The Discussion of the Results and Conclusions

In light of the analysis and the experimental results described previously we may attempt to give some cautious answers to the questions set forth in the Introduction to this paper:

Figs. 9, 10, and 13 and the related experiments clearly indicate that, in ideal brittle materials, the so-called "sliding" and "tearing" modes of crack extension do not take place. The mode of fracture seems to be always a crack opening.

The simple explanation of the crack extension in two-dimensional problems given in paragraph 2 is based on the three hypotheses: (a) crack grows radially; (b) growth direction is perpendicular to maximum tension; (c) maximum stress theory is applicable, and may be considered as a satisfactory model for brittle materials. The first one is quite plausible and does not require any supporting argument. In support of (b), we mention the results given in Table 1 and Fig. 5. Table 1 gives an average fracture angle of 70 deg as compared to the calculated value of 70.5 deg. In the case of biaxial loading, even though there is more scatter, the agreement between observed and calculated fracture angles given in Fig. 5 seems to be satisfactory.

The results shown in Fig. 6 indicate that the maximum stress hypothesis (c) should be regarded as a practical design criterion only. At least for the material under consideration, it seems to be a conservative theory. Unless Barenblatt's suggestions concerning the finiteness of the stresses at the crack tip is blended into Griffith's energy concepts, it is difficult to explain the presence of the maximum stress theory in fracture mechanics. Obviously, more work is needed to be done in this area. The test results of Fig. 6 suggest somewhat more strongly the validity of a more plausible criterion as given by equation (10). Here the difficulty is mathematical in nature and lies in the evaluation of the coefficients a_{ij} corresponding to maximum elastic energy release per unit crack extension $\Delta U/\delta$.

As pointed out in the reasoning which led to equation (10), $\Delta U/\delta$ is a quadratic form in the stress intensity factors irrespective of the direction of the crack extension. Hence, in simple cases like pure shear or pure twist $\Delta U/\delta$ will be proportional to the square of the respective stress intensity factors, k_2 or K_2, and since $\Delta U/\delta = $ constant initiates the crack growth, one may talk about critical stress intensity factors. For example, in plane shear $k_1 = 0$ and $\Delta U/\delta = a_{22}k_{2c}{}^2 = $ constant would initiate the crack extension. It has to be noted that the constant a_{22} is not equal to $\pi(\kappa + 1)/(4G)$ as given in equation (9) and for plexiglass $a_{22} > \pi(\kappa + 1)/(4G)$. The results given in Figs. 8 and 12 seem to support this conclusion. The wide scatter in Fig. 12 is mostly due to the difficulty in the detection of the initiation of crack extension. The cracks shown in Fig. 13 are very stable and at a reasonable loading rate (say $1/2$ hour for k_{2c}), they grow very slowly.

The bending problem needs to be studied theoretically first. Its physical picture also seems to be very complicated. We may mention an easily observable peculiarity which is that the plane tangent to the crack extension is not perpendicular to the plane of the plate, it is inclined toward the crack itself on the tension side. In plane loading, all the fracture surfaces were perpendicular to the plane of the plate.

In conclusion, it should be pointed out that most of the foregoing conclusions cannot be expected to hold if any plastic zone develops around the crack tip. Preliminary tests with aluminum plates indicate that the only conclusion which may be applicable to ductile materials is the existence of a possible fracture criterion as given by equation (10); in fact, the scatter for aluminum is considerably smaller than that for plexiglass shown in Fig. 6.

Acknowledgments

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DISCUSSION

F. A. McClintock [4]

The authors need not apologize for studying fracture in terms of local stress and strain concentrations instead of energy release rates, for Griffith himself stated the "general condition for rupture will be the attainment of a specific tensile stress at the edge of one of the cracks" [11].[5] Orowan [12] developed this idea for elliptical cracks in tension, pointing out that it gave the following expression for the fracture stress in terms of the theoretical or ideal cohesive strength, σ_i, the radius of curvature at the tip, ρ, and the half length of the crack, a:

$$\sigma = \sigma_i \sqrt{\rho/4a}. \tag{15}$$

He also showed that a reasonable approximation to the surface energy converts the usual equation based on surface energy to

$$\sigma = \sigma_i \sqrt{\rho/\pi a}. \tag{16}$$

These equations differ only by the factor $\sqrt{\pi/4}$. The theory used by the authors can be cast in a similar form, using a kind of Neuber criterion [13, 14] by requiring that the ideal strength of the material, σ_i, be attained at a distance ρ from the tip of the crack, giving

$$\sigma = \sigma_i \sqrt{2\rho/a}. \tag{17}$$

This differs from the two previous equations by a larger numerical factor, but still is of the proper form. Because of this similarity of the theories based on the ellipse and on the slit for the tensile case, it is interesting to compare the two for the case of combined tension and shear discussed by the authors as well as for the case of combined compressive stress discussed by Griffith and modified by McClintock and Walsh [15] to include the effects of friction acting across the crack face.

