TABLE 1
Biases and confidence region coverage levels for quadratic regression model (nominal coverage 95%).

	Bias			Coverage			Coverage
	β_0	θ	β_2	β_0	θ	β_2	$(\beta_0, \theta, \beta_2)$
			(1) $\beta_0 = 0$, $\theta = 8$, $\beta_2 = -0.25$; no outlier				
MLE	−0.00266	0.07541	−0.00012	89.9	88.9	89.9	76.1
LQ	−0.00266	0.07541	−0.00012	85.3	77.4	83.5	55.2
$J(1)$	0.28099	0.16291	0.01394	88.1	85.6	87.7	61.4
$J(1)M$	0.05570	−0.17410	0.00441	89.6	86.6	88.3	56.4
RLQM	−0.00093	0.07661	0.00008	96.4	94.4	96.5	79.4
			(2) $\beta_0 = 0$, $\theta = 8$, $\beta_2 = -0.25$; outlier				
MLE	−0.03359	0.45568	−0.00166	82.9	65.8	65.2	44.5
LQ	−0.03359	0.45568	−0.00166	77.7	55.6	58.3	31.6
$J(1)$	0.52607	0.45261	0.03350	81.6	69.7	74.1	52.7
$J(1)M$	(*)	(*)	(*)	83.5	72.1	79.5	59.1
RLQM	−0.03037	−0.05202	−0.00154	92.1	80.0	85.1	54.2

most effective approach, however (being robust to both curvature and an outlier), is RLQM. The poor results for simultaneous confidence regions are due to severe nonlinearity.

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I congratulate Professor Wu for this important contribution on resampling procedures for regression analysis. The representations reported in Section 3 are elegant and useful. It is interesting to note that the representations arise out of some fairly standard results in matrix algebra. I was slightly disappointed to note that one needs the nonsingularity of $X_s'X_s$ for every s in order to define the weighted jackknife variance estimates in Section 4. It does not appear to be a minor restriction. Another general comment I want to make is that it would be probably nicer to see consistency results for the various estimates of variance rather than just the asymptotic unbiasedness.

From here on I confine myself to interval estimates based on the percentiles of the distributional limits and resampling histograms. While summarizing the simulation study of Section 10, Professor Wu remarks that he found the undercoverage of the bootstrap percentile interval very disappointing in view of the second-order asymptotics on the bootstrap. I am forced to say here that there is a misunderstanding on Wu's part concerning the second-order asymptotics. I do not blame Wu very much for this since the second-order development is scattered over a number of articles. A unified presentation in a survey paper or a monograph will certainly help. I make my point clear in terms of the one-dimensional mean in the following paragraph:

Consider the following three confidence intervals: $[L_\Phi, U_\Phi]$, $[L_B, U_B]$ and $[L_{B,st}, U_{B,st}]$. The first interval is based on the simple normal approximation, i.e., $L_\Phi = \overline{X} - z_{1-\alpha/2}s_n/\sqrt{n}$ and $U_\Phi = \overline{X} + z_{1-\alpha/2}s_n/\sqrt{n}$, where $z_t = \Phi^{-1}(t)$, $\overline{X}$ is the sample mean and s_n is the sample standard deviation. The second interval is based on the bootstrap distribution of $\sqrt{n}(\overline{Y} - \overline{X})$ (say H_n), where $\overline{Y}$ denotes the mean of a bootstrap sample. Thus, $L_B = \overline{X} - b_{1-\alpha/2}/\sqrt{n}$ and $U_B = \overline{X} - b_{\alpha/2}/\sqrt{n}$, where b_t is the tth quantile of H_n. The third interval is based on the bootstrap distribution of $\sqrt{n}(\overline{Y} - \overline{X})/s_n^*$ (say H_n^*) where s_n^* denotes the s.d. of a bootstrap sample. Thus $L_{B,st} = \overline{X} - q_{1-\alpha/2}s_n/\sqrt{n}$ and $U_{B,st} = \overline{X} - q_{\alpha/2}s_n/\sqrt{n}$ where $q_t =$ the tth quantile of H_n^*. If $E_F|X|^6 < \infty$ and F is continuous, then one has the following expansions:

