2, i = 1,2$ is granted access to the network.

Let

$$i_k^* = \min\left\{\arg\max_{i\in\{1,2\}}\left|\sqrt{Q}_i\left(\hat{y}^i(s_{k-1}) - y^i(s_k)\right)\right|^2\right\} \tag{21}$$

be the index of the active output node at the sampling instant s_k. This can be rewritten as

$$\left|\sqrt{Q}_{i_k^*} e_{i_k^*}(t)\right|^2 \geqslant \left|\sqrt{Q}_i e_i(t)_{|i\neq i_k^*}\right|^2,\ t \in [t_k, t_{k+1}-1],\ t \in \mathbb{Z}^+. \tag{22}$$

Because of the control bounds defined in (2), the effective control signal to be applied to system (1) is given by

$$u(t) = sat\left[K^{i_k^*} y^{i_k^*}(t_k - \eta_k) + K^i \hat{y}^i(t_{k-1} - \eta_{k-1})_{|i\neq i_k^*}\right],\ t \in [t_k, t_{k+1}-1],\ t \in \mathbb{Z}^+.$$

If the control is such that $x(t) \in \mathcal{L}\left(K^1, \bar{u}\right) \cap \mathcal{L}(K^2, \bar{u})$, then $\left|(K^1C^1)_i x + (K^2C^2)_i x(t)\right| \leqslant \bar{u}_i$. The closed-loop system can be presented as

$$\begin{aligned} x(t+1) &= Ax(t) + A_1 x(t_k - \eta_k) + B_i e_i(t)_{|i\neq i_k^*}, \\ e(t+1) &= e(t),\ t \in [t_k, t_{k+1}-2],\ t \in \mathbb{Z}^+, \end{aligned} \tag{23}$$

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with the delayed reset system for $t = t_{k+1} - 1$

$$
\begin{aligned}
x(t_{k+1}) &= Ax(t_{k+1}-1) + A_1 x(t_k - \eta_k) + B_i e_i(t_k)_{|i \neq i_k^*}, \\
e_i(t_{k+1}) &= C^i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})],\ i = i_k^*, \\
e_i(t_{k+1}) &= e_i(t_k) + C^i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})],\ i \neq i_k^*,
\end{aligned} \tag{24}
$$

where $A_1 = BKC,\ B_i = BK^i,\ K = [K^1\ K^2],\ i = 1, 2$. The initial condition for (22)–(24) has the form of $e(t_0) = -Cx(t_0 - \eta_0) = -Cx_0$ and

$$
x(t+1) = Ax(t),\ t = 0, 1, \ldots, t_0 - 1,\ \ t \in \mathbb{Z}^+. \tag{25}
$$

Definition 1
Hybrid systems (22)–(24) are said to be partially exponentially stable with respect to x if there exist constants $b > 0, 0 < \kappa < 1$ such that the following holds

$$
|x(t)|^2 \leqslant b\kappa^{t-t_0}\left[|x_0|^2 + |e(t_0)|^2\right], \quad t \geqslant t_0
$$

for the solutions of the hybrid system initialized with (25) and $e(t_0) \in \mathbb{R}^{n_y}$.

Given K^1, K^2 and positive integers $0 \leqslant \eta_m \leqslant \eta_M < \tau_M$, our objective is to obtain an estimate $\mathcal{X}_\beta \subset \mathcal{A}$ (as large as we can get) on the domain of attraction, for which exponential stability of the closed-loop systems (22)–(24) with respect to variable of interest x is ensured, where $\mathcal{X}_\beta$ is given by (11). In [15], the notion of partial stability was also used.

