# Discrete-Time Process Algebra with Empty Process

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**Abstract**

We introduce an ACP-style discrete-time process algebra with relative timing, that features the empty process. Extensions to this algebra are described, and ample attention is paid to the considerations and problems that arise when introducing the empty process. We prove time determinacy, soundness, completeness, and the axioms of standard concurrency.

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# Contents

# 1 Introduction

## 1.1 Motivation

One of the main features of ACP-style algebraic process theory [12, 13, 14] is its modularity: it is relatively easy to add features for specific purposes, and, given two of such extensions, it is often straightforward to combine them into one theory.

In the past, the feature of the *empty process*, the process that does nothing, and terminates successfully (in contrast with the *deadlocked process*, which does nothing, and terminates unsuccessfully), has been studied extensively [9, 19, 30, 31]. Recently, there has been described an elegant and coherent way to incorporate *discrete-time extensions* into the ACP-framework [5, 7, 8, 11, 26], and a case study has been made [15].

Given the fact that both extensions, empty process and discrete time, have been researched very well, it seemed reasonable to study how these two extensions go together. This is especially important, as the empty process does have great advantages in the specification of protocols. For example, a stack over a infinite data type can only be specified in finitely many equations (three, to be exact), when we have the empty process at our disposal [19]. Furthermore, the empty process is needed to give a process-algebra based semantics to the specification languages SDL and MSC, and hence a timed empty process is needed to give such a semantics to timed variants of these languages.

But, as introduction of the empty process has historically often been viewed as troublesome, the modular combination of it with other features has hardly been studied. In this paper, we remedy this situation for discrete-time process algebra with relative timing.

As far as related work is concerned: most other discrete-time process algebras, such as TCCS [17, 20], ATP [21], have only action prefixing, not general sequential composition, and only one mode of termination. Thus, successful and unsuccessful termination cannot be distinguished. Timed CSP does have general sequential composition and two modes of termination: it has the process $\mathsf{SKIP}$ for successful termination (cf. our delayable process $\varepsilon$) and $\mathsf{STOP}$ for unsuccessful termination (cf. our delayable process $\delta$). For an operational semantics of timed CSP, see [27]. Differences with this paper are that only failures semantics is considered, not bisimulation semantics, and, as a consequence, successful termination can be modeled as an internal step, avoiding all the difficulties that we encounter. Given the timed empty process described here, it becomes easier to compare timed CSP with timed ACP, and furthermore, it allows for a bisimulation semantics to be given to timed CSP.

## 1.2 Discrete-time Process Algebra with Relative Timing

For readers not familiar with ACP-style process algebra, we will first give the intuitive meaning behind the principles of discrete-time process algebra with relative timing.

First, we assume time to be divided into *slices*, one for every natural number. A process that is started in a certain time-slice, can do zero or more actions in that time-slice, move on to the following time-slice, repeat these steps, and eventually terminate. This is why we use the term *discrete time*; the observation of time (as far as our theory is concerned) is not continuous, but discrete. The opposite notion, where actions are indexed by real numbers, is called *real time* [3].

There is a finite alphabet $A$ whose symbols will stand for actions. For example, we could choose $A = \{\texttt{insert-money}, \texttt{push-button}, \texttt{pour-coffee}\}$ if we wanted to describe a coffee-machine. For every symbol $a \in A$, we have the *undelayable action* $\underline{\underline{a}}$ (also denoted $\mathrm{cts}(a)$, see [26]). The process $\underline{\underline{a}}$ intuitively means: "execute $a$ in the current time-slice". So, if the process $\underline{\underline{a}}$ is started in time-slice 3, it does an $a$ in time-slice 3, if it is started in slice 4, the $a$ executes in slice 4. This is why we use the term *relative timing*; a process executes its actions relative to the time at which the process is started. The opposite notion, where we explicitly indicate at what moment an action should execute, irrespective of the time it is started, is called *absolute timing* [7, 8].

As elementary operators, we have the *alternative composition*, also called choice, notation $x + y$, and the sequential composition, notation $x \cdot y$. The process $x + y$ can either perform the process $x$, or the process $y$. The choice is deterministic in the sense that it may be influenced by the outside world, but if both arguments start with the same action, the choice becomes non-deterministic. The process $x \cdot y$ first performs the process $x$, and if and when $x$ terminates successfully, continues by performing process $y$.

We have a special constant *undelayable deadlock*, notation $\underline{\underline{\delta}}$, that stands for the process that cannot do anything. In particular, it cannot even terminate successfully, so $\underline{\underline{\delta}} \cdot x$ is equal to $\underline{\underline{\delta}}$.

Then, we have the *time-unit delay* operator, notation $\sigma(x)$ (also notated $\sigma_{\mathrm{rel}}(x)$, see Remark 2.1.3 on page 8), that allows a process to move on to the next time-slice. So, the process $\sigma(\underline{\underline{a}})$, when initialized in time-slice 5, will execute an $a$ in time-slice 6. An important concept we use is the concept of *time factorization* [21], which enforces that moving into the next time-slice does not determine a choice (unless it has to). So, when we initialize the process $\sigma(\underline{\underline{a}}) + \sigma(\underline{\underline{b}})$ in the first time-slice, the choice between $\underline{\underline{a}}$ and $\underline{\underline{b}}$ is still open after the process has moved into the second time-slice, as both $\sigma(\underline{\underline{a}})$ and $\sigma(\underline{\underline{b}})$ can move there. However, when we initialize the process $\underline{\underline{a}} + \sigma(\underline{\underline{b}})$ in the first time-slice, the $\underline{\underline{a}}$ will be lost when move into the second time-slice, leaving only $\underline{\underline{b}}$, as $\underline{\underline{a}}$ cannot move beyond the first time-slice, while $\sigma(\underline{\underline{b}})$ can.

Finally, we have the "*now*" operator, notation $\nu(x)$ (also notated $\nu_{\mathrm{rel}}(x)$, see Remark 2.1.3 on page 8), that restricts a process to the parts of it that can start in the current time-slice. In other words, $\nu(x)$ is just like $x$, except that it must first do an action before it may move on to the next time-slice. So, $\nu(\underline{\underline{a}} + \sigma(\underline{\underline{b}}))$ must execute its $a$; it can never do a $b$. The process $\nu(\underline{\underline{a}} \cdot \sigma(\underline{\underline{b}}))$, however, is not prohibited from executing its $b$ (in the second time-slice), since it has already done an $a$ in the first time-slice.

## 1.3 Design goals for the empty process in discrete-time process algebra

Often, when extending an already existing process algebra with some additional constant or operator, one has a lot of "maneuvering room" to make choices regarding the exact implementation and the fine details of its behavior. For example, a very broad class of "deadlock-like" processes has been described in the literature. To name a few variants (in order of increasing degree of catastrophe): "classic" ($\delta$, see [13]), "immediate" ($\dot{\delta}$, see [7, 8]), "should not be here" ($\perp$, see [6]), and "true zero" (0, see [2]). A similar observation holds for the implementation of the concept "choice": half a dozen versions, ranging from completely deterministic to completely non-deterministic have been described.

It appears that the concept "empty process" is different; it leaves little room for differing implementations.

When we started thinking about the process that "does nothing" and terminates successfully (in the context of discrete-time ACP), we had a three design goals in mind for the empty process:

(i). it should be a unit element with respect to the $\cdot$ and $\|$ operators,

(ii). it ought not to destroy the commutativity and associativity of the $\|$ and $|$ operators,

(iii). it ought not to destroy the time-factorization property.

These properties we deemed essential; violating one of them would render our empty process useless. And, as it turned out, these goals very much restricted our freedom in choosing our definitions. In this article, we will motivate our choices on the grounds of the above design goals.

Finally, we had a few other design goals of lesser importance; we wanted that the existing "smaller" process algebras (without empty process, without time, or without either) could be embedded in the new process algebras in a simple way (i.e., in the terminology of [4] the old theories should be Subalgebras of a Reduced Model (SRMs) of the new theories).

## 1.4 Disclaimer

This paper should be read in conjunction with [26], as the proofs of some of the theorems given here strongly lean on results and techniques described there.

# 2 Theories with Undelayable Actions

In this section, we will define the process algebra $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID, and step-by-step extend it with the empty process, the free merge, and the merge. For each extension we will give an axiomatization, an operational semantics, and a description of all concepts that are introduced. Furthermore, we give the considerations that have led us to construct these algebras in the way we have done. We will restrict ourselves to *undelayable actions*, as opposed to delayable actions, which will be introduced in Section 3.

## 2.1 The Basis

In this section, we will define the process algebra $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID, which will serve as the basis for all the process algebras we will construct.

A short note on nomenclature: the name $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID should be interpreted as follows. BPA indicates we only have the elementary operators $+$ and $\cdot$, and no operators to deal with parallelism. The subscript "$_{\mathrm{drt}}$" stands for "discrete, relative timing". The superscript "$^{-}$" indicates we have no delayable actions (to be treated in Section 3). The suffix "–ID", finally, indicates we have no "immediate deadlock" (see Section 2.5 on page 36). For a full account of ACP nomenclature in general, see [13], for ACP nomenclature with respect to discrete time, see [7, 8, 26].

**Remark 2.1.1 (Alphabet)**
For this section, and all sections to come, we presume the existence of a fixed, finite alphabet $A$, that can be considered a parameter of the respective theories. Furthermore, we define $A_\delta$ as $A \cup \{\delta\}$ and $A_\sigma$ as $A \cup \{\sigma\}$, where $\delta$ and $\sigma$ are still to be treated symbols that are not contained in $A$.

**Definition 2.1.2 (Signature of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID)**
The signature of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *time-unit delay operator* $\sigma$, and the *"now" operator* $\nu$.

**Remark 2.1.3 (Signature of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID)**
In [7, 8, 26] the $\sigma$ and $\nu$ operators were denoted as $\sigma_{\mathrm{rel}}$ and $\nu_{\mathrm{rel}}$, in order to distinguish them from their absolute-time counterparts $\sigma_{\mathrm{abs}}$ and $\nu_{\mathrm{abs}}$. As absolute time does not play a role here, we leave out the "$_{\mathrm{rel}}$" subscript, since it clutters up the formulae.

**Remark 2.1.4 (Notation of Atoms)**
When we write "$a$" (or "$b$", or "$c$") in the context of an axiom, we mean this $a$ to range over $A_\delta$, and when we write it in the context of a deduction rule, we mean it to range over $A$. In all other cases, we explicitly state whether it ranges over $A$ or $A_\delta$.

**Definition 2.1.5 (Operator precedence)**
Throughout this paper we adhere to the following operator precedence scheme, which consist of four categories of operators. The four categories, from strongly binding to weakly binding, are:

(i). all unary operators,

(ii). the sequential composition operator "$\cdot$",

(iii). all binary operators, except the "+" and the "$\cdot$",

(iv). the alternative composition operator "+".

Within one category, all operators bind equally strong.

**Definition 2.1.6 (Axioms of $\text{BPA}^{-}_{\text{drt}}$–ID)**
The process algebra $\text{BPA}^{-}_{\text{drt}}$–ID is axiomatized by Axioms A1–A7 shown in Table 1 and Axioms DRT1–DRT2 and DCS1–DCS4 show in Table 2: $\text{BPA}^{-}_{\text{drt}}$–ID = A1–A7 + DRT1–DRT2 + DCS1–DCS4.

| | |
|---|---|
| $x + y = y + x$ | A1 |
| $(x + y) + z = x + (y + z)$ | A2 |
| $x + x = x$ | A3 |
| $(x + y) \cdot z = x \cdot z + y \cdot z$ | A4 |
| $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ | A5 |
| $x + \underline{\underline{\delta}} = x$ | A6 |
| $\underline{\underline{\delta}} \cdot x = \underline{\underline{\delta}}$ | A7 |

Table 1: Axioms of $\text{BPA}_\delta$.

| | |
|---|---|
| $\sigma(x) + \sigma(y) = \sigma(x + y)$ | DRT1 |
| $\sigma(x) \cdot y = \sigma(x \cdot y)$ | DRT2 |
| $\nu(\underline{\underline{a}}) = \underline{\underline{a}}$ | DCS1 |
| $\nu(x + y) = \nu(x) + \nu(y)$ | DCS2 |
| $\nu(x \cdot y) = \nu(x) \cdot y$ | DCS3 |
| $\nu(\sigma(x)) = \underline{\underline{\delta}}$ | DCS4 |

Table 2: Additional axioms for $\text{BPA}^{-}_{\text{drt}}$–ID

**Remark 2.1.7 (Axioms of $\text{BPA}^{-}_{\text{drt}}$–ID)**
The axioms of $\text{BPA}^{-}_{\text{drt}}$–ID serve the following purposes. Axioms A1–A3 express the commutative, associative, and idempotent character of the choice. Axiom A4 expresses the right distributivity of the $\cdot$ over the $+$; note that we do not have left distributivity, as that would lead to so called *trace semantics*, while we are interested in *bisimulation semantics*. Axiom A5 expresses the associativity of the sequential composition. Axioms A6–A7 express the fact that given the choice between something and deadlock, the $+$ always

avoids the deadlock, and the fact that once you are stuck in a deadlock, you cannot go on to the second part of a sequential composition.

DRT1 expresses the *time-factorization property* mentioned before: moving on to the next time-slice, when no actions are enabled, does not determine a choice. DRT2 expresses the *relative-time character* of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID: once a process has moved to a following time-slice, the complete remaining part of the process has also moved on, irrespective of the scope of the $\sigma$ operator.

DCS1 expresses that an undelayable action always starts in the current time-slice. DCS2 expresses that the part of $x + y$ that starts in the current time-slice consists of the parts of $x$ and $y$ that start in the current time-slice. DCS3 expresses that the part of $x \cdot y$ that starts in the current time-slice consists of the part of $x$ that starts in the current time-slice, followed by $y$ (which need not start in the current time-slice). DCS4, finally, expresses that $\sigma(x)$ cannot start in the current time-slice.

**Definition 2.1.8 (Notation regarding Semantics)**
In order to define a semantics, we will use term deduction system semantics in the style of Section 2.2.3 of [12] (also called "Structured Operational Semantics" or "Plotkin-style semantics"). We use the notation $x \xrightarrow{a} x'$ to denote that $x$ can do an $a$-step to $x'$, $x \xrightarrow{a} \surd$ to denote that $x$ can do an $a$-step and then terminate, $x \overset{a}{\nrightarrow}$ to denote that $x$ cannot do an $a$-step, $x \overset{\sigma}{\nrightarrow}$ to denote that $x$ cannot do a time step, and $x \nrightarrow$ to denote that $x$ cannot do any step at all. Finally, $x{\downarrow}$ denotes that $x$ can successfully terminate in the current time-slice, and $x{\not\downarrow}$ that it cannot.

For each process algebra we define, we will give a term deduction system. By using the concept of bisimulation (to be defined in Definition 2.1.13 on the next page), we then turn the term deduction system into a model of the given axioms.

**Definition 2.1.9 (Semantics of $\mathbf{BPA^-_{drt}}$–ID)**
The semantics of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID are given by the term deduction system $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID})$, induced by the deduction rules for $\mathrm{BPA}^-_{\mathrm{drt}}$–ID shown in Table 3 on the following page.

**Theorem 2.1.10 (Time Determinacy for $\mathbf{BPA^-_{drt}}$–ID)**
*Let $x$, $y$, and $y'$ be closed $BPA^-_{drt}$-ID terms. Then we have:*

$$T(BPA^-_{drt}\text{-}ID) \models x \xrightarrow{\sigma} y,\ x \xrightarrow{\sigma} y' \implies y \equiv y'$$

**Proof** Suppose that $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y,\ x \xrightarrow{\sigma} y'$. We proceed by case distinction on the form of $x$. For every case we will either derive, by inspection of the deduction rules, that $y \equiv y'$, or arrive at a contradiction, indicating that the case under consideration does not occur.

(i). $x \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. In contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$.

(ii). $x \equiv s \cdot t$ for closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID terms $s$ and $t$. We proceed by case distinction on the transitions of $s$.

(a) $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models s \xrightarrow{\sigma} s'$. Then $y \equiv y' \equiv s' \cdot t$.

(b) $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models s \overset{\sigma}{\nrightarrow}$. In contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$.

$$\underline{\underline{a}} \xrightarrow{a} \surd \qquad \sigma(x) \xrightarrow{\sigma} x$$

$$\frac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'} \qquad \frac{y \xrightarrow{a} y'}{x + y \xrightarrow{a} y'} \qquad \frac{x \xrightarrow{a} \surd}{x + y \xrightarrow{a} \surd} \qquad \frac{y \xrightarrow{a} \surd}{x + y \xrightarrow{a} \surd}$$

$$\frac{x \xrightarrow{a} x'}{x \cdot y \xrightarrow{a} x' \cdot y} \qquad \frac{x \xrightarrow{a} \surd}{x \cdot y \xrightarrow{a} y}$$

$$\frac{x \xrightarrow{\sigma} x',\ y \xrightarrow{\sigma} y'}{x + y \xrightarrow{\sigma} x' + y'} \qquad \frac{x \xrightarrow{\sigma} x',\ y \overset{\sigma}{\nrightarrow}}{x + y \xrightarrow{\sigma} x'} \qquad \frac{x \overset{\sigma}{\nrightarrow},\ y \xrightarrow{\sigma} y'}{x + y \xrightarrow{\sigma} y'} \qquad \frac{x \xrightarrow{\sigma} x'}{x \cdot y \xrightarrow{\sigma} x' \cdot y}$$

$$\frac{x \xrightarrow{a} x'}{\nu(x) \xrightarrow{a} x'} \qquad \frac{x \xrightarrow{a} \surd}{\nu(x) \xrightarrow{a} \surd}$$

Table 3: Deduction rules for BPA$^-_{\mathrm{drt}}$–ID.

(iii). $x \equiv s + t$ for closed BPA$^-_{\mathrm{drt}}$–ID terms $s$ and $t$. We proceed by case distinction on the transitions of $s$ and $t$.

(a) $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models s \xrightarrow{\sigma} s',\ t \xrightarrow{\sigma} t'$. Then $y \equiv y' \equiv s' + t'$.
(b) $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models s \xrightarrow{\sigma} s',\ t \overset{\sigma}{\nrightarrow}$. Then $y \equiv y' \equiv s'$.
(c) $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models s \overset{\sigma}{\nrightarrow},\ t \xrightarrow{\sigma} t'$. Then $y \equiv y' \equiv t'$.
(d) $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models s \overset{\sigma}{\nrightarrow},\ t \overset{\sigma}{\nrightarrow}$. In contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$.

(iv). $x \equiv \sigma(t)$ for a closed BPA$^-_{\mathrm{drt}}$–ID term t. Then $y \equiv y' \equiv t$.

Having inspected all possible cases, we may now conclude that $y \equiv y'$. ∎

**Remark 2.1.11 (Time Determinacy for BPA$^-_{\mathrm{drt}}$–ID)**
Theorem 2.1.10 on the page before embodies the *time-factorization property* we discussed earlier: when a process moves on to the next time-slice, the outcome is uniquely determined. Note that this is not always the case for doing an action. For example, we have $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models \underline{\underline{a}} \cdot \underline{\underline{b}} + \underline{\underline{a}} \cdot \underline{\underline{c}} \xrightarrow{a} \underline{\underline{b}}$, but also $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models \underline{\underline{a}} \cdot \underline{\underline{b}} + \underline{\underline{a}} \cdot \underline{\underline{c}} \xrightarrow{a} \underline{\underline{c}}$.

**Definition 2.1.12 (Symmetric Closure)**
For a binary relation $R$, we denote its *symmetric closure* by $R^{\mathrm{S}}$:

$$R^{\mathrm{S}} = R \cup \{(y,x) \mid (x,y) \in R\}$$

**Definition 2.1.13 (Bisimulation for BPA$^-_{\mathrm{drt}}$–ID)**
Bisimulation for BPA$^-_{\mathrm{drt}}$–ID is defined as follows; a binary relation $R$ on process terms is a bisimulation iff the following transfer conditions hold for all process terms $p$ and $q$:

(i). If $R^{\mathrm{S}}(p,q)$ and $p \xrightarrow{u} p'$, where $u \in A_\sigma$, then there exists a process term $q'$ such that $q \xrightarrow{u} q'$ and $R^{\mathrm{S}}(p',q')$,

(ii). If $R^{\mathrm{S}}(p,q)$ and $p \xrightarrow{a} \surd$, where $a \in A$, then $q \xrightarrow{a} \surd$.

