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Discreteness and Rationality of F-Thresholds

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Dedicated to Mel Hochster on the occasion of his sixty-fifth birthday

1. Introduction

In recent years, multiplier ideals have played an increasingly important role in higher-dimensional birational geometry. For a given ideal $\mathfrak{a}$ on a smooth variety X and a real parameter $c > 0$, the multiplier ideal $\mathcal{J}(\mathfrak{a}^c)$ is defined via a log resolution of the pair $(X, \mathfrak{a})$. Recall that a log resolution is a proper birational map $\pi : X' \to X$, with X' smooth such that $\mathfrak{a}\mathcal{O}_{X'}$ defines a simple normal crossing divisor $A = \sum_{i=1}^{r} a_i E_i$. Then, by definition,

$$\mathcal{J}(\mathfrak{a}^c) := \pi_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor cA \rfloor), \tag{1}$$

and this is an ideal of $\mathcal{O}_X$ that does not depend on the chosen log resolution. A *jumping coefficient* (also called a jumping *number* or a jumping *exponent*) of $\mathfrak{a}$ is a positive real number c such that $\mathcal{J}(\mathfrak{a}^c) \neq \mathcal{J}(\mathfrak{a}^{c-\varepsilon})$ for every $\varepsilon > 0$. These invariants were introduced and studied in [ELSV]. It follows from formula (1) that, if c is a jumping coefficient, then $c \cdot a_i$ is an integer for some i. In particular, every jumping coefficient is a rational number, and the set of jumping coefficients of a given ideal is discrete.

Hara and Yoshida [HaY] introduced a positive characteristic analogue of multiplier ideals, denoted by $\tau(\mathfrak{a}^c)$. This is a generalized test ideal for a tight closure theory with respect to the pair $(X, \mathfrak{a}^c)$. Similarly, one can define jumping numbers for such test ideals. These invariants were studied under the name of *F-thresholds* in [MTW], where it was shown that they satisfy many of the formal properties of the jumping coefficients in characteristic 0.

We emphasize that the test ideals are not determined by a log resolution of singularities—even in cases where such a resolution is known to exist. Instead, the definition uses the Frobenius morphism and requires a priori infinitely many conditions to be checked. This lack of built-in finiteness makes the question of rationality and discreteness of the F-thresholds nontrivial, and in fact these properties were left open in [MTW].

In this paper we settle these questions in the case of a regular ring R that is essentially of finite type over an F-finite field. More precisely, we show that, for

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every ideal in such a ring, all F-thresholds are rational and they form a discrete set. This statement has been proved recently by Hara and Monsky [HaMo] for $\mathfrak{a}$ a principal ideal in $R = k[\![x, y]\!]$, where k is a finite field. Our key result, which implies discreteness and rationality of F-thresholds, is a finiteness statement for test ideals in a polynomial ring. We show in Proposition 3.2 that, if $\mathfrak{a}$ is an ideal in $k[x_1, \dots, x_n]$ that is generated in degree $\le d$, then the test ideal $\tau(\mathfrak{a}^c)$ is generated in degree $\le \lfloor c \cdot d \rfloor$.

The structure of the paper is as follows. In Section 2 we recall the definition of test ideals and F-thresholds in a slightly different setup from that in [HaY] and [MTW]. In fact, we prefer to work with a different definition of test ideals that is more suitable for our purpose; we show later that this is equivalent to the definition from [HaY]. Because of this alternative definition and the slightly different setup, we decided to develop the theory from scratch. We hope this will prove beneficial for the reader, since given the alternative definition, the basic results become particularly transparent. After this initial setup, we prove our main results in Section 3.

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2. Generalized Test Ideals and F-Thresholds

In this section we collect some basic results about generalized test ideals as introduced by Hara and Yoshida [HaY] (see also [HaT; T1]). We restrict ourselves to working over an F-finite regular ring, which allows us to use an alternative description of these ideals. We subsequently prove that our ideals agree with the ones defined in [HaY]. An advantage of our definition is that the basic properties are easy to prove. We include some of the characteristic-p analogues of some of the basic results on multiplier ideals—such as the restriction theorem (Remark 2.28), Skoda's theorem (Proposition 2.25), and the subadditivity theorem (Proposition 2.11(4))—with likewise elementary proofs.

We also recall the definition of F-thresholds from [MTW], though we do not restrict to the case of a *local* ring as in that work. In this more general framework we show that the set of F-thresholds coincides with the set of jumping exponents for the test ideals.

Let us fix the following notation: R denotes a regular F-finite ring of positive characteristic p. We stress here that R is not assumed to be local. The regularity of R implies that the Frobenius morphism $F\colon R \to R$ sending $x \in R$ to x^p is flat [K]. This is equivalent to saying that the functor $(F^e)^*$ of extending scalars via F^e is exact. By our assumption that R is F-finite we just mean that F is a finite morphism. Since F is also flat, it follows that R is a finitely generated locally free module over its subring R^p of pth powers.

For an ideal J of R and a positive integer e we set $J^{[p^e]} := (u^{p^e} \mid u \in J)$. If $J = (u_1, \dots, u_r)$, then $J^{[p^e]} = (u_1^{p^e}, \dots, u_r^{p^e})$. Note that we have $(J^{[p^e]})^{[p^{e'}]} = J^{[p^{e+e'}]}$. One easily verifies that $(F^e)^*(R/J) \cong R/(J^{[p^e]})$. In particular, since F is (faithfully) flat we see that, if u is in R, then $u^p \in J^{[p]}$ if and only if $u \in J$.

EXAMPLE 2.1. For a field k, being F-finite means precisely that $[k : k^p] < \infty$. One easily verifies, by explicitly giving a basis of R over R^p, that for such fields the polynomial ring $k[x_1, \ldots, x_n]$ and even the power series ring $k[\![x_1, \ldots, x_n]\!]$ in finitely many variables are F-finite.

More generally: if A is an F-finite ring, then every A-algebra that is essentially of finite type over A is again F-finite. To check this, observe that if S is a multiplicative system in a ring R and if $a_1, \ldots, a_N$ generate R over R^p, then $\frac{a_1}{1}, \ldots, \frac{a_N}{1}$ generate $S^{-1}R$ over $(S^{-1}R)^p$.

Conversely, if k is a field and if R is an F-finite k-algebra essentially of finite type over k, then $[k : k^p] < \infty$. Indeed, if $\mathfrak{m}$ is a maximal ideal of R and if $K = R/\mathfrak{m}$, then $[K : K^p] < \infty$. Since $[K : k] < \infty$, we deduce that $[k : k^p] < \infty$.

If $(R, \mathfrak{m})$ is a regular local ring that is F-finite, then its completion $\hat{R}$ is also F-finite (and regular). In fact, since R is local, R is free over R^p. If $a_1, \ldots, a_N$ give a basis of R over R^p, then we claim that these elements also give a basis of $\hat{R}$ over $\hat{R}^p$. Indeed, we have canonical isomorphisms

$$F^*(\hat{R}) = F^*\Big(\operatorname{proj\,lim}_{\ell} R/\mathfrak{m}^{\ell}\Big) \simeq \operatorname{proj\,lim}_{\ell} F^*(R/\mathfrak{m}^{\ell}) \simeq \operatorname{proj\,lim}_{\ell} R/(\mathfrak{m}^{\ell})^{[p]} = \hat{R}$$

(observe that, since F is finite, F^* commutes with the projective limit). Hence the Frobenius morphism on $\hat{R}$ is obtained by base extension from the Frobenius morphism on R, which implies our claim.

2.1. The Ideals $\mathfrak{b}^{[1/q]}$

Let $\mathfrak{b}$ be an ideal of R and let $q = p^e$. We now introduce certain ideals $\mathfrak{b}^{[1/q]}$ that are crucial for our definition of the generalized test ideals. More precisely, the test ideals of $\mathfrak{a}$ are constructed using all $(\mathfrak{a}^r)^{[1/p^e]}$. The ideals $(\mathfrak{a}^r)^{[1/p^e]}$ were introduced in the case of a principal ideal $\mathfrak{a}$ in [ABLy] to study the D-module structure of a localization of R. It was predicted in [ABLy] that these ideals might be related to tight closure theory, and this paper confirms that prediction. Furthermore, similar ideas also appear in the description of generalized test ideals in [HaT, Lemma 2.1].

DEFINITION 2.2. For an ideal $\mathfrak{b}$ of R and $q = p^e$, where e is a positive integer, let $\mathfrak{b}^{[1/q]}$ denote the unique smallest ideal J of R with respect to inclusion and such that

$$\mathfrak{b} \subseteq J^{[q]}.$$

Because R is a finitely generated flat (hence projective) R-module via F^e, the following well-known lemma implies that, for every family of ideals $\{J_i\}_i$ of R, we have $\big(\bigcap_i J_i\big)^{[q]} = \bigcap_i J_i^{[q]}$. Therefore, $\mathfrak{b}^{[1/q]}$ is well-defined. We make the convention that $\mathfrak{b}^{[1/p^0]} = \mathfrak{b}$. Note in particular that $\mathfrak{b}^{[q'/q]}$ makes sense for every $q = p^e$ and $q' = p^{e'}$.