It is convenient to discuss elliptical cracks in terms of elliptical coordinates. The one describing the surface of the ellipse is defined in terms of the crack half length, a, and the radius of curvature at the tip, ρ, by:

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[5] Numbers in brackets designate Additional References at end of discussion.

$$\alpha_e = \sqrt{\rho/a}. \tag{18}$$

The coordinate describing position along the length of the ellipse is given in terms of the local slope of the surface. If the x_1 coordinate is parallel to the major axis, and the x_2 coordinate parallel to the minor axis, the local slope is dx_2/dx_1 and the second elliptical coordinate is

$$\beta_e = -\tan^{-1}[\alpha_e/(dx_2/dx_1)]. \tag{19}$$

The general stress distribution, given by Inglis [16], is quite complicated, but for the case of sharp cracks, the stress parallel to the surface of the crack is given closely in terms of the normal and shear stress by

$$\sigma_{\beta\beta} = \frac{2\alpha_e\sigma_{22\infty} - 2\beta_e\sigma_{12\infty}}{\alpha_e{}^2 + \beta_e{}^2}. \tag{20}$$

Differentiation of this with respect to β_e to find the location of maximum stress for given applied stresses, substitution into equation (19) to find the local slope of the ellipse, and taking the local normal, dx_1/dx_2, gives the fracture angle Θ_0, at which the crack first leaves the original crack in terms of the angle β between the direction of the crack and the applied stress:

$$\Theta_0 = (90 - \beta)/2. \tag{21}$$

Similarly, substitution of the critical value of the coordinate β_e into equation (20) gives the following fracture locus:

$$\frac{\sigma}{\alpha_e \sigma_i} = \frac{1 + \dfrac{\sigma^2}{\tau^2}\left[1 - \sqrt{1 + \tau^2/\sigma^2}\right]}{\sqrt{1 + \tau^2/\sigma^2}} \tag{22}$$

Equations (21) and (22) are plotted in Figs. 14 and 15, along with the authors' theoretical and experimental results. It will be seen that they, based on the maximum stress on an ellipse, give by no means as good a fit as the authors' analysis, based on the stress near a slit.

On the other hand, consider the case of fracture under combined compressive stress, assuming that a negligible normal stress across the crack surface will cause it to close, and that shear across the crack is impeded by a coefficient of friction of approximately unity, as discussed by McClintock and Walsh [15]. For a crack making an angle β with the direction of maximum compressive stress, $-\sigma_2$, and with a minimum compressive stress, $-\sigma_1$, the residual shear stress available to produce cracking after subtracting out the effect of friction is given by

$$\tau = (\sigma_1 - \sigma_2)\sin\beta\cos\beta - \mu(-\sigma_1\cos^2\beta - \sigma_2\sin^2\beta). \tag{23}$$

Differentiation with respect to β, to find the crack with the most residual shear stress, gives the result that for a coefficient of friction of unity, the angle β is $\pi/8$, and the residual shear stress is given in terms of the principal applied stresses by:

$$\tau = \frac{\sigma_1 - \sigma_2}{\sqrt{2}} + \frac{\sigma_1 + \sigma_2}{2}. \tag{24}$$

From Fig. 15 of this discussion, the residual strength under shear is twice that under pure tension for the elliptically shaped crack, and $\sqrt{3}/2$ times that under tension for the slit. Thus the strength under combined compression will be much less for the slit theory. For example, in the case of uniaxial compression, $\sigma_2 = -C$, $\sigma_1 = 0$, the ratio of compressive to tensile strength is 9.7 for the ellipse, but only 4.2 for the slit. Since the usual compressive strength is of the order of 8 times the tensile strength for isotropic materials, the ellipse gives a much better fit than the slit. An unrealistically large coefficient of friction would be required to bring the slit theory into line with the data, and then the strengths under biaxial compression would be much too high.

