Discriminating the Weyl type in higher dimensions using scalar curvature invariants

Alan Coley[♡] **and Sigbjørn Hervik**[✐]

[♡]Department of Mathematics and Statistics,
Dalhousie University, Halifax, Nova Scotia,
Canada B3H 3J5

[✐]Faculty of Science and Technology,
University of Stavanger,
N-4036 Stavanger, Norway

aac@mathstat.dal.ca, sigbjorn.hervik@uis.no

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Abstract

Higher dimensional Lorentzian spacetimes are of considerable interest in current theoretical physics. It is possible to algebraically classify any tensor in a Lorentzian spacetime of arbitrary dimensions using alignment theory. In the case of the Weyl tensor, and using bivector theory, the associated Weyl curvature operator will have a restricted eigenvector structure for algebraic types **II** and **D**, which leads to necessary conditions on the discriminants of the associated characteristic equation which can be manifestly expressed in terms of polynomial scalar curvature invariants. The use of such necessary conditions in terms of scalar curvature invariants will be of great utility in the study and classification of black hole solutions in more than four dimensions.

1 Introduction

Higher dimensional Lorentzian spacetimes are of considerable interest in current theoretical physics. Lorentzian spacetimes for which all polynomial scalar invariants constructed from the Riemann tensor and its covariant derivatives are constant are called CSI spacetimes [1]. All curvature invariants of all orders vanish in an n-dimensional Lorentzian VSI spacetime [2]. The higher dimensional VSI and CSI degenerate Kundt spacetimes are examples of spacetimes that are of fundamental importance since they are solutions of supergravity or superstring theory, when supported by appropriate bosonic fields [3]. Higher dimensional black hole solutions are also of current interest [4].

The introduction of *alignment theory* [5] has made it possible to algebraically classify any tensor in a Lorentzian spacetime of arbitrary dimensions by boost weight. In particular, using the notions of an aligned null direction and alignment order in Lorentzian geometry, the dimension-independent theory of alignment can be applied to the tensor classification problem for the Weyl tensor (and the Ricci tensor) in higher dimensions (thus generalizing the Petrov classification in four dimensions (4D)) [5]. We note that alignment type suffices for the classification of 4D Weyl tensors, but the situation for higher-dimensional Weyl tensors is more complicated (and different classifications in 4D are not equivalent in higher dimensions).

Another classification can be obtained by introducing *bivectors*. In [6], the Weyl bivector operator was defined in a manner consistent with its boost-weight decomposition. The Weyl tensor can then be algebraically classified (based, in part, on the eigenbivector problem), which gives rise to a refinement in dimensions higher than four of the usual alignment (boost-weight) classification, in terms of the irreducible representations of the spins. In particular, the classification in 5D was discussed in some detail [6].

A *scalar polynomial curvature invariant of order* n (or, in short, a scalar invariant) is a scalar obtained by contraction from a polynomial in the Riemann tensor and its covariant derivatives up to the order n. Scalar invariants have been extensively used in the study of VSI and CSI spacetimes [1, 2, 3]. In arbitrary dimensions, demanding that all of the zeroth polynomial Weyl invariants vanish implies that the Weyl type is **III**, **N**, or **O** (similarly for the Ricci type). It would be particularly useful to find necessary conditions in terms of simple scalar invariants in order for the Weyl type (or the Ricci type) to be of type **II** or **D**. *The main goal of this Letter is the determination of necessary conditions in higher dimensions for algebraic Weyl type, and particularly type* **II** *or* **D** *using scalar polynomial curvature invariants.*

For a tensor of a particular algebraic type, the associated operator will have a restricted eigenvector structure. For a given curvature operator in arbitrary dimensions [9], we can then consider the eigenvalues of this operator to obtain necessary conditions. In principle the analysis can be used to study the various subclasses in more detail. In particular, requiring the algebraic type to be **II** or **D** will impose restrictions on the structure of the eigenvalues of the operator. We shall describe an analysis of the discriminants of the associated characteristic equation to determine the conditions on a tensor for a given algebraic type. Since the characteristic equation has coefficients which can be expressed in terms of the scalar polynomial curvature invariants of the operator, we can consequently give conditions on the eigenvalue structure (in terms of degeneracies in a set of discriminants ${}^nD_i$), which can be manifestly expressed in terms of these polynomial curvature invariants [14].