LEMMA 2.3. *If M is a finitely generated projective module over the Noetherian ring R, then for every family $\{J_i\}_i$ of ideals of R we have*

$$\bigcap_i J_i M = \Big(\bigcap_i J_i\Big) M.$$

Proof. The assertion is clear when M is free. The general case follows because we can find a finitely generated free module P such that M is a direct summand of P. $\square$

The following lemma collects some basic properties of the ideals $\mathfrak{b}^{[1/q]}$.

LEMMA 2.4. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R. Let $q = p^e$ and $q' = p^{e'}$, where e and e' are positive integers. Then the following statements hold.*

(i) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathfrak{a}^{[1/q]} \subseteq \mathfrak{b}^{[1/q]}$.*
(ii) $(\mathfrak{a} \cap \mathfrak{b})^{[1/q]} \subseteq \mathfrak{a}^{[1/q]} \cap \mathfrak{b}^{[1/q]}$ *and* $\mathfrak{a}^{[1/q]} + \mathfrak{b}^{[1/q]} = (\mathfrak{a} + \mathfrak{b})^{[1/q]}$.
(iii) $(\mathfrak{a} \cdot \mathfrak{b})^{[1/q]} \subseteq \mathfrak{a}^{[1/q]} \cdot \mathfrak{b}^{[1/q]}$.
(iv) $(\mathfrak{b}^{[q']})^{[1/q]} = \mathfrak{b}^{[q'/q]} \subseteq (\mathfrak{b}^{[1/q]})^{[q']}$.
(v) $\mathfrak{b}^{[1/qq']} \subseteq (\mathfrak{b}^{[1/q]})^{[1/q']}$.
(vi) $\mathfrak{b}^{[1/q]} \subseteq (\mathfrak{b}^{q'})^{[1/qq']}$.

Proof. Statement (i) is straightforward from the definition, and both inclusions "$\subseteq$" in (ii) follow from (i). On the other hand, in order to show that $(\mathfrak{a} + \mathfrak{b})^{[1/q]} \subseteq \mathfrak{a}^{[1/q]} + \mathfrak{b}^{[1/q]}$, it is enough to use

$$\mathfrak{a} + \mathfrak{b} \subseteq (\mathfrak{a}^{[1/q]})^{[q]} + (\mathfrak{b}^{[1/q]})^{[q]} = (\mathfrak{a}^{[1/q]} + \mathfrak{b}^{[1/q]})^{[q]}$$

together with the minimality in the definition of $(\mathfrak{a} + \mathfrak{b})^{[1/q]}$.

For (iii) we have $\mathfrak{a} \subseteq (\mathfrak{a}^{[1/q]})^{[q]}$ and $\mathfrak{b} \subseteq (\mathfrak{b}^{[1/q]})^{[q]}$, so we deduce that

$$\mathfrak{a} \cdot \mathfrak{b} \subseteq (\mathfrak{a}^{[1/q]})^{[q]} \cdot (\mathfrak{b}^{[1/q]})^{[q]} = (\mathfrak{a}^{[1/q]} \cdot \mathfrak{b}^{[1/q]})^{[q]}.$$

The inclusion in (iii) then follows from the minimality in the definition of $(\mathfrak{a} \cdot \mathfrak{b})^{[1/q]}$.

Statement (iv) is straightforward when $q' \geq q$, so we assume that $q = p^e \geq q' = p^{e'}$. For any ideal J in a regular ring, we have $\mathfrak{b}^{[p^{e'}]} \subseteq J^{[p^e]}$ if and only if $\mathfrak{b} \subseteq J^{[p^{e-e'}]}$. On the other hand, since

$$\mathfrak{b} \subseteq (\mathfrak{b}^{[1/p^e]})^{[p^e]} = \left((\mathfrak{b}^{[1/p^e]})^{[p^{e'}]}\right)^{[p^{e-e'}]},$$

it follows that $\mathfrak{b}^{[1/p^{e-e'}]} \subseteq (\mathfrak{b}^{[1/p^e]})^{[p^{e'}]}$ by the minimality in the definition of $\mathfrak{b}^{[1/p^{e-e'}]}$.

Similarly, for (v) we use the minimality in the definition of $\mathfrak{b}^{[1/qq']}$ together with

$$\mathfrak{b} \subseteq (\mathfrak{b}^{[1/q]})^{[q]} \subseteq ((\mathfrak{b}^{[1/q]})^{[1/q']})^{[qq']}.$$

Finally, for (vi) observe that (iv) implies $\mathfrak{b}^{[1/q]} = (\mathfrak{b}^{[q']})^{[1/qq']}$, and this is contained in $(\mathfrak{b}^{q'})^{[1/qq']}$ by (i). $\square$

When R is free over R^{p^e}, one has the following alternative description of these newly defined ideals. Again, in the case of a principal ideal this was already observed in [ABLy].

PROPOSITION 2.5. *Suppose that R is free over R^q, and let $e_1, \ldots, e_N$ be a basis of R over R^q. Let $h_1, \ldots, h_s$ be generators of an ideal $\mathfrak{b}$ of R, and for every $i = 1, \ldots, s$ let*

$$h_i = \sum_{j=1}^{N} a_{i,j}^q e_j \tag{2}$$

with $a_{i,j} \in R$. *Then* $\mathfrak{b}^{[1/q]} = (a_{i,j} \mid i \leq s,\ j \leq N)$.

Note that the proposition implies in particular that the description therein does not depend on the chosen basis of R over R^q or on the generators of $\mathfrak{b}$.

Proof of Proposition 2.5. It follows from (2) that $(h_1, \ldots, h_s) \subseteq (a_{i,j}^q \mid i \leq s,\ j \leq N)$ and therefore $\mathfrak{b} \subseteq (a_{i,j} \mid i, j)^{[q]}$. Hence the inclusion "'$\subseteq$" in the proposition follows immediately from the definition of $\mathfrak{b}^{[1/q]}$.

For the reverse inclusion, suppose that $\mathfrak{b} \subseteq J^{[q]}$. If $g_1, \ldots, g_m$ generate J, we may write

$$h_i = \sum_{\ell=1}^{m} b_\ell g_\ell^q \tag{3}$$

for some $b_\ell \in R$. Recalling that R is assumed free over R^q, consider the dual basis $e_1^*, \ldots, e_N^*$ for $\mathrm{Hom}_{R^q}(R, R^q)$, so $e_i^*(e_j) = \delta_{i,j}$. It follows from (2) that $e_j^*(h_i) = a_{i,j}^q$. On the other hand, (3) shows that

$$e_j^*(h_i) = \sum_\ell g_\ell^q e_j^*(b_\ell) \in J^{[q]}.$$

Therefore, $a_{i,j} \in J$ for every i and j, which gives $(a_{i,j} \mid i \leq s,\ j \leq N) \subseteq \mathfrak{b}^{[1/q]}$. □

REMARK 2.6. In [ABLy] it is shown that the ideals $\mathfrak{b}^{[1/q]}$ have the following D-module theoretic description. If $D_R^{(e)}$ denotes the subring of the ring of differential operators D_R that are linear over R^q (recall that $q = p^e$), then $(\mathfrak{b}^{[1/q]})^{[q]}$ is equal to the $D_R^{(e)}$-submodule of R generated by $\mathfrak{b}$. This interesting viewpoint will not be exploited further here. It is, however, used in our preprint [BMS] to derive rationality and discreteness of F-thresholds of principal ideals in rings of formal power series, thereby generalizing the two-variable case treated in [HaMo].

The following lemma shows that the formation of the ideals $\mathfrak{b}^{[1/q]}$ commutes with localization and completion. In particular, in order to compute $\mathfrak{b}^{[1/q]}$ we may always localize so that R is free over R^q and then use Proposition 2.5.

LEMMA 2.7. *Let* $\mathfrak{b}$ *be an ideal in* R.

(i) *If* S *is a multiplicative system in* R, *then* $(S^{-1}\mathfrak{b})^{[1/q]} = S^{-1}(\mathfrak{b}^{[1/q]})$.
(ii) *If* R *is local and* $\hat{R}$ *is the completion of* R, *then* $(\mathfrak{b}\hat{R})^{[1/q]} = (\mathfrak{b}^{[1/q]})\hat{R}$.

Proof. For (i), note that $\mathfrak{b} \subseteq (\mathfrak{b}^{[1/q]})^{[q]}$ implies

$$S^{-1}\mathfrak{b} \subseteq S^{-1}((\mathfrak{b}^{[1/q]})^{[q]}) = (S^{-1}\mathfrak{b}^{[1/q]})^{[q]}.$$

Therefore, $(S^{-1}\mathfrak{b})^{[1/q]} \subseteq S^{-1}(\mathfrak{b}^{[1/q]})$.

For the reverse inclusion, write $(S^{-1}\mathfrak{b})^{[1/q]} = S^{-1}J$ for some ideal J such that $(J : s) = J$ for every $s \in S$. Using the flatness of F, we see that $(J^{[q]} : s^q) = J^{[q]}$ for every $s \in S$ and hence $(J^{[q]} : s) = J^{[q]}$. Since

$$S^{-1}\mathfrak{b} \subseteq (S^{-1}J)^{[q]} = S^{-1}(J^{[q]}),$$

it follows that $\mathfrak{b} \subseteq J^{[q]}$. Therefore $\mathfrak{b}^{[1/q]} \subseteq J$, which gives "$\supseteq$" in (i).

