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ANTHONY CARBERY, TIMO S. HÄNNINEN
AND STEFÁN INGI VALDIMARSSON

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DISENTANGLEMENT, MULTILINEAR DUALITY AND FACTORISATION FOR NONPOSITIVE OPERATORS

ANTHONY CARBERY, TIMO S. HÄNNINEN AND STEFÁN INGI VALDIMARSSON

In a previous work we established a multilinear duality and factorisation theory for norm inequalities for pointwise weighted geometric means of positive linear operators defined on normed lattices. In this paper we extend the reach of the theory for the first time to the setting of *general* linear operators defined on normed spaces. The scope of this theory includes multilinear Fourier restriction-type inequalities. We also sharpen our previous theory of positive operators.

Our results all share a common theme: estimates on a weighted geometric mean of linear operators can be *disentangled* into quantitatively linked estimates on each operator separately. The concept of disentanglement recurs throughout the paper.

The methods we used in the previous work — principally convex optimisation — relied strongly on positivity. In contrast, in this paper we use a vector-valued reformulation of disentanglement, geometric properties (Rademacher-type) of the underlying normed spaces, and probabilistic considerations related to p-stable random variables.

1. Introduction

In our previous work [Carbery et al. 2022] we introduced and developed a general functional-analytic principle concerning norm inequalities for pointwise weighted geometric means

$$\prod_{j=1}^{d} |T_j f_j(x)|^{\alpha_j}$$

of *positive* linear operators T_j defined on suitable spaces, where $\alpha_j \geq 0$ and $\sum_{j=1}^{d} \alpha_j = 1$. In this paper we extend our study to the situation in which the linear operators T_j are no longer assumed to be positive.

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Keywords: multilinear inequalities, duality, factorisation, disentanglement, p-convexity, Rademacher-type.

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The techniques of [Carbery et al. 2022] relied strongly on positivity, so it will be necessary to involve a new set of ideas.

In order to set the scene for this, it will be helpful to recall the main theorem of [Carbery et al. 2022], but we first we need to set up some notation. Let $(X, \mathrm{d}\mu)$ be a measure space and let $\mathcal{M}(X)$ be the class of measurable functions on X. Let $\mathcal{Y}$ be a real or complex normed space. (For example, if Y is a measure space, $\mathcal{Y}$ could be the class $\mathcal{S}(Y)$ of simple functions with an L^p-norm for some $p \geq 1$.) We say that a linear map $T : \mathcal{Y} \to \mathcal{M}(X)$ *saturates* X if, for each subset $E \subseteq X$ of positive measure, there exists a subset $E' \subseteq E$ with $\mu(E') > 0$ and an $h \in \mathcal{Y}$ such that $|Th| > 0$ a.e. on E'. For reasons explained in [Carbery et al. 2022], such a condition is needed for the result which follows to hold.

Theorem 1.1 [Carbery et al. 2022]. *Suppose that X is a σ-finite measure space and that $\mathcal{Y}_j$, for $j = 1, \dots, d$, are normed lattices. Suppose that the linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are positive and that each T_j saturates X. Suppose that $0 < q \leq \infty$ and $\sum_{j=1}^d \alpha_j = 1$. Finally, suppose that*

$$\left\| \prod_{j=1}^d (T_j f_j)^{\alpha_j} \right\|_{L^q(X)} \leq A \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{\alpha_j} \tag{1}$$

for all nonnegative $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d$.

***Case I*:** (*disentanglement*) *If $q = 1$, then there exist nonnegative measurable functions g_j on X such that*

$$1 \leq \prod_{j=1}^d g_j(x)^{\alpha_j} \quad \text{a.e. on } X \tag{2}$$

and such that, for each j,

$$\int_X g_j(x) T_j f_j(x)\, \mathrm{d}\mu(x) \leq A \|f_j\|_{\mathcal{Y}_j} \tag{3}$$

for all $f_j \in \mathcal{Y}_j$, with the same constant A as in (1).