Thus the slit gives a better fit for the polymethylmethacrylate under combined tension and shear, whereas the ellipse gives a better fit for compressive strengths of rocks under uniaxial or biaxial compression. It is possible that a theory based on the
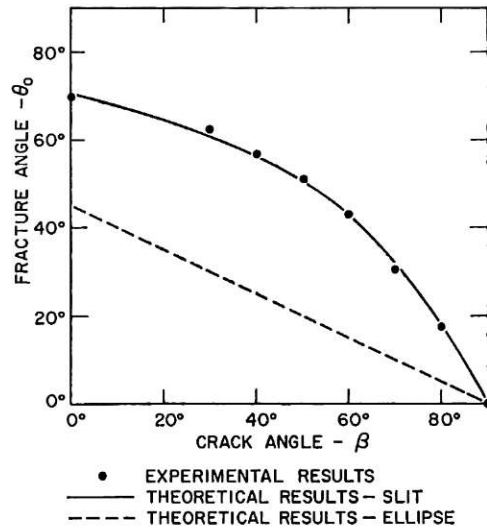

Fig. 14 Fracture angle versus crack angle in a cracked plate under uniform tension
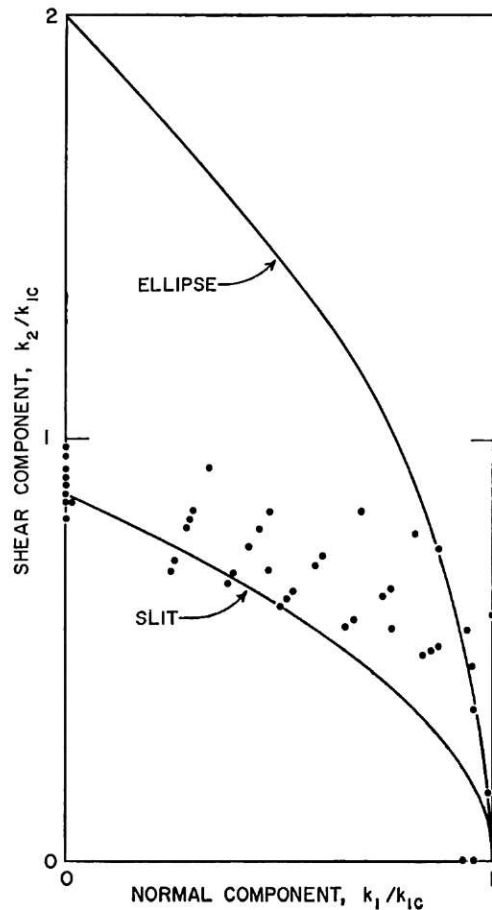

Fig. 15 k_1 versus k_2 at the beginning of crack extension in a cracked plate under plane loading

ellipse is more realistic, and that the deviation from this theory in the case of the polymethylmethacrylate is due to localized plastic flow, which has certainly been observed under compression tests. On the other hand, it is also possible that the slit theory gives the better fit and that modifications are required to the theory for fracture of materials under combined stress. Tests on a more brittle material, such as glass, would help to resolve this question. In the meantime, the two different theories should each be used in the regions in which they have been experimentally shown to give good correlations.
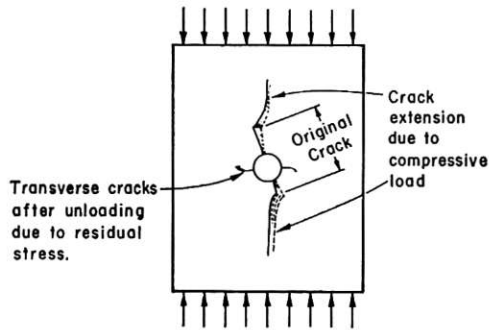
Fig. 16 Crack propagation in PMMA under compression

Another example of the authors' point about the importance of the local stress rather than an overall energy criterion is that under uniaxial compression, the cracks tend to run parallel to the compression axis and to suffer a decreasing stress concentration as they do so, as shown in Fig. 16 [17]. The circular hole was drilled to allow the insertion of a wedge to produce the diagonal crack. The curved crack from the diagonal arose on compression loading (the transverse crack, after release of load, was due to residual stress). The presence of any transverse compression would stabilize the vertical crack and cause it to stop. If, on the other hand, the crack were to continue growing in the original diagonal plane, a release of strain energy would still be possible. In the case of tension, this distinction does not appear since the local state of stress tends to make the crack grow in the direction in which the large scale strain energy release rate is also a maximum.