For (ii) we will use Proposition 2.5. Since R is local, R is free over R^q. Moreover, one shows as in Example 2.1 that, if $e_1, \dots, e_N$ give a basis of R over R^q, then these elements also give a basis of $\hat{R}$ over $(\hat{R})^q$. Then the assertion in (ii) follows from Proposition 2.5. □

2.2. *Generalized Test Ideals*

We will now use the ideals $(\mathfrak{a}^r)^{[1/p^e]}$ to define the generalized test ideals of Hara and Yoshida [HaY]. In Proposition 2.22 it is shown that our definition coincides with that in [HaY].

LEMMA 2.8. *Let $\mathfrak{a}$ be an ideal in R. If r, r', e and e' are such that $r/p^e \geq r'/p^{e'}$ and $e' \geq e$, then*

$$(\mathfrak{a}^r)^{[1/p^e]} \subseteq (\mathfrak{a}^{r'})^{[1/p^{e'}]}.$$

Proof. Observe that $r' \leq rp^{e'-e}$ and so $\mathfrak{a}^{r'} \supseteq \mathfrak{a}^{rp^{e'-e}}$. It follows from parts (i) and (vi) of Lemma 2.4 that

$$(\mathfrak{a}^{r'})^{[1/p^{e'}]} \supseteq (\mathfrak{a}^{rp^{e'-e}})^{[1/p^{e'}]} \supseteq (\mathfrak{a}^r)^{[1/p^e]}.$$ □

Let $\mathfrak{a}$ be an ideal in R and let c be a positive real number. If we denote by $\lceil x \rceil$ the smallest integer greater than or equal to x, then for every e we have $\lceil cp^e \rceil/p^e \geq \lceil cp^{e+1} \rceil/p^{e+1}$. It follows from Lemma 2.8 that

$$(\mathfrak{a}^{\lceil cp^e \rceil})^{[1/p^e]} \subseteq (\mathfrak{a}^{\lceil cp^{e+1} \rceil})^{[1/p^{e+1}]}.$$

DEFINITION 2.9. With notation as before, the generalized test ideal of $\mathfrak{a}$ with exponent c is defined to be

$$\tau(\mathfrak{a}^c) = \bigcup_{e>0} (\mathfrak{a}^{\lceil cp^e \rceil})^{[1/p^e]}.$$

Since R is Noetherian, this union stabilizes after finitely many steps. In particular, the test ideal $\tau(\mathfrak{a}^c)$ is equal to $(\mathfrak{a}^{\lceil cp^e \rceil})^{[1/p^e]}$ for all sufficiently large e.

REMARK 2.10. We can write $R = R_1 \times \cdots \times R_m$, where all R_i are (F-finite) regular domains. An ideal $\mathfrak{a}$ in R can be written as $\mathfrak{a} = \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_m$, and it is clear that for every c we have

$$\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}_1^c) \times \cdots \times \tau(\mathfrak{a}_m^c). \tag{4}$$

This allows us to assume that R is a domain whenever this is convenient.

We make the convention that if R is a domain then $\tau(\mathfrak{a}^0) = R$ unless $\mathfrak{a} = (0)$, in which case $\tau(\mathfrak{a}^0) = (0)$. When R is not necessarily a domain, we define $\tau(\mathfrak{a}^0)$ such that the decomposition (4) holds also when $c = 0$.

PROPOSITION 2.11. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R.*

(i) *If $c_1 < c_2$, then $\tau(\mathfrak{a}^{c_2}) \subseteq \tau(\mathfrak{a}^{c_1})$.*
(ii) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\tau(\mathfrak{a}^c) \subseteq \tau(\mathfrak{b}^c)$.*
(iii) $\tau((\mathfrak{a} \cap \mathfrak{b})^c) \subseteq \tau(\mathfrak{a}^c) \cap \tau(\mathfrak{b}^c)$ *and* $\tau(\mathfrak{a}^c) + \tau(\mathfrak{b}^c) \subseteq \tau((\mathfrak{a} + \mathfrak{b})^c)$.
(iv) $\tau((\mathfrak{a} \cdot \mathfrak{b})^c) \subseteq \tau(\mathfrak{a}^c) \cdot \tau(\mathfrak{b}^c)$.

Proof. Lemma 2.4(i) implies that $(\mathfrak{a}^{\lceil c_2 p^e \rceil})^{[1/p^e]} \subseteq (\mathfrak{a}^{\lceil c_1 p^e \rceil})^{[1/p^e]}$. By taking $e \gg 0$, we obtain assertion (i). Assertions (ii)–(iv) follow similarly by taking the limit in the corresponding assertions from Lemma 2.4. □

REMARK 2.12. The inclusion in Proposition 2.11(iv) is the analogue of the subadditivity theorem for multiplier ideals in characteristic 0 (see [L, Thm. 9.5.20]). See also [HaY, Thm. 6.10] for a different approach.

A direct application of Lemma 2.7 (for $\mathfrak{b} = \mathfrak{a}^{\lceil cp^e \rceil}$ with $e \gg 0$) shows that the formation of test ideals commutes with localization and completion (see also [HaT]).

PROPOSITION 2.13. *Let $\mathfrak{a}$ be an ideal in R and let c be a nonnegative real number.*

(i) *If S is a multiplicative system in R, then $\tau((S^{-1}\mathfrak{a})^c) = S^{-1}\tau(\mathfrak{a}^c)$.*
(ii) *If R is local and $\hat{R}$ is the completion of R, then $\tau((\mathfrak{a}\hat{R})^c) = \tau(\mathfrak{a}^c)\hat{R}$.*

We now show that the family of test ideals $\tau(\mathfrak{a}^c)$ of a fixed ideal $\mathfrak{a}$ is right continuous in c.

PROPOSITION 2.14. *If $\mathfrak{a}$ is an ideal in R and if c is a nonnegative real number, then there exists an $\varepsilon > 0$ such that $\tau(\mathfrak{a}^c) = (\mathfrak{a}^r)^{[1/p^e]}$ whenever $c < r/p^e < c + \varepsilon$. That is, $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}^{c'})$, where c' is a rational number of the form r/p^e that approximates c from above sufficiently well.*

Proof. We show first that there is an $\varepsilon > 0$ and an ideal I in R such that $(\mathfrak{a}^r)^{[1/p^e]} = I$ whenever $c < r/p^e < c + \varepsilon$. Indeed, otherwise we can find r_m and e_m for $m \geq 1$ such that r_m/p^{e_m} form a strictly decreasing sequence converging to c and $(\mathfrak{a}^{r_m})^{[1/p^{e_m}]} \neq (\mathfrak{a}^{r_{m+1}})^{[1/p^{e_{m+1}}]}$ for every m. After replacing this sequence by a subsequence if necessary, we may assume that $e_m \leq e_{m+1}$ for every m. By Lemma 2.8, $(\mathfrak{a}^{r_m})^{[1/p^{e_m}]} \subsetneq (\mathfrak{a}^{r_{m+1}})^{[1/p^{e_{m+1}}]}$ for every m. Since this sequence of ideals does not stabilize, we contradict the fact that R is Noetherian. Hence we can find an ideal I as claimed.

We show now that $I = \tau(\mathfrak{a}^c)$. By Remark 2.10 we may assume that R is a domain. If $\mathfrak{a} = (0)$, then $(\mathfrak{a}^r)^{[1/p^e]} = (0)$ for every r and e, and our assertion is trivial. We assume henceforth that $\mathfrak{a} \neq (0)$. Let e be large enough such that $\tau(\mathfrak{a}^c) = (\mathfrak{a}^{\lceil cp^e \rceil})^{[1/p^e]}$ and $\lceil cp^e \rceil/p^e < c + \varepsilon$. If cp^e is not an integer, then $\lceil cp^e \rceil/p^e > c$ and we have $I = \tau(\mathfrak{a}^c)$.

Suppose now that cp^e is an integer. After possibly replacing e by a larger value, we may assume that $c+1/p^e < c+\varepsilon$ and hence $I = (\mathfrak{a}^{cp^e+1})^{[1/p^e]} \subseteq (\mathfrak{a}^{cp^e})^{[1/p^e]} = \tau(\mathfrak{a}^c)$. For the reverse inclusion we need to show that $\mathfrak{a}^{cp^e} \subseteq I^{[p^e]}$. Let $u \in \mathfrak{a}^{cp^e}$. If $e' \geq e$, then $(cp^{e'}+1)/p^{e'} < c+\varepsilon$ and so $\mathfrak{a}^{cp^{e'}+1} \subseteq I^{[p^{e'}]}$. We deduce that if v is a nonzero element in $\mathfrak{a}$ then, for every $e' \gg e$, we have $vu^{p^{e'-e}} \in (I^{[p^e]})^{[p^{e'-e}]}$. This means that u is in the tight closure of the ideal $I^{[p^e]}$, which is equal to $I^{[p^e]}$ because R is a regular ring (see [HoH]). We thus conclude that $u \in I^{[p^e]}$, and the proof is complete. (One could avoid the explicit appearance of tight closure by reducing to the local case and using the freeness of R over $R^{p^{e'-e}}$ to find, for $e' \gg e$, a splitting $\varphi\colon R \to R$ of the $(e'-e)$th iterate of the Frobenius $F^{e'-e}\colon R \to R$ such that $\varphi(v) = 1$. Applying φ to an equation witnessing the membership $vu^{p^{e'-e}} \in (I^{[p^e]})^{[p^{e'-e}]}$, it follows that $u \in I^{[p^e]}$.) □