3.2. Partial exponential stability of the hybrid delayed system without actuator saturation

Consider the LKF of the form

$$
\begin{aligned}
V_e(t) &= V_{TOD}(t) + \frac{t_{k+1} - t}{\tau_M - \eta_m + 1} e_i^T(t_k) Q_i e_i(t_k)_{|i \neq i_k^*}, \\
V_{TOD}(t) &= \tilde{V}(t) + V_Q(t), \\
V_Q(t) &= (\tau_M - \eta_m) \sum_{s=t_k-\eta_k}^{t-1} \lambda^{t-s-1} \eta^T(s) Q \eta(s), \\
\tilde{V}(t) &= x^T(t) P x(t) + \sum_{s=t-\eta_m}^{t-1} \lambda^{t-s-1} x^T(s) S_0 x(s) + \sum_{s=t-\tau_M}^{t-\eta_m-1} \lambda^{t-s-1} x^T(s) S_1 x(s) \\
&\quad + \eta_m \sum_{j=-\eta_m}^{-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_0 \eta(s) \\
&\quad + (\tau_M - \eta_m) \sum_{j=-\tau_M}^{-\eta_m-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_1 \eta(s), \\
&\quad P > 0, S_i > 0, R_i > 0, Q > 0, Q_j > 0, 0 < \lambda < 1, i = 0, 1, j = 1, 2, \\
&\quad t \in [t_k, t_{k+1} - 1], t \in \mathbb{Z}^+, k \in \mathbb{Z}^+,
\end{aligned} \tag{26}
$$

where we define $x(t) = x_0,\ t \leqslant 0$.

Our objective is to guarantee that

$$
V_e(t+1) - \lambda V_e(t) \leqslant 0,\ t \in [t_k, t_{k+1} - 1],\ t \in \mathbb{Z}^+ \tag{27}
$$

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holds along (22)–(24). The inequality (27) implies the following bound

$$V_{TOD}(t) \leqslant V_e(t) \leqslant \lambda^{t-t_0} V_e(t_0),\ t \geqslant t_0,\ t \in \mathbb{Z}^+, \\ V_e(t_0) \leqslant V_{TOD}(t_0) + \min_{i=1,2}\left\{\left|\sqrt{Q_i}e_i(t_0)\right|^2\right\} \tag{28}$$

for the solution of (22)–(24) with the initial condition (25) and $e(t_0) \in \mathbb{R}^{n_y}$. Here, we took into account that for the case of two sensor nodes

$$\left|\sqrt{Q_i}e_i(t_0)\right|^2_{|i\neq i_k^*} = \min_{i=1,2}\left\{\left|\sqrt{Q_i}e_i(t_0)\right|^2\right\}.$$

From (28), it follows that systems (22)–(24) is exponentially stable with respect to x.

The novel term $V_Q(t)$ of LKF is inserted to cope with the delays in the reset conditions

$$\begin{aligned} &V_Q(t_{k+1}) - \lambda V_Q(t_{k+1}-1) \\ &= (\tau_M - \eta_m)\left[\sum_{s=t_{k+1}-\eta_{k+1}}^{t_{k+1}-1} \lambda^{t_{k+1}-s-1}\eta^T(s)Q\eta(s) - \sum_{s=t_k-\eta_k}^{t_{k+1}-2} \lambda^{t_{k+1}-s-1}\eta^T(s)Q\eta(s)\right] \\ &\leqslant (\tau_M - \eta_m)\eta^T(t_{k+1}-1)Q\eta(t_{k+1}-1) - (\tau_M - \eta_m)\lambda^{\tau_M} \sum_{s=t_k-\eta_k}^{t_{k+1}-\eta_{k+1}-1} \eta^T(s)Q\eta(s) \\ &\leqslant (\tau_M - \eta_m)\eta^T(t_{k+1}-1)Q\eta(t_{k+1}-1) - \lambda^{\tau_M}\left|\sqrt{Q}[x(t_{k+1}-\eta_{k+1}) - x(t_k-\eta_k)]\right|^2, \end{aligned} \tag{29}$$

where we applied Cauchy–Schwartz inequality (see e.g., [16]). The term $\frac{t_{k+1}-t}{\tau_M-\eta_m+1}e_i^T(t_k)Q_ie_i(t_k)$ is inspired by the similar construction of LKF for the sampled-data systems [6].