In particular, we use the technique to study the necessary conditions in arbitrary dimensions for the Weyl and Ricci curvature operators (and hence the higher dimensional Weyl and Ricci tensors) to be of algebraic type **II** or **D**. We are consequently able to determine the necessary condition(s) in terms of simple scalar polynomial curvature invariant for the higher dimensional Weyl and Ricci tensors to be of type **II** or **D**.

2 Discriminant Analysis

We can analyse the discriminants of the associated characteristic equation to determine the conditions on a tensor for a given algebraic type [14]. For a tensor of a particular algebraic type, the associated operator will have a restricted eigenvector structure. For a given curvature operator, R, we can consider the eigenvalues of this operator to obtain necessary conditions. In particular, requiring the algebraic type to be **II** or **D** (**II**/**D**) will impose restrictions of the eigenvalues on the operator (e.g.,, the eigenvalue type, or 'Segre type', will have to be of a particular form). Crucial in this discussion is the eigenvalue equation or characteristic equation:

$$\det(\mathsf{R} - \lambda \mathsf{1}) = 0. \tag{1}$$

This equation is a polynomial equation in λ and the eigenvalues are the roots of this equation. Since the characteristic equation has coefficients which can be expressed in terms of the invariants of R, we can give conditions on the eigenvalue structure expressed manifestly in terms of the invariants of R. The invariants of R are polynomial curvature invariants of spacetime, and will be referred to as syzygies.

The characteristic equation can be expanded as a polynomial equation:

$$f(\lambda) = \det(\lambda \mathsf{1} - \mathsf{R}) = a_0\lambda^n + a_1\lambda^{n-1} + \dots a_i\lambda^{n-i} + \dots + a_n. \tag{2}$$

Defining the polynomial invariants of R,

$$R_1 \equiv \mathrm{Tr}(\mathsf{R}), \quad R_2 \equiv \mathrm{Tr}(\mathsf{R}^2), \quad R_3 \equiv \mathrm{Tr}(\mathsf{R}^3), \quad \text{etc}, \tag{3}$$

we can write the coefficients a_i as a determinant of an $i \times i$ matrix as follows:

$$a_i = \frac{(-1)^i}{i!} \det \begin{bmatrix} R_1 & 1 & 0 & \cdots & 0 \\ R_2 & R_1 & 2 & \ddots & \vdots \\ R_3 & R_2 & R_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (i-1) \\ R_i & \dots & R_3 & R_2 & R_1 \end{bmatrix}, \tag{4}$$

where $a_0 \equiv 1$. It is often convenient to analyse the algebraic structure of the trace-free part of the curvature operator R, denoted S, where now $S_1 = 0$.

The given polynomical can now be analysed and the criteria for the various 'Segre types' can be given. The resulting syzygies are special polynomial invariants which can be used to characterise the various eigenvalue cases; i.e., they are *discriminants*. A complete set of discriminants can be found algorithmically [11]. The resulting discriminants will be denoted ${}^nD_i$, ${}^nE_i$, ${}^nF_i$ etc., where n denotes order of the polynomial, and i is a running index. These discriminants can be given in terms of the coefficients a_i; however, using Newton's identities, we can express them explicitely in terms of the polynomial invariants R_1, R_2, etc.

For the polynomial (2), we define the $(2n+1) \times (2n+1)$ discrimination matrix $Disc(f)$:

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 & 0 \\ 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & 0 & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & & 0 & 0 \\ 0 & 0 & na_0 & \cdots & 2a_{n-2} & a_{n-1} & & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n \end{bmatrix} \tag{5}$$

Consider now the principal minor series, $\{d_1, d_2, d_3, ..., d_{2n+1}\}$, where d_k is the determinant of the submatrix of $Disc(f)$ formed by the first k rows and k columns. Let ${}^nD_i = d_{2i}$, $i = 1, ..., n$, then the *discriminant sequence* of the polynomial $f(x)$ is given by: $\{{}^nD_1, {}^nD_2, {}^nD_3, ..., {}^nD_n\}$. By expressing the ${}^nD_i$ in terms of the curvature invariants, R_1, R_2, etc, we can obtain the *primary syzygies* ${}^nD_i$ for the operator R. Note that the order of the ${}^nD_i$ are $\mathcal{O}({}^nD_i) = R^{i(i-1)}$.