Two $\mathrm{BPA}^-_{\mathrm{drt}}$–ID terms $p$ and $q$ are bisimilar, notation $p \sim_{\mathrm{BPA}^-_{\mathrm{drt}}\text{-ID}} q$, if there exists a bisimulation relation $R$ such that $R(p,q)$. Where there can be no confusion, we abbreviate this by $p \sim q$.

**Definition 2.1.14 (Bisimulation Model for $\mathbf{BPA}^-_{\mathbf{drt}}$–ID)**
Using bisimulation, we can now construct a model of the axioms of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID. In order to do this, we first need to know that bisimulation is a congruence with respect to all operators. In [28] it is proven that a sufficient condition for this is that:

(i). The deduction rules satisfy the so-called *panth* format,

(ii). the deduction rules are *well-founded*,

(iii). a *stratification* can be given for the deduction rules.

It is easy to check that these three conditions are indeed satisfied.

We now construct the bisimulation model for $\mathrm{BPA}^-_{\mathrm{drt}}$–ID by taking the equivalence classes of the set of all closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID terms with respect to bisimulation equivalence. As bisimulation is a congruence, the operators can be trivially defined on the equivalence classes. For example for the + operator:

$$[x]_\sim + [y]_\sim = [x+y]_\sim$$

Here $[x]_\sim$ denotes the equivalence class of $x$ with respect to the equivalence relation $\sim$. The other operators are defined in the same way.

**Definition 2.1.15 (Basic Terms of $\mathbf{BPA}^-_{\mathbf{drt}}$–ID)**
We define $(\sigma, \underline{\underline{\delta}})$*-basic terms* inductively as follows:

(i). For every $a \in A_\delta$, $\underline{\underline{a}}$ is a $(\sigma, \underline{\underline{\delta}})$-basic term,

(ii). if $a \in A_\delta$ and $t$ is a $(\sigma, \underline{\underline{\delta}})$-basic term, then $\underline{\underline{a}} \cdot t$ is a $(\sigma, \underline{\underline{\delta}})$-basic term,

(iii). if $s$ and $t$ are $(\sigma, \underline{\underline{\delta}})$-basic terms, then $s + t$ is a $(\sigma, \underline{\underline{\delta}})$-basic term,

(iv). if $t$ is a $(\sigma, \underline{\underline{\delta}})$-basic term, then $\sigma(t)$ is a $(\sigma, \underline{\underline{\delta}})$-basic term.

From now on, when we speak of basic terms in the context of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID, we mean $(\sigma, \underline{\underline{\delta}})$-basic terms.

**Theorem 2.1.16 (Elimination for $\mathbf{BPA}^-_{\mathbf{drt}}$–ID)**
*Let $t$ be a closed $BPA^-_{drt}$–ID term. Then there is a basic term $s$ such that $BPA^-_{drt}$–ID $\vdash s = t$.*

**Proof** This is Theorem 2.5.12 on page 26 of [26]; see the proof given there. ■

**Theorem 2.1.17 (Soundness of $\mathbf{BPA}^-_{\mathbf{drt}}$–ID)**
*The set of closed $BPA^-_{drt}$–ID terms modulo bisimulation equivalence is a model of the axioms of $BPA^-_{drt}$–ID.*

**Proof** This is Theorem 2.5.14 on page 27 of [26]; see the proof given there. ∎

**Theorem 2.1.18 (Completeness of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID)**
*The axiom system $\mathrm{BPA}^{-}_{drt}$–ID is a complete axiomatization of the set of closed $\mathrm{BPA}^{-}_{drt}$–ID terms modulo bisimulation equivalence.*

**Proof** This is Theorem 2.5.17 on page 31 of [26]; see the proof given there. ∎

## 2.2 The Empty-Process Extension

In this section, we will define $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID, which is basically $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID extended with the empty process (indicated by the subscript "$_\varepsilon$").

**Definition 2.2.1 (Signature of $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
The signature of $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *time-unit delay constant* $\underline{\underline{\sigma}}$, the *undelayable empty process constant* $\underline{\underline{\varepsilon}}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, and the *"now" operator* $\nu$.

**Definition 2.2.2 (Axioms of $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
The process algebra $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID is axiomatized by Axioms A1–A7 shown in Table 1 on page 9, and Axioms A8–A9, TF, and DCSE1–DCSE4 shown in Table 4: $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID = A1–A9 + TF + DCSE1–DCSE4.

| | |
|---|---|
| $x \cdot \underline{\underline{\varepsilon}} = x$ | A8 |
| $\underline{\underline{\varepsilon}} \cdot x = x$ | A9 |
| $\underline{\underline{\sigma}} \cdot x + \underline{\underline{\sigma}} \cdot y = \underline{\underline{\sigma}} \cdot (x + y)$ | TF |
| $\nu(\underline{\underline{\varepsilon}}) = \underline{\underline{\varepsilon}}$ | DCSE1 |
| $\nu(x + y) = \nu(x) + \nu(y)$ | DCSE2 |
| $\nu(\underline{\underline{a}} \cdot x) = \underline{\underline{a}} \cdot x$ | DCSE3 |
| $\nu(\underline{\underline{\sigma}} \cdot x) = \underline{\underline{\delta}}$ | DCSE4 |

Table 4: Additional axioms for $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID.

**Remark 2.2.3 (The processes $\underline{\underline{\varepsilon}}$ and $\underline{\underline{\sigma}}$)**
The *undelayable empty process* $\underline{\underline{\varepsilon}}$ we have just defined has the intuitive meaning of doing nothing, and then terminating successfully. As such, it is a proper 1-element for the sequential composition: $\underline{\underline{\varepsilon}} \cdot x = x \cdot \underline{\underline{\varepsilon}} = x$. Note that now we have a 1-element, we do not need the $\sigma$ operator anymore. This is because we can replace it by the constant $\underline{\underline{\sigma}}$ (intuitively: $\underline{\underline{\sigma}} = \sigma(\underline{\underline{\varepsilon}})$), the process that can move on to the following time-slice, and terminate there. The process formerly expressed as $\sigma(x)$ can now be represented by $\underline{\underline{\sigma}} \cdot x$, as, intuitively, $\sigma(x) = \sigma(\underline{\underline{\varepsilon}} \cdot x) = \sigma(\underline{\underline{\varepsilon}}) \cdot x = \underline{\underline{\sigma}} \cdot x$.

**Remark 2.2.4 (Axioms of $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
Axioms A8–A9 express the 1-element property of $\underline{\underline{\varepsilon}}$. As we do not have the $\sigma$ operator anymore, we can drop Axioms DRT1–DRT2.

To express the time-factorization property associated with $\underline{\underline{\sigma}}$, formerly done by Axiom DRT1, we now use Axiom TF. The purpose of DRT2 disappears, as it becomes a special case of Axiom A5, the associativity of sequential composition: $(\underline{\underline{\sigma}} \cdot x) \cdot y = \underline{\underline{\sigma}} \cdot (x \cdot y)$.

Finally, Axioms DCSE1–DCSE4 express the properties of the $\nu$ operator in the context of $\underline{\underline{\varepsilon}}$. Note that we do not anymore have that $\nu(x \cdot y) = \nu(x) \cdot y$, as that would lead to $\nu(\underline{\underline{\sigma}}) = \nu(\underline{\underline{\varepsilon}} \cdot \underline{\underline{\sigma}}) = \nu(\underline{\underline{\varepsilon}}) \cdot \underline{\underline{\sigma}} = \underline{\underline{\varepsilon}} \cdot \underline{\underline{\sigma}} = \underline{\underline{\sigma}}$, which is of course undesirable.

**Definition 2.2.5 (Semantics of $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
The semantics of $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID are given by the term deduction system $T(\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID})$, induced by the deduction rules for $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID shown in Table 5.

$$\underline{\underline{a}} \xrightarrow{a} \underline{\underline{\varepsilon}} \qquad \underline{\underline{\sigma}} \xrightarrow{\sigma} \underline{\underline{\varepsilon}} \qquad \underline{\underline{\varepsilon}}\downarrow$$

$$\frac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'} \qquad \frac{y \xrightarrow{a} y'}{x + y \xrightarrow{a} y'} \qquad \frac{x\downarrow}{(x+y)\downarrow} \qquad \frac{y\downarrow}{(x+y)\downarrow}$$

$$\frac{x \xrightarrow{a} x'}{x \cdot y \xrightarrow{a} x' \cdot y} \qquad \frac{x\downarrow,\ y \xrightarrow{a} y'}{x \cdot y \xrightarrow{a} y'} \qquad \frac{x\downarrow,\ y\downarrow}{(x \cdot y)\downarrow}$$

$$\frac{x \xrightarrow{\sigma} x',\ y \xrightarrow{\sigma} y'}{x + y \xrightarrow{\sigma} x' + y'} \qquad \frac{x \xrightarrow{\sigma} x',\ y \stackrel{\sigma}{\nrightarrow}}{x + y \xrightarrow{\sigma} x'} \qquad \frac{x \stackrel{\sigma}{\nrightarrow},\ y \xrightarrow{\sigma} y'}{x + y \xrightarrow{\sigma} y'}$$

$$\frac{x \xrightarrow{\sigma} x',\ x\not\downarrow}{x \cdot y \xrightarrow{\sigma} x' \cdot y} \qquad \frac{x \xrightarrow{\sigma} x',\ y \stackrel{\sigma}{\nrightarrow}}{x \cdot y \xrightarrow{\sigma} x' \cdot y} \qquad \frac{x \stackrel{\sigma}{\nrightarrow},\ x\downarrow,\ y \xrightarrow{\sigma} y'}{x \cdot y \xrightarrow{\sigma} y'} \qquad \frac{x \xrightarrow{\sigma} x',\ x\downarrow,\ y \xrightarrow{\sigma} y'}{x \cdot y \xrightarrow{\sigma} x' \cdot y + y'}$$

$$\frac{x \xrightarrow{a} x'}{\nu(x) \xrightarrow{a} x'} \qquad \frac{x\downarrow}{\nu(x)\downarrow}$$

Table 5: Deduction rules for $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID.

**Remark 2.2.6 (Deduction rules for $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
The deduction rules for the sequential composition with respect to $\sigma$-transitions, shown in Table 5, are very subtle. Consider for example the expression $(\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\underline{\varepsilon}}) \cdot (\underline{\underline{\sigma}} \cdot \underline{\underline{b}})$.

Algebraically, we have:

$$\begin{aligned}(\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\underline{\varepsilon}}) \cdot (\underline{\underline{\sigma}} \cdot \underline{\underline{b}}) &= \underline{\underline{\sigma}} \cdot \underline{\underline{a}} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\varepsilon}} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{b}} \\ &= \underline{\underline{\sigma}} \cdot \underline{\underline{a}} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\sigma}} \cdot \underline{\underline{b}} \\ &= \underline{\underline{\sigma}} \cdot (\underline{\underline{a}} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{b}})\end{aligned}$$

therefore, in our model we should have $(\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\underline{\varepsilon}}) \cdot (\underline{\underline{\sigma}} \cdot \underline{\underline{b}}) \xrightarrow{\sigma} \underline{\underline{a}} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{b}}$ and in particular we should *not* have $(\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\underline{\varepsilon}}) \cdot (\underline{\underline{\sigma}} \cdot \underline{\underline{b}}) \xrightarrow{\sigma} \underline{\underline{a}} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{b}}$, as such a transition would lose the option to do just a $b$, while the principle of time factorization mentioned before explicitly forbids that a $\sigma$-transition determines a choice here.

**Theorem 2.2.7 (Time Determinacy for $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID)**
*Let $x$, $y$, and $y'$ be closed $BPA^-_{drt,\varepsilon}$-ID terms. Then we have:*

$$T(BPA^-_{drt,\varepsilon}\text{-}ID) \models x \xrightarrow{\sigma} y,\ x \xrightarrow{\sigma} y' \implies y \equiv y'$$

**Proof** Suppose that $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}) \models x \xrightarrow{\sigma} y,\ x \xrightarrow{\sigma} y'$. We proceed by case distinction on the form of $x$. For every case we will either derive, by inspection of the deduction rules, that $y \equiv y'$, or arrive at a contradiction, indicating that the case under consideration does not occur.

(i). $x \equiv \underline{\underline{\varepsilon}}$. In contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}) \models x \xrightarrow{\sigma} y$.

(ii). $x \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. In contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}) \models x \xrightarrow{\sigma} y$.

(iii). $x \equiv \underline{\underline{\sigma}}$. Then $y \equiv y' \equiv \underline{\underline{\varepsilon}}$.

(iv). $x \equiv s \cdot t$ for closed $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID terms $s$ and $t$. We proceed by case distinction on the transitions of $s$ and $t$.

(a) $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \xrightarrow{\sigma} s',\ s\downarrow,\ t \xrightarrow{\sigma} t'$. Then $y \equiv y' \equiv s' \cdot t + t'$.
(b) $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \xrightarrow{\sigma} s'$ otherwise. Then $y \equiv y' \equiv s' \cdot t$.
(c) $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \stackrel{\sigma}{\not\to},\ s\downarrow,\ t \xrightarrow{\sigma} t'$. Then $y \equiv y' \equiv t'$.
(d) $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \stackrel{\sigma}{\not\to}$ otherwise. In contradiction with $x \xrightarrow{\sigma} y$.

(v). $x \equiv s + t$ for closed $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID terms $s$ and $t$. We proceed by case distinction on the transitions of $s$ and $t$.

(a) $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \xrightarrow{\sigma} s',\ t \xrightarrow{\sigma} t'$. Then $y \equiv y' \equiv s' + t'$.
(b) $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \xrightarrow{\sigma} s',\ t \stackrel{\sigma}{\not\to}$. Then $y \equiv y' \equiv s'$.
(c) $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \stackrel{\sigma}{\not\to},\ t \xrightarrow{\sigma} t'$. Then $y \equiv y' \equiv t'$.
(d) $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \stackrel{\sigma}{\not\to},\ t \stackrel{\sigma}{\not\to}$. In contradiction with $x \xrightarrow{\sigma} y$.

Having inspected all possible cases, we may now conclude that $y \equiv y'$. ∎

**Definition 2.2.8 (Bisimulation for $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID)**
Bisimulation for $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID is defined as follows; a binary relation $R$ on process terms is a bisimulation iff the following transfer conditions hold for all process terms $p$ and $q$:

(i). If $R^{S}(p,q)$ and $p \xrightarrow{u} p'$, where $u \in A_\sigma$, then there exists a process term $q'$ such that $q \xrightarrow{u} q'$ and $R^{S}(p',q')$,

(ii). If $R^{S}(p,q)$ and $p\downarrow$, then $q\downarrow$.

**Definition 2.2.9 (Bisimulation Model for BPA$^-_{\mathbf{drt},\varepsilon}$–ID)**
The bisimulation model for BPA$^-_{\mathrm{drt},\varepsilon}$–ID is defined in the same way as for BPA$^-_{\mathrm{drt}}$–ID. Replace "BPA$^-_{\mathrm{drt}}$–ID" by "BPA$^-_{\mathrm{drt},\varepsilon}$–ID" in Definition 2.1.14 on page 12.

**Definition 2.2.10 (Basic Terms of BPA$^-_{\mathbf{drt},\varepsilon}$–ID)**
We define $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}})$*-basic terms* inductively as follows:

(i). The constant $\underline{\underline{\varepsilon}}$ is a $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}})$-basic term,

(ii). if $a \in A_\delta$ and $t$ is a $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}})$-basic term, then $\underline{\underline{a}} \cdot t$ is a $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}})$-basic term,

(iii). if $s$ and $t$ are $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}})$-basic terms, then $s + t$ is a $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}})$-basic term,

(iv). if $t$ is a $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}})$-basic term, then $\underline{\underline{\sigma}} \cdot t$ is a $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}})$-basic term.

From now on, when we speak of basic terms in the context of BPA$^-_{\mathrm{drt},\varepsilon}$–ID, we mean $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}})$-basic terms.

**Example 2.2.11 (Basic Terms of BPA$^-_{\mathbf{drt},\varepsilon}$–ID)**
The following are basic terms of BPA$^-_{\mathrm{drt},\varepsilon}$–ID: $\underline{\underline{\delta}} \cdot \underline{\underline{\varepsilon}}, \underline{\underline{a}} \cdot \underline{\underline{\varepsilon}}, \underline{\underline{a}} \cdot (\underline{\underline{b}} \cdot \underline{\underline{c}} \cdot \underline{\underline{\varepsilon}} + \underline{\underline{d}} \cdot \underline{\underline{\varepsilon}})$. Note, however, that the following are *not* basic terms: $\underline{\underline{\delta}}, \underline{\underline{a}}, \underline{\underline{a}} \cdot (\underline{\underline{b}} + \underline{\underline{c}})$.

**Definition 2.2.12 (Number of Symbols of a BPA$^-_{\mathbf{drt},\varepsilon}$–ID Term)**
We define $n(x)$, the number of symbols of $x$, inductively as follows:

(i). We define $n(\underline{\underline{\varepsilon}}) = n(\underline{\underline{\sigma}}) = 1$,

(ii). for $a \in A_\delta$, we define $n(\underline{\underline{a}}) = 1$,

(iii). for closed BPA$^-_{\mathrm{drt},\varepsilon}$–ID terms $x$ and $y$, we define $n(x+y) = n(x \cdot y) = n(x) + n(y) + 1$.

**Definition 2.2.13 (Summation Convention)**
We will use the meta-notation $\sum_{i \in I}$ to denote the summation over some finite index set $I$. The summation over the empty set yields the undelayable deadlock:

$$\sum_{i \in \emptyset} t_i \equiv \underline{\underline{\delta}}$$

Note however, that we use the convention that an empty summation disappears in the presence of other summands. So:

$$s + \sum_{i \in \emptyset} t_i \equiv s$$

Here $s$ and $t_i$ denote arbitrary closed terms.

**Theorem 2.2.14 (General Form of Basic Terms of $\mathbf{BPA^-_{drt,\varepsilon}}$–ID)**
*Modulo the commutativity and associativity of the $+$, all basic terms $t$ of $BPA^-_{drt,\varepsilon}$–ID are of the form:*

$$t \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j + \sum_{k<p} \underline{\underline{\varepsilon}}$$

*for $m, n, p \in \mathbb{N}$, $m + n + p \geq 1$, $a_i \in A_\delta$, and basic terms $s_i$ and $u_j$.*

**Proof** Trivial, by inspection of the definition of basic terms, Definition 2.2.10. Observe that the general form of basic terms is closed under the formation rules given in Definition 2.2.10. Note that by our summation convention, Definition 2.2.13 on the page before, the above general form never contains a summand $\underline{\underline{\delta}}$. ∎

**Lemma 2.2.15 (Representation of $\mathbf{BPA^-_{drt,\varepsilon}}$–ID Terms)**
*Let $t$ be a basic term. Then either $BPA^-_{drt,\varepsilon}$–ID $\vdash t = \nu(t)$, or there exists a basic term $s$ such that $BPA^-_{drt,\varepsilon}$–ID $\vdash t = \nu(t) + \underline{\underline{\sigma}} \cdot s$ and $n(s) < n(t)$.*

**Proof** Let $t$ be a basic term. By Theorem 2.2.14, we may now proceed by case analysis on the general form of basic terms:

(i). Either we have no $\underline{\underline{\sigma}} \cdot u_j$ summands ($n = 0$ in Definition 2.2.14):

$$t \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{k<p} \underline{\underline{\varepsilon}}$$

for $m, p \in \mathbb{N}$, $m + p \geq 1$, $a_i \in A_\delta$, and basic terms $s_i$. Then we have the following computation:

$$\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash t = \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{k<p} \underline{\underline{\varepsilon}} = \sum_{i<m} \nu(\underline{\underline{a_i}} \cdot s_i) + \sum_{k<p} \nu(\underline{\underline{\varepsilon}}) =$$
$$\nu\left(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{k<p} \underline{\underline{\varepsilon}}\right) = \nu(t)$$

(ii). Or we have at least one $\underline{\underline{\sigma}} \cdot u_j$ summand ($n \geq 1$ in Definition 2.2.14):

$$t \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j + \sum_{k<p} \underline{\underline{\varepsilon}}$$

for $m, n, p \in \mathbb{N}$, $m + n + p \geq 1$, $a_i \in A_\delta$, and basic terms $s_i$ and $u_j$. Then we have the following computation:

$$\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash t = \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j + \sum_{k<p} \underline{\underline{\varepsilon}} =$$
$$\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \underline{\underline{\delta}} + \sum_{k<p} \underline{\underline{\varepsilon}} + \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j =$$
$$\sum_{i<m} \nu(\underline{\underline{a_i}} \cdot s_i) + \sum_{j<n} \nu(\underline{\underline{\sigma}} \cdot u_j) + \sum_{k<p} \nu(\underline{\underline{\varepsilon}}) + \underline{\underline{\sigma}} \cdot \left(\sum_{j<n} u_j\right) =$$

$$\nu\left(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j + \sum_{k<p} \underline{\underline{\varepsilon}}\right) + \underline{\underline{\sigma}} \cdot \left(\sum_{j<n} u_j\right) =$$
$$\nu(t) + \underline{\underline{\sigma}} \cdot s$$

Where we define:

$$s \equiv \sum_{j<n} u_j$$

Note that $n(s) < n(t)$ is now trivially satisfied, as for every summand $u_j$ of $s$, there is a corresponding summand $\underline{\underline{\sigma}} \cdot u_j$ of $t$, and at least one such summand exists as $n \geq 1$.