We have

$$\begin{aligned} V_e(t_{k+1}) - \lambda V_e(t_{k+1}-1) &= \tilde V(t_{k+1}) - \lambda \tilde V(t_{k+1}-1) \\ &+ \tfrac{t_{k+2}-t_{k+1}}{\tau_M-\eta_m+1}e_i^T(t_{k+1})Q_ie_i(t_{k+1})_{|i\neq i_{k+1}^*} - \tfrac{\lambda}{\tau_M-\eta_m+1}e_i^T(t_k)Q_ie_i(t_k)_{|i\neq i_k^*} \\ &+ (\tau_M-\eta_m)\eta^T(t_{k+1}-1)Q\eta(t_{k+1}-1) - \lambda^{\tau_M}\left|\sqrt{Q}[x(t_{k+1}-\eta_{k+1}) - x(t_k-\eta_k)]\right|^2. \end{aligned} \tag{30}$$

Note that under TOD protocol for $i_{k+1}^* = i_k^*$,

$$e_i^T(t_{k+1})Q_ie_i(t_{k+1})_{|i\neq i_{k+1}^*} \leqslant e_{i_k^*}^T(t_{k+1})Q_{i_k^*}e_{i_k^*}(t_{k+1}),$$

whereas for $i_{k+1}^* \neq i_k^*$, the latter relation holds with the equality. Hence,

$$\begin{aligned} \frac{t_{k+2}-t_{k+1}}{\tau_M-\eta_m+1}e_i^T(t_{k+1})Q_ie_i(t_{k+1})_{|i\neq i_{k+1}^*} &\leqslant e_{i_k^*}^T(t_{k+1})Q_ie_{i_k^*}(t_{k+1}) \\ &= |\sqrt{Q_{i_k^*}}C_{i_k^*}[x(t_{k+1}-\eta_{k+1}) - x(t_k-\eta_k)]|^2. \end{aligned}$$

Assume that

$$\lambda^{\tau_M}Q > C^{iT}Q_iC^i,\ i = 1, 2. \tag{31}$$

Then for $t = t_{k+1}-1$, we obtain

$$\begin{aligned} V_e(t+1) - \lambda V_e(t) \leqslant\ & \tilde V(t+1) - \lambda \tilde V(t) + (\tau_M-\eta_m)\eta^T(t)Q\eta(t) \\ & - \tfrac{\lambda}{\tau_M-\eta_m+1}e_i^T(t_k)Q_ie_i(t_k)_{|i\neq i_k^*}. \end{aligned}$$

Furthermore, because of

$$-\frac{\lambda}{\tau_M-\eta_m+1}=-\frac{1}{\tau_M-\eta_m+1}+\frac{1-\lambda}{\tau_M-\eta_m+1}\leqslant-\frac{1}{\tau_M-\eta_m+1}+1-\lambda,$$

for $t=t_{k+1}-1$, we arrive at

$$\begin{aligned}V_e(t+1)-\lambda V_e(t)\leqslant\ &\tilde{V}(t+1)-\lambda\tilde{V}(t)+(\tau_M-\eta_m)\eta^T(t)Q\eta(t)\\&-\left[\tfrac{1}{\tau_M-\eta_m+1}-(1-\lambda)\right]e_i^T(t_k)Q_ie_i(t_k)_{|i\neq i_k^*}\triangleq\Psi(t).\end{aligned}\tag{32}$$

For $t\in[t_k,t_{k+1}-2]$, we have

$$\begin{aligned}V_e(t+1)-\lambda V_e(t)\leqslant\ &\tilde{V}(t+1)-\lambda\tilde{V}(t)+(\tau_M-\eta_m)\eta^T(t)Q\eta(t)\\&+\left[\tfrac{t_{k+1}-t-1}{\tau_M-\eta_m+1}-\lambda\tfrac{t_{k+1}-t}{\tau_M-\eta_m+1}\right]e_i^T(t_k)Q_ie_i(t_k)_{|i\neq i_k^*}.\end{aligned}$$