It should be noted that a factor σ appears to be missing from the right-hand sides of the authors' equation (4).

The writer hopes these remarks will further clarify the authors' interesting and worthwhile work on fracture under combined stress.

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Authors' Closure

The authors wish to express their thanks to Professor Mc-Clintock for the very interesting discussion of their paper.

As pointed out by Griffith himself [11] in applying his ideas concerning the fracture of brittle materials to bodies subjected to general three-dimensional loading and containing randomly oriented small cracks, the solution to the three-dimensional problem in elasto-statics is needed. The solutions given by Inglis [16] and that of the present paper are for the case of generalized plane stress.

The solution to the general problem of infinite elastic medium subjected to three-dimensional stress state at infinity and containing a plane cavity of zero thickness bounded by a convex closed curve is not available. However, from Sneddon's solu-

tion[6] for a penny-shaped crack in an infinite elastic medium subjected to uniform pressure on the crack surfaces, it may easily be seen that "bulk constraining" improves the strength of the material (the ratio of tensile stresses initiating the crack extension in materials with penny-shaped crack and through crack, i.e., the case of plane strain, is $\pi/2$). In the case of pure shear of the materials with "contained" cracks, the phenomenon of crack growth is more complicated and on the physical grounds it is reasonable to expect that the influence of the bulk constraining will be more than that for the tension case. Since most of the available compression test data are obtained from the materials containing internal cracks, a direct application of the results of plane stress, to these cases may not be valid.

The authors also would like to point out that equation (22) (in which, incidentally, the bracket should be squared and a factor of 2 in the numerator of left-hand side is missing), derived by Professor McClintock should be applied with some caution. In the argument leading to (22) and thereafter it is assumed that the parameter α_e is constant and has the same value for all cases which are being compared. Theoretically, in the case of slit, α_e is zero and according to some school of thought (notably, Barenblatt [9]), it remains zero while the crack is propagating. However, even if we assume that α_e is small but finite, i.e., the crack tip initially has a small but finite radius of curvature, in most brittle materials, the variation in α_e as a result of loading is of the same order of magnitude as its initial value. From (22) we may write

$$\tau = 0 : \sigma_{\beta\beta} = 2\sigma/\alpha'_e$$
$$\sigma = 0 : \sigma_{\beta\beta} = \tau/\alpha''_e \quad (25)$$

where α_e' and α_e'' are the values of α_e in the two limiting cases considered.

On the other hand, starting with a theoretical slit, the radii of curvature (or the change in the radii of curvature) after deformations, for the case of symmetric tension and shear may be obtained as

$$\tau = 0 : \rho' = 4\sigma^2 a/E^2$$
$$\sigma = 0 : \rho'' = 16\tau^4 a/E^4 \quad (26)$$

With (18), (25), and (26) points to the fact that, if the initial radius of curvature, ρ, is very small and as a result if we have to take into account the variations in ρ due to the loading, because of the large difference in the order of magnitudes of the variations in ρ, we will have $\alpha_e'' < \alpha_e'$. For the same $\sigma_{\beta\beta}$ (25) gives

$$\tau_c = 2 \frac{\alpha''_e}{\alpha'_e} \sigma_c \quad (27)$$

where subscript c refers to the critical values, i..e, to those stresses initiating the crack extension.

Since Professor McClintock is using essentially the same criterion, one would expect a better agreement between the results obtained from the solutions based on the slit and the ellipse. It may be that the explanation of the difference lies in (27). The ellipse approach is further complicated by the fact that, sometimes it may even lead to paradoxical conclusions, as for example in the case of combined compression and shear, where the presence of compression tends to reduce α_e at a greater rate than its increase due to shear [see equation (26)] and hence may increase $\sigma_{\beta\beta}$, causing a reduction in compressive strength.

The authors agree with Professor McClintock in his emphasis on the influence of friction in compression tests. It may be pointed out that in the case of natural cracks the coefficient of friction may vary very widely even for the same material, as the main resistance to the relative displacements on the crack surfaces may come from the macroscopic jaggedness of these surfaces.

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