■

**Remark 2.2.16 (Representation of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID Terms)**
The main use of Lemma 2.2.15 will be in induction proofs regarding the (not yet treated) theories $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID and $\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID (see Sections 2.3 and 2.4).

**Definition 2.2.17 (Alternative Normal Form for $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID)**
We define the *alternative normal form (ANF) terms* of $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID inductively as follows:

(i). The constant $\underline{\underline{\delta}}$ is an ANF term,

(ii). the constant $\underline{\underline{\varepsilon}}$ is an ANF term,

(iii). if $a \in A_\delta$ and $s$ and $t$ are ANF terms, then $\underline{\underline{a}} \cdot s + t$ is an ANF term,

(iv). if $t$ is an ANF term, then $\underline{\underline{\sigma}} \cdot t$ is an ANF term,

(v). if $t$ is an ANF term, then $\underline{\underline{\sigma}} \cdot t + \underline{\underline{\varepsilon}}$ is an ANF term.

**Remark 2.2.18 (Alternative Normal Form for $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID)**
We will need ANF terms in order to axiomize the *left merge operator* to be treated in Section 2.3. As we will show in Theorem 2.2.19, basic terms and ANF terms are equivalent in the sense that any basic term can be rewritten into a ANF term, and vice versa.

**Theorem 2.2.19 (Equivalence of Basic and ANF Terms)**
*For every ANF term $x$ of $BPA^-_{drt,\varepsilon}$-ID, there exists a basic term $x'$ such that $BPA^-_{drt,\varepsilon}$-ID $\vdash x = x'$, and, vice versa, for every basic term $x$ of $BPA^-_{drt,\varepsilon}$-ID, there exists an ANF term $x'$ such that $BPA^-_{drt,\varepsilon}$-ID $\vdash x = x'$.*

**Proof** The first part is simple. Suppose that $x$ is an ANF-term. We use induction on the structure of ANF terms, given in Definition 2.2.17. Suppose that $x \equiv \underline{\underline{\delta}}$, then we take $x' \equiv \underline{\underline{\delta}} \cdot \underline{\underline{\varepsilon}}$. Now the property trivially holds for $x \equiv \underline{\underline{\varepsilon}}$, $x \equiv \underline{\underline{a}} \cdot s + t$, $x \equiv \underline{\underline{\sigma}} \cdot t$, and $x \equiv \underline{\underline{\sigma}} \cdot t + \underline{\underline{\varepsilon}}$, as by the induction hypothesis the property already holds for the ANF terms $s$ and $t$.

The second part is harder. Suppose that $x$ is a basic term. Then, by Theorem 2.2.14 on the page before, we may assume that $x$ is of the following form:

$$x \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j + \sum_{k<p} \underline{\underline{\varepsilon}}$$

for $m, n, p \in \mathbb{N}$, $m+n+p \geq 1$, $a_i \in A_\delta$, and basic terms $s_i$ and $u_j$. We now distinguish six mutually exclusive cases:

(i). There are only $\underline{\underline{\varepsilon}}$ summands: $m = n = 0$ and $p \geq 1$.

(ii). There are only $\underline{\underline{\sigma}} \cdot u_j$ summands: $m = 0$, $n \geq 1$, and $p = 0$.

(iii). There are both $\underline{\underline{\sigma}} \cdot u_j$ and $\underline{\underline{\varepsilon}}$ summands, but no other summands: $m = 0$, $n \geq 1$, and $p \geq 1$.

(iv). There is exactly one $\underline{\underline{a_i}} \cdot s_i$ summand, and no other summands: $m = 1$ and $n = p = 0$.

(v). There is exactly one $\underline{\underline{a_i}} \cdot s_i$ summand, and at least one other summand: $m = 1$ and $n + p \geq 1$.

(vi). There are at least two $\underline{\underline{a_i}} \cdot s_i$ summands: $m \geq 2$.

As can be easily seen, this covers all cases. We now prove the property for all six cases, using induction on the number of symbols in $x$:

(i). We have $m = n = 0$ and $p \geq 1$. Then:

$$\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x = \sum_{k<p} \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}$$

and $\underline{\underline{\varepsilon}}$ is an ANF term. Note that the sum over $k$ cannot be empty, as $p \geq 1$.

(ii). We have $m = 0$, $n \geq 1$, and $p = 0$. Then:

$$\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x = \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j = \underline{\underline{\sigma}} \cdot \left( \sum_{j<n} u_j \right)$$

and this last term is an ANF term, since by the induction hypothesis the property already holds for $\sum_{j<n} u_j$. Note that the sum over $j$ cannot be empty, as $n \geq 1$.

(iii). We have $m = 0$, $n \geq 1$, and $p \geq 1$. Then:

$$\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x = \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j + \sum_{k<p} \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot \left( \sum_{j<n} u_j \right) + \underline{\underline{\varepsilon}}$$

and this last term is an ANF term, since by the induction hypothesis the property already holds for $\sum_{j<n} u_j$. Note that neither the sum over $j$ nor the one over $k$ can be empty, as $n \geq 1$ and $p \geq 1$.

(iv). We have $m = 1$ and $n = p = 0$. Then:

$$\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x = \underline{\underline{a_1}} \cdot s_1 = \underline{\underline{a_1}} \cdot s_1 + \underline{\underline{\delta}}$$

and $\underline{\underline{a_1}} \cdot s_1 + \underline{\underline{\delta}}$ is an ANF term since by the induction hypothesis the property already holds for $s_1$.

(v). We have $m = 1$ and $n + p \geq 1$. Then:

$$\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x = \underline{\underline{a_1}} \cdot s_1 + \left( \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j + \sum_{k<p} \underline{\underline{\varepsilon}} \right)$$

and this last term is an ANF term, since by the induction hypothesis the property already holds for $s_1$ and the second summand. Note that the second summand can not be empty, as $n + p \geq 1$.

(vi). We have $m \geq 2$. Then:

$$\begin{aligned}\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x = \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j + \sum_{k<p} \underline{\underline{\varepsilon}} = \\ \underline{\underline{a_1}} \cdot s_1 + \left( \sum_{2\leq i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j + \sum_{k<p} \underline{\underline{\varepsilon}} \right)\end{aligned}$$

and this last term is an ANF term, since by the induction hypothesis the property already holds for $s_1$ and the second summand. Note that the second summand can not be empty, as $m \geq 2$.

■

**Corollary 2.2.20 (Structural Induction over ANF)**
*Any proposition of the form "For all basic terms ... ", that we would normally prove by structural induction over the definition of basic terms, may now also be proven by induction over the definition of ANF terms.*

**Proof** This follows directly from Theorem 2.2.19 on page 18. ■

**Theorem 2.2.21 (Elimination for $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
*Let $t$ be a closed $BPA^{-}_{drt,\varepsilon}$-ID term. Then there is a basic term $s$ such that $BPA^{-}_{drt,\varepsilon}$-ID $\vdash s = t$.*

**Proof** Turn the axioms into a term rewriting system, and then apply the lexicographical path ordering technique. We give no details here, see [26] for more information. ■

**Theorem 2.2.22 (Soundness of $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
*The set of closed $BPA^{-}_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence is a model of the axioms of $BPA^{-}_{drt,\varepsilon}$-ID.*

**Proof** For every axiom, derive a concrete bisimulation relation that relates both sides of the axiom for all closed instantiations of the free variables. We give no details here, see [26] for more information. ■

**Lemma 2.2.23 (Towards Completeness of $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
*Let $x$ be a closed $BPA^{-}_{drt,\varepsilon}$-ID term and let $a \in A$. Then we have:*

*(i). $T(BPA^{-}_{drt,\varepsilon}\text{-}ID) \models x \xrightarrow{a} y \implies BPA^{-}_{drt,\varepsilon}\text{-}ID \vdash x = \underline{\underline{a}} \cdot y + x$,*

*(ii). $T(BPA^{-}_{drt,\varepsilon}\text{-}ID) \models x\downarrow \implies BPA^{-}_{drt,\varepsilon}\text{-}ID \vdash x = \underline{\underline{\varepsilon}} + x$,*

*(iii).* $T(BPA^-_{drt,\varepsilon}\text{-}ID) \models x \overset{\sigma}{\nrightarrow} \implies BPA^-_{drt,\varepsilon}\text{-}ID \vdash x = \nu(x)$,

*(iv).* $T(BPA^-_{drt,\varepsilon}\text{-}ID) \models x \xrightarrow{\sigma} y \implies BPA^-_{drt,\varepsilon}\text{-}ID \vdash x = \underline{\underline{\sigma}} \cdot y + \nu(x)$,

*(v).* $T(BPA^-_{drt,\varepsilon}\text{-}ID) \models x \xrightarrow{a} y \implies n(x) > n(y)$,

*(vi).* $T(BPA^-_{drt,\varepsilon}\text{-}ID) \models x \xrightarrow{\sigma} y \implies n(x) > n(y)$.

**Proof** By induction on the structure of $x$. We give no details here, see [26] for more information. ∎

**Theorem 2.2.24 (Completeness of $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID)**
*The axiom system $BPA^-_{drt,\varepsilon}$-ID is a complete axiomatization of the set of closed $BPA^-_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence.*

**Proof** We use, without further explanation, the methods outlined in [26]. Suppose $x + y \sim_{\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}} y$. We will prove, with induction on the structure of basic term $x$, that $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID $\vdash x + y = y$.

(i). $x \equiv \underline{\underline{\varepsilon}}$. From the deduction rules we have $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models x{\downarrow}$, and $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models (x+y){\downarrow}$. Since $x + y \sim_{\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}} y$, we then also have $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models y{\downarrow}$. Then, using Lemma 2.2.23(ii), we have $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID $\vdash x + y = \underline{\underline{\varepsilon}} + y = y$.

(ii). $x \equiv \underline{\underline{\delta}} \cdot t$, where $t$ is a basic term. Then we have $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID $\vdash x + y = \underline{\underline{\delta}} \cdot t + y = \underline{\underline{\delta}} + y = y$.

(iii). $x \equiv \underline{\underline{a}} \cdot t$, where $a \in A$ and $t$ is a basic term. From the deduction rules we obtain $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models x \xrightarrow{a} t$ and $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models x+y \xrightarrow{a} t$. Since $x+y \sim_{\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}} y$, we then also have $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models y \xrightarrow{a} s$ for some $s$ such that $t \sim_{\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}} s$. By the induction hypothesis we have $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID $\vdash s = t$. From Lemma 2.2.23(i) we have $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID $\vdash y = \underline{\underline{a}} \cdot s + y$. So, $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID $\vdash x + y = \underline{\underline{a}} \cdot t + y = \underline{\underline{a}} \cdot s + y = y$.

(iv). $x \equiv s + t$, where $s$ and $t$ are basic terms. Since $x + y \sim_{\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}} y$, we also have $s + y \sim_{\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}} y$ and $t + y \sim_{\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}} y$. By the induction hypothesis we then have $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID $\vdash s + y = y, t + y = y$. So, $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID $\vdash x + y = s + t + y = s + y = y$.

(v). $x \equiv \underline{\underline{\sigma}} \cdot t$, where $t$ is a basic term. From the deduction rules we now have that $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models x \xrightarrow{\sigma} t$ and since $x + y \sim_{\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{-ID}} y$ we also have $T(\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models y \xrightarrow{\sigma} s, x + y \xrightarrow{\sigma} t + s$ for some $s$ such that $t + s \sim_{\mathrm{BPA}^-_{\mathrm{drt}}\text{-ID}} s$. By Lemma 2.2.23(iv) we have $\mathrm{BPA}^-_{\mathrm{drt}}$–ID $\vdash y = \underline{\underline{\sigma}} \cdot s + \nu(y)$. By the induction hypothesis we have $\mathrm{BPA}^-_{\mathrm{drt}}$–ID $\vdash t + s = s$. So, $\mathrm{BPA}^-_{\mathrm{drt}}$–ID $\vdash x + y = \underline{\underline{\sigma}} \cdot t + y = \underline{\underline{\sigma}} \cdot t + \underline{\underline{\sigma}} \cdot s + \nu(y) = \underline{\underline{\sigma}} \cdot (t + s) + \nu(y) = \underline{\underline{\sigma}} \cdot s + \nu(y) = y$.

∎

## 2.3 The Free-Merge Extension

In this section, we will define $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID, which is basically $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID extended with the free merge (indicated by the letters "PA" instead of "BPA"). The free merge allows us to write down the parallel composition of two processes.

**Definition 2.3.1 (Signature of $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID)**
The signature of $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID consists of the *undelayable atomic actions* $\{\underline{\underline{a}}|a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *time-unit delay constant* $\underline{\underline{\sigma}}$, the *undelayable empty process constant* $\underline{\underline{\varepsilon}}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *"now" operator* $\nu$, the *free merge operator* $\|$, and the *left merge operator* $\lfloor\!\lfloor$ .

**Definition 2.3.2 (Axioms of $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID)**
The process algebra $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID is axiomatized by the axioms of $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID given in Definition 2.2.2 on page 13, and the Axioms DRTEM1–DRTEM12 shown in Table 6: $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID = A1–A9 + TF + DCSE1–DCSE4 + DRTEM1–DRTEM12. Note that the merge operators bind weaker than the $\cdot$, but stronger than the $+$.

| | |
|---|---|
| $x \parallel y = x \lfloor\!\lfloor y + y \lfloor\!\lfloor x$ | DRTEM1 |
| $\underline{\underline{a}} \cdot x \lfloor\!\lfloor y = \underline{\underline{a}} \cdot (x \parallel y)$ | DRTEM2 |
| $(x + y) \lfloor\!\lfloor z = x \lfloor\!\lfloor z + y \lfloor\!\lfloor z$ | DRTEM3 |
| $\underline{\underline{\varepsilon}} \lfloor\!\lfloor \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}$ | DRTEM4 |
| $\underline{\underline{\varepsilon}} \lfloor\!\lfloor \underline{\underline{a}} \cdot x = \underline{\underline{\delta}}$ | DRTEM5 |
| $\underline{\underline{\varepsilon}} \lfloor\!\lfloor \underline{\underline{\sigma}} \cdot x = \underline{\underline{\delta}}$ | DRTEM6 |
| $\underline{\underline{\varepsilon}} \lfloor\!\lfloor (x + y) = \underline{\underline{\varepsilon}} \lfloor\!\lfloor x + \underline{\underline{\varepsilon}} \lfloor\!\lfloor y$ | DRTEM7 |
| $\underline{\underline{\sigma}} \cdot x \lfloor\!\lfloor (\underline{\underline{a}} \cdot y + z) = \underline{\underline{\sigma}} \cdot x \lfloor\!\lfloor z$ | DRTEM8 |
| $\underline{\underline{\sigma}} \cdot x \lfloor\!\lfloor \underline{\underline{\delta}} = \underline{\underline{\delta}}$ | DRTEM9 |
| $\underline{\underline{\sigma}} \cdot x \lfloor\!\lfloor \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot x$ | DRTEM10 |
| $\underline{\underline{\sigma}} \cdot x \lfloor\!\lfloor \underline{\underline{\sigma}} \cdot y = \underline{\underline{\sigma}} \cdot (x \lfloor\!\lfloor y)$ | DRTEM11 |
| $\underline{\underline{\sigma}} \cdot x \lfloor\!\lfloor (\underline{\underline{\sigma}} \cdot y + \underline{\underline{\varepsilon}}) = \underline{\underline{\sigma}} \cdot (x \lfloor\!\lfloor y)$ | DRTEM12 |

Table 6: Additional axioms for $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID.

**Remark 2.3.3 (Axioms of $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID, Part I)**
The *free merge* of two processes $x$ and $y$, notation $x \parallel y$, should be interpreted as the simultaneous, interleaving execution of $x$ and $y$. The two processes $x$ and $y$ do in no way communicate or synchronize with each other (hence the merge is called *free*). The *left merge*, notation $x \lfloor\!\lfloor y$, is an auxiliary operator that is introduced for the purpose of the axiomatization only; it is not intended to be used in actual specifications. The process $x \lfloor\!\lfloor y$ is like $x \parallel y$, but always executes at least one action from $x$ before it executes an action from $y$.

Regarding the inner workings of the axioms: DRTEM4–DRTEM7 ensure that $x \parallel y$ has a summand $\underline{\underline{\varepsilon}}$ iff $x$ and $y$ both have a summand $\underline{\underline{\varepsilon}}$. Furthermore, DRTEM8–DRTEM12 work by first eliminating all summands $\underline{\underline{a}} \cdot y$ on the right-hand side of the left merge (DRTEM8), and then distinguishing between terms that only have a $\underline{\underline{\sigma}} \cdot y$ summand left (DRTEM11), only a $\underline{\underline{\varepsilon}}$ summand left (DRTEM10), or both (DRTEM12), or none (DRTEM9). Notice that this is exactly the structure of ANF terms given in Definition 2.2.17 on page 18.

For more details concerning axioms for merge operators, see [9, 14].

**Remark 2.3.4 (Axioms of $\mathrm{PA}^{-}_{\mathbf{drt},\varepsilon}$–ID, Part II)**
Note that, unlike in a setting without empty process, we do not need an axiom $\underline{\underline{a}} \mathbin{\lfloor\!\lfloor} x = \underline{\underline{a}} \cdot x$, as that equality is derivable for all closed terms $x$ (we will prove this in Proposition 2.3.16 on page 27). For example:

$$\begin{aligned}
\underline{\underline{a}} \mathbin{\lfloor\!\lfloor} \underline{\underline{b}} &= \underline{\underline{a}} \cdot \underline{\underline{\varepsilon}} \mathbin{\lfloor\!\lfloor} \underline{\underline{b}} = \underline{\underline{a}} \cdot (\underline{\underline{\varepsilon}} \parallel \underline{\underline{b}}) = \underline{\underline{a}} \cdot (\underline{\underline{\varepsilon}} \mathbin{\lfloor\!\lfloor} \underline{\underline{b}} + \underline{\underline{b}} \mathbin{\lfloor\!\lfloor} \underline{\underline{\varepsilon}}) = \underline{\underline{a}} \cdot (\underline{\underline{\delta}} + \underline{\underline{b}} \cdot \underline{\underline{\varepsilon}} \mathbin{\lfloor\!\lfloor} \underline{\underline{\varepsilon}}) \\
&= \underline{\underline{a}} \cdot \underline{\underline{b}} \cdot (\underline{\underline{\varepsilon}} \parallel \underline{\underline{\varepsilon}}) = \underline{\underline{a}} \cdot \underline{\underline{b}} \cdot (\underline{\underline{\varepsilon}} \mathbin{\lfloor\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\lfloor\!\lfloor} \underline{\underline{\varepsilon}}) = \underline{\underline{a}} \cdot \underline{\underline{b}} \cdot (\underline{\underline{\varepsilon}} \mathbin{\lfloor\!\lfloor} \underline{\underline{\varepsilon}}) = \underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{\varepsilon}} \\
&= \underline{\underline{a}} \cdot \underline{\underline{b}}
\end{aligned}$$

**Remark 2.3.5 (Axioms of $\mathrm{PA}^{-}_{\mathbf{drt},\varepsilon}$–ID, Part III)**
The axiomatization as given in Definition 2.3.2 on the page before has some consequences that may not be obvious at first sight. For example, consider what we get if we eliminate the merge operator from the expression $(\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \parallel \underline{\underline{b}}$ using the axioms:

$$\begin{aligned}
(\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \parallel \underline{\underline{b}} &= (\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \mathbin{\lfloor\!\lfloor} \underline{\underline{b}} + \underline{\underline{b}} \mathbin{\lfloor\!\lfloor} (\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \\
&= \underline{\underline{a}} \mathbin{\lfloor\!\lfloor} \underline{\underline{b}} + \underline{\underline{\varepsilon}} \mathbin{\lfloor\!\lfloor} \underline{\underline{b}} + \underline{\underline{b}} \cdot (\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \\
&= \underline{\underline{a}} \cdot \underline{\underline{b}} + \underline{\underline{\delta}} + \underline{\underline{b}} \cdot (\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \\
&= \underline{\underline{a}} \cdot \underline{\underline{b}} + \underline{\underline{b}} \cdot (\underline{\underline{a}} + \underline{\underline{\varepsilon}})
\end{aligned}$$

So, if we execute a $b$ first, then an $a$ can always still follow. This may seem counterintuitive; apparently it is not possible for the $\underline{\underline{\varepsilon}}$ to the left of the merge to execute before the $b$ does, in which case $a$ would *not* be enabled after the execution of the $b$.