Because

$$\tfrac{t_{k+1}-t-1}{\tau_M-\eta_m+1}-\lambda\tfrac{t_{k+1}-t}{\tau_M-\eta_m+1}=-\tfrac{1}{\tau_M-\eta_m+1}+(1-\lambda)\tfrac{t_{k+1}-t}{\tau_M-\eta_m+1}\leqslant-\tfrac{1}{\tau_M-\eta_m+1}+1-\lambda,$$

we conclude that (32) is valid also for $t\in[t_k,t_{k+1}-2]$. Therefore, (27) holds if

$$\Psi(t)\leqslant 0,\quad t\in[t_k,t_{k+1}-1].\tag{33}$$

Note that $i\neq i_k^*$ for $i=1,2$ is the same as $i=3-i_k^*$. By using the standard arguments for the delay-dependent analysis [14], we derive the following conditions for (33) (and, thus for (28)):

Lemma 2
Given scalar $0<\lambda<1$, positive integers $0\leqslant\eta_m\leqslant\eta_M<\tau_M$, and K^1,K^2, if there exist $n\times n$ matrices $P>0,Q>0,S_j>0,R_j>0(j=0,1),S_{12},n_i\times n_i$ matrices $Q_i>0(i=1,2)$ such that (31) and

$$\hat{\Omega}=\begin{bmatrix}R_1 & S_{12}\\ * & R_1\end{bmatrix}\geqslant 0,\tag{34}$$

$$\hat{F}_0^TP\hat{F}_0+\hat{\Sigma}+\hat{F}_{01}^TW\hat{F}_{01}-\lambda^{\eta_m}F_{12}^TR_0F_{12}-\lambda^{\tau_M}\hat{F}^T\hat{\Omega}\hat{F}<0,\tag{35}$$

are feasible, where

$$\begin{aligned}&\hat{F}_0=[A\ 0\ A_1\ 0\ B_{3-i}],\ \hat{F}_{01}=[A-I\ 0\ A_1\ 0\ B_{3-i}],\\&\hat{F}=\begin{bmatrix}0 & I & -I & 0 & 0\\ 0 & 0 & I & -I & 0\end{bmatrix},\\&\hat{\Sigma}=\text{diag}\left\{S_0-\lambda P,-\lambda^{\eta_m}(S_0-S_1),0,-\lambda^{\tau_M}S_1,\varphi\right\},\\&\varphi=-\left[\frac{1}{\tau_M-\eta_m+1}-(1-\lambda)\right]Q_{3-i},\\&W=\eta_m^2R_0+(\tau_M-\eta_m)^2R_1+(\tau_M-\eta_m)Q,\ \ i=1,2.\end{aligned}\tag{36}$$

Then solutions of the hybrid systems (22)–(24) satisfy the bound (28) and are exponentially stable with respect to x. Moreover, if the aforementioned inequalities are feasible with $\lambda=1$, then the bound (28) holds with $\lambda=1-\varepsilon$, where $\varepsilon>0$ is small enough.

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Remark 1
The inequality $V_e(t) \leqslant \lambda^{t-t_0} V_e(t_0), t \geqslant t_0, t \in \mathbb{Z}^+$ in (28) guarantees that

$$\frac{t_{k+1} - t_k}{\tau_M - \eta_m + 1} e_i^T(t_k) Q_i e_i(t_k)|_{i \neq i_k^*}$$

is bounded, and it does not guarantee that $e(t_k)$ is bounded. That is why (28) implies only partial stability with respect to x.

3.3. Partial exponential stability of the hybrid delayed system with actuator saturation

Our objective is to derive the bound on $V_{TOD}(t_0)$ in terms of x_0. By using arguments similar to Lemma 1, we arrive at the following.