$$P(\mu < L_\Phi) = \frac{\alpha}{2} - \frac{\mu_3}{6\sigma^3\sqrt{n}}\left(2z_{\alpha/2}^2 + 1\right)\phi(z_{\alpha/2}) + o(n^{-1/2}),$$

$$P(\mu > U_\Phi) = \frac{\alpha}{2} + \frac{\mu_3}{6\sigma^3\sqrt{n}}\left(2z_{\alpha/2}^2 + 1\right)\phi(z_{\alpha/2}) + o(n^{-1/2}),$$

$$P(\mu < L_B) = \frac{\alpha}{2} - \frac{\mu_3}{2\sigma^3\sqrt{n}} z_{\alpha/2}^2\phi(z_{\alpha/2}) + o(n^{-1/2}),$$

$$P(\mu > U_B) = \frac{\alpha}{2} + \frac{\mu_3}{2\sigma^3\sqrt{n}} z_{\alpha/2}^2\phi(z_{\alpha/2}) + o(n^{-1/2}),$$

$$P(\mu < L_{B,st}) = \frac{\alpha}{2} + o(n^{-1/2})$$

and $$P(\mu > U_{B,st}) = \frac{\alpha}{2} + o(n^{-1/2}).$$

Thus all three intervals above have coverage probability $1 - \alpha + o(n^{-1/2})$. However, in terms of distributing α equally in the two sides by taking the skewness into account, the third interval is asymptotically the best. The same comment holds if one looks at the one-sided intervals for the sake of hypothesis testing. It is not known to me at this point how the three confidence intervals compare in terms of the third-order asymptotics. The following result, which follows easily from the previous expansions, provides a comparison between the one-sided intervals based on the normal approximation and the nonstudentized bootstrap:

If $z_{1-\alpha/2} > 1$,

$$\text{(A)} \qquad \left|P(\mu > U_B) - \frac{\alpha}{2}\right| = \left|P(\mu > U_\Phi) - \frac{\alpha}{2}\right| + \left|z_{\alpha/2}^2 - 1\right| \frac{|\mu_3|}{6\sigma^3\sqrt{n}} \phi(z_{\alpha/2}) + o(n^{-1/2}).$$

Thus, the normal approximation does better than the nonstudentized bootstrap if $z_{1-\alpha/2} > 1$, which is typically the case. The same comment holds for the lower side probability. The result (A) seems to be new. This finding does look a little surprising in view of the fact that H_n approximates the true distribution of $\sqrt{n}(\bar{X} - \mu)$ better than $\Phi(x/s_n)$ does, although there is no mathematical contradiction. In any case, I see no reason to expect that $[L_B, U_B]$ will have its coverage probability closer to $1 - \alpha$, than its competitors.

The idea of forming histograms of properly normalized delete-k (or add-k) jackknife values in order to estimate sampling distributions has been on my mind too for a couple of years. Except for the first order consistency, which uses the Erdös–Rényi formula for characteristic functions, I have not been able to find any important property of these histograms. Let us note that jackknifing is a resampling procedure that corresponds to random sampling without replacement, not with replacement. In fact, the bootstrap that has been proposed in the literature for sampling without replacement from a finite population can in fact be regarded as a jackknife procedure. Thus, I am not very enthusiastic about the jackknife histograms proposed by Wu for independent r.v.'s. However, the weighted bootstrap proposed in Section 7 did catch my attention. It may turn out to be an important resampling method for regression analysis. I think further theoretical study on the proposed procedure will be worthwhile.

I conclude this discussion by saying that, though I have reservations about certain parts of the article, it does seem that this work has broadened the scope of the jackknife for regression analysis. I thank the Editor of *The Annals of Statistics* for giving me the opportunity to make the preceding comments.

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