COROLLARY 2.15. *If m is a positive integer, then for every $c \in \mathbb{R}_{\geq 0}$ we have*
$$\tau((\mathfrak{a}^m)^c) = \tau(\mathfrak{a}^{cm}).$$

Proof. It is clear that $((\mathfrak{a}^m)^r)^{[1/p^e]} = (\mathfrak{a}^{rm})^{[1/p^e]}$ for every positive integer r and e. Let e be large enough such that
$$\tau((\mathfrak{a}^m)^c) = (\mathfrak{a}^{\lceil cp^e \rceil m})^{[1/p^e]} \quad \text{and} \quad \tau(\mathfrak{a}^{cm}) = (\mathfrak{a}^{\lceil cmp^e \rceil})^{[1/p^e]}.$$
If for some e as before we have $cp^e \in \mathbb{Z}$, then our assertion is clear. If this is not the case, then for $e \gg 0$ we have $\lceil cp^e \rceil m/p^e$ larger than cm, but arbitrarily close to cm as e becomes large, and the statement follows from Proposition 2.14. □

COROLLARY 2.16. *For every ideal $\mathfrak{a}$ in R and every nonnegative real number c, there is an $\varepsilon > 0$ such that $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}^{c'})$ for every $c' \in [c, c+\varepsilon)$.*

Proof. It is clear that we may take ε as given by Proposition 2.14. □

DEFINITION 2.17. A positive real number c is an *F-jumping exponent* for $\mathfrak{a}$ if $\tau(\mathfrak{a}^c) \neq \tau(\mathfrak{a}^{c-\varepsilon})$ for every positive ε.

Unless explicitly mentioned otherwise, we make the convention that 0 is also an F-jumping exponent. We will study the basic properties of these numbers in the next section.

REMARK 2.18. Suppose that K/k is an extension of perfect fields, and consider the ring extension $R = k[x_1, \dots, x_n] \subseteq S = K[x_1, \dots, x_n]$. If $\mathfrak{a}$ is an ideal in R, then $\tau(\mathfrak{a}^c) \cdot S = \tau((\mathfrak{a} \cdot S)^c)$. Indeed, the monomials of degree at most $p^e - 1$ in each variable give a basis of both R and S over R^{p^e} and S^{p^e}, respectively. It follows from Proposition 2.5 that, for every r and e, we have $(\mathfrak{a}^r)^{[1/p^e]} \cdot S = ((\mathfrak{a}S)^r)^{[1/p^e]}$. In particular, we see that $\mathfrak{a}$ and $\mathfrak{a} \cdot S$ have the same F-jumping exponents.

2.3. The Connection with the Hara–Yoshida Definition

We show now that the ideals $\tau(\mathfrak{a}^c)$ we have defined coincide with the $\mathfrak{a}^c$-test ideals introduced in [HaY], temporarily denoted by $\tau'(\mathfrak{a}^c)$. To state Hara and Yoshida's definition, we need to recall their notion of $\mathfrak{a}^c$-tight closure.

For every $e \geq 1$, let R^e denote the $(R\text{–}R)$-bimodule on R, where the left R-module structure is the usual structure while the right one is induced by $F^e \colon R \to R$. For any inclusion $N \subset M$ of R-modules, there is an induced map of left R-modules

$$R^e \otimes N \to R^e \otimes M$$

whose image we denote by $N_M^{[p^e]}$. An element $m \in M$ is said to be *in the* $\mathfrak{a}^c$*-tight closure of* N *in* M if there exists $u \in R$, not in any minimal prime, such that

$$\operatorname{image}(u\mathfrak{a}^{\lceil cp^e \rceil} \otimes m) \in N_M^{[p^e]}$$

for all $e \gg 0$. The set of all such elements forms a submodule $N_M^{*\mathfrak{a}^c}$ of M.

We are mainly interested in the case where R is regular, N is zero, and

$$M = E := \bigoplus_{\mathfrak{m} \in \operatorname{maxspec} R} E_R(R/\mathfrak{m}) \tag{5}$$

is the direct sum, over all the maximal ideals of R, of the injective hulls at the corresponding residue fields. With this notation, Hara and Yoshida's definition can be stated as follows.

Definition 2.19. Let R be a regular ring of prime characteristic p, $\mathfrak{a}$ an ideal of R, and c any positive real number. The $\mathfrak{a}^c$*-test ideal,* denoted $\tau'(\mathfrak{a}^c)$, is the annihilator in R of the R-module $0_E^{*\mathfrak{a}^c}$, where E is as defined in (5).

Remark 2.20. More generally, Hara and Yoshida define the ideals $\tau'(\mathfrak{a}^c)$ even when R is not regular as the annihilator of the R-module $\bigcup_M 0_M^{*\mathfrak{a}^c}$, where the union is taken over all finitely generated submodules of E. (In the parlance of [HoH], this union could be called the *finitistic* $\mathfrak{a}^c$*-tight closure* of zero in E.) In the regular case, however, it is easy to check that $\bigcup_M 0_M^{*\mathfrak{a}^c} = 0_E^{*\mathfrak{a}^c}$, since the flatness of Frobenius implies $R^e \otimes M \subseteq R^e \otimes E$ for all submodules M of E.

Remark 2.21. Hara and Yoshida's definition recovers the classical definition of the test ideal, as introduced by Hochster and Huneke in [HoH], by taking $\mathfrak{a}$ to be the unit ideal and c arbitrary; see [HoH, Prop. 8.23]. However, we caution the reader that the name "test ideal" is misleading if $\mathfrak{a} \neq (1)$, because the elements of $\tau'(\mathfrak{a}^c)$ are *not the test elements* for $\mathfrak{a}^c$-tight closure. See [HaY, Def. 1.6, Rem. 1.10].

The next proposition states that these two notions of test ideals agree.

Proposition 2.22. *If* $\mathfrak{a}$ *is an ideal in a regular F-finite ring* R, *then for every nonnegative real number* c *we have* $\tau(\mathfrak{a}^c) = \tau'(\mathfrak{a}^c)$.

Before proving this, we recall a few well-known facts. First, if $(R, \mathfrak{m})$ is a regular local ring of dimension n, then the injective hull of the residue field $E = E_R(R/\mathfrak{m})$ can be identified with the local cohomology module $H_\mathfrak{m}^n(R)$. The Frobenius morphism on R induces a Frobenius action on $H_\mathfrak{m}^n(R)$ and hence also on E. This action, also denoted by F, can be described as follows. If $x_1, \ldots, x_n$ form a minimal system of generators of $\mathfrak{m}$, then

$$E \simeq \frac{R_{x_1\cdots x_n}}{\sum_{i=1}^n R_{x_1\cdots\widehat{x_i}\cdots x_n}}.$$

Under this isomorphism, the Frobenius morphism on E is induced by the Frobenius morphism on $R_{x_1\cdots x_n}$. In particular, an element $\eta \in E$ represented by the fraction $w/(x_1\cdots x_n)^N$ will be sent under Frobenius to the element $\eta^{p^e} \in E$ represented by the fraction

$$\frac{w^{p^e}}{(x_1\cdots x_n)^{Np^e}}.$$

Again using this representation of $\eta \in E$, we have $\operatorname{Ann}_R \eta = (x_1^N,\dots,x_n^N) :_R w$. In particular, for all $\eta \in E$,

$$\operatorname{Ann}_R(\eta^{p^e}) = (\operatorname{Ann}_R \eta)^{[p^e]} \quad \text{for all } e \geq 1. \tag{6}$$

Indeed, $\operatorname{Ann}_R \eta^{p^e} = (x_1^{p^eN},\dots,x_n^{p^eN}) :_R w^{p^e} = ((x_1^N,\dots,x_n^N) :_R w)^{[p^e]}$, with the last equality following from the flatness of Frobenius.

The Frobenius action on E induces an R-linear map $\nu^e\colon R^e \otimes_R E \to E$ given by $\nu^e(a\otimes\eta) = a\eta^{p^e}$. If R is regular then this map is an isomorphism for all $e \geq 1$. (See e.g. [B, p. 20; Ly, Ex. 1.2].) Thus we can conveniently compute the $\mathfrak{a}^c$-tight closure of zero in E as the set of all $\eta \in E$ for which there exists a $u \in R$ (not in any minimal prime) such that

$$u\mathfrak{a}^{\lceil cp^e\rceil}\eta^{p^e} = 0 \quad \text{for all } e \gg 0. \tag{7}$$

We now state two lemmas, both well known to practitioners of tight closure, whose proofs we include to keep the paper self-contained. In the language of tight closure, the first lemma says that, for a regular local ring, zero is tightly closed in the injective hull of the residue field.