Utilizing the notion of a sign list and a multiple factor sequence, we can establish an algorithm which enables us to completely determine the eigenvalue structure of R using the invariants $\mathrm{Tr}(\mathsf{R}^k)$ (up to degeneracies). Note that this procedure will provide us with the discriminants (or syzygies) to study the necessary conditions on any curvature operator of any specific algebraic type [14]. A complete set of discriminants will not only involve the primary syzygies, ${}^nD_i$, but also secondary syzygies ${}^nE_i$, ${}^nF_i$ etc. for the case of multiple eigenvalues.

In particular, we can use the technique to study the necessary conditions in order for the Weyl (and Ricci) curvature operators to be of algebraic type **II/D**. For example, we note that the condition ${}^nD_n = 0$ will signal a double eigenvalue since the number of eigenvalues is maximum $(n-1)$. If ${}^nD_{n-1} = 0$ also, then we have maximum $(n-2)$ eigenvalues, etc. We can utilise this to create syzygies which are necessary for the special algebraic type to be fulfilled.

3 Type II/D

For the Weyl or Ricci tensor to be of type **II** (or **D**) then the eigenvalues of the corresponding operator need to be of a special form. Since the invariants of a type **II** are the same as for type **D**, we will assume type **D** in this discussion.

For example, in 4D if the complex zeroth order quadratic and cubic Weyl invariants I and J satisfy $27J^2 = I^3$, then the Weyl tensor is of type **II** (or more special; e.g., type **D**) [7, 10] . The real and imaginary parts of this complex syzygy can be expressed using invariants of the Weyl tensor not containing duals. This is equivalent to the (12th order) real syzygies given in [14] from the associated six dimensional (bivector) system with ${}^6D_6 = 0$ and ${}^6D_5 = 0$. Applying the condition ${}^4D_4 = 0$ to the 4D trace-free Ricci tensor we obtain the (12th order) syzygy for the trace-free Ricci tensor to be of type **II/D** [10, 14].

3.1 Ricci tensor

For the Ricci tensor, we note that a type **II**/**D** tensor is of Segre type $\{(1,1)11...1\}$, or simpler. This implies that the *Ricci operator has at least one eigenvalue of (at least) multiplicity 2. Furthermore, all the eigenvalues are real.*

3.1.1 Ricci tensor in 5D

For a 5D trace-free operator ($S_1 = 0$), the discriminants (given explicitly in [14]) are: ${}^5D_2, {}^5D_3, {}^5D_4, {}^5D_5, {}^5E_2, {}^5F_2$. In particular, defining $\mathcal{D} \equiv {}^5D_5$, we have that