Lemma 3
Consider LKF $V_{TOD}(t)$ given by (26) and denote $V_0(t) = x^T(t)Px(t)$. Under (8), let there exist $0 < \lambda < 1$ and $\mu > 1$ such that

$$V_0(t+1) - \mu V_0(t) \leqslant 0, \ t = 0, 1, \ldots, t_0 - 1, \tag{37a}$$

$$V_{TOD}(t+1) - \lambda V_{TOD}(t) - (\mu - 1)V_0(t) \leqslant 0, \ t = 0, 1, \ldots, t_0 - 1, \tag{37b}$$

hold along (25). Then we have

$$V_{TOD}(t_0) \leqslant x_0^T \left[\lambda^{\eta_m} \Xi_{TOD} + (\mu^{\eta_M} - 1)P\right] x_0, \tag{38}$$

where

$$\Xi_{TOD} = P + \eta_m S_0 + \lambda^{\eta_m}(\tau_M - \eta_m)S_1. \tag{39}$$

Proof
From (37a), $V_0(t) \leqslant \mu^t V_0(0)$ for $t = 0, 1, \ldots, t_0$. Under the constant initial condition (8) and $V_{TOD}(t)$ of (26), we have for $t = 0$

$$V_{TOD}(0) = x_0^T P x_0 + \sum\nolimits_{s=-\eta_m}^{-1} \lambda^{-s-1} x_0^T S_0 x_0 + \sum\nolimits_{s=-\tau_M}^{-\eta_m - 1} \lambda^{-s-1} x_0^T S_1 x_0.$$

Thus, $V_{TOD}(0) \leqslant x_0^T \Xi_{TOD} x_0$. Noting that $\eta_m \leqslant t_0 = \eta_0 \leqslant \eta_M$, the inequality (37b) implies

$$\begin{aligned} V_{TOD}(t_0) \leqslant \lambda^{t_0} V_{TOD}(0) + (\mu^{t_0} - 1) x_0^T P x_0 &\leqslant x_0^T \left[\lambda^{t_0} \Xi_{TOD} + (\mu^{t_0} - 1)P\right] x_0 \\ &\leqslant x_0^T \left[\lambda^{\eta_m} \Xi_{TOD} + (\mu^{\eta_M} - 1)P\right] x_0. \end{aligned} \tag{40}$$

□

Lemmas 2 and 3 imply the following result:

Theorem 2
Given scalars $0 < \lambda < 1, \beta > 0, \mu > 1, \sigma > 0, \bar{\sigma} > 0$, positive integers $0 \leqslant \eta_m \leqslant \eta_M < \tau_M$, and K^1, K^2, if there exist $n \times n$ matrices $P > 0, Q > 0, S_l > 0, R_l > 0 (l = 0, 1), S_{12}, n_i \times n_i$ matrices $Q_i > 0 (i = 1, 2)$ such that (16), (31), (34), (35), and

$$\eta_m S_0 + \lambda^{\eta_m}(\tau_M - \eta_m)S_1 \leqslant \sigma P, \tag{41}$$

$$\lambda^{\tau_M} Q \leqslant \bar{\sigma} P, \tag{42}$$

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$$\begin{bmatrix} P\rho_{TOD}^{-1} & (K^iC^i)_j^T \\ * & \frac{1}{4}\beta\bar{u}_j^2 \end{bmatrix} \geqslant 0, \; j = 1, \ldots, n_u, \tag{43}$$

$$\tilde{F}_0^T P \tilde{F}_0 + \tilde{\Sigma} + \tilde{F}_{01}^T W \tilde{F}_{01} - \lambda^{\eta_m} \tilde{F}_{12}^T R_0 \tilde{F}_{12} - \lambda^{\tau_M} \tilde{F}^T \hat{\Omega} \tilde{F} < 0, \tag{44}$$

are feasible, where notations W and $\hat{\Omega}$ are given by (36) and (34), respectively, and where

$$\begin{aligned} \tilde{F}_0 &= [A \; 0 \; 0 \; 0], \; \tilde{F}_{01} = [A - I \; 0 \; 0 \; 0], \; \tilde{F}_{12} = [I \; -I \; 0 \; 0], \\ \tilde{F} &= \begin{bmatrix} 0 & I & -I & 0 \\ 0 & 0 & I & -I \end{bmatrix}, \\ \tilde{\Sigma} &= \text{diag}\{S_0 - \lambda P + (1-\mu)P, -\lambda^{\eta_m}(S_0 - S_1), 0, -\lambda^{\tau_M} S_1\}, \\ \rho_{TOD} &= \lambda^{\eta_m}(1+\sigma) + (\mu^{\eta_M} - 1) + \bar{\sigma}, \; i = 1, 2. \end{aligned} \tag{45}$$