LEMMA 2.23. *With notation as before, let u be a nonzero element in a regular local ring $(R,\mathfrak{m})$. If $\eta \in E = E_R(R/\mathfrak{m})$ satisfies $u\eta^{p^e} = 0$ for every $e \gg 0$, then $\eta = 0$.*

Proof. The hypothesis implies that $u \in \operatorname{Ann}_R(\eta^{p^e})$ for all $e \gg 0$. But then, by (6), we see that $u \in (\operatorname{Ann}_R(\eta))^{[p^e]}$ for every $e \gg 0$. Since u is nonzero, it follows from Nakayama's lemma that $\operatorname{Ann}_R(\eta) = R$. □

The second lemma is essentially a special case of the statement that 1 is a $\mathfrak{a}^c$-test element in a regular local ring (see [HaY, Thm. 1.7]).

LEMMA 2.24. *If R is a regular local ring, then*

$$0_E^{*\mathfrak{a}^c} = \{\eta \in E \mid \mathfrak{a}^{\lceil cp^e\rceil}\eta^{p^e} = 0 \text{ for all } e \geq 1\}.$$

Proof. Clearly $\supseteq$ holds, so consider any $\eta \in 0_E^{*\mathfrak{a}^c}$. We know that there is a nonzero element $u \in R$ such that $u\mathfrak{a}^{\lceil cp^e\rceil}\eta^{p^e} = 0$.

Now, fix any $e' \geq 1$ and any $h \in \mathfrak{a}^{\lceil cp^{e'}\rceil}$. For all $e \geq 1$, we have

$$h^{p^e} \in (\mathfrak{a}^{\lceil cp^{e'}\rceil})^{p^e} \subseteq \mathfrak{a}^{\lceil cp^{e'+e}\rceil}.$$

Thus

$$u(h\eta^{p^{e'}})^{p^e} = uh^{p^e}\eta^{p^{e'+e}} \in u\mathfrak{a}^{\lceil cp^{e'+e}\rceil}\eta^{p^{e+e'}},$$

which is zero for $e \gg 0$ by hypothesis. By Lemma 2.23, it follows that $h\eta^{p^{e'}} = 0$ for all $e' \geq 1$. Since h was an arbitrary element of $\mathfrak{a}^{\lceil cp^{e'}\rceil}$, we conclude that $\mathfrak{a}^{\lceil cp^{e'}\rceil}\eta^{p^{e'}} = 0$ for all $e' \geq 1$. The proof is complete. □

Proof of Proposition 2.22. The proof reduces immediately to the case where R is local. Indeed, since $E_R(R/\mathfrak{m})$ is already an $R_\mathfrak{m}$-module, the $\mathfrak{a}^c$-tight closure of zero in $E_R(R/\mathfrak{m})$ is the same as the $(\mathfrak{a}R_\mathfrak{m})^c$-tight closure of zero in $E_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m})$, whence

$$\tau'(\mathfrak{a}^c) = \bigcap_{\mathfrak{m}}\{f \in R \mid f/1 \in \tau((\mathfrak{a}R_\mathfrak{m})^c)\}. \tag{8}$$

This implies that $\tau'(\mathfrak{a}^c)$ commutes with localization at maximal ideals, whereas Lemma 2.7 implies that $\tau(\mathfrak{a}^c)$ does.

Now, we assume $(R, \mathfrak{m})$ is a regular local ring. To show that the ideals $\tau'(\mathfrak{a}^c)$ and $\tau(\mathfrak{a}^c)$ coincide, it is enough, by Matlis duality, to show that their annihilators in E coincide. By definition (and Matlis duality), the annihilator of $\tau'(\mathfrak{a}^c)$ in E is precisely $0_E^{*\mathfrak{a}^c}$. It remains to describe the annihilator in E of $\tau(\mathfrak{a}^c)$.

By definition, $\tau(\mathfrak{a}^c) = (\mathfrak{a}^{\lceil cp^e\rceil})^{[1/p^e]}$ for all $e \gg 0$. Hence an element η represented by a fraction $w/(x_1\cdots x_n)^N$ is annihilated by $\tau(\mathfrak{a}^c)$ if and only if

$$(\mathfrak{a}^{\lceil cp^e\rceil})^{[1/p^e]} \subseteq (x_1^N, \dots, x_n^N) : w.$$

By definition of $\mathfrak{b}^{[1/q]}$, this holds if and only if

$$\mathfrak{a}^{\lceil cp^e\rceil} \subseteq ((x_1^N, \dots, x_n^N) : w)^{[p^e]} = (x_1^{Np^e}, \dots, x_n^{Np^e}) : w^{p^e},$$

which in turn means exactly that $\mathfrak{a}^{\lceil cp^e\rceil}\eta^{p^e} = 0$ for all $e \gg 0$. We conclude, using Lemma 2.24, that the annihilator of $\tau(\mathfrak{a}^c)$ in E is precisely $0_E^{*\mathfrak{a}^c}$. This completes the proof. □

2.4. Skoda's Theorem

For future reference, we include the following characteristic-p version of Skoda's theorem due to Hara and Takagi [HaT]. Since the result there is not stated in the generality we need, we include a proof for the benefit of the reader.

Proposition 2.25. *If $\mathfrak{a}$ is an ideal generated by m elements, then for every $c \geq m$ we have*

$$\tau(\mathfrak{a}^c) = \mathfrak{a}\cdot\tau(\mathfrak{a}^{c-1}).$$

Proof. If e is large enough, then

$$\tau(\mathfrak{a}^c) = (\mathfrak{a}^{\lceil cp^e\rceil})^{[1/p^e]} \quad \text{and} \quad \tau(\mathfrak{a}^{c-1}) = (\mathfrak{a}^{\lceil cp^e\rceil - p^e})^{[1/p^e]}.$$

It is therefore enough to show that, for every $r \geq mp^e$,

$$(\mathfrak{a}^r)^{[1/p^e]} = \mathfrak{a}\cdot(\mathfrak{a}^{r-p^e})^{[1/p^e]}.$$

The inclusion $\mathfrak{a} \cdot (\mathfrak{a}^{r-p^e})^{[1/p^e]} \subseteq (\mathfrak{a}^r)^{[1/p^e]}$ actually holds for every $r \geq p^e$. Indeed, this says that $(\mathfrak{a}^{r-p^e})^{[1/p^e]} \subseteq ((\mathfrak{a}^r)^{[1/p^e]} : \mathfrak{a})$, which is equivalent to

$$\mathfrak{a}^{r-p^e} \subseteq ((\mathfrak{a}^r)^{[1/p^e]} : \mathfrak{a})^{[p^e]} = \left(((\mathfrak{a}^r)^{[1/p^e]})^{[p^e]} : \mathfrak{a}^{[p^e]}\right).$$

This holds because $\mathfrak{a}^{r-p^e} \cdot \mathfrak{a}^{[p^e]} \subseteq \mathfrak{a}^r \subseteq ((\mathfrak{a}^r)^{[1/p^e]})^{[p^e]}$.

Suppose now that $\mathfrak{a} = (h_1, \dots, h_m)$. In order to prove the reverse inclusion, note that if $r \geq m(p^e - 1) + 1$ then, in the product of r of the h_i, at least one of these appears p^e times. Therefore, $\mathfrak{a}^r = \mathfrak{a}^{[p^e]} \cdot \mathfrak{a}^{r-p^e}$. We deduce that

$$\mathfrak{a}^r \subseteq \mathfrak{a}^{[p^e]} \cdot \mathfrak{a}^{r-p^e} \subseteq \mathfrak{a}^{[p^e]} \cdot ((\mathfrak{a}^{r-p^e})^{[1/p^e]})^{[p^e]} = (\mathfrak{a} \cdot (\mathfrak{a}^{r-p^e})^{[1/p^e]})^{[p^e]},$$

which implies $(\mathfrak{a}^r)^{[1/p^e]} \subseteq \mathfrak{a} \cdot (\mathfrak{a}^{r-p^e})^{[1/p^e]}$. $\square$

REMARK 2.26. In Proposition 2.25, rather than taking m to be the minimal number of generators for $\mathfrak{a}$, we can take m to be the *analytic spread* of $\mathfrak{a}$—that is, the minimal number of generators for any subideal with the same integral closure as $\mathfrak{a}$. Indeed, this is immediate from the following lemma, which shows that the test ideals of $\mathfrak{a}$ depend only on its integral closure (see also [HaY] and [HaT]).

LEMMA 2.27. *If $\bar{\mathfrak{a}}$ denotes the integral closure of $\mathfrak{a}$, then $\tau(\mathfrak{a}^c) = \tau(\bar{\mathfrak{a}}^c)$ for every c.*

Proof. The inclusion $\tau(\mathfrak{a}^c) \subseteq \tau(\bar{\mathfrak{a}}^c)$ is immediate from Proposition 2.11(ii). For the reverse inclusion, observe that by usual properties of integral closure there exists an m such that $\bar{\mathfrak{a}}^{m+\ell} \subseteq \mathfrak{a}^\ell$ for every ℓ. Corollary 2.16 gives $c' > c$ such that $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}^{c'})$ and $\tau(\bar{\mathfrak{a}}^c) = \tau(\bar{\mathfrak{a}}^{c'})$.