$$\begin{aligned}
\mathcal{D} \;=\;& \frac{21}{2}\,{S_2}^2{S_3}^2{S_5}^2 - \frac{539}{120}\,{S_2}^3{S_3}^3S_5 - \frac{91}{72}\,S_2\,{S_3}^2{S_4}^3 - \frac{31}{96}\,{S_2}^3{S_3}^2{S_4}^2\\
&+\frac{41}{96}\,{S_2}^5{S_3}^2S_4 - \frac{5}{2}\,S_2\,{S_5}^2{S_4}^2 + \frac{11}{8}\,{S_5}^2{S_2}^3S_4 - \frac{59}{48}\,{S_3}^4S_4\,{S_2}^2\\
&+\frac{11}{48}\,{S_2}^6S_3\,S_5 + \frac{9}{4}\,S_3\,S_5\,{S_2}^2{S_4}^2 - \frac{31}{20}\,S_3\,S_5\,{S_2}^4S_4 + \frac{4}{45}\,{S_3}^5S_5\\
&-\frac{5}{2}\,{S_3}^2S_4\,{S_5}^2 - \frac{2}{27}\,{S_3}^6S_2 - \frac{35}{3}\,S_2\,S_3\,{S_5}^3 + \frac{1}{512}\,{S_2}^{10}\\
&-\frac{1}{48}\,{S_3}^4{S_4}^2 - \frac{79}{400}\,{S_2}^5{S_5}^2 - \frac{79}{1152}\,{S_2}^7{S_3}^2\\
&+\frac{151}{192}\,{S_2}^4{S_3}^4 - \frac{7}{256}\,{S_2}^8S_4 + \frac{19}{128}\,{S_2}^6{S_4}^2 + \frac{5}{3}\,S_3\,S_5\,{S_4}^3\\
&-\frac{25}{64}\,{S_2}^4{S_4}^3 + \frac{1}{2}\,{S_2}^2{S_4}^4 - \frac{1}{4}\,{S_4}^5 + 5\,{S_5}^4 + \frac{43}{12}\,{S_3}^3S_4\,S_2\,S_5
\end{aligned}$$

For the trace-free Ricci tensor, we note that type **II**/**D** has to be of Segre type $\{(1,1)111\}$ or simpler. This implies that 2 eigenvalues are equal, while the remaining eigenvalue has to be real:

$${}^5D_5 = 0, \quad {}^5D_4 \geq 0, \quad {}^5D_3 \geq 0, \quad {}^5D_2 \geq 0.$$

Result: *The necessary condition for the trace-free Ricci tensor, S, to be of algebraic type* **II** *or* **D** *in 5D is that the discriminant* ${}^5D_5$ *is zero, so that the related scalar polynomial curvature invariant* $\mathcal{D} = 0$ (20th order syzygy).

3.2 Weyl tensor

Let us next consider the the Weyl tensor in n dimensions. Let $V \equiv \wedge^2 T_p^* M$ be the vector space of bivectors at a point p. The symmetric ($C_{MN} = C_{NM}$) bivector operator $\mathsf{C} = (C_N^{\ M}) : V \mapsto V$, can be written in an appropriate $(n{+}1{+}m{+}n)$-block form [6]. The eigenbivector problem can now be formulated as follows. A bivector F_A is an eigenbivector of C if and only if

$$C_N^{\ M} F_M = \lambda F_N, \quad \lambda \in \mathbb{C}.$$

In particular, for type **D**, the canonical form is given

$$\mathsf{C} = \mathrm{blockdiag}(M, \Psi, M^t), \tag{6}$$

where M is an $(n-2)\times(n-2)$ matrix and Ψ is a square matrix (in terms of boost weight 0 components which can be specified using the irreducible compositions

$(\bar{R}, \bar{S}_{ij}, A_{ij}, \bar{C}_{ijkl})$, which can be explicitly written in terms of the components of the Weyl tensor) [6]. For a Weyl tensor of type **II** there are also negative boost weights terms that do not affect the discriminants (or the scalar polynomial curvature invariants).

Since the eigenvalues of M and M^t are the same, we have that the *Weyl operator has at least* $(n-2)$ *eigenvalues of (at least) multiplicity 2.* This observation connects the algebraic types to the eigenvalue structure and enables us to construct the necessary discriminants.

3.2.1 Weyl tensor in 5D

In 5D ($n = 3$), the structure of (6) simplifies; $\bar{C}_{ijkl} = 0$, and thus we only have the following boost weight 0 components: $\bar{R}$, $\bar{S}_{ij}$, A_{ij} (where $i, j = 3, 4, 5$, and we can use the spins to diagonalise $S^i{}_j$), where $C_{1i0j} = -\frac{1}{2}\bar{R}_{ij} - \frac{1}{2}A_{ij}$, $\quad C_{01ij} = A_{ij}$, $\quad C_{0101} = -\frac{1}{2}\bar{R}$, $\quad C_{ijkl} = \bar{R}_{ijkl}$ (and $\bar{R}^k{}_{ikj} = \bar{R}_{ij} = \frac{1}{3}\bar{R}\delta_{ij} + \bar{S}_{ij}$) [14].