The rationale behind this, is that the $\underline{\underline{\varepsilon}}$ is not something that "executes"; it merely represents an *option* to terminate. Hence, the option remains open until the process actually *does terminate* or performs an action. It does not "just get lost". So for example the process $\underline{\underline{a}} + \underline{\underline{\varepsilon}}$ can not drop the $\underline{\underline{\varepsilon}}$ to turn itself into the process $\underline{\underline{a}}$. If it wants to lose the $\underline{\underline{\varepsilon}}$, it has to execute an $a$.

In the context of a (multiple) merge, this means that an $\underline{\underline{\varepsilon}}$ summand in one of the merge components does not manifest itself as long as the other components of the merge still have to execute one or more actions before they can terminate or move on to the following time-slice.

One could be tempted to "repair" this behavior, for example by dropping DRTEM4–DRTEM7, and adding the axiom $\underline{\underline{\varepsilon}} \mathbin{\lfloor\!\lfloor} x = x$ (i.e., treating $\underline{\underline{\varepsilon}}$ like it was an ordinary action). Although this does give us the "desired" equality $(\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \parallel \underline{\underline{b}} = (\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \cdot \underline{\underline{b}} + \underline{\underline{b}} \cdot (\underline{\underline{a}} + \underline{\underline{\varepsilon}})$, it backfires immediately, as it destroys the associativity of the merge. This can be easily checked by expanding $((\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \parallel \underline{\underline{b}}) \parallel \underline{\underline{c}}$ and $(\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \parallel (\underline{\underline{b}} \parallel \underline{\underline{c}})$; it turns out that the second process has a summand $\underline{\underline{c}} \cdot \underline{\underline{b}}$, while the first process does not. This is of course unacceptable, as we already stated in our design goals. Any attempt at such "repairs" appears to have this consequence, unless one is willing to sacrifice the right-distributivity of the $\cdot$ over the $+$ (Axiom A4), which, again, would violate our design goals.

For other discussions of the above dilemma, see [9, 31].

**Remark 2.3.6 (Axioms of $\mathrm{PA}^{-}_{\mathbf{drt},\varepsilon}$–ID, Part IV)**
The behavior of $\underline{\underline{\varepsilon}}$ with respect to the merge manifests itself even more unexpectedly when time comes into play. For example, consider the expression $(\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\underline{\varepsilon}}) \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{b}}$.

Applying the axioms we get:

$$
\begin{aligned}
(\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\underline{\varepsilon}}) \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{b}} &= (\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\underline{\varepsilon}}) \mathbin{\|\!\!\!\lfloor} \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\sigma}} \cdot \underline{\underline{b}} \mathbin{\|\!\!\!\lfloor} (\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\underline{\varepsilon}}) \\
&= \underline{\underline{\sigma}} \cdot \underline{\underline{a}} \mathbin{\|\!\!\!\lfloor} \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\varepsilon}} \mathbin{\|\!\!\!\lfloor} \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\sigma}} \cdot \underline{\underline{b}} \mathbin{\|\!\!\!\lfloor} (\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\underline{\varepsilon}}) \\
&= \underline{\underline{\sigma}} \cdot \underline{\underline{a}} \mathbin{\|\!\!\!\lfloor} \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\delta}} + \underline{\underline{\sigma}} \cdot \underline{\underline{b}} \mathbin{\|\!\!\!\lfloor} \underline{\underline{\sigma}} \cdot \underline{\underline{a}} \\
&= \underline{\underline{\sigma}} \cdot \underline{\underline{a}} \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{b}}
\end{aligned}
$$

So, the $\underline{\underline{\varepsilon}}$-summand on the left side of the merge is immaterial! How should we interpret this?

Again, by viewing $\underline{\underline{\varepsilon}}$ as the option to terminate (in the first time-slice, in this case), we can see that this option can never be exercised. That is to say, because the right component of the merge cannot terminate in the first time-slice, the option to terminate of the left component remains open till we enter the second time-slice. But as we leave the first time-slice, the $\underline{\underline{\varepsilon}}$ disappears, as it only presents an option to terminate *in that time-slice.* Hence, it vanishes into thin air.

As we have argued before, attempting to "repair" this behavior by treating $\underline{\underline{\varepsilon}}$ as an action, means we lose the associativity of the merge. Alternatively, we could replace DRTEM12 with the following:

$$
\underline{\underline{\sigma}} \cdot x \mathbin{\|\!\!\!\lfloor} (\underline{\underline{\sigma}} \cdot y + \underline{\underline{\varepsilon}}) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\!\!\lfloor} \underline{\underline{\sigma}} \cdot y + \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\!\!\lfloor} \underline{\underline{\varepsilon}}
$$

Then the $\underline{\underline{\varepsilon}}$ does not vanish, but leads to an extra summand $\underline{\underline{\sigma}} \cdot b$ in the above example. This alternative, however, violates the time-determinacy principle: $\underline{\underline{\sigma}} \cdot \underline{\underline{a}} \mathbin{\|\!\!\!\lfloor} (\underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\varepsilon}})$ can do a time step to $\underline{\underline{a}} \parallel \underline{\underline{b}}$, while $\underline{\underline{\sigma}} \cdot \underline{\underline{a}} \mathbin{\|\!\!\!\lfloor} \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\sigma}} \cdot \underline{\underline{b}}$ can do a time step to $\underline{\underline{a}} \parallel \underline{\underline{b}} + \underline{\underline{b}}$. As stated in our design goals, this is not acceptable. Again, we are stuck.

Now look at the following example. Applying our axioms, we have:

$$
\begin{aligned}
(\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{b}} &= (\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \mathbin{\|\!\!\!\lfloor} \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\sigma}} \cdot \underline{\underline{b}} \mathbin{\|\!\!\!\lfloor} (\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \\
&= \underline{\underline{a}} \mathbin{\|\!\!\!\lfloor} \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\varepsilon}} \mathbin{\|\!\!\!\lfloor} \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\sigma}} \cdot \underline{\underline{b}} \mathbin{\|\!\!\!\lfloor} (\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \\
&= \underline{\underline{a}} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\delta}} + \underline{\underline{\sigma}} \cdot \underline{\underline{b}} \mathbin{\|\!\!\!\lfloor} \underline{\underline{\varepsilon}} \\
&= \underline{\underline{a}} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{b}} + \underline{\underline{\sigma}} \cdot \underline{\underline{b}} \\
&= (\underline{\underline{a}} + \underline{\underline{\varepsilon}}) \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{b}}
\end{aligned}
$$

Here, the $\underline{\underline{\varepsilon}}$-summand on the left side of the merge does materialize. This is due to the fact that $\underline{\underline{a}} + \underline{\underline{\varepsilon}}$ cannot move on to the second time-slice, while $\underline{\underline{\sigma}} \cdot \underline{\underline{b}}$ can. So, $\underline{\underline{a}} + \underline{\underline{\varepsilon}}$ is forced to terminate before $\underline{\underline{\sigma}} \cdot \underline{\underline{b}}$ can move on, and the $\underline{\underline{\varepsilon}}$ does not vanish.

If, for the sake of orthogonality, we would want the $\underline{\underline{\varepsilon}}$-summand to disappear in this case also, we could for example replace DRTEM10 by $\underline{\underline{\sigma}} \cdot x \mathbin{\|\!\!\!\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{\delta}}$. That however, would lead to $\underline{\underline{\sigma}} \parallel \underline{\underline{\varepsilon}} = \underline{\underline{\delta}}$, destroying the unit property of $\underline{\underline{\varepsilon}}$ with respect to the $\parallel$. Furthermore, this leads to $\underline{\underline{\sigma}} \parallel \underline{\underline{a}} = \underline{\underline{a}} \cdot \underline{\underline{\delta}}$, in itself enough reason to dismiss this alternative. If we attempt to save what can be saved, for example by putting $\underline{\underline{\varepsilon}} \mathbin{\|\!\!\!\lfloor} x = x$, we again lose the associativity of the merge.

Concluding: when $x$ can terminate, and $y$ can do a time step, then the termination option of $x$ materializes in $x \parallel y$ *if and only if* $x$ cannot do a time step. This may seem, and probably is, counter intuitive. However, a simpler way is not conceivable. If the termination option would *always* materialize, we either lose the associativity of the merge or time determinacy. If it would *never* materialize, we lose the unit property with respect to the merge and the associativity of the merge. So, both simple ways are too simple, and lead to violations of our design goals.

**Remark 2.3.7 (Axioms of $\mathbf{PA}^{-}_{\mathbf{drt},\varepsilon}$–ID, Part V)**
In the case where one of the $n$ components in a multiple merge has *only* the option to terminate (so the component has no enabled actions or time steps), while the other components do not all have that option, the $\underline{\varepsilon}$ again vanishes, collapsing the merge to $n-1$ components. So for example, $\underline{\varepsilon} \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{a}} \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{b}} = \underline{\underline{\sigma}} \cdot \underline{\underline{a}} \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{b}}$. Or, more generally, $\underline{\varepsilon} \parallel x = x \parallel \underline{\varepsilon} = x$ for any closed process term $x$: with respect to closed terms, $\underline{\varepsilon}$ is also a proper 1-element for the merge. We will prove this in Proposition 2.3.16 on page 27.

**Definition 2.3.8 (Semantics of $\mathbf{PA}^{-}_{\mathbf{drt},\varepsilon}$–ID)**
The semantics of $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID are given by the term deduction system $T(\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID})$, induced by the deduction rules for $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID given in Definition 2.2.5 on page 14, and the additional deduction rules for $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID shown in Table 7.

$$\frac{x \xrightarrow{a} x'}{x \parallel y \xrightarrow{a} x' \parallel y} \qquad \frac{y \xrightarrow{a} y'}{x \parallel y \xrightarrow{a} x \parallel y'} \qquad \frac{x \xrightarrow{a} x'}{x \mathbin{\lfloor\!\lfloor} y \xrightarrow{a} x' \parallel y}$$

$$\frac{x \xrightarrow{\sigma} x',\ y \xrightarrow{\sigma} y'}{x \parallel y \xrightarrow{\sigma} x' \parallel y'} \qquad \frac{x \xrightarrow{\sigma} x',\ y \xrightarrow{\sigma} y'}{x \mathbin{\lfloor\!\lfloor} y \xrightarrow{\sigma} x' \mathbin{\lfloor\!\lfloor} y'}$$

$$\frac{x \not\xrightarrow{\sigma},\ x{\downarrow},\ y \xrightarrow{\sigma} y'}{x \parallel y \xrightarrow{\sigma} y'} \qquad \frac{x \xrightarrow{\sigma} x',\ y \not\xrightarrow{\sigma},\ y{\downarrow}}{x \parallel y \xrightarrow{\sigma} x'} \qquad \frac{x \xrightarrow{\sigma} x',\ y \not\xrightarrow{\sigma},\ y{\downarrow}}{x \mathbin{\lfloor\!\lfloor} y \xrightarrow{\sigma} x'}$$

$$\frac{x{\downarrow},\ y{\downarrow}}{(x \parallel y){\downarrow}} \qquad \frac{x{\downarrow},\ y{\downarrow}}{(x \mathbin{\lfloor\!\lfloor} y){\downarrow}}$$

Table 7: Additional deduction rules for $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID.

**Remark 2.3.9 (Deduction rules for $\mathbf{PA}^{-}_{\mathbf{drt},\varepsilon}$–ID)**
The deduction rules for the parallel composition with respect to $\sigma$-transitions, shown in Table 7, may need some clarification. Consider the following examples.

As we have $\underline{\varepsilon} \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{a}} = \underline{\underline{\sigma}} \cdot \underline{\underline{a}}$, we should have $\underline{\varepsilon} \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{a}} \xrightarrow{\sigma} \underline{\underline{a}}$. Likewise, as $(\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\varepsilon}) \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{b}} = \underline{\underline{\sigma}} \cdot (\underline{\underline{a}} \parallel \underline{\underline{b}})$, we should have $(\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\varepsilon}) \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{b}} \xrightarrow{\sigma} \underline{\underline{a}} \parallel \underline{\underline{b}}$, and not $(\underline{\underline{\sigma}} \cdot \underline{\underline{a}} + \underline{\varepsilon}) \parallel \underline{\underline{\sigma}} \cdot \underline{\underline{b}} \xrightarrow{\sigma} \underline{\underline{b}}$.

**Theorem 2.3.10 (Time Determinacy for $\mathbf{PA}^{-}_{\mathbf{drt},\varepsilon}$–ID)**
*Let $x$, $y$, and $y'$ be closed $PA^{-}_{drt,\varepsilon}$–ID terms. Then we have:*

$$T(PA^{-}_{drt,\varepsilon}\text{–}ID) \models x \xrightarrow{\sigma} y,\ x \xrightarrow{\sigma} y' \implies y \equiv y'$$

**Proof** We proceed in the manner outlined in Theorem 2.2.7 on page 15. As the only new deduction rules are for the $\|$ and the $\lfloor\!\lfloor$ , we do not repeat the cases (i)–(v) already listed in Theorem 2.2.7.

(vi). $x \equiv s \| t$ for closed $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$-ID terms $s$ and $t$. We proceed by case distinction on the transitions of $s$ and $t$.

(a) $T(\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \xrightarrow{\sigma} s',\ t \xrightarrow{\sigma} t'$. Then $y \equiv y' \equiv s' \| t'$.

(b) $T(\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \overset{\sigma}{\nrightarrow},\ s{\downarrow},\ t \xrightarrow{\sigma} t'$. Then $y \equiv y' \equiv t'$.

(c) $T(\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \xrightarrow{\sigma} s',\ t \overset{\sigma}{\nrightarrow},\ t{\downarrow}$. Then $y \equiv y' \equiv s'$.

(d) Otherwise. In contradiction with $T(\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{-ID}) \models x \xrightarrow{\sigma} y$.

(vii). $x \equiv s \lfloor\!\lfloor t$ for closed $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$-ID terms $s$ and $t$. We proceed by case distinction on the transitions of $s$ and $t$.

(a) $T(\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \xrightarrow{\sigma} s',\ t \xrightarrow{\sigma} t'$. Then $y \equiv y' \equiv s' \lfloor\!\lfloor t'$.

(b) $T(\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{-ID}) \models s \xrightarrow{\sigma} s',\ t \overset{\sigma}{\nrightarrow},\ t{\downarrow}$. Then $y \equiv y' \equiv s'$.

(c) Otherwise. In contradiction with $T(\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{-ID}) \models x \xrightarrow{\sigma} y$.

Having inspected all possible cases, we may now conclude that $y \equiv y'$. ∎

**Definition 2.3.11 (Bisimulation and Bisimulation Model for $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
Bisimulation for $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID. Replace "$\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID" by "$\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID" in Definition 2.2.8 on page 15 and Definition 2.2.9 on page 16.

**Definition 2.3.12 (Basic Terms of $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
When we speak of basic terms in the context of $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID, we mean $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}})$-basic terms as defined in Definition 2.2.10 on page 16.

**Theorem 2.3.13 (Elimination for $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
*Let t be a closed $PA^{-}_{drt,\varepsilon}$-ID term. Then there is a basic term s such that $PA^{-}_{drt,\varepsilon}$-ID $\vdash s = t$.*

**Proof** By structural induction on $t$ prove that every closed $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID term can be rewritten into a $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID term, and then apply Theorem 2.2.21 on page 20. We give no details. ∎

**Theorem 2.3.14 (Soundness of $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
*The set of closed $PA^{-}_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence is a model of the axioms of $PA^{-}_{drt,\varepsilon}$-ID.*

**Proof** Extending the proof of Theorem 2.2.22 on page 20, derive, for every new axiom, a concrete bisimulation relation that relates both sides of the axiom for all closed instantiations of the free variables. We give no details. ∎

**Theorem 2.3.15 (Completeness of $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
*The axiom system $PA^{-}_{drt,\varepsilon}$-ID is a complete axiomatization of the set of closed $PA^{-}_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence.*

**Proof** Use Verhoef's General Completeness Theorem [29]. We give no details. ∎

**Proposition 2.3.16 (Properties of $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID, Part I)**
*Let $x$ be a basic term and $a \in A_\delta$. Then the following properties hold:*

*(i).* $PA^-_{drt,\varepsilon}\text{-}ID \vdash x \parallel \underline{\underline{\varepsilon}} = x$

*(ii).* $PA^-_{drt,\varepsilon}\text{-}ID \vdash \underline{\underline{\varepsilon}} \parallel x = x$

*(iii).* $PA^-_{drt,\varepsilon}\text{-}ID \vdash \underline{\underline{a}} \mathbin{\|\mkern-9mu\lfloor} x = \underline{\underline{a}} \cdot x$

*(iv).* $PA^-_{drt,\varepsilon}\text{-}ID \vdash \underline{\underline{\delta}} \mathbin{\|\mkern-9mu\lfloor} x = \underline{\underline{\delta}}$

*(v).* $PA^-_{drt,\varepsilon}\text{-}ID \vdash x \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} = x$

**Proof**

(i). We prove this by induction on the structure of basic terms (Definition 2.3.12 on the preceding page).

(a) $x \equiv \underline{\underline{\varepsilon}}$. Then $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \parallel \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} \parallel \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} = x$.

(b) $x \equiv \underline{\underline{a}} \cdot t$ for some $a \in A_\delta$ and basic term $t$. Then $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \parallel \underline{\underline{\varepsilon}} = \underline{\underline{a}} \cdot t \parallel \underline{\underline{\varepsilon}} = \underline{\underline{a}} \cdot t \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{a}} \cdot t = \underline{\underline{a}} \cdot (t \parallel \underline{\underline{\varepsilon}}) + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot t + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot t = x$.

(c) $x \equiv s + t$ for some basic terms $s$ and $t$. Then $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \parallel \underline{\underline{\varepsilon}} = (s+t) \parallel \underline{\underline{\varepsilon}} = (s+t) \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} (s+t) = s \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} + t \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} s + \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} t = s \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} s + t \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} t = s \parallel \underline{\underline{\varepsilon}} + t \parallel \underline{\underline{\varepsilon}} = s + t = x$.

(d) $x \equiv \underline{\underline{\sigma}} \cdot t$ for some basic term $t$. Then $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \parallel \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot t \parallel \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot t \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\sigma}} \cdot t = \underline{\underline{\sigma}} \cdot t + \underline{\underline{\delta}} = \underline{\underline{\sigma}} \cdot t = x$.

(ii). Using (i), we have $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \underline{\underline{\varepsilon}} \parallel x = \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} x + x \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} = x \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} x = x \parallel \underline{\underline{\varepsilon}} = x$.

(iii). Using (ii), we have $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \underline{\underline{a}} \mathbin{\|\mkern-9mu\lfloor} x = \underline{\underline{a}} \cdot \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} x = \underline{\underline{a}} \cdot (\underline{\underline{\varepsilon}} \parallel x) = \underline{\underline{a}} \cdot x$.

(iv). Using (iii), we have $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \underline{\underline{\delta}} \mathbin{\|\mkern-9mu\lfloor} x = \underline{\underline{\delta}} \cdot x = \underline{\underline{\delta}}$.

(v). We again prove this by induction on the structure of basic terms.

(a) $x \equiv \underline{\underline{\varepsilon}}$. Then $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} = x$.

(b) $x \equiv \underline{\underline{a}} \cdot t$ for some $a \in A_\delta$ and basic term $t$. Then $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{a}} \cdot t \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{a}} \cdot (t \parallel \underline{\underline{\varepsilon}}) = \underline{\underline{a}} \cdot t = x$.

(c) $x \equiv s + t$ for some basic terms $s$ and $t$. Then $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash (s+t) \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} = s \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} + t \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} = s + t = x$.

(d) $x \equiv \underline{\underline{\sigma}} \cdot t$ for some basic term $t$. Then $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x = \underline{\underline{\sigma}} \cdot t \mathbin{\|\mkern-9mu\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot t = x$.