Then, for all initial conditions x_0 belonging to $\mathcal{X}_\beta$, the closed-loop systems (22)–(24) are exponentially stable with respect to x. Moreover, if the aforementioned inequalities hold with $\lambda = 1$, then they are feasible for $\lambda = 1 - \varepsilon$, where $\varepsilon > 0$ is small enough.

Proof

Suppose that $x(t) \in \mathcal{L}(K^1, \bar{u}) \cap \mathcal{L}(K^2, \bar{u})$. As shown in Lemma 2, (31), (34), and (35) lead to (28) for $t = t_0, t_0 + 1, \ldots$. Following the proof of Theorem 1, for $t = 0, 1, \ldots, t_0 - 1$, (16) and (44) with (34) guarantee (37a) and (37b), respectively.

Next, noting that (31) and (42), we have

$$x_0^T C^{iT} Q_i C^i x_0 < \lambda^{\tau_M} x_0^T Q x_0 \leqslant \bar{\sigma} x_0^T P x_0, \tag{46}$$

which implies that $\left|\sqrt{Q_i} e_i(t_0)\right|^2 = \left|-\sqrt{Q_i} C^i x_0\right|^2 < \bar{\sigma} x_0^T P x_0, \; i = 1, 2.$

Therefore, taking into (28), (38), (41), and (46), we obtain

$$\begin{aligned} x^T(t) P x(t) &\leqslant V_{TOD}(t) \leqslant \lambda^{t-t_0} V_e(t_0) \\ &\leqslant \lambda^{t-t_0} \left\{V_{TOD}(t_0) + \min_{i=1,2}\left\{e_i^T(t_0) Q_i e_i(t_0)\right\}\right\} \\ &\leqslant x_0^T \left[\lambda^{\eta_m} \Xi_{TOD} + (\mu^{\eta_M} - 1)P + \bar{\sigma} P\right] x_0 \\ &\leqslant x_0^T \left[\lambda^{\eta_m}(1+\sigma) + (\mu^{\eta_M} - 1) + \bar{\sigma}\right] P x_0 \\ &= \rho_{TOD} x_0^T P x_0, \quad t = t_0, t_0 + 1, \ldots. \end{aligned}$$

So for all $x(t) : x^T(t) P x(t) \leqslant \rho_{TOD}\beta^{-1} \Rightarrow x^T(t)(K^iC^i)_j^T (K^iC^i)_j x(t) \leqslant \frac{1}{4}\bar{u}_j^2$, if

$$x^T(t)(K^iC^i)_j^T(K^iC^i)_j x(t) \leqslant \frac{1}{4}\beta\rho_{TOD}^{-1} x^T(t) P x(t) \bar{u}_j^2, \; j = 1, \ldots, n_u, \; i = 1, 2.$$

The latter inequality is guaranteed if $\frac{1}{4}\beta\rho_{TOD}^{-1} P \bar{u}_j^2 - (K^iC^i)_j^T (K^iC^i)_j \geqslant 0$, and, thus, by Schur complements if (43) is feasible. Hence, solutions of the hybrid systems (22)–(24) initialized with (25) and $e(t_0) \in \mathbb{R}^{n_y}$ converge to the origin exponentially with respect to x, provided that $x_0 \in \mathcal{X}_\beta$. □

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Remark 2
Note that for the stability analysis of discrete-time systems with time-varying delay in the state, a switched system transformation approach can be used in addition to Lyapunov–Krasovskii method. See more details in [17].

Remark 3
Note that

$$x_0^T P x_0 \leqslant \lambda_{max}(P)|x_0|^2 < \beta^{-1}, \tag{47}$$

where $\lambda_{max}(P)$ denotes the largest eigenvalue of P. Hence, the following initial region $|x_0|^2 < \beta^{-1}/\lambda_{max}(P)$ is inside of $\mathcal{X}_\beta$. In order to maximize the initial ball, we can add the condition

$$P - \gamma I < 0, \tag{48}$$

to Theorems 1 and 2, where $\gamma > 0$ is minimized.