For the 5D Weyl tensor, the bivector space is 10-dimensional. The discriminants of a 10-dimensional trace-free operator ($S_1 = 0$) were given in [14]. In particular, the type **II** operator has 3 eigenvalues of (at least) multiplicity 2. Therefore, we get the necessary conditions (syzygies):

$$^{10}D_{10} = {}^{10}D_9 = {}^{10}D_8 = 0.$$

Since these polynomial invariants are of particular importance, we will denote $^{10}D_{10}, {}^{10}D_9, {}^{10}D_8$ by $\mathcal{C}, \mathcal{H}$ and $\mathcal{P}$, respectively (the $\mathcal{CHP}$ Weyl invariants).

Result: *The necessary condition for the Weyl tensor to be of type* **II** *or* **D** *in 5D is that the scalar polynomial curvature invariants* $\mathcal{C} = \mathcal{H} = \mathcal{P} = 0$.

These syzygies are of order 90, 72 and 56, respectively. In principle, they can be computed symbolically (e.g., the discriminant $\mathcal{P}$, which contains 13377 terms, was written symbolically in [14] and is available at website:www). In practice, these expressions may not be very useful, although for specific metrics they may be computable using MAPLE and useful results may be possible. Sometimes a more indirect approach may be more fruitful (see the note below).

We stress that the conditions determined are necessary conditions. Indeed, these conditions may not be sufficient. We also note that for the solvmanifold considered in [14]: $^{10}D_{10} > 0$, $^{10}D_9 > 0$, $^{10}D_8 > 0$, showing that this metric is not type **II** or **D** (and that the $\mathcal{CHP}$ invariants are not trivial).

Practical Note: Necessary conditions can also be found from considering various combinations of the Weyl tensor. for example, in 5D the 5-dimensional (trace-free part of the) operator $T^\alpha_\beta = C^{\alpha\mu\nu\rho}C_{\beta\mu\nu\rho}$ can be considered. If the Weyl tensor is of type **II**/**D**, then so is T^α_β, and we can again obtain the necessary conditions:

$${}^5_TD_5 = 0, \quad {}^5_TD_4 \geq 0, \quad {}^5_TD_3 \geq 0, \quad {}^5_TD_2 \geq 0.$$

Note that ${}^5_TD_5 = 0$ is a 40th order syzygy. Therefore, a useful strategy in practical computations (for example, determined the algebraic type of a 5D Weyl tensor) might be to test for necessicity using an operator like T, which is relatively simple. If the syzygy is not satisfied we have a definitive result. It is possible that the syzygy can only be satified for certain coordinate values (or parameter values), whence the $\mathcal{CHP}$ syzygies can be tested in these simpler particular cases. See the example below for an illustration.

3.2.2 Type II/D in higher dimensions

In higher dimensions we will obtain similar syzygies for type **II/D** tensors. In n dimensions, the Ricci and Weyl type **II/D** conditions are the corresponding syzygies ($m = n(n-1)/2$):

$$\text{Ricci:} \qquad {}^{n}D_n = 0, \tag{7}$$

$$\text{Weyl:} \qquad {}^{m}D_m = {}^{m}D_{m-1} = ... = {}^{m}D_{m-n+2} = 0. \tag{8}$$

Note that the Ricci syzygy is of order $n(n-1)$, while the highest Weyl syzygy is of order $n(n^2-1)(n-2)/4$.

4 Example: The Rotating Black Ring.

The 5D Rotating Black Ring (RBR) is generally of type **G** or $\mathbf{I_i}$, but can also be of type **II** or **D** at different locations and for particular values of the parameters λ, μ [12]. Assuming that the form of the metric given by eqn. (9) in [12] (in terms of the parameters λ, μ , where R has been set to unity), we consider the coordinate ranges $-1 \leq x \leq 1$ and $1 \leq y < \infty$, corresponding to the regions B, A_2, A_3 in [12] in order to retain the correct (Lorentzian) signature. Let us consider the algebraic type of the 5D Weyl tensor. Calculating the polynomial invariants $\mathrm{Tr}(\mathsf{C}^k)$ and evaluating at the 'target' point $x = 0$ and $y = 2$ in the region under consideration, all of the R_i and hence the resulting discriminants are functions of the parameters λ, μ only. Then, at the 'target' point, in general the metric is type **G** or $\mathbf{I_i}$; however, the case $\mu = 1/2$ corresponds to the RBR horizon ($y = 1/\mu$, type **II**), and $y = 1/\lambda$ corresponds to a curvature singularity.