■

**Proposition 2.3.17 (Properties of $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID, Part II)**
*Let $x$, $y$, $z$ be basic terms. Then the following properties hold:*

*(i).* $PA^-_{drt,\varepsilon}\text{-}ID \vdash \underline{\underline{\varepsilon}} \mathbin{\|\mkern-9mu\lfloor} x = \begin{cases} \underline{\underline{\varepsilon}} & \text{if } x\!\downarrow \\ \underline{\underline{\delta}} & \text{if } x\!\not\downarrow \end{cases}$

*(ii).* $PA^{-}_{drt,\varepsilon}\text{-}ID \vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(y) = \begin{cases} \underline{\underline{\sigma}} \cdot x & \mathit{if}\ y\downarrow \\ \underline{\underline{\delta}} & \mathit{if}\ y\updownarrow \end{cases}$

*(iii).* $PA^{-}_{drt,\varepsilon}\text{-}ID \vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} (\nu(y) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}}(x \mathbin{\|\!\lfloor} z)$

*(iv).* $PA^{-}_{drt,\varepsilon}\text{-}ID \vdash \nu(x) \mathbin{\|\!\lfloor} y = \nu(\nu(x) \mathbin{\|\!\lfloor} y)$.

*(v).* $PA^{-}_{drt,\varepsilon}\text{-}ID \vdash \nu(x) \parallel \nu(y) = \nu(\nu(x) \parallel \nu(y))$.

**Proof**

(i). We use induction on the structure of $x$.

(a) $x \equiv \underline{\underline{\varepsilon}}$. Then $x\downarrow$, and we have $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} x = \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}$.

(b) $x \equiv \underline{\underline{a}} \cdot t$ for some $a \in A_\delta$ and basic term $t$. Then $x\updownarrow$, and we have $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} x = \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} \underline{\underline{a}} \cdot t = \underline{\underline{\delta}}$.

(c) $x \equiv s+t$ for some basic terms $s$ and $t$. First, suppose that $x\updownarrow$. Then, $s\updownarrow$ and $t\updownarrow$. So, by the induction hypothesis, $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} x = \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} (s+t) = \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} s + \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} t = \underline{\underline{\delta}} + \underline{\underline{\delta}} = \underline{\underline{\delta}}$. Secondly, suppose that $x\downarrow$. Then, $s\downarrow$, or $t\downarrow$, or both. Assume that $s\downarrow$ and $t\updownarrow$. Then, by the induction hypothesis $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} x = \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} (s+t) = \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} s + \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} t = \underline{\underline{\varepsilon}} + \underline{\underline{\delta}} = \underline{\underline{\varepsilon}}$. The cases $s\updownarrow$, $t\downarrow$ and $s\downarrow$, $t\downarrow$ are handled in the same way.

(d) $x \equiv \underline{\underline{\sigma}} \cdot t$ for some basic term $t$. Then $x\updownarrow$, and we have $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} x = \underline{\underline{\varepsilon}} \cdot \underline{\underline{\sigma}} \cdot t = \underline{\underline{\delta}}$.

(ii). As $y$ is a basic term, by Corollary 2.2.20, we may use induction on the structure of ANF terms

(a) $y \equiv \underline{\underline{\delta}}$. Then $y\updownarrow$, and we have $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(y) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(\underline{\underline{\delta}}) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(\underline{\underline{\delta}} \cdot \underline{\underline{\varepsilon}}) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \underline{\underline{\delta}} \cdot \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \underline{\underline{\delta}} = \underline{\underline{\delta}}$.

(b) $y \equiv \underline{\underline{\varepsilon}}$. Then $y\downarrow$, and we have $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(y) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(\underline{\underline{\varepsilon}}) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot x$.

(c) $y \equiv \underline{\underline{a}} \cdot s + t$ for some $a \in A_\delta$ and basic terms $s$ and $t$. Then $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(y) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(\underline{\underline{a}} \cdot s + t) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} (\nu(\underline{\underline{a}} \cdot s) + \nu(t)) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} (\underline{\underline{a}} \cdot s + \nu(t)) = \underline{\underline{\sigma}} \cdot \nu(t)$. Now, using the induction hypothesis, we get $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} n(t) = \underline{\underline{\sigma}} \cdot x$ when $t\downarrow$, hence $y\downarrow$, and $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} n(t) = \underline{\underline{\delta}}$ when $t\updownarrow$, hence $y\updownarrow$.

(d) $y \equiv \underline{\underline{\sigma}} \cdot t$ for some basic term $t$. Then $y\updownarrow$, and we have $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(y) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(\underline{\underline{\sigma}} \cdot t) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \underline{\underline{\delta}} = \underline{\underline{\delta}}$.

(e) $y \equiv \underline{\underline{\sigma}} \cdot t + \underline{\underline{\varepsilon}}$ for some basic term $t$. Then $y\downarrow$, and we have $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(y) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \nu(\underline{\underline{\sigma}} \cdot t + \underline{\underline{\varepsilon}}) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} (\nu(\underline{\underline{\sigma}} \cdot t) + \nu(\underline{\underline{\varepsilon}})) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} (\underline{\underline{\delta}} + \underline{\underline{\varepsilon}}) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot x$.

(iii). As $y$ is a basic term, by Corollary 2.2.20, we may use induction on the structure of ANF terms.

(a) $y \equiv \underline{\underline{\delta}}$. Then $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} (\nu(y) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} (\nu(\underline{\underline{\delta}}) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} (\underline{\underline{\delta}} + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\lfloor} \underline{\underline{\sigma}} \cdot z = \underline{\underline{\sigma}} \cdot (x \mathbin{\|\!\lfloor} z)$.

(b) $y \equiv \underline{\underline{\varepsilon}}$. Then $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(y) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(\underline{\underline{\varepsilon}}) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot (x \mathbin{\|\!\_} z)$.

(c) $y \equiv \underline{\underline{a}} \cdot s + t$ for some $a \in A_\delta$ and basic terms $s$ and $t$. Then $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(y) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(\underline{\underline{a}} \cdot s + t) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(\underline{\underline{a}} \cdot s) + \nu(t) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\underline{\underline{a}} \cdot s + \nu(t) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(t) + \underline{\underline{\sigma}} \cdot z)$. Now, using the induction hypothesis, $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(t) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot (x \mathbin{\|\!\_} z)$.

(d) $y \equiv \underline{\underline{\sigma}} \cdot t$ for some basic term $t$. Then $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(y) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(\underline{\underline{\sigma}} \cdot t) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\underline{\underline{\delta}} + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} \underline{\underline{\sigma}} \cdot z = \underline{\underline{\sigma}} \cdot (x \mathbin{\|\!\_} z)$.

(e) $y \equiv \underline{\underline{\sigma}} \cdot t + \underline{\underline{\varepsilon}}$ for some basic term $t$. Then $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(y) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(\underline{\underline{\sigma}} \cdot t + \underline{\underline{\varepsilon}}) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\nu(\underline{\underline{\sigma}} \cdot t) + \nu(\underline{\underline{\varepsilon}}) + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\underline{\underline{\delta}} + \underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot x \mathbin{\|\!\_} (\underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} \cdot z) = \underline{\underline{\sigma}} \cdot (x \mathbin{\|\!\_} z)$.

(iv). We use induction on the structure of $x$.

(a) $x \equiv \underline{\underline{\varepsilon}}$. Suppose that $y\downarrow$. Using (i), we then have:

$$\begin{aligned}\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \nu(x) \mathbin{\|\!\_} y &= \nu(\underline{\underline{\varepsilon}}) \mathbin{\|\!\_} y = \underline{\underline{\varepsilon}} \mathbin{\|\!\_} y = \underline{\underline{\varepsilon}} = \nu(\underline{\underline{\varepsilon}}) = \nu(\underline{\underline{\varepsilon}} \mathbin{\|\!\_} y) = \\ &\nu(\nu(\underline{\underline{\varepsilon}}) \mathbin{\|\!\_} y) = \nu(\nu(x) \mathbin{\|\!\_} y)\end{aligned}$$

The case where $y\updownarrow$ is handled in the same way.

(b) $x \equiv \underline{\underline{a}} \cdot t$ for some $a \in A_\delta$ and basic term $t$. Using Proposition 2.3.16(iii), we then have:

$$\begin{aligned}\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \nu(x) \mathbin{\|\!\_} y &= \nu(\underline{\underline{a}} \cdot t) \mathbin{\|\!\_} y = \underline{\underline{a}} \cdot t \mathbin{\|\!\_} y = \underline{\underline{a}} \cdot (t \parallel y) = \\ &\nu(\underline{\underline{a}} \cdot (t \parallel y)) = \nu(\underline{\underline{a}} \cdot t \mathbin{\|\!\_} y) = \nu(\nu(\underline{\underline{a}} \cdot t) \mathbin{\|\!\_} y) = \\ &\nu(\nu(x) \mathbin{\|\!\_} y)\end{aligned}$$

(c) $x \equiv s + t$ for some basic terms $s$ and $t$. Using the induction hypothesis, we then have:

$$\begin{aligned}\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \nu(x) \mathbin{\|\!\_} y &= \nu(s+t) \mathbin{\|\!\_} y = (\nu(s) + \nu(t)) \mathbin{\|\!\_} y = \\ &\nu(s) \mathbin{\|\!\_} y + \nu(t) \mathbin{\|\!\_} y = \nu(\nu(s) \mathbin{\|\!\_} y) + \nu(\nu(t) \mathbin{\|\!\_} y) = \\ &\nu(\nu(s) \mathbin{\|\!\_} y + \nu(t) \mathbin{\|\!\_} y) = \nu((\nu(s) + \nu(t)) \mathbin{\|\!\_} y) = \\ &\nu(\nu(s+t) \mathbin{\|\!\_} y) = \nu(\nu(x) \mathbin{\|\!\_} y)\end{aligned}$$

(d) $x \equiv \underline{\underline{\sigma}} \cdot t$ for some basic term $t$. Using Proposition 2.3.16(iv), we then have:

$$\begin{aligned}\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \nu(x) \mathbin{\|\!\_} y &= \nu(\underline{\underline{\sigma}} \cdot t) \mathbin{\|\!\_} y = \underline{\underline{\delta}} \mathbin{\|\!\_} y = \underline{\underline{\delta}} = \underline{\underline{\delta}} \cdot \underline{\underline{\varepsilon}} = \nu(\underline{\underline{\delta}} \cdot \underline{\underline{\varepsilon}}) = \\ &\nu(\underline{\underline{\delta}}) = \nu(\underline{\underline{\delta}} \mathbin{\|\!\_} y) = \nu(\nu(\underline{\underline{\sigma}} \cdot t) \mathbin{\|\!\_} y) = \nu(\nu(x) \mathbin{\|\!\_} y)\end{aligned}$$

(v). Using (iv), we have:

$$\begin{aligned}\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \nu(x) \parallel \nu(y) &= \nu(x) \mathbin{\|\!\_} \nu(y) + \nu(y) \mathbin{\|\!\_} \nu(x) = \\ &\nu(\nu(x) \mathbin{\|\!\_} \nu(y)) + \nu(\nu(y) \mathbin{\|\!\_} \nu(x)) = \\ &\nu(\nu(x) \mathbin{\|\!\_} \nu(y) + \nu(y) \mathbin{\|\!\_} \nu(x)) = \nu(\nu(x) \parallel \nu(y))\end{aligned}$$

$\blacksquare$

**Theorem 2.3.18 (Axioms of Standard Concurrency for $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
*Let $x$, $y$, $z$ be closed $PA^{-}_{drt,\varepsilon}$-ID terms. Then the following properties hold:*

*(i).* $PA^{-}_{drt,\varepsilon}\text{-}ID \vdash x \parallel y = y \parallel x$

*(ii).* $PA^{-}_{drt,\varepsilon}\text{-}ID \vdash (x \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z = x \mathbin{\|\!\!\lfloor} (y \parallel z)$

*(iii).* $PA^{-}_{drt,\varepsilon}\text{-}ID \vdash (x \parallel y) \parallel z = x \parallel (y \parallel z)$

**Proof** By Theorem 2.3.13 on page 26, we may assume that $x$, $y$ and $z$ are basic terms.

(i). This follows directly from the axioms: $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \parallel y = x \mathbin{\|\!\!\lfloor} y + y \parallel x = y \mathbin{\|\!\!\lfloor} x + x \mathbin{\|\!\!\lfloor} y = y \parallel x$ (note that this holds for open terms also).

(ii). We use induction on the structure of basic term $x$, combined with simultaneous induction with (iii):

(a) $x \equiv \underline{\underline{\varepsilon}}$. Using Proposition 2.3.17(i), we then have:

$$\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash (x \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z = (\underline{\underline{\varepsilon}} \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z = \begin{cases} \underline{\underline{\varepsilon}} \mathbin{\|\!\!\lfloor} z \text{ if } y\downarrow \\ \underline{\underline{\delta}} \mathbin{\|\!\!\lfloor} z \text{ if } y\updownarrow \end{cases} = \begin{cases} \underline{\underline{\varepsilon}} \text{ if } y\downarrow,\ z\downarrow \\ \underline{\underline{\delta}} \text{ otherwise} \end{cases} =$$
$$\begin{cases} \underline{\underline{\varepsilon}} \text{ if } (y \parallel z)\downarrow \\ \underline{\underline{\delta}} \text{ if } (y \parallel z)\updownarrow \end{cases} = \underline{\underline{\varepsilon}} \mathbin{\|\!\!\lfloor} (y \parallel z) = x \mathbin{\|\!\!\lfloor} (y \parallel z)$$

(b) $x \equiv \underline{\underline{a}} \cdot t$ for some $a \in A_\delta$ and basic term $t$. Using simultaneous induction with (iii), and using Proposition 2.3.16(iii) we then have:

$$\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash (x \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z = (\underline{\underline{a}} \cdot t \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z = \underline{\underline{a}} \cdot (t \parallel y) \mathbin{\|\!\!\lfloor} z =$$
$$\underline{\underline{a}} \cdot ((t \parallel y) \parallel z) = \underline{\underline{a}} \cdot (t \parallel (y \parallel z)) = \underline{\underline{a}} \cdot t \mathbin{\|\!\!\lfloor} (y \parallel z) = x \mathbin{\|\!\!\lfloor} (y \parallel z)$$

(c) $x \equiv s + t$ for some basic terms $s$ and $t$. Using the induction hypothesis, we then have:

$$\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash (x \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z = ((s + t) \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z = (s \mathbin{\|\!\!\lfloor} y + t \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z =$$
$$(s \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z + (t \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z = s \mathbin{\|\!\!\lfloor} (y \parallel z) + t \mathbin{\|\!\!\lfloor} (y \parallel z) =$$
$$(s + t) \mathbin{\|\!\!\lfloor} (y \parallel z) = x \mathbin{\|\!\!\lfloor} (y \parallel z)$$

(d) $x \equiv \underline{\underline{\sigma}} \cdot t$ for some basic term $t$. We assume, by Lemma 2.2.15 on page 17, that either $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash y = \nu(y)$, or that there exists a basic term $y'$, such that $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash y = \nu(y) + \underline{\underline{\sigma}} \cdot y'$ and $n(y') < n(y)$, and assume mutatis mutandis the same for $z$. We now distinguish four cases:

1. $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash y = \nu(y)$ and $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash z = \nu(z)$. Then we have, using Proposition 2.3.17:

$$\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash (x \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z = (\underline{\underline{\sigma}} \cdot t \mathbin{\|\!\!\lfloor} \nu(y)) \mathbin{\|\!\!\lfloor} \nu(z) =$$
$$\begin{cases} (\underline{\underline{\sigma}} \cdot t) \mathbin{\|\!\!\lfloor} \nu(z) & \text{if } y\downarrow \\ \underline{\underline{\delta}} \mathbin{\|\!\!\lfloor} \nu(z) & \text{if } y\updownarrow \end{cases} = \begin{cases} (\underline{\underline{\sigma}} \cdot t) \mathbin{\|\!\!\lfloor} \nu(z) & \text{if } y\downarrow \\ \underline{\underline{\delta}} & \text{if } y\updownarrow \end{cases} =$$

$$\begin{cases} \underline{\underline{\sigma}} \cdot t \text{ if } y{\downarrow},\ z{\downarrow} \\ \underline{\underline{\delta}} \quad \text{otherwise} \end{cases} = \begin{cases} \underline{\underline{\sigma}} \cdot t \text{ if } (y \parallel z){\downarrow} \\ \underline{\underline{\delta}} \quad \text{otherwise} \end{cases} =$$

$$\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} \nu(y \parallel z) = \underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} \nu(\nu(y) \parallel \nu(z)) =$$
$$\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(y) \parallel \nu(z)) = x \mathbin{|\!\!\lfloor} (y \parallel z)$$

2. $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$-ID $\vdash y = \nu(y)$ and $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$-ID $\vdash z = \nu(z) + \underline{\underline{\sigma}} \cdot z'$. Then we have, using Proposition 2.3.17:

$$\begin{aligned}
&\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{-ID} \vdash \\
&(x \mathbin{|\!\!\lfloor} y) \mathbin{|\!\!\lfloor} z = (\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} \nu(y)) \mathbin{|\!\!\lfloor} (\nu(z) + \underline{\underline{\sigma}} \cdot z') = \\
&\begin{cases} (\underline{\underline{\sigma}} \cdot t) \mathbin{|\!\!\lfloor} (\nu(z) + \underline{\underline{\sigma}} \cdot z') & \text{if } y{\downarrow} \\ \underline{\underline{\delta}} \mathbin{|\!\!\lfloor} (\nu(z) + \underline{\underline{\sigma}} \cdot z') & \text{if } y{\updownarrow} \end{cases} = \begin{cases} \underline{\underline{\sigma}} \cdot (t \mathbin{|\!\!\lfloor} z') & \text{if } y{\downarrow} \\ \underline{\underline{\delta}} & \text{if } y{\updownarrow} \end{cases} = \\
&\begin{cases} \underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(s) + \underline{\underline{\sigma}} \cdot z') & \text{if } y{\downarrow} \\ \underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} \nu(s) & \text{if } y{\updownarrow} \end{cases} = \begin{cases} \underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(s) + \underline{\underline{\sigma}} \cdot z') & \text{if } y{\downarrow} \\ \underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(s) + \underline{\underline{\delta}}) & \text{if } y{\updownarrow} \end{cases} = \\
&\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(s) + \underline{\underline{\sigma}} \cdot z' \mathbin{|\!\!\lfloor} \nu(y)) = \\
&\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(\nu(y) \mathbin{|\!\!\lfloor} z + \nu(z) \mathbin{|\!\!\lfloor} y) + \underline{\underline{\sigma}} \cdot z' \mathbin{|\!\!\lfloor} \nu(y)) = \\
&\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(\nu(y) \mathbin{|\!\!\lfloor} z) + \nu(\nu(z) \mathbin{|\!\!\lfloor} y) + \underline{\underline{\sigma}} \cdot z' \mathbin{|\!\!\lfloor} \nu(y)) = \\
&\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(y) \mathbin{|\!\!\lfloor} z + \nu(z) \mathbin{|\!\!\lfloor} y + \underline{\underline{\sigma}} \cdot z' \mathbin{|\!\!\lfloor} \nu(y)) = \\
&\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(y) \mathbin{|\!\!\lfloor} (\nu(z) + \underline{\underline{\sigma}} \cdot z') + (\nu(z) + \underline{\underline{\sigma}} \cdot z') \mathbin{|\!\!\lfloor} \nu(y)) = \\
&\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(y) \parallel (\nu(z) + \underline{\underline{\sigma}} \cdot z')) = \\
&x \mathbin{|\!\!\lfloor} (y \parallel z)
\end{aligned}$$

where we define:

$$s \equiv \nu(y) \mathbin{|\!\!\lfloor} z + \nu(z) \mathbin{|\!\!\lfloor} y$$

and use that if $y{\updownarrow}$, then $s{\updownarrow}$.