Using Corollary 2.15, we see that

$$\tau(\bar{\mathfrak{a}}^{c'}) = \tau((\bar{\mathfrak{a}}^{m+\ell})^{c'/(m+\ell)}) \subseteq \tau((\mathfrak{a}^\ell)^{c'/(m+\ell)}) = \tau(\mathfrak{a}^{c'-c'm/(m+\ell)}).$$

If $\ell \gg 0$, then $c < c' - c'm/(m+\ell) < c'$. Hence, by our choice of c', we obtain $\tau(\bar{\mathfrak{a}}^c) \subseteq \tau(\mathfrak{a}^c)$. $\square$

REMARK 2.28. If $\varphi\colon R \to S$ is a morphism of regular F-finite rings of positive characteristic and if $\mathfrak{b}$ is an ideal in R, then $(\mathfrak{b} \cdot S)^{[1/p^e]} \subseteq \mathfrak{b}^{[1/p^e]} \cdot S$ for every e. Indeed, since $\mathfrak{b} \subseteq (\mathfrak{b}^{[1/p^e]})^{[p^e]}$ it follows that $\mathfrak{b} \cdot S \subseteq (\mathfrak{b}^{[1/p^e]} \cdot S)^{[p^e]}$.

We deduce that $\tau((\mathfrak{a} \cdot S)^c) \subseteq \tau(\mathfrak{a}^c) \cdot S$ for every nonnegative c. This is an analogue of the restriction theorem for multiplier ideals in characteristic 0 (see [L, Exs. 9.5.4 and 9.5.8]). For a different argument in a more general (characteristic-p) framework, see [HaY, Thms. 4.1 and 6.10].

2.5. *F-Jumping Exponents and F-Thresholds*

In [MTW], F-jumping exponents of an ideal $\mathfrak{a}$ were described as F-thresholds. Because the statements and proofs in [MTW] were given for the local case, we review them here for the reader's convenience.

Let $\mathfrak{a}$ be an ideal in R. For a fixed ideal J in R such that $\mathfrak{a} \subseteq \operatorname{rad}(J)$ and for an integer $e > 0$, we define $\nu_{\mathfrak{a}}^J(p^e)$ to be the largest nonnegative integer r such that

$\mathfrak{a}^r \not\subseteq J^{[p^e]}$ (if there is no such r, then we put $\nu_{\mathfrak{a}}^J(p^e) = 0$). If $\mathfrak{a}^r \not\subseteq J^{[p^e]}$, then $\mathfrak{a}^{pr} \not\subseteq J^{[p^{e+1}]}$. Indeed, otherwise we get $(\mathfrak{a}^r)^{[p]} \subseteq J^{[p^{e+1}]}$ and hence $\mathfrak{a}^r \subseteq J^{[p^e]}$, a contradiction. Therefore,

$$\frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} \leq \frac{\nu_{\mathfrak{a}}^J(p^{e+1})}{p^{e+1}},$$

so we may define the *F-threshold* of $\mathfrak{a}$ with respect to J as

$$c^J(\mathfrak{a}) := \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} = \sup_{e\geq 1} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}.$$

Note that if $\mathfrak{a}$ is generated by s elements then $\mathfrak{a}^{s(p^e-1)+1} \subseteq \mathfrak{a}^{[p^e]}$. If $\mathfrak{a}^\ell \subseteq J$, then $\nu_{\mathfrak{a}}^J(p^e) \leq \ell(s(p^e-1)+1)-1$ for every e. Hence $c^J(\mathfrak{a}) \leq s\ell$; in particular, $c^J(\mathfrak{a})$ is finite.

The following proposition relates the F-thresholds to the generalized test ideals of $\mathfrak{a}$.

PROPOSITION 2.29. *Let $\mathfrak{a}$ be an ideal in R.*

(i) *If J is an ideal in R such that $\mathfrak{a} \subseteq \mathrm{rad}(J)$, then*

$$\tau(\mathfrak{a}^{c^J(\mathfrak{a})}) \subseteq J.$$

(ii) *If c is a nonnegative real number, then $\mathfrak{a} \subseteq \mathrm{rad}(\tau(\mathfrak{a}^c))$ and*

$$c^{\tau(\mathfrak{a}^c)}(\mathfrak{a}) \leq c.$$

Proof. For (i), observe that by Corollary 2.16 there exists a $c' > c^J(\mathfrak{a})$ such that $I := \tau(\mathfrak{a}^{c^J(\mathfrak{a})}) = \tau(\mathfrak{a}^{c'})$. Suppose now that $e \gg 0$, so $\tau(\mathfrak{a}^{c'}) = (\mathfrak{a}^{\lceil c'p^e\rceil})^{[1/p^e]}$.

Since $c' > c^J(\mathfrak{a})$ and e is large enough, we have $\lceil c'p^e\rceil \geq \nu_{\mathfrak{a}}^J(p^e)+1$; hence

$$\mathfrak{a}^{\lceil c'p^e\rceil} \subseteq J^{[p^e]}.$$

This implies that $I \subseteq J$, as required.

For (ii), let e be large enough that $\tau(\mathfrak{a}^c) = (\mathfrak{a}^{\lceil cp^e\rceil})^{[1/p^e]}$. By definition, we have $\mathfrak{a}^{\lceil cp^e\rceil} \subseteq \tau(\mathfrak{a}^c)^{[p^e]}$, which implies

$$\nu_{\mathfrak{a}}^{\tau(\mathfrak{a}^c)}(p^e) \leq \lceil cp^e\rceil - 1.$$

Dividing by p^e and letting e go to infinity, we obtain the required inequality. $\square$

COROLLARY 2.30. *For every ideal $\mathfrak{a}$ in R, the set of F-jumping exponents for $\mathfrak{a}$ is equal to the set of F-thresholds of $\mathfrak{a}$ (as we range over all possible ideals J).*

Proof. We show first that, if α is an F-jumping exponent for $\mathfrak{a}$, then $\alpha = c^J(\mathfrak{a})$ for $J = \tau(\mathfrak{a}^\alpha)$. Indeed, Proposition 2.29(ii) gives $c^J(\mathfrak{a}) \leq \alpha$ and so, by Proposition 2.11(i), $J = \tau(\mathfrak{a}^\alpha) \subseteq \tau(\mathfrak{a}^{c^J(\mathfrak{a})})$. We also have the reverse inclusion by Proposition 2.29(i). Since α is an F-jumping exponent, we must have $\alpha = c^J(\mathfrak{a})$.

Suppose now that $\alpha = c^J(\mathfrak{a})$ for some J whose radical contains $\mathfrak{a}$. We need to show that α is a jumping exponent. If this is not the case, then there is an $\alpha' < \alpha$ such that $\tau(\mathfrak{a}^\alpha) = \tau(\mathfrak{a}^{\alpha'})$. Using Proposition 2.29(i) yields $\tau(\mathfrak{a}^{\alpha'}) \subseteq J$. If e is large enough, then $\tau(\mathfrak{a}^{\alpha'}) = (\mathfrak{a}^{\lceil \alpha' p^e \rceil})^{[1/p^e]}$. Therefore, $\mathfrak{a}^{\lceil \alpha' p^e \rceil} \subseteq J^{[p^e]}$ and $\nu^J(\mathfrak{a}) \leq \lceil \alpha' p^e \rceil - 1$. Dividing by p^e and letting e go to infinity, we get $c^J(\mathfrak{a}) \leq \alpha'$, a contradiction. This completes the proof of the corollary. □

2.6. *Mixed Generalized Test Ideals*

With our approach it is also easy to define "mixed" test ideals as in [HaY] and [HaT]. For example, suppose that $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are ideals in the regular F-finite ring R, and let $c_1, \dots, c_r \in \mathbb{R}_+$. For every $e \geq 1$, consider

$$I_e := (\mathfrak{a}_1^{\lceil c_1 p^e \rceil} \cdots \mathfrak{a}_r^{\lceil c_r p^e \rceil})^{[1/p^e]}.$$

As before, one can check easily that $I_e \subseteq I_{e+1}$ for every e; and, since R is Noetherian, this sequence of ideals stabilizes. Its limit is the test ideal $\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_r^{c_r})$. It is straightforward to check that the basic properties we have discussed so far generalize to this setting. For example, the subadditivity formula from Proposition 2.11 can be generalized with the same proof to the following statement: if $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are ideals in R and if $\lambda_1, \dots, \lambda_r \in \mathbb{R}_+$, then

$$\tau(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r}) \subseteq \tau(\mathfrak{a}_1^{\lambda_1}) \cdots \tau(\mathfrak{a}_r^{\lambda_r})$$

(see also [HaY, Thm. 6.10]). Similarly, the summation formula due to Takagi [T2, Thm. 3.1] can be easily proved using our definition.

3. Discreteness and Rationality

In this section we prove our main result.

THEOREM 3.1. *Let k be a field of characteristic $p > 0$ and let R be a regular F-finite ring, essentially of finite type over k. Suppose that $\mathfrak{a}$ is an ideal in R.*

(i) *The set of F-jumping exponents of $\mathfrak{a}$ is discrete (i.e., in every finite interval there are only finitely many such numbers).*
(ii) *Every F-jumping exponent of $\mathfrak{a}$ is a rational number.*

We will reduce the proof of the theorem to the case $R = k[x_1, \dots, x_n]$. We start with some preliminary results. The first proposition, of independent interest, gives an effective bound for the degrees of the generators of the ideals $\tau(\mathfrak{a}^c)$ in terms of the degrees of the generators of $\mathfrak{a}$. It is our main ingredient for the proof of the theorem in the polynomial ring case. For a real number t we denote by $\lfloor t \rfloor$ the largest integer less than or equal to t, and by $\{t\}$ we denote the fractional part $t - \lfloor t \rfloor$.