4. EXAMPLE: DISCRETE-TIME CART–PENDULUM

Consider the following linearized model of the inverted pendulum on a cart [9]:

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{a}{M} \\ 0 \\ \frac{-a}{Ml} \end{bmatrix} u(t), \ t \in \mathbb{R}^+ \tag{49}$$

with $M = 3.9249$ Kg, $m = 0.2047$ Kg, $l = 0.2302$ m, $g = 9.81$ N/Kg, $a = 25.3$ N/V, and $\bar{u} = 50$. In the model, x and θ represent cart position coordinate and pendulum angle from vertical, respectively. Such a model is discretized with a sampling time $\mathrm{T}_s = 0.001s$

$$\begin{bmatrix} x(t+1) \\ \Delta x(t+1) \\ \theta(t+1) \\ \Delta\theta(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.001 & 0 & 0 \\ 0 & 1 & -0.0005 & 0 \\ 0 & 0 & 1.00 & 0.001 \\ 0 & 0 & 0.0448 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ \Delta x(t) \\ \theta(t) \\ \Delta\theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0064 \\ 0 \\ -0.0280 \end{bmatrix} u(t), \ t \in \mathbb{Z}^+ \tag{50}$$

with $\bar{u} = 50$. The pendulum can be stabilized by a state feedback $u(t) = K \begin{bmatrix} x & \Delta x & \theta & \Delta\theta \end{bmatrix}^T$ with the gain $K = \begin{bmatrix} K^1 & K^2 \end{bmatrix}$

$$K^1 = \begin{bmatrix} 5.825 & 5.883 \end{bmatrix}, \quad K^2 = \begin{bmatrix} 24.941 & 5.140 \end{bmatrix}, \tag{51}$$

which leads to the closed-loop system eigenvalues $\{0.8997, 0.9980 + 0.0020i, 0.9980 - 0.0020i, 0.9980\}$. Suppose the variables $\theta, \Delta\theta$ and $x, \Delta x$ are not accessible simultaneously. We consider measurements $y^i(t) = C^i x(t), t \in \mathbb{Z}^+$, where

$$C^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{52}$$

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Table I. Largest ball of admissible initial conditions for different μ.

μ	1.02	1.2	1.5	2	3
$\|x_0\|$(Theorem 1, RR)	0.4964	0.3557	0.2304	0.1297	0.1339
$\|x_0\|$(Theorem 2,TOD)	0.3342	0.3069	0.2674	0.2168	0.1541

RR, round-robin; TOD, try-once-discard.

Choose $\lambda = 1,\ \beta = 1,\ \sigma = 1.0 \times 10^{-2}$ and $\eta_m = 1,\ \eta_M = 2,\ \tau_M = 3$. Applying Theorem 2 with $\mu = 1.02, \bar{\sigma} = 1.1$ and Remark 3, the closed-loop systems (22)–(24) are exponentially stable with respect to x starting from the initial ball $|x_0| < 0.3342$ by the presented TOD protocol. Applying Theorem 1 with $\mu = 1.02$ and Remark 3, the closed-loop systems (6), (10) are asymptotically stable and the largest ball of admissible initial conditions is $|x_0| < 0.4964$.

Then for different μ, by Theorems 1, 2 with $\lambda = 1$ and Remark 3, we give the corresponding largest ball of admissible initial conditions (see Table I).

5. CONCLUSIONS

In this paper, a time-delay approach was developed for the stability analysis of discrete-time NCSs in the presence of actuator saturation under the RR or under a weighted TOD scheduling. A Lyapunov-based method was presented for finding the domain of attraction under both scheduling protocols. The conditions are given in terms of LMIs. Polytopic uncertainties in the system model can be easily included in the analysis. Numerical example illustrates the efficiency of our method.

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