3. $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$-ID $\vdash y = \nu(y) + \underline{\underline{\sigma}} \cdot y'$ and $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$-ID $\vdash z = \nu(z) + \underline{\underline{\sigma}} \cdot z'$. This case is handled in the same way as the previous case.
4. $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$-ID $\vdash y = \nu(y) + \underline{\underline{\sigma}} \cdot y'$ and $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$-ID $\vdash z = \nu(z) + \underline{\underline{\sigma}} \cdot z'$. Then we have, using Proposition 2.3.17, and simultaneous induction with (iii):

$$\begin{aligned}
\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{-ID} \vdash (x \mathbin{|\!\!\lfloor} y) \mathbin{|\!\!\lfloor} z = \qquad & \\
&(\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(y) + \underline{\underline{\sigma}} \cdot y')) \mathbin{|\!\!\lfloor} (\nu(z) + \underline{\underline{\sigma}} \cdot z') = \\
&(\underline{\underline{\sigma}} \cdot (t \mathbin{|\!\!\lfloor} y')) \mathbin{|\!\!\lfloor} (\nu(z) + \underline{\underline{\sigma}} \cdot z') = \\
&\underline{\underline{\sigma}} \cdot ((t \mathbin{|\!\!\lfloor} y') \mathbin{|\!\!\lfloor} z') = \\
&\underline{\underline{\sigma}} \cdot (t \mathbin{|\!\!\lfloor} (y' \parallel z')) = \\
&\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(\nu(y) \mathbin{|\!\!\lfloor} z + \nu(z) \mathbin{|\!\!\lfloor} y) + \underline{\underline{\sigma}} \cdot (y' \parallel z')) = \\
&\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} (\nu(\nu(y) \mathbin{|\!\!\lfloor} z + \nu(z) \mathbin{|\!\!\lfloor} y) + \underline{\underline{\sigma}} \cdot (y' \mathbin{|\!\!\lfloor} z' + z' \mathbin{|\!\!\lfloor} y')) = \\
&\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} \begin{pmatrix} \nu(\nu(y) \mathbin{|\!\!\lfloor} z) + \underline{\underline{\sigma}} \cdot (y' \mathbin{|\!\!\lfloor} z') + \\ \nu(\nu(z) \mathbin{|\!\!\lfloor} y) + \underline{\underline{\sigma}} \cdot (z' \mathbin{|\!\!\lfloor} y') \end{pmatrix} = \\
&\underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} \begin{pmatrix} \nu(y) \mathbin{|\!\!\lfloor} z + \underline{\underline{\sigma}} \cdot y' \mathbin{|\!\!\lfloor} (\nu(z) + \underline{\underline{\sigma}} \cdot z') + \\ \nu(z) \mathbin{|\!\!\lfloor} y + \underline{\underline{\sigma}} \cdot z' \mathbin{|\!\!\lfloor} (\nu(y) + \underline{\underline{\sigma}} \cdot y') \end{pmatrix} =
\end{aligned}$$

$$\underline{\underline{\sigma}} \cdot t \mathbin{⫟} \begin{pmatrix} (\nu(y) + \underline{\underline{\sigma}} \cdot y') \mathbin{⫟} (\nu(z) + \underline{\underline{\sigma}} \cdot z'') + \\ (\nu(z) + \underline{\underline{\sigma}} \cdot z') \mathbin{⫟} (\nu(y) + \underline{\underline{\sigma}} \cdot y'') \end{pmatrix} =$$
$$\underline{\underline{\sigma}} \cdot t \mathbin{⫟} ((\nu(y) + \underline{\underline{\sigma}} \cdot y') \parallel (\nu(z) + \underline{\underline{\sigma}} \cdot z'')) =$$
$$x \mathbin{⫟} (y \parallel z)$$

(iii). We use simultaneous induction with (ii). Using (i), we then have:

$$\begin{aligned} \mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}\text{-ID} \vdash\ & (x \parallel y) \parallel z = \\ & (x \parallel y) \mathbin{⫟} z + z \mathbin{⫟} (x \parallel y) = \\ & (x \mathbin{⫟} y + y \mathbin{⫟} x) \mathbin{⫟} z + z \mathbin{⫟} (x \parallel y) = \\ & (x \mathbin{⫟} y) \mathbin{⫟} z + (y \mathbin{⫟} x) \mathbin{⫟} z + z \mathbin{⫟} (y \parallel x) = \\ & x \mathbin{⫟} (y \parallel z) + y \mathbin{⫟} (x \parallel z) + (z \mathbin{⫟} y) \mathbin{⫟} x = \\ & x \mathbin{⫟} (y \parallel z) + y \mathbin{⫟} (z \parallel x) + (z \mathbin{⫟} y) \mathbin{⫟} x = \\ & x \mathbin{⫟} (y \parallel z) + (y \mathbin{⫟} z) \mathbin{⫟} x + (z \mathbin{⫟} y) \mathbin{⫟} x = \\ & x \mathbin{⫟} (y \parallel z) + (y \mathbin{⫟} z + z \mathbin{⫟} y) \mathbin{⫟} x = \\ & x \mathbin{⫟} (y \parallel z) + (y \parallel z) \mathbin{⫟} x = \\ & x \parallel (y \parallel z) \end{aligned}$$

■

**Remark 2.3.19 (Axioms of Standard Concurrency)**
The properties of Theorem 2.3.18 on page 30, together with the additional properties of Theorem 2.4.14 on page 36, are historically known as the *axioms of standard concurrency* (see for example page 97 of [13]). However, as their status is not that of *axioms*, but that of from the axioms derivable *properties*, the name is quite misplaced and misleading.

**Corollary 2.3.20 (Commutativity and Associativity of the Merge)**
*For closed terms, the free merge operator of $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$-ID is commutative and associative.*

**Proof** This follows directly from Theorem 2.3.18 on page 30. ■

## 2.4 The Merge Extension

In this section, we will define $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$-ID which is basically $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$-ID with a (full) merge instead of a free merge (indicated by the letters "ACP" instead of "PA"). The (full) merge is similar to the free merge, but is also allows the processes to communicate or synchronize.

**Definition 2.4.1 (Communication Function)**
For this section, and all sections to come, we presume the existence of a fixed, commutative, associative, complete function $\gamma : A_\delta \times A_\delta \to A_\delta$, that can be considered a parameter of the respective theories. The function $\gamma$ has to be *strict* in the sense that it should always evaluate to $\delta$ when one or both of its parameters is $\delta$.

**Definition 2.4.2 (Signature of $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$-ID)**
The signature of $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$-ID consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *time-unit delay constant* $\underline{\underline{\sigma}}$, the *undelayable empty*

*process constant* $\underline{\underline{\varepsilon}}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the the *"now" operator* $\nu$, *merge operator* $\|$, the *left merge operator* $\mathbin{\|\!\_}$ , and the *communication merge operator* $\mid$ .

**Definition 2.4.3 (Axioms of $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
The process algebra $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID is axiomatized by the axioms of $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID given in Definition 2.3.2 on page 22, *minus* Axiom DRTEM1, *plus* Axioms DRTECM1–DRTECM9 and DRTCF shown in Table 8: $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID = A1–A9 + TF + DCSE1–DCSE4 + DRTEM2–DRTEM12 + DRTECM1–DRTECM9 + DRTCF.

| | | |
|---|---|---|
| $x \parallel y = x \mathbin{\|\!\_} y + y \mathbin{\|\!\_} x + x \mid y$ | | DRTECM1 |
| $\underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot (x \parallel y)$ | | DRTECM2 |
| $\underline{\underline{\sigma}} \cdot x \mid \underline{\underline{\sigma}} \cdot y = \underline{\underline{\sigma}} \cdot (x \mid y)$ | | DRTECM3 |
| $\underline{\underline{\sigma}} \cdot x \mid \underline{\underline{a}} \cdot y = \underline{\underline{\delta}}$ | | DRTECM4 |
| $\underline{\underline{a}} \cdot x \mid \underline{\underline{\sigma}} \cdot y = \underline{\underline{\delta}}$ | | DRTECM5 |
| $x \mid \underline{\underline{\varepsilon}} = \underline{\underline{\delta}}$ | | DRTECM6 |
| $\underline{\underline{\varepsilon}} \mid x = \underline{\underline{\delta}}$ | | DRTECM7 |
| $(x + y) \mid z = x \mid z + y \mid z$ | | DRTECM8 |
| $x \mid (y + z) = x \mid y + x \mid z$ | | DRTECM9 |
| $\underline{\underline{a}} \mid \underline{\underline{b}} = \underline{\underline{c}}$ | if $\gamma(a,b) = c$ | DRTCF |

Table 8: Additional axioms for $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID.

**Remark 2.4.4 (Axioms of $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
The *merge* of two processes $x$ and $y$, notation $x \parallel y$, introduced in $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID is very much like the free merge of $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID, but it also allows $x$ and $y$ to communicate. The nature of this communication is determined by a function $\gamma$ of the form defined in Definition 2.4.1 on the preceding page. When we have $\gamma(a,b) = c$, this should be interpreted as "the execution of $c$ results if $a$ and $b$ are simultaneously executed". Then, for example $\underline{\underline{a}} \parallel \underline{\underline{b}} = \underline{\underline{a}} \cdot \underline{\underline{b}} + \underline{\underline{b}} \cdot \underline{\underline{a}} + \underline{\underline{c}}$: either the $a$ executes first, followed by the $b$, or the other way around, or both $a$ and $b$ execute at the same time, leading to the execution of $c$.

The *communication merge*, notation $x \mid y$, is an auxiliary operator that is introduced for the purpose of the axiomatization only; it is not intended to be used in actual specifications. The process $x \mid y$ is like $x \parallel y$, but always executes two actions from $x$ and $y$ simultaneously before continuing as a (full) merge.

**Definition 2.4.5 (Semantics of $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID)**
The semantics of $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID are given by the term deduction system $T(\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}\text{–ID})$, induced by the deduction rules for $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID given in Definition 2.3.8 on page 25, and the additional deduction rules for $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID shown in Table 9 on the following page.

$$\frac{x \xrightarrow{a} x',\ y \xrightarrow{b} y',\ \gamma(a,b) = c}{x \parallel y \xrightarrow{c} x' \parallel y'} \qquad \frac{x \xrightarrow{a} x',\ y \xrightarrow{b} y',\ \gamma(a,b) = c}{x \mid y \xrightarrow{c} x' \parallel y'} \qquad \frac{x \xrightarrow{\sigma} x',\ y \xrightarrow{\sigma} y'}{x \mid y \xrightarrow{\sigma} x' \mid y'}$$

Table 9: Additional deduction rules for $\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID.

**Remark 2.4.6 (Deduction rules for $\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID)**
Note that there is no deduction rule of the form:

$$\frac{x\downarrow,\ y\downarrow}{(x \mid y)\downarrow}$$

because the process $\underline{\underline{\varepsilon}} \mid \underline{\underline{\varepsilon}}$ cannot successfully terminate: we have that $\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID $\vdash \underline{\underline{\varepsilon}} \mid \underline{\underline{\varepsilon}} = \underline{\underline{\delta}}$, and $T(\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models \underline{\underline{\delta}}\not\downarrow$, hence $T(\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models (\underline{\underline{\varepsilon}} \mid \underline{\underline{\varepsilon}})\not\downarrow$.

**Theorem 2.4.7 (Time Determinacy for $\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID)**
*Let x, y, and y′ be closed $ACP^-_{drt,\varepsilon}$-ID terms. Then we have:*

$$T(ACP^-_{drt,\varepsilon}\text{-}ID) \models x \xrightarrow{\sigma} y,\ x \xrightarrow{\sigma} y' \implies y \equiv y'$$

**Proof** We proceed in the manner outlined in Theorem 2.2.7 on page 15. As the only new deduction rule regarding $\sigma$-transitions is for the $\mid$, we do not repeat the cases (i)–(vii) already listed in Theorem 2.2.7 and Theorem 2.3.10.

(viii). $x \equiv s \mid t$ for closed $\mathrm{PA}^-_{\mathrm{drt},\varepsilon}$–ID terms $s$ and $t$. We proceed by case distinction on the transitions of $s$ and $t$.

(a) $T(\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models s \xrightarrow{\sigma} s',\ t \xrightarrow{\sigma} t'$. Then $y \equiv y' \equiv s' \mid t'$.

(b) Otherwise. In contradiction with $T(\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}\text{–ID}) \models x \xrightarrow{\sigma} y$.

Having inspected all possible cases, we may now conclude that $y \equiv y'$. ■

**Definition 2.4.8 (Bisimulation and Bisimulation Model for $\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID)**
Bisimulation for $\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID. Replace "$\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID" by "$\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID" in Definition 2.2.8 on page 15 and Definition 2.2.9 on page 16.

**Definition 2.4.9 (Basic Terms of $\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID)**
When we speak of basic terms in the context of $\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID, we mean $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}})$-basic terms as defined in Definition 2.2.10 on page 16.

**Theorem 2.4.10 (Elimination for $\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID)**
*Let t be a closed $ACP^-_{drt,\varepsilon}$-ID term. Then there is a basic term s such that $BPA^-_{drt,\varepsilon}$-ID $\vdash s = t$.*

**Proof** By structural induction on $t$ prove that every closed $\mathrm{ACP}^-_{\mathrm{drt},\varepsilon}$–ID term can be rewritten into a $\mathrm{BPA}^-_{\mathrm{drt},\varepsilon}$–ID term, and then apply Theorem 2.2.21 on page 20. We give no details. ■

**Theorem 2.4.11 (Soundness of $\mathbf{ACP}^{-}_{\mathbf{drt},\varepsilon}$–ID)**
*The set of closed $ACP^{-}_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence is a model of the axioms of $ACP^{-}_{drt,\varepsilon}$-ID.*

**Proof** Extending the proof of Theorem 2.3.14 on page 26, derive, for every new axiom, a concrete bisimulation relation that relates both sides of the axiom for all closed instantiations of the free variables. We give no details. ■

**Theorem 2.4.12 (Completeness of $\mathbf{ACP}^{-}_{\mathbf{drt},\varepsilon}$–ID)**
*The axiom system $ACP^{-}_{drt,\varepsilon}$-ID is a complete axiomatization of the set of closed $ACP^{-}_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence.*

**Proof** Use Verhoef's General Completeness Theorem [29]. We give no details. ■

**Proposition 2.4.13 (Properties of $\mathbf{ACP}^{-}_{\mathbf{drt},\varepsilon}$–ID)**
*Let $x$ be a basic term and $a \in A_\delta$. Then the following properties hold:*

*(i).* $ACP^{-}_{drt,\varepsilon}\text{-ID} \vdash x \parallel \underline{\underline{\varepsilon}} = x$

*(ii).* $ACP^{-}_{drt,\varepsilon}\text{-ID} \vdash \underline{\underline{\varepsilon}} \parallel x = x$

*(iii).* $ACP^{-}_{drt,\varepsilon}\text{-ID} \vdash \underline{\underline{a}} \mathbin{\|\!\lfloor} x = \underline{\underline{a}} \cdot x$

*(iv).* $ACP^{-}_{drt,\varepsilon}\text{-ID} \vdash \underline{\underline{\delta}} \mathbin{\|\!\lfloor} x = \underline{\underline{\delta}}$

*(v).* $ACP^{-}_{drt,\varepsilon}\text{-ID} \vdash x \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} = x$

**Proof**

(i). We prove this by induction on the structure of basic terms.

(a) $x \equiv \underline{\underline{\varepsilon}}$. Then $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \parallel \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} \parallel \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mid \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} + \underline{\underline{\delta}} = \underline{\underline{\varepsilon}} = x$.

(b) $x \equiv \underline{\underline{a}} \cdot t$ for $a \in A_\delta$ and a basic term $t$. Then $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \parallel \underline{\underline{\varepsilon}} = \underline{\underline{a}} \cdot t \parallel \underline{\underline{\varepsilon}} = \underline{\underline{a}} \cdot t \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} \underline{\underline{a}} \cdot t + \underline{\underline{a}} \cdot t \mid \underline{\underline{\varepsilon}} = \underline{\underline{a}} \cdot (t \parallel \underline{\underline{\varepsilon}}) + \underline{\underline{\delta}} + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot t + \underline{\underline{\delta}} + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot t = x$.

(c) $x \equiv s + t$ for basic terms $s$ and $t$. Then $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \parallel \underline{\underline{\varepsilon}} = (s+t) \parallel \underline{\underline{\varepsilon}} = (s+t) \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} (s+t) + (s+t) \mid \underline{\underline{\varepsilon}} = s \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + t \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} s + \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} t + s \mid \underline{\underline{\varepsilon}} + t \mid \underline{\underline{\varepsilon}} = s \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} s + s \mid \underline{\underline{\varepsilon}} + t \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} t + t \mid \underline{\underline{\varepsilon}} = s \parallel \underline{\underline{\varepsilon}} + t \parallel \underline{\underline{\varepsilon}} = s + t = x$.

(d) $x \equiv \underline{\underline{\sigma}} \cdot t$ for a basic term $t$. Then $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \parallel \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot t \parallel \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot t \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} \underline{\underline{\sigma}} \cdot t + \underline{\underline{\sigma}} \cdot t \mid \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot t + \underline{\underline{\delta}} + \underline{\underline{\delta}} = \underline{\underline{\sigma}} \cdot t = x$.

(ii). Using (i), we have $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \underline{\underline{\varepsilon}} \parallel x = \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} x + x \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mid x = \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} x + x \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\delta}} = x \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} x + x \mid \underline{\underline{\varepsilon}} = x \parallel \underline{\underline{\varepsilon}} = x$.

(iii). Using (ii), we have $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \underline{\underline{a}} \mathbin{\|\!\lfloor} x = \underline{\underline{a}} \cdot \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} x = \underline{\underline{a}} \cdot (\underline{\underline{\varepsilon}} \parallel x) = \underline{\underline{a}} \cdot x$.

(iv). Using (iii), we have $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash \underline{\underline{\delta}} \mathbin{\|\!\lfloor} x = \underline{\underline{\delta}} \cdot x = \underline{\underline{\delta}}$.

(v). We again prove this by induction on the structure of basic terms.

(a) $x \equiv \underline{\underline{\varepsilon}}$. Then $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} = x$.

(b) $x \equiv \underline{\underline{a}} \cdot t$ for $a \in A_\delta$ and a basic term $t$. Then $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}\text{–ID} \vdash x \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{a}} \cdot t \mathbin{\|\!\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{a}} \cdot (t \parallel \underline{\underline{\varepsilon}}) = \underline{\underline{a}} \cdot t = x$.

(c) $x \equiv s+t$ for basic terms $s$ and $t$. Then $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$-ID $\vdash (s+t) \mathbin{|\!\!\lfloor} \underline{\underline{\varepsilon}} = s \mathbin{|\!\!\lfloor} \underline{\underline{\varepsilon}} + t \mathbin{|\!\!\lfloor} \underline{\underline{\varepsilon}} = s + t = x$.

(d) $x \equiv \underline{\underline{\sigma}} \cdot t$ for a basic term $t$. Then $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$-ID $\vdash x = \underline{\underline{\sigma}} \cdot t \mathbin{|\!\!\lfloor} \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} \cdot t = x$.

■

**Theorem 2.4.14 (Axioms of Standard Concurrency for $\mathbf{ACP}^{-}_{\mathbf{drt},\varepsilon}$–ID)**
*Let $x$, $y$, $z$ be closed $ACP^{-}_{drt,\varepsilon}$-ID terms. Then the following properties hold:*

*(i).* $ACP^{-}_{drt,\varepsilon}$*-ID* $\vdash x \mid y = y \mid x$

*(ii).* $ACP^{-}_{drt,\varepsilon}$*-ID* $\vdash x \parallel y = y \parallel x$

*(iii).* $ACP^{-}_{drt,\varepsilon}$*-ID* $\vdash (x \mid y) \mid z = x \mid (y \mid z)$

*(iv).* $ACP^{-}_{drt,\varepsilon}$*-ID* $\vdash (x \mathbin{|\!\!\lfloor} y) \mathbin{|\!\!\lfloor} z = x \mathbin{|\!\!\lfloor} (y \parallel z)$

*(v).* $ACP^{-}_{drt,\varepsilon}$*-ID* $\vdash x \mid (y \mathbin{|\!\!\lfloor} z) = (x \parallel y) \mathbin{|\!\!\lfloor} z$

*(vi).* $ACP^{-}_{drt,\varepsilon}$*-ID* $\vdash (x \parallel y) \parallel z = x \parallel (y \parallel z)$

**Proof** By 2.4.10 assume that $x$, $y$, and $z$ are basic terms. Prove (i) by structural induction on both $x$ and $y$. Then, using (i), we have that $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$-ID $\vdash x \parallel y = x \mathbin{|\!\!\lfloor} y + y \mathbin{|\!\!\lfloor} x + x \mid y = y \mathbin{|\!\!\lfloor} x + x \mathbin{|\!\!\lfloor} y + y \mid x = y \parallel x$, which proves (ii). Items (iii)–(vi) are finally proven together, using simultaneous induction in the manner of the proof of item (ii) and (iii) of Theorem 2.3.18 on page 30. We give no details. ■

**Corollary 2.4.15 (Commutativity and Associativity of the Merge)**
*For closed terms, the merge and communication merge operators of $ACP^{-}_{drt,\varepsilon}$-ID are commutative and associative.*

**Proof** This follows directly from Theorem 2.4.14. ■

## 2.5 Immediate deadlock

In this section, we make a small detour into (the absence of) the theory $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$. This theory is like $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$-ID, with a special constant $\dot{\delta}$ added. As we will argue, the addition of this so called *immediate deadlock constant* to $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$-ID is not very well conceivable, i.e., appears to lead inescapably to inconsistencies.