PROPOSITION 3.2. *Let $\mathfrak{a}$ be an ideal in the polynomial ring $k[x_1, \dots, x_n]$, where k is a field of characteristic p such that $[k : k^p] < \infty$. If $\mathfrak{a}$ can be generated by polynomials of degree at most d then, for every nonnegative real number c, the ideal $\tau(\mathfrak{a}^c)$ can be generated by polynomials of degree at most $\lfloor cd \rfloor$.*

Proof. Fix first r and e. The ideal $\mathfrak{a}^r$ is generated by polynomials of degree at most rd. Choose such generators $h_1, \dots, h_s$ for $\mathfrak{a}^r$.

Let $b_1, \dots, b_m$ be a basis of k over k^{p^e}, and consider the basis of $R = k[x_1, \dots, x_n]$ over R^{p^e} given by

$$\{b_i x^u \mid i \le m,\ u \in \mathbb{N}^n,\ 0 \le u_j \le p^e - 1 \text{ for all } j\}$$

(if $u = (u_1, \dots, u_n)$ then we put $x^u = x_1^{u_1} \cdots x_n^{u_n}$). Let

$$h_j = \sum_{i,u} a_{i,u}^{p^e} b_i x^u,$$

with $a_{i,u} \in R$. Then for every i and u we have $\deg(a_{i,u}^{p^e}) \le \deg(h_j) \le rd$. It follows from Proposition 2.5 that $(\mathfrak{a}^r)^{[1/p^e]}$ can be generated by polynomials of degree at most rd/p^e. But by Proposition 2.14, $\tau(\mathfrak{a}^c) = (\mathfrak{a}^r)^{[1/p^e]}$ for some $r/p^e \ge c$ closely approximating c. Thus, for sufficiently close approximations r/p^e, $\tau(\mathfrak{a}^c)$ is generated by polynomials of degree at most $\lfloor rd/p^e \rfloor = \lfloor cd \rfloor$. The proof is complete. $\square$

Remark 3.3. As Lazarsfeld has pointed out, the analogue of Proposition 3.2 for multiplier ideals can be deduced from a vanishing statement. Indeed, suppose that $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ is an ideal generated in degree at most d, and denote by $\mathcal{J}(\mathfrak{a}^c)$ its multiplier ideal. After homogenizing $\mathfrak{a}$ and taking the associated sheaf on $\mathbf{P}^n$, we obtain a sheaf of ideals $\tilde{\mathfrak{a}}$ whose restriction to $\mathbb{A}^n = (x_0 \ne 0) \subseteq \mathbf{P}^n$ is $\mathfrak{a}$ and such that $\tilde{\mathfrak{a}} \otimes \mathcal{O}_{\mathbf{P}}(d)$ is globally generated. Let H denote the hyperplane class on $\mathbf{P}^n$; then $\lfloor cd \rfloor H \sim K_{\mathbf{P}^n} + (n + 1 + \lfloor cd \rfloor)H$. For $1 \le i \le n$ we have $(n + 1 + \lfloor cd \rfloor - i - cd)H$ ample, so

$$H^i(\mathbf{P}^n, \mathcal{J}(\tilde{\mathfrak{a}}^c) \otimes \mathcal{O}_{\mathbf{P}}(\lfloor cd \rfloor - i)) = 0$$

by the Nadel vanishing theorem (see [L, Cor. 9.4.15]). Therefore, $\mathcal{J}(\tilde{\mathfrak{a}}^c) \otimes \mathcal{O}_{\mathbf{P}}(\lfloor cd \rfloor)$ is 0-regular in the sense of Castelnuovo–Mumford regularity; hence it is generated by global sections. This implies that $\mathcal{J}(\mathfrak{a}^c)$ is generated in degree at most $\lfloor cd \rfloor$.

Proposition 3.4. *Let $\mathfrak{a}$ be an ideal in a regular F-finite ring R.*

(i) *If α is an F-jumping exponent for $\mathfrak{a}$, then also $p\alpha$ is an F-jumping exponent.*
(ii) *If $\mathfrak{a}$ can be generated by m elements and if $\alpha > m$ is an F-jumping exponent for $\mathfrak{a}$, then also $\alpha - 1$ is an F-jumping exponent.*

Remark 3.5. For a stronger statement in (ii), we can take m to be any integer at least as large as the analytic spread of $\mathfrak{a}$. (This follows from Remark 2.26.) For example, if R is local with an infinite residue field, then we can take $m = \dim R$.

Proof of Proposition 3.4. For (i), note that by Corollary 2.30 there is an ideal J containing $\mathfrak{a}$ in its radical such that $\alpha = c^J(\mathfrak{a})$. It is clear that $\nu_{\mathfrak{a}}^{J^{[p]}}(p^e) = \nu_{\mathfrak{a}}^J(p^{e+1})$, so $c^{J^{[p]}}(\mathfrak{a}) = p \cdot c^J(\mathfrak{a})$. Hence $p\alpha$ is an F-jumping exponent of $\mathfrak{a}$ by Corollary 2.30.

For (ii), suppose that $\alpha - 1$ is not an F-jumping exponent. Let $\varepsilon > 0$ be such that $\tau(\mathfrak{a}^{\alpha-1}) = \tau(\mathfrak{a}^{\alpha-1-\varepsilon})$ and $\alpha - \varepsilon > m$. It follows from Proposition 2.25 that $\tau(\mathfrak{a}^\alpha) = \tau(\mathfrak{a}^{\alpha-\varepsilon})$ and hence α is not an F-jumping exponent—a contradiction. $\square$

The following proposition relates the generalized test ideals of two different ideals defining the same scheme. For the characteristic-0 analogue in the context of multiplier ideals, see [M, Prop. 2.3].

PROPOSITION 3.6. *Let R be a regular F-finite ring of positive characteristic, and let I be an ideal in R of pure codimension r such that $S = R/I$ is regular. If $\mathfrak{a}$ is an ideal in R containing I, then for every nonnegative real number c we have*

$$\tau((\mathfrak{a}/I)^c) = \tau(\mathfrak{a}^{c+r}) \cdot S.$$

Proof. By Proposition 2.13, forming generalized test ideals commutes with localization and completion. Therefore it is enough to prove the case where R is local and complete. The ideal I is generated by part of a regular system of parameters for R, so induction on r shows that we need only prove the case $r = 1$. Hence we assume that $R = k[\![x_1, \dots, x_n]\!]$ and $I = (x_n)$. Note that $[k : k^p] < \infty$, since R is F-finite.

We claim that

$$(\mathfrak{a}^{r+p^e})^{[1/p^e]} \cdot S = \big((\mathfrak{a}/(x_n))^{r+1}\big)^{[1/p^e]} \tag{9}$$

for any integer r and all $e \gg 0$. This implies the desired statement. Indeed, if e is large enough then

$$\tau(\mathfrak{a}^{c+1}) \cdot S = (\mathfrak{a}^{\lceil cp^e \rceil + p^e})^{[1/p^e]} \cdot S = \big((\mathfrak{a}/(x_n))^{\lceil cp^e \rceil + 1}\big)^{[1/p^e]} = \tau\big((\mathfrak{a}/(x_n))^c\big).$$

The last equality follows from Proposition 2.14, since $(\lceil cp^e \rceil + 1)/p^e$ is larger than c but arbitrarily close to c as e gets very large.

We now prove (9) with the aid of Proposition 2.5. Let $a_1, \dots, a_m$ be a basis of k over k^{p^e}, and consider the basis of R over R^{p^e} given by

$$\{a_i x^u \mid i \le m,\ u = (u_j) \in \mathbb{N}^n,\ 0 \le u_j \le p^e - 1 \text{ for all } j\}.$$

Write $\mathfrak{a} = (x_n) + \mathfrak{b}$, where $\mathfrak{b}$ is generated by power series in $k[\![x_1, \dots, x_{n-1}]\!]$. Then

$$\mathfrak{a}^{r+p^e} = \sum_{i=0}^{r+p^e} x_n^i \mathfrak{b}^{r+p^e-i}.$$

The generators of $(\mathfrak{a}^{r+p^e})^{[1/p^e]}$ that come from writing in this basis the generators of $x_n^i \mathfrak{b}^{r+p^e-i}$ are divisible by x_n if $i \ge p^e$, and hence they map to zero in S. On the other hand, the generators coming from $x_n^i \mathfrak{b}^{r+p^e-i}$ for $i \le p^e - 1$ are the same as the ones obtained from writing the generators of $\mathfrak{b}^{r+p^e-i}$ in the corresponding basis of $k[\![x_1, \dots, x_{n-1}]\!]$ over $k^{p^e}[\![x_1^{p^e}, \dots, x_{n-1}^{p^e}]\!]$. It is clearly enough to consider only the largest such ideal, namely $\mathfrak{b}^{r+1}$. This shows that $(\mathfrak{a}^{r+p^e})^{[1/p^e]} \cdot S = \big((\mathfrak{a}/(x_n))^{r+1}\big)^{[1/p^e]}$, as claimed. □

COROLLARY 3.7. *If R, S, and $\mathfrak{a}$ are as in Proposition 3.6 and if $c > 0$ is a jumping exponent for $\mathfrak{a} \cdot S$, then $c + r$ is a jumping exponent for $\mathfrak{a}$.*

We can give now the proof of our main result.