Let us first consider the trace-free part of the operator $T^{\alpha}_{\ \beta} = C^{\alpha\mu\nu\rho}C_{\beta\mu\nu\rho}$, which gives us the discriminant:

$${}^{5}_{T}D_5 = \frac{\lambda^{12}(\lambda-\mu)^{12}(2\mu-1)^2(1-\lambda)^4(1+\lambda)^4}{(1-2\lambda)^{113}}F(\mu,\lambda), \tag{9}$$

where $F(\mu, \lambda)$ is a polynomial which is generically not zero. On the horizon $\mu = 1/2$, we see that ${}^{5}_{T}D_5 = 0$, and computing ${}^{5}_{T}D_4$ we get ${}^{5}_{T}D_4 > 0$ except for special values of λ. This is a signal that the metric is of type **II** on the horizon. Indeed, at the horizon, $\mu = 1/2$, the computation simplifies and we can compute the $\mathcal{CHP}$ invariants. Since, $\mathcal{C} = \mathcal{H} = \mathcal{P} = 0$ (for $\mu = 1/2$), this gives further evidence that the the metric is of type **II** on the horizon. Note that we actually get further contraints from the secondary discriminants (e.g.; if ${}^{10}_{W}D_7 \neq 0$, then ${}^{10}_{W}F_3 \neq 0$ etc. for this to be of type **II**). In principle, we can also check for type $\mathbf{I_i}$. Another interesting special case is the Myers-Perry solution with $\lambda = 1$ (type **D**), for which both ${}^{5}_{T}D_5 = {}^{5}_{T}D_4 = 0$, and ${}^{5}_{T}D_2 > 0$, and all of the $\mathcal{CHP}$ invariants are zero.

4.0.3 Differential invariants

It was proven that a 4D Lorentzian spacetime metric is either *$\mathcal{I}$-non-degenerate* or *degenerate Kundt* [8]. The $\mathcal{I}$-non-degenerate theorem deals with not only zeroth order invariants but also differential scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives. These results were generalized to higher dimensions in [13].

Therefore, a metric that is not characterized by its scalar invariants must be of degenerate Kundt form. Thus, the *necessary* conditions in order for a spacetime not to be $\mathcal{I}$-non-degenerate [8] are that the Riemann tensor and all of its covariant derivatives must be of types **II** or **D**. By constructing the appropriate curvature operators, these necessary conditions (syzygies) can be obtained using discrimants. In [8] two higher order syzygies were given as *sufficient* conditions for $\mathcal{I}$-non-degeneracy, which can be expressed in terms of discriminants [14].

4.0.4 Discussion

Recently, there has been considerable interest in black holes in more than four dimensions [4]. While the study of black holes in higher dimensions was perhaps originally motivated by supergravity and string theory, now the physical properties of such black holes are of interest in their own right. Indeed, studies have shown that even at the classical level gravity in higher dimensions exhibits much richer dynamics than in 4D, and one of the most remarkable features of higher dimensions is the non-uniqueness of black holes [4]

There now exist a number of different higher dimensional black hole solutions [4], including the rotating black rings [12], that are the subject of ongoing research in classical relativity and string theory. Some of these new spacetimes have be classified algebraically [5, 12]. However, in order to make further progress it is absolutely necessary to be able to develop new techniques for solving the vacuum field equations in higher dimensions and to be able to comprehensively classify such solutions, and the algebraic techniques introduced to date will be of fundemental importance in this development [5, 6, 14]. However, the algebraic techniques used up until now are rather difficult to apply, and the development of simpler criteria, including the use of necessary conditions in terms of scalar curvature invariants introduced here, will hopefully prove to be of great utility.

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