The immediate deadlock can be described as being similar to the undelayable deadlock, with the added property that it prohibits the passage of time. Hence, we get the axioms shown in Table 10 on the next page (taken from Section 2.7 of [26]).

Here A6ID and A7ID express the deadlock property of $\dot{\delta}$, and DRTESID expresses the fact that $\dot{\delta}$ prohibits the further passage of time: stopping time at the beginning of the following time-slice (the process $\underline{\underline{\sigma}} \cdot \dot{\delta}$) is the same as stopping time at the end of this time-slice (the process $\underline{\underline{\delta}}$).

The introduction of A6ID makes it necessary to drop Axiom A6: as $\dot{\delta}$ now plays the role of the 0-element for the choice, the equality $x + \underline{\underline{\delta}} = x$ does not hold anymore for $x = \dot{\delta}$, as we have $\underline{\underline{\delta}} + \dot{\delta} = \underline{\underline{\delta}}$ by Axiom A6ID. To compensate for this, we will have to introduce

| | |
|---|---|
| $\underline{\underline{\sigma}} \cdot \dot{\delta} = \underline{\underline{\delta}}$ | DRTESID |
| $x + \dot{\delta} = x$ | A6ID |
| $\dot{\delta} \cdot x = \dot{\delta}$ | A7ID |

Table 10: Axioms for $\dot{\delta}$.

| | |
|---|---|
| $\underline{\underline{a}} + \underline{\underline{\delta}} = \underline{\underline{a}}$ | DUD1 |
| $\underline{\underline{\sigma}} + \underline{\underline{\delta}} = \underline{\underline{\sigma}}$ | DUD2 |
| $\underline{\underline{\varepsilon}} + \underline{\underline{\delta}} = \underline{\underline{\varepsilon}}$ | DUD3 |

Table 11: Axioms for $\underline{\underline{\delta}}$.

axioms to the effect that $x + \underline{\underline{\delta}} = x$ for $x \neq \dot{\delta}$. So, for example, we could introduce the axioms shown in Table 11.

Unfortunately, this leads already to an inconsistency, as we now have:

$$\dot{\delta} = \underline{\underline{\varepsilon}} \cdot \dot{\delta} = (\underline{\underline{\varepsilon}} + \underline{\underline{\delta}}) \cdot \dot{\delta} = \underline{\underline{\varepsilon}} \cdot \dot{\delta} + \underline{\underline{\delta}} \cdot \dot{\delta} = \dot{\delta} + \underline{\underline{\delta}} = \underline{\underline{\delta}}$$

Hence, $\dot{\delta} = \underline{\underline{\delta}}$, and the whole exercise is in vain.

What to do? We have several options. We could for example weaken Axiom A8 and A9 so that $x$ is not allowed to be $\dot{\delta}$ anymore. This could be achieved by replacing Axiom A8 and A9 by Axiom A8ID and A9ID from Table 12.

| | |
|---|---|
| $(x + \underline{\underline{\delta}}) \cdot \underline{\underline{\varepsilon}} = x + \underline{\underline{\delta}}$ | A8ID |
| $\underline{\underline{\varepsilon}} \cdot (x + \underline{\underline{\delta}}) = x + \underline{\underline{\delta}}$ | A9ID |

Table 12: Axiom A8ID and A9ID.

However, this does not get us far, as we now have the decide what the process $\underline{\underline{\varepsilon}} \cdot \dot{\delta}$ should be. As shown above, we cannot let it be $\dot{\delta}$, as that leads to $\underline{\underline{\delta}} = \dot{\delta}$. Then, the next logical candidate is $\underline{\underline{\delta}}$. However, putting $\underline{\underline{\varepsilon}} \cdot \dot{\delta} = \underline{\underline{\delta}}$ leads to another inconsistency, as we then have:

$$\underline{\underline{a}} \cdot \dot{\delta} = (\underline{\underline{a}} \cdot \underline{\underline{\varepsilon}}) \cdot \dot{\delta} = \underline{\underline{a}} \cdot (\underline{\underline{\varepsilon}} \cdot \dot{\delta}) = \underline{\underline{a}} \cdot \underline{\underline{\delta}}$$

So, $\underline{\underline{a}} \cdot \dot{\delta} = \underline{\underline{a}} \cdot \underline{\underline{\delta}}$, and that is not what we want either.

Our last resort now is to introduce, next to $\dot{\delta}$, a brand new constant $\dot{\varepsilon}$, that denotes "the immediate successful termination option". The idea is that $\dot{\varepsilon}$ can terminate (like $\underline{\underline{\varepsilon}}$), but cannot idle (like $\dot{\delta}$). This $\dot{\varepsilon}$ would then be a true 1-element for the sequential composition. However, the role of the undelayable actions would be seriously changed, as we now need to consider the idling behavior within a time-slice. So, for example, the process $\underline{\underline{a}}$ now differs from the process $\underline{\underline{a}} \cdot \underline{\underline{\varepsilon}}$, as the second can still idle after it has executed an $a$, while the first can not. If we fail to make this distinction, we run into the same problems as before. As a result, we have to construct quite different SOS-rules, leading to a different model. Since one of our design goals was to build on existing discrete-time ACP theories, and not invent totally different ones, we will not further explore the $\dot{\varepsilon}$ option sketched here.

Taking into account the above, we conclude that in the current framework $\underline{\underline{\varepsilon}}$ and $\dot{\delta}$ do not combine in a any useful way, and we will not consider $\dot{\delta}$ any further in this paper.

This is very unfortunate, as $\dot{\delta}$ could potentially be quite useful. Consider for example Lemma 2.2.15 on page 17. If we would have a consistent way to define a theory $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$, for all terms $t$ there would be a term $s$ such that $t + \underline{\underline{\delta}} = \nu(t) + \underline{\underline{\sigma}} \cdot s$ (in those cases where $t = \nu(t)$, we would take $s \equiv \dot{\delta}$, as then $\nu(t) + \underline{\underline{\sigma}} \cdot \dot{\delta} = \nu(t) + \underline{\underline{\delta}} = t + \underline{\underline{\delta}}$). So, Lemma 2.2.15 would only have one case instead of two, and as a result the proof on page 30 would only need to examine *one* case under item (i)(d), instead of *four* as it does now.

In this way, the presence of $\dot{\delta}$ would enable us to collapse quite a few of the case distinctions with which our proofs are ridden. Strangely enough, in the conclusions of [26] it is argued (on equally valid grounds), that indeed the presence of $\underline{\underline{\varepsilon}}$ would collapse a lot of the case distinctions made there. It seems very ironical you cannot have them both.

## 2.6 Embeddings

The process algebras $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID, $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID described in Section 2 can reduced to simpler ones already know from the literature. Or, in other words, some already known algebras can be injectively embedded our process algebras. We name a few, and give the embeddings.

The process algebras BPA, $\mathrm{BPA}_{\delta}$, and $\mathrm{BPA}_{\delta\varepsilon}$ of [13] can be embedded in $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID by projecting every action $a$ onto the undelayable action $\underline{\underline{a}}$, $\delta$ onto $\underline{\underline{\delta}}$, $\varepsilon$ onto $\underline{\underline{\varepsilon}}$, and $+$ and $\cdot$ onto themselves. The process algebras PA, $\mathrm{PA}_{\delta}$, and ACP of [13] can be embedded in $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID, $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID, and $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID respectively in the same way. Using this embedding, we project every untimed process onto the first time slice of $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID, $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID, or $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID. Finally, note that $\mathrm{PA}_{\varepsilon}$ and $\mathrm{ACP}_{\varepsilon}$ of [13] cannot easily be embedded in $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID and $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID in their entirety, due to their use of the $\surd$ operator for handling the interaction between the empty step and the merge operator. We can, however, construct an embedding if we do not consider the auxiliary operators $\mathbin{|\!\!\lfloor}$, $|$, and $\surd$; project every action $a$ onto the undelayable action $\underline{\underline{a}}$, $\delta$ onto $\underline{\underline{\delta}}$, $\varepsilon$ onto $\underline{\underline{\varepsilon}}$, and $+$, $\cdot$, and $\|$ onto themselves.

The process algebras $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ and $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID of [26] can be embedded in $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID by projecting $\sigma_{\mathrm{rel}}(x)$ onto $\underline{\underline{\sigma}} \cdot x$, and everything else onto itself. The process algebras $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID and $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID from [26] can be projected onto $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID and $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID in the same way.

# 3 Theories with Delayable Actions

In this section, we will introduce the concept of *delayable actions*. We will define the process algebra $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID, and step-by-step extend it with the the free merge and the (full) merge. For each extension we will give an axiomatization, an operational semantics, and a description of all concepts that are introduced. Furthermore, we give the considerations that have led us to construct these algebras in the way we have done.

## 3.1 The Delayable-Actions Extension

In this section, we will define $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID, which is basically $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID extended with delayable actions (note the absence of the superscript "$^-$"). A delayable action can be interpreted as an action that may delay its execution for a arbitrary number of time-slices.

**Definition 3.1.1 (Signature of $\mathbf{BPA}_{\mathbf{drt},\varepsilon}$–ID)**
The signature of $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *delayable atomic actions* $\{a | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *delayable deadlock constant* $\delta$, the *time-unit delay constant* $\underline{\underline{\sigma}}$, the *undelayable empty process constant* $\underline{\underline{\varepsilon}}$, the *delayable empty process constant* $\varepsilon$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, and the *"now" operator* $\nu$.

**Remark 3.1.2 (Symbol versus Action)**
Note that in Definition 3.1.1, in the expression $\{a | a \in A\}$, the second $a$ refers to the *symbol a*, while the first one refers to the *action a*. This distinction should be clearly made, and it can be considered a tragic historical incident that these different notions have received the same notation. (Note, however, that in [7, 8] the notation $\mathrm{ats}(a)$ is used for the delayable action $a$. This has the advantage of avoiding the confusion described above, but the disadvantage of cluttering up the formulae.)

**Definition 3.1.3 (Axioms of $\mathbf{BPA}_{\mathbf{drt},\varepsilon}$–ID)**
The process algebra $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID is axiomatized by the axioms of $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID given in Definition 2.2.2 on page 13, and Axioms DEP and DA shown in Table 13: $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID = A1–A9 + TF + DCSE1–DCSE4 + DEP + DA.

| | | | |
|---|---|---|---|
| $\varepsilon = \underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} \cdot \varepsilon$ | DEP | $a = \varepsilon \cdot \underline{\underline{a}}$ | DA |

Table 13: Additional axioms for $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID.

**Remark 3.1.4 (Axioms of $\mathbf{BPA}_{\mathbf{drt},\varepsilon}$–ID)**
The process $\varepsilon$ introduced in this section is the *delayable* counterpart of $\underline{\underline{\varepsilon}}$: it can terminate successfully in the current time-slice, but it can also move on to a following time-slice, and terminate there. The *delayable action a* introduced for every symbol $a \in A$, is the counterpart of the undelayable action $\underline{\underline{a}}$ introduced in Section 2 in the same sense: it can either perform $a$, or move on and perform $a$ in some future time-slice.

Axiom DEP defines the $\varepsilon$: the choice between terminating now, or moving on to the next time-slice. Axiom DA then uses $\varepsilon$ to define the delayable actions: a delayable action corresponds to moving to some time-slice, followed by the execution of an $a$.

**Definition 3.1.5 (Recursion Principle for $\varepsilon$)**
Next to the axioms mentioned in Definition 3.1.3, the system $\mathrm{BPA}^{+}_{\mathrm{drt},\varepsilon}$–ID also contains the *recursion principle* RSP(DEP) shown in Table 14. For more information on recursion principles and their status with respect to axioms, see [13, 26].

$$y = x + \underline{\underline{\sigma}} \cdot y \qquad \Longrightarrow \qquad y = \varepsilon \cdot x \qquad \mathrm{RSP(DEP)}$$

Table 14: Recursive Specification Principle for the Delayable Empty Process.

**Remark 3.1.6 (The Recursion Principle RSP(DEP))**
The recursion principle RSP(DEP) will be used to derive equalities between terms that contain delayable actions. As it turns out, RSP(DEP) is very powerful: for every new operator we add to $\mathrm{BPA}^{+}_{\mathrm{drt},\varepsilon}$–ID, we only have to add axioms to eliminate this operator from terms that contain only *undelayable* actions. The elimination of the new operator from terms that also contain *delayable* actions is then possible using RSP(DEP).

The price to be paid for this power, is the fact that RSP(DEP) is formulated as a conditional axiom. Hence, we lose the strict equationality of our theory. If so desired, it is possible to maintain strict equationality, but that requires the addition of new axioms to deal with delayable actions for every new operator added. For a description and examples on how to do this in a structured manner, see [26].

In Example 3.1.7 we show how to apply RSP(DEP) to derive simple equalities in $\mathrm{BPA}^{+}_{\mathrm{drt},\varepsilon}$–ID. In Example 3.2.10 on page 43 we show how to apply it to derive equalities involving a newly introduced operator, namely the free merge.

**Example 3.1.7 (Use of RSP(DEP))**
First, we show how to derive the equality $a = \varepsilon \cdot a$ in $\mathrm{BPA}^{+}_{\mathrm{drt},\varepsilon}$–ID. To begin with, we derive $a = \varepsilon \cdot \underline{\underline{a}} = (\underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} \cdot \varepsilon) \cdot \underline{\underline{a}} = \underline{\underline{\varepsilon}} \cdot \underline{\underline{a}} + \underline{\underline{\sigma}} \cdot \varepsilon \cdot \underline{\underline{a}} = \underline{\underline{a}} + \underline{\underline{\sigma}} \cdot a$. Using this equality, we can then derive $a = \underline{\underline{a}} + \underline{\underline{\sigma}} \cdot a = \underline{\underline{a}} + \underline{\underline{\sigma}} \cdot a + \underline{\underline{\sigma}} \cdot a = a + \underline{\underline{\sigma}} \cdot a$. Applying RSP(DEP), we now get the desired result $a = \varepsilon \cdot a$. Note that the converse, namely $a = a \cdot \varepsilon$, does not hold, as $a \cdot \varepsilon$ can still idle after it has done an $a$, while $a$ cannot.

Secondly, we show how to derive the equality $\varepsilon \cdot x + \varepsilon \cdot y = \varepsilon \cdot (x + y)$. Applying the axioms, we get $\varepsilon \cdot x + \varepsilon \cdot y = (\underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} \cdot \varepsilon) \cdot x + (\underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} \cdot \varepsilon) \cdot y = \underline{\underline{\varepsilon}} \cdot x + \underline{\underline{\sigma}} \cdot \varepsilon \cdot x + \underline{\underline{\varepsilon}} \cdot y + \underline{\underline{\sigma}} \cdot \varepsilon \cdot y = x + y + \underline{\underline{\sigma}} \cdot (\varepsilon \cdot x + \varepsilon \cdot y)$. Applying RSP(DEP), we now get the desired result $\varepsilon \cdot x + \varepsilon \cdot y = \varepsilon \cdot (x + y)$.

Thirdly, we show how to derive the equality $\underline{\underline{\sigma}} \cdot \varepsilon = \varepsilon \cdot \underline{\underline{\sigma}}$. Applying the axioms, we get $\underline{\underline{\sigma}} \cdot \varepsilon = \underline{\underline{\sigma}} \cdot (\underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} \cdot \varepsilon) = \underline{\underline{\sigma}} \cdot \underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} \cdot \underline{\underline{\sigma}} \cdot \varepsilon = \underline{\underline{\sigma}} + \underline{\underline{\sigma}} \cdot \underline{\underline{\sigma}} \cdot \varepsilon$. Applying RSP(DEP), we now get the desired result $\underline{\underline{\sigma}} \cdot \varepsilon = \varepsilon \cdot \underline{\underline{\sigma}}$.

**Definition 3.1.8 (Semantics of $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID)**
The semantics of $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID are given by the term deduction system $T(\mathrm{BPA}_{\mathrm{drt},\varepsilon}\text{–ID})$, induced by the deduction rules for $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID given in Definition 2.2.5 on page 14, and the additional deduction rules for $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID shown in Table 15.

$$\underline{\underline{a}} \xrightarrow{a} \underline{\underline{\varepsilon}} \qquad \underline{\underline{a}} \xrightarrow{\sigma} \underline{\underline{a}} \qquad \underline{\underline{\delta}} \xrightarrow{\sigma} \underline{\underline{\delta}} \qquad \underline{\underline{\varepsilon}} \xrightarrow{\sigma} \underline{\underline{\varepsilon}} \qquad \underline{\underline{\varepsilon}}\downarrow$$

Table 15: Additional deduction rules for $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID.

**Theorem 3.1.9 (Time Determinacy for $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID)**
*Let $x$, $y$, and $y'$ be closed $BPA_{drt,\varepsilon}$-ID terms. Then we have:*

$$T(BPA_{drt,\varepsilon}\text{-}ID) \models x \xrightarrow{\sigma} y,\ x \xrightarrow{\sigma} y' \implies y \equiv y'$$

**Proof** Next to the cases treated in Theorem 2.2.7 on page 15, we only have to examine the cases $x \equiv \underline{\underline{a}}$ for $a \in A_\delta$ and $x \equiv \underline{\underline{\varepsilon}}$. For all these cases we have that $T(\mathrm{BPA}_{\mathrm{drt},\varepsilon}\text{–ID}) \models x \xrightarrow{\sigma} y,\ x \xrightarrow{\sigma} y'$ implies that $y \equiv y' \equiv x$, and we are done. ∎

**Definition 3.1.10 (Bisimulation and Bisimulation Model for $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID)**
Bisimulation for $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID. Replace "$\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$–ID" by "$\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID" in Definition 2.2.8 on page 15 and Definition 2.2.9 on page 16.

**Definition 3.1.11 (Basic Terms of $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID)**
We define $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}}, \delta, \varepsilon)$*-basic terms* inductively as follows:

(i). The constant $\underline{\underline{\varepsilon}}$ is a $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}}, \delta, \varepsilon)$-basic term,

(ii). if $a \in A_\delta$ and $t$ is a $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}}, \delta, \varepsilon)$-basic term, then $\underline{\underline{a}} \cdot t$ is a $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}}, \delta, \varepsilon)$-basic term,

(iii). if $t$ and $s$ are $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}}, \delta, \varepsilon)$-basic terms, then $t + s$ is a $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}}, \delta, \varepsilon)$-basic term,

(iv). if $t$ is a $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}}, \delta, \varepsilon)$-basic term, then $\underline{\underline{\sigma}} \cdot t$ and $\varepsilon \cdot t$ are $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}}, \delta, \varepsilon)$-basic terms.

From now on, when we speak of basic terms in the context of $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$-ID, we mean $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}}, \delta, \varepsilon)$-basic terms.

**Theorem 3.1.12 (General Form of Basic Terms of $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID)**
*Modulo the commutativity and associativity of the $+$, all basic terms $t$ of $BPA^{-}_{drt,\varepsilon}$-ID are of the form:*

$$t \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{\sigma}} \cdot u_j + \sum_{k<p} \varepsilon \cdot v_k + \sum_{l<q} \underline{\underline{\varepsilon}}$$

*for $m, n, p, q \in \mathbb{N}$, $m + n + p + q \geq 1$, $a_i \in A_\delta$, and basic terms $s_i$, $u_j$, and $v_k$.*

**Proof** Trivial, by inspection of the definition of basic terms, Definition 3.1.11 on the preceding page. Observe that the general form of basic terms is closed under the formation rules given in Definition 3.1.11 on the page before. ■

**Theorem 3.1.13 (Elimination for $\mathbf{BPA}^{+}_{\mathbf{drt},\varepsilon}$–ID)**
*Let $t$ be a closed $BPA_{drt,\varepsilon}$-ID term. Then there is a basic term $s$ such that $BPA^{+}_{drt,\varepsilon}$-ID $\vdash s = t$.*

**Proof** Extending the proof of Theorem 2.2.21 on page 20, turn the new axioms into new term rewriting rules, and then apply the lexicographical path ordering technique. We give no details. ■

**Theorem 3.1.14 (Soundness of $\mathbf{BPA}^{+}_{\mathbf{drt},\varepsilon}$–ID)**
*The set of closed $BPA_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence is a model of the axioms of $BPA^{+}_{drt,\varepsilon}$-ID.*

**Proof** Extending the proof of Theorem 2.2.22 on page 20, for every new axiom, derive a concrete bisimulation relation that relates both sides of the axiom for all closed instantiations of the free variables. Furthermore, prove the soundness of the recursion principle RSP(DEP) separately. We give no details. ■

**Theorem 3.1.15 (Completeness of $\mathbf{BPA}^{+}_{\mathbf{drt},\varepsilon}$–ID)**
*The axiom system $BPA^{+}_{drt,\varepsilon}$-ID is a complete axiomatization of the set of closed $BPA_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence.*

**Proof** Use the same technique as in Lemma 2.2.23 and Theorem 2.2.24 on page 21. We give no details. ■

## 3.2 The Free-Merge Extension

In this section, we will define $\mathrm{PA}_{\mathrm{drt},\varepsilon}$–ID, which is basically $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID extended with the free merge. Adding the free merge to $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID to get $\mathrm{PA}_{\mathrm{drt},\varepsilon}$–ID is entirely similar to adding the free merge in the case without delayable actions (treated Section 2.3 on page 21), so we suffice by only giving one example to illustrate the new theory.