Proof of Theorem 3.1. For (i), suppose we have a sequence of F-jumping exponents $\{\alpha_m\}_m$ for $\mathfrak{a}$ with a finite accumulation point α. By Corollary 2.16, $\alpha_m < \alpha$ for $m \gg 0$. After replacing this sequence by a subsequence, we may assume that $\alpha_m < \alpha_{m+1}$ for every m.

Let $R = R_1 \times \cdots \times R_s$, where all R_i are domains. We have $\mathfrak{a} = \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_s$, and for every m there is j such that α_m is a jumping exponent for $\mathfrak{a}_j$. After replacing our sequence by a subsequence, we may replace R by some R_j and hence assume that R is a domain.

By hypothesis, we can write $R \simeq S^{-1}(k[x_1, \dots, x_n]/I)$ for some ideal I and some multiplicative system S. Observe that $[k : k^p] < \infty$ (see Example 2.1). Let $\mathfrak{a} = S^{-1}(\mathfrak{b}/I)$ for some ideal $\mathfrak{b} \subset k[x_1, \dots, x_n]$ containing I. Note that $S^{-1}I$ is a prime ideal and therefore has pure codimension, say r, in $S^{-1}k[x_1, \dots, x_n]$. It follows from Corollary 3.7 that $r + \alpha$ is an accumulation point for the jumping exponents of $S^{-1}\mathfrak{b}$. Moreover, Proposition 2.13(i) implies that $r + \alpha$ is an accumulation point for the jumping exponents of $\mathfrak{b}$. Hence, in order to achieve a contradiction, we may assume that $R = k[x_1, \dots, x_n]$.

Suppose now that $\mathfrak{a}$ is generated by polynomials of degree at most d. In this case, Proposition 3.2 implies that every $\tau(\mathfrak{a}^{\alpha_m})$ is generated by polynomials of degree at most $\lfloor \alpha d \rfloor$. Because the ideals $\tau(\mathfrak{a}^{\alpha_m})$ form a strictly decreasing sequence of ideals, their k-subvector spaces of polynomials of degree at most $\lfloor \alpha d \rfloor$ form a strictly decreasing sequence of subspaces of the finite-dimensional vector space $k[x_1, \dots, x_n]_{\le \lfloor \alpha d \rfloor}$. This contradiction completes the proof of Theorem 3.1(i).

For part (ii), suppose that $\alpha > 0$ is an F-jumping exponent for $\mathfrak{a}$. Proposition 3.4(i) implies that all $p^e\alpha$ are F-jumping exponents for $\mathfrak{a}$. We may assume that no $p^e\alpha$ is an integer, since otherwise α is clearly rational. Suppose that $\mathfrak{a}$ is generated by m elements and let e_0 be such that $p^{e_0}\alpha > m$. We deduce from Proposition 3.4(ii) that $\{p^e\alpha\} + m - 1$ is a jumping exponent for every $e \ge e_0$. Since all these numbers lie in $[m-1, m)$, it follows from part (i) of the theorem that there are only finitely many such numbers. As a result, we can find $e_1 \ne e_2$ such that $p^{e_1}\alpha - p^{e_2}\alpha$ is an integer, and so α is a rational number. □

The ideas in the foregoing proof can be used to estimate explicitly the F-jumping exponents of an ideal in a polynomial ring in terms of the degrees of its generators. Since we know that all F-jumping exponents are rational, it is enough to bound their denominators.

PROPOSITION 3.8. *Let $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$ be an ideal generated by m polynomials of degree at most d. If e_0 is such that $p^{e_0} > md$ and $N = \binom{md+n}{n}$, then for every F-jumping exponent α of $\mathfrak{a}$ we have $p^a(p^b - 1)\alpha \in \mathbb{N}$ for some $a \le e_0 + N$ and $b \le N$.*

Proof. By Proposition 3.4(ii) it is enough to consider the case $\alpha \le m$. Since $\tau(\mathfrak{a}^c)$ is generated by polynomials of degree at most md for every $c \le m$, it follows that there are at most $N = \dim_k k[x_1, \dots, x_n]_{\le md}$ jumping exponents of $\mathfrak{a}$ in $(0, m]$.

Assume that for every $e \le e_0 + N$ we have $p^e\alpha \notin \mathbb{N}$, since otherwise our assertion is clear. In particular, $\alpha > 0$ and therefore $\alpha \ge 1/d$ (note that $\tau(\mathfrak{a}^c) = R$ for $c < 1/d$ by Proposition 3.2). Consider now e, with $e_0 \le e \le e_0 + N$. Because $p^e\alpha \ge p^{e_0}/d > m$, Proposition 3.4 implies that $\{p^e\alpha\} + m - 1$ is an F-jumping exponent of $\mathfrak{a}$. This gives $N+1$ numbers in $(m-1, m)$ that are F-jumping exponents. We deduce that there exist $e_1 < e_2$ in $\{e_0, \dots, e_0 + N\}$ such that $p^{e_1}(p^{e_2-e_1} - 1)\alpha \in \mathbb{N}$, which completes the proof. □

REMARK 3.9. We can derive a bound that is independent of m in Proposition 3.8 by taking $m = n$. Indeed, note first that we may assume k is infinite; otherwise, take an infinite perfect extension K of k and replace $\mathfrak{a}$ by $\mathfrak{a} \cdot K[x_1, \dots, x_n]$ using Remark 2.18.

Now, for any maximal ideal $\mathfrak{m} \subset k[x_1, \dots, x_n] = R$, it is clear (by Proposition 2.13) that the set of jumping exponents for $\mathfrak{a}R_{\mathfrak{m}}$ is a subset of those for $\mathfrak{a}$. On the other hand, for each of the finitely many jumping exponents α in the range $[0, m]$, there exists a maximal ideal $\mathfrak{m}_\alpha$ such that the inclusion $\tau(\mathfrak{a}^c) \subsetneq \tau(\mathfrak{a}^{c-\varepsilon})$ is preserved under localization at $\mathfrak{m}_\alpha$. It follows from Proposition 3.4(ii) that the set of jumping exponents for $\mathfrak{a}$ is contained in (and hence equal to) the union of the sets of jumping exponents for the $\mathfrak{m}_\alpha$ as we range over these finitely many $\mathfrak{m}_\alpha$.

Next, we fix generators $h_1, \dots, h_m$ for $\mathfrak{a}$ of degree at most d, and let $\mathfrak{b}$ be the ideal generated by $g_i = \sum_{j=1}^m a_{i,j} h_j$ for $1 \le i \le n$, where $a_{i,j} \in k$. It is well known that, for every maximal ideal $\mathfrak{m}$ in R, the ideals $\mathfrak{b}R_{\mathfrak{m}}$ and $\mathfrak{a}R_{\mathfrak{m}}$ have the same integral closure if the $a_{i,j}$ are general in k (see e.g. [L, Ex. 9.6.19], whose proof does not assume that the base field is $\mathbb{C}$). Hence, by choosing the $a_{i,j}$ sufficiently general, we may assume that $\mathfrak{b}R_{\mathfrak{m}_\alpha}$ and $\mathfrak{a}R_{\mathfrak{m}_\alpha}$ have the same integral closure, and hence the same test ideals, for each of our finitely many relevant maximal ideals $\mathfrak{m}_\alpha$. It follows that the set of jumping exponents for $\mathfrak{a}$ is contained in the set of jumping exponents for the n-generated ideal $\mathfrak{b}$. Thus we obtain the desired statement for $\mathfrak{a}$ by applying Proposition 3.8 to $\mathfrak{b}$.

REMARK 3.10. Consider several ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ in $k[x_1, \dots, x_n]$, where k is F-finite. The same argument used in the proof of Proposition 3.2 shows that if each $\mathfrak{a}_i$ is generated in degree at most d_i then, for every $c_1, \dots, c_r \in \mathbb{R}_+$, the mixed test ideal $\tau(\mathfrak{a}^{c_1} \cdots \mathfrak{a}^{c_r})$ can be generated in degree at most $\sum_{i=1}^r c_i d_i$. However, this assertion does not seem to have such strong consequences in the case of several ideals. The most optimistic expectation in this case is that, for every $b_1, \dots, b_r$, the region

$$\{c \in \mathbb{R}_+^n \mid c_i \le b_i \text{ for all } i\}$$

can be decomposed into a finite set of rational polytopes with non-overlapping interiors such that, on the interior of each face of such a polytope, the mixed test ideal $\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_r^{c_r})$ is constant.

If k is contained in the algebraic closure of a finite field, then we can find a finite field k_0 such that every $\mathfrak{a}_i$ is generated by polynomials in $k_0[x_1, \dots, x_n]$. In this case, it follows that every $\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_r^{c_r})$ is generated by polynomials with coefficients in k_0. In particular, if the c_i are bounded above then the generators of our mixed test ideal lie in a finite-dimensional vector space over the finite field k_0. Therefore, given $\mathfrak{a}_1, \dots, \mathfrak{a}_r$, we can have only finitely many possible test ideals $\tau(\mathfrak{a}^{c_1} \cdots \mathfrak{a}^{c_r})$ if we bound the c_i.

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