**Definition 3.2.1 (Signature of $\mathbf{PA}_{\mathbf{drt},\varepsilon}$–ID)**
The signature of $\mathrm{PA}_{\mathrm{drt},\varepsilon}$–ID consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *delayable atomic actions* $\{a | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *delayable deadlock constant* $\delta$, the *time-unit delay constant* $\underline{\underline{\sigma}}$, the *undelayable empty process constant* $\underline{\underline{\varepsilon}}$, the *delayable empty process constant* $\varepsilon$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *"now" operator* $\nu$, the *free merge operator* $\|$, and the *left merge operator* $\mathbin{\|\!\_}$ .

**Definition 3.2.2 (Axioms of $\mathbf{PA}_{\mathbf{drt},\varepsilon}$–ID)**
The process algebra $\mathrm{PA}_{\mathrm{drt},\varepsilon}$–ID is axiomatized by the axioms of $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID given in Definition 2.3.2 on page 22, and Axioms DEP and DA shown in Table 13 on page 39: $\mathrm{PA}_{\mathrm{drt},\varepsilon}$–ID = A1–A9 + TF + DCSE1–DCSE4 + DEP + DA + DRTEM1–DRTEM12.

**Definition 3.2.3 (Semantics of $\mathbf{PA}_{\mathbf{drt},\varepsilon}$–ID)**
The semantics of $\mathrm{PA}_{\mathrm{drt},\varepsilon}$–ID are given by the term deduction system $T(\mathrm{PA}_{\mathrm{drt},\varepsilon}\text{–ID})$, induced by the deduction rules for $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$–ID given in Definition 2.3.8 on page 25, and the additional deduction rules for $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID shown in Table 15 on the page before.

**Theorem 3.2.4 (Time Determinacy for $\mathbf{PA_{drt,\varepsilon}}$–ID)**
*Let $x$, $y$, and $y'$ be closed $PA_{drt,\varepsilon}$-ID terms. Then we have:*

$$T(PA_{drt,\varepsilon}\text{-}ID) \vDash x \xrightarrow{\sigma} y,\ x \xrightarrow{\sigma} y' \implies y \equiv y'$$

**Proof** Proven in the same way as Theorem 3.1.9 on page 41, extending it with the extra cases given in the proof of Theorem 2.3.10 on page 25. We give no details. ■

**Definition 3.2.5 (Bisimulation and Bisimulation Model for $\mathbf{PA_{drt,\varepsilon}}$–ID)**
Bisimulation for $\mathrm{PA_{drt,\varepsilon}}$–ID and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA^{-}_{drt,\varepsilon}}$–ID. Replace "$\mathrm{BPA^{-}_{drt,\varepsilon}}$–ID" by "$\mathrm{PA_{drt,\varepsilon}}$–ID" in Definition 2.2.8 on page 15 and Definition 2.2.9 on page 16.

**Definition 3.2.6 (Basic Terms of $\mathbf{PA_{drt,\varepsilon}}$–ID)**
When we speak of basic terms in the context of $\mathrm{PA_{drt,\varepsilon}}$–ID, we mean $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}}, \delta, \varepsilon)$-basic terms as defined in Definition 3.1.11 on page 41.

**Theorem 3.2.7 (Elimination for $\mathbf{PA^{+}_{drt,\varepsilon}}$–ID)**
*Let $t$ be a closed $PA_{drt,\varepsilon}$-ID term. Then there is a basic term $s$ such that $PA^{+}_{drt,\varepsilon}$-ID $\vdash s = t$.*

**Proof** By structural induction on $t$ prove that every closed $\mathrm{PA_{drt,\varepsilon}}$–ID term can be rewritten into a $\mathrm{BPA_{drt,\varepsilon}}$–ID term, and then apply Theorem 3.1.13 on the page before. We give no details. ■

**Theorem 3.2.8 (Soundness of $\mathbf{PA^{+}_{drt,\varepsilon}}$–ID)**
*The set of closed $PA_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence is a model of the axioms of $PA^{+}_{drt,\varepsilon}$-ID.*

**Proof** Extending the proof of Theorem 3.1.14 on the preceding page, for every new axiom, derive a concrete bisimulation relation that relates both sides of the axiom for all closed instantiations of the free variables. We give no details. ■

**Theorem 3.2.9 (Completeness of $\mathbf{PA^{+}_{drt,\varepsilon}}$–ID)**
*The axiom system $PA^{+}_{drt,\varepsilon}$-ID is a complete axiomatization of the set of closed $PA_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence.*

**Proof** Use Verhoef's General Completeness Theorem [29]. We give no details. ■

**Example 3.2.10 (Use of RSP(DEP) with $\mathbf{PA_{drt,\varepsilon}}$–ID)**
We show how to derive the equality $a \parallel b = a \cdot b + b \cdot a$. Applying the axioms, we get $a \parallel b = a \mathbin{\|\!\_} b + b \mathbin{\|\!\_} a = (\underline{\underline{a}} + \underline{\underline{\sigma}} \cdot a) \mathbin{\|\!\_} b + (\underline{\underline{b}} + \underline{\underline{\sigma}} \cdot b) \mathbin{\|\!\_} a = \underline{\underline{a}} \mathbin{\|\!\_} b + \underline{\underline{\sigma}} \cdot a \mathbin{\|\!\_} (\underline{\underline{b}} + \underline{\underline{\sigma}} \cdot b) + \underline{\underline{b}} \mathbin{\|\!\_} a + \underline{\underline{\sigma}} \cdot b \mathbin{\|\!\_} (\underline{\underline{a}} + \underline{\underline{\sigma}} \cdot a) = \underline{\underline{a}} \cdot b + \underline{\underline{\sigma}} \cdot (a \mathbin{\|\!\_} b) + \underline{\underline{b}} \cdot a + \underline{\underline{\sigma}} \cdot (b \mathbin{\|\!\_} a) = \underline{\underline{a}} \cdot b + \underline{\underline{b}} \cdot a + \underline{\underline{\sigma}} \cdot (a \parallel b)$. Applying RSP(DEP), and using $\varepsilon \cdot (x + y) = \varepsilon \cdot x + \varepsilon \cdot y$ (see Example 3.1.7 on page 40), we now get $a \parallel b = \varepsilon \cdot (\underline{\underline{a}} \cdot b + \underline{\underline{b}} \cdot a) = \varepsilon \cdot \underline{\underline{a}} \cdot b + \varepsilon \cdot \underline{\underline{b}} \cdot a = a \cdot b + b \cdot a$.

**Proposition 3.2.11 (Properties of $\mathbf{PA_{drt,\varepsilon}}$–ID)**
*Let $x$ be a basic term and $a \in A_\delta$. Then the following properties hold:*

*(i).* $PA_{drt,\varepsilon}$-ID $\vdash x \parallel \underline{\underline{\varepsilon}} = x$

*(ii).* $PA_{drt,\varepsilon}$-ID $\vdash \underline{\underline{\varepsilon}} \parallel x = x$

(iii). $PA_{drt,\varepsilon}$-$ID \vdash \underline{\underline{a}} \mathbin{\underline{\|}} x = \underline{\underline{a}} \cdot x$

(iv). $PA_{drt,\varepsilon}$-$ID \vdash \underline{\underline{\delta}} \mathbin{\underline{\|}} x = \underline{\underline{\delta}}$

(v). $PA_{drt,\varepsilon}$-$ID \vdash x \mathbin{\underline{\|}} \underline{\underline{\varepsilon}} = x$

**Proof** In the same manner as Proposition 2.3.16 on page 27. We give no details. ■

**Theorem 3.2.12 (Axioms of Standard Concurrency for $\mathbf{PA^{+}_{drt,\varepsilon}}$–ID)**
*Let $x$, $y$, $z$ be closed $PA_{drt,\varepsilon}$-ID terms. Then the following properties hold:*

(i). $PA^{+}_{drt,\varepsilon}$-$ID \vdash x \parallel y = y \parallel x$

(ii). $PA^{+}_{drt,\varepsilon}$-$ID \vdash (x \mathbin{\underline{\|}} y) \mathbin{\underline{\|}} z = x \mathbin{\underline{\|}} (y \parallel z)$

(iii). $PA^{+}_{drt,\varepsilon}$-$ID \vdash (x \parallel y) \parallel z = x \parallel (y \parallel z)$

**Proof** In the same manner as Theorem 2.3.18 on page 30. We give no details. ■

**Corollary 3.2.13 (Commutativity and Associativity of the Merge)**
*For closed terms, the free merge operator of $PA^{+}_{drt,\varepsilon}$-ID is commutative and associative.*

**Proof** This follows directly from Theorem 3.2.12. ■

## 3.3 The Merge Extension

In this section, we will define $\mathrm{ACP}_{\mathrm{drt},\varepsilon}$–ID, which is basically $\mathrm{PA}_{\mathrm{drt},\varepsilon}$–ID extended with the merge. Again, adding the merge is entirely similar to adding the merge in the case without delayable actions (treated in Section 2.4 on page 32).

**Definition 3.3.1 (Signature of $\mathbf{ACP_{drt,\varepsilon}}$–ID)**
The signature of $\mathrm{ACP}_{\mathrm{drt},\varepsilon}$–ID consists of the *undelayable atomic actions* $\{\underline{\underline{a}}|a \in A\}$, the *delayable atomic actions* $\{a|a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *delayable deadlock constant* $\delta$, the *time-unit delay constant* $\underline{\underline{\sigma}}$, the *undelayable empty process constant* $\underline{\underline{\varepsilon}}$, the *delayable empty process constant* $\varepsilon$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *"now" operator* $\nu$, the *merge operator* $\parallel$, the *left merge operator* $\mathbin{\underline{\|}}$, and the *communication merge operator* $|$.

**Definition 3.3.2 (Axioms of $\mathbf{ACP_{drt,\varepsilon}}$–ID)**
The process algebra $\mathrm{ACP}_{\mathrm{drt},\varepsilon}$–ID is axiomatized by the axioms of $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID given in Definition 2.4.3 on page 33, and the Axioms DEP and DA shown in Table 13 on page 39: $\mathrm{ACP}_{\mathrm{drt},\varepsilon}$–ID = A1–A9 + TF + DCSE1–DCSE4 + DEP + DA + DRTEM2–DRTEM12 + DRTECM1–DRTECM9 + DRTCF.

**Definition 3.3.3 (Semantics of $\mathbf{ACP_{drt,\varepsilon}}$–ID)**
The semantics of $\mathrm{ACP}_{\mathrm{drt},\varepsilon}$–ID are given by the term deduction system $T(\mathrm{ACP}_{\mathrm{drt},\varepsilon}\text{–ID})$, induced by the deduction rules for $\mathrm{ACP}^{-}_{\mathrm{drt},\varepsilon}$–ID given in Definition 2.4.5 on page 33, and the additional deduction rules for $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$–ID shown in Table 15 on page 41.

**Theorem 3.3.4 (Time Determinacy for $\mathbf{ACP_{drt,\varepsilon}}$–ID)**
*Let $x$, $y$, and $y'$ be closed $ACP_{drt,\varepsilon}$-ID terms. Then we have:*

$$T(ACP_{drt,\varepsilon}\text{-}ID) \models x \xrightarrow{\sigma} y,\ x \xrightarrow{\sigma} y' \implies y \equiv y'$$

**Proof** Proven in the same way as Theorem 3.1.9 on page 41, extending it with the extra cases given in the proofs of Theorem 2.3.10 on page 25 and Theorem 2.4.7 on page 34. We give no details. ■

**Definition 3.3.5 (Bisimulation and Bisimulation Model for $\mathbf{ACP_{drt,\varepsilon}}$–ID)**
Bisimulation for $\mathrm{ACP_{drt,\varepsilon}}$–ID and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA^-_{drt,\varepsilon}}$–ID. Replace "$\mathrm{BPA^-_{drt,\varepsilon}}$–ID" by "$\mathrm{ACP_{drt,\varepsilon}}$–ID" in Definition 2.2.8 on page 15 and Definition 2.2.9 on page 16.

**Definition 3.3.6 (Basic Terms of $\mathbf{ACP_{drt,\varepsilon}}$–ID)**
When we speak of basic terms in the context of $\mathrm{ACP_{drt,\varepsilon}}$–ID, we mean $(\underline{\underline{\sigma}}, \underline{\underline{\delta}}, \underline{\underline{\varepsilon}}, \delta, \varepsilon)$-basic terms as defined in Definition 3.1.11 on page 41.

**Theorem 3.3.7 (Elimination for $\mathbf{ACP^+_{drt,\varepsilon}}$–ID)**
*Let $t$ be a closed $ACP_{drt,\varepsilon}$-ID term. Then there is a basic term $s$ such that $ACP^+_{drt,\varepsilon}$-ID $\vdash s = t$.*

**Proof** By structural induction on $t$ prove that every closed $\mathrm{ACP_{drt,\varepsilon}}$–ID term can be rewritten into a $\mathrm{BPA_{drt,\varepsilon}}$–ID term, and then apply Theorem 3.1.13 on page 42. We give no details. ■

**Theorem 3.3.8 (Soundness of $\mathbf{ACP^+_{drt,\varepsilon}}$–ID)**
*The set of closed $ACP_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence is a model of the axioms of $ACP^+_{drt,\varepsilon}$-ID.*

**Proof** Extending the proof of Theorem 3.2.8 on page 43, for every new axiom, derive a concrete bisimulation relation that relates both sides of the axiom for all closed instantiations of the free variables. We give no details. ■

**Theorem 3.3.9 (Completeness of $\mathbf{ACP^+_{drt,\varepsilon}}$–ID)**
*The axiom system $ACP^+_{drt,\varepsilon}$-ID is a complete axiomatization of the set of closed $ACP_{drt,\varepsilon}$-ID terms modulo bisimulation equivalence.*

**Proof** Use Verhoef's General Completeness Theorem [29]. We give no details. ■

**Proposition 3.3.10 (Properties of $\mathbf{ACP_{drt,\varepsilon}}$–ID)**
*Let $x$ be a basic term and $a \in A_\delta$. Then the following properties hold:*

*(i).* $ACP_{drt,\varepsilon}\text{-}ID \vdash x \parallel \underline{\underline{\varepsilon}} = x$

*(ii).* $ACP_{drt,\varepsilon}\text{-}ID \vdash \underline{\underline{\varepsilon}} \parallel x = x$

*(iii).* $ACP_{drt,\varepsilon}\text{-}ID \vdash \underline{\underline{a}} \mathbin{⌊\!⌊} x = \underline{\underline{a}} \cdot x$

*(iv).* $ACP_{drt,\varepsilon}\text{-}ID \vdash \underline{\underline{\delta}} \mathbin{⌊\!⌊} x = \underline{\underline{\delta}}$

*(v).* $ACP_{drt,\varepsilon}\text{-}ID \vdash x \mathbin{⌊\!⌊} \underline{\underline{\varepsilon}} = x$

**Proof** In the same manner as Proposition 2.4.13 on page 35. We give no details. ■

**Theorem 3.3.11 (Axioms of Standard Concurrency for $\mathbf{ACP^+_{drt,\varepsilon}}$–ID)**
*Let $x$, $y$, $z$ be closed $ACP_{drt,\varepsilon}$-ID terms. Then the following properties hold:*

*(i).* $ACP^+_{drt,\varepsilon}\text{-}ID \vdash x \mid y = y \mid x$

*(ii).* $ACP^{+}_{drt,\varepsilon}$*-ID* $\vdash x \parallel y = y \parallel x$

*(iii).* $ACP^{+}_{drt,\varepsilon}$*-ID* $\vdash (x \mid y) \mid z = x \mid (y \mid z)$

*(iv).* $ACP^{+}_{drt,\varepsilon}$*-ID* $\vdash (x \mathbin{⌊\!⌊} y) \mathbin{⌊\!⌊} z = x \mathbin{⌊\!⌊} (y \parallel z)$

*(v).* $ACP^{+}_{drt,\varepsilon}$*-ID* $\vdash x \mid (y \mathbin{⌊\!⌊} z) = (x \parallel y) \mathbin{⌊\!⌊} z$

*(vi).* $ACP^{+}_{drt,\varepsilon}$*-ID* $\vdash (x \parallel y) \parallel z = x \parallel (y \parallel z)$

**Proof** In the same manner as Theorem 2.4.14 on page 36. We give no details. ■

**Corollary 3.3.12 (Commutativity and Associativity of the Merge)**
*For closed terms, the merge and communication merge operators of* $ACP^{+}_{drt,\varepsilon}$*-ID are commutative and associative.*

**Proof** This follows directly from Theorem 3.3.11 on the preceding page. ■

## 3.4 Embeddings

The embeddings into $\mathrm{BPA}^{-}_{\mathrm{drt},\varepsilon}$-ID, $\mathrm{PA}^{-}_{\mathrm{drt},\varepsilon}$-ID, and $\mathrm{ACP}^{-}_{\mathrm{drt}}$-ID given in Section 2.6 also can be used as embeddings into $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$-ID, $\mathrm{PA}_{\mathrm{drt},\varepsilon}$-ID, and $\mathrm{ACP}_{\mathrm{drt}}$-ID respectively. Furthermore, the presence of delayable actions makes new embeddings possible.

The process algebras BPA, $\mathrm{BPA}_{\delta}$, and $\mathrm{BPA}_{\delta\varepsilon}$ of [13] can be embedded in $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$-ID by projecting every action $a$ onto the delayable process $a \cdot \varepsilon$, and $\delta$, $\varepsilon$, $+$, and $\cdot$ onto themselves. The process algebras PA, $\mathrm{PA}_{\delta}$, and ACP of [13] can be embedded in $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$-ID, $\mathrm{PA}_{\mathrm{drt},\varepsilon}$-ID, and $\mathrm{ACP}_{\mathrm{drt},\varepsilon}$-ID respectively in the same way. Using this embedding, we project every untimed process onto a $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$-ID, $\mathrm{PA}_{\mathrm{drt},\varepsilon}$-ID, or $\mathrm{ACP}_{\mathrm{drt},\varepsilon}$-ID process that is completely delayable, i.e., can do a time step to itself.

Finally, the process algebra $\mathrm{BPA}_{\mathrm{drt}}$-ID of [26] can be embedded in $\mathrm{BPA}_{\mathrm{drt},\varepsilon}$-ID by projecting $\sigma_{\mathrm{rel}}(x)$ onto $\underline{\underline{\sigma}} \cdot x$, and everything else onto itself.

## 4 Conclusions

We have successfully introduced the empty process in the context of discrete-time process algebra with relative timing. In doing so, we found that there is not much room for choice: the constraints of the unit-element property with respect to sequential composition and merge, associativity of the merge, time determinacy, and taking $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID as a basis almost completely determine which course to take. We also found that the empty process cannot straightforwardly be combined with the immediate deadlock process of Baeten and Bergstra [7, 8].

The axioms we have given lead to a sound and complete axiomatization of our bisimulation model. For closed terms, the axioms of standard concurrency are derivable.

As the behavior of the empty process is not always into accordance with one's first intuition (see Remark 2.3.5 and 2.3.6), one should be very careful when verifying protocols, to make sure that the protocol that is coded in process algebra, is indeed the same as the one that is supposed to be under study.

The discrete-time empty process makes for a worthwhile addition to the theory of process algebra. It can potentially be very useful in giving a formal semantics to specification languages such as SDL and MSC. It also extends the class of processes that can be finitely specified.

The usefulness of the empty process with respect to real-life protocol verification remains to be determined.

## 5 Acknowledgments

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