Conversely, if the T_j are positive linear operators such that there exist nonnegative measurable functions g_j on X such that (2) holds and such that (3) holds for all $f_j \in \mathcal{Y}_j$, then (1) holds for all nonnegative $f_j \in \mathcal{Y}_j$.

***Case II*:** (*multilinear duality*) *If $q > 1$, then for every nonnegative $G \in L^{q'}(X)$ there exist nonnegative measurable functions g_j on X such that*

$$G(x) \leq \prod_{j=1}^d g_j(x)^{\alpha_j} \quad \text{a.e. on } X \tag{4}$$

and such that, for each j,

$$\int_X g_j(x) T_j f_j(x)\, \mathrm{d}\mu(x) \leq A \|G\|_{L^{q'}} \|f_j\|_{\mathcal{Y}_j} \tag{5}$$

for all $f_j \in \mathcal{Y}_j$, with the same constant A as in (1).

Conversely, if the T_j are positive linear operators such that for every nonnegative $G \in L^{q'}(X)$ there exist nonnegative measurable functions g_j on X such that (4) holds and such that (5) holds for all $f_j \in \mathcal{Y}_j$, then (1) holds for all nonnegative $f_j \in \mathcal{Y}_j$.

Case III: (*multilinear Maurey factorisation*) *If* $0 < q < 1$, *then there exist nonnegative measurable functions* g_j *on* X *such that*[1]

$$\Big\| \prod_{j=1}^{d} g_j(x)^{\alpha_j} \Big\|_{q'} = 1 \tag{6}$$

and such that, for each j, (3) *holds for all* $f_j \in \mathcal{Y}_j$, *with the same constant* A *as in* (1).

Conversely, if the T_j *are positive linear operators such that there exist nonnegative measurable functions* g_j *on* X *such that* (6) *holds and such that* (3) *holds for all* $f_j \in \mathcal{Y}_j$, *then* (1) *holds for all nonnegative* $f_j \in \mathcal{Y}_j$.

Numerous illustrations and applications of this theorem were given in [Carbery et al. 2022]. It should be stressed that this result is a general one, applying to the class of positive operators broadly.

The forward parts of this result are the difficult ones; the converses follow easily by applying Hölder's inequality. When $d = 1$, Case II reduces to an elementary duality statement concerning the operator $T : \mathcal{Y} \to L^q$ and this gives rise to the sobriquet "multilinear duality" in the case of general d. The term "factorisation" relates both to the pointwise factorisation expressed by (4) and to the condition (5), which is a statement that each operator T_j factorises through a certain weighted L^1-space.

Case I, corresponding to $q = 1$, plays a special role, and indeed the remaining cases corresponding to $q \neq 1$ can be deduced from it without too much difficulty — see Section 5 for arguments of this type. We describe the case $q = 1$ as a "disentanglement" result since it disentangles a bound (1) on the pointwise combination of the T_j's into bounds (3) on each T_j separately, with the individual bounds linked via (2).

Notice that, when suitably modified, the statement of Theorem 1.1 makes perfectly good sense in principle *without* the hypothesis of positivity of the operators T_j; nevertheless, as we have mentioned, the arguments from [Carbery et al. 2022] rely very heavily on positivity. In this paper we use vector-valued techniques to develop an analogue of Theorem 1.1 which applies to general linear operators defined on normed spaces. See Theorems 1.5, 1.7, 4.3 and 5.2 below.

In what follows we shall primarily focus on the case of L^1 norms of pointwise weighted products $\prod_{j=1}^{d} |T_j f_j|^{\gamma_j}$ in our pursuit of extending Theorem 1.1 to general linear operators T_j. We return to the case of general L^q-norms of such expressions in Section 5, and there we see that it is relatively straightforward to derive the results for general q, which even in the positive case significantly generalise Theorem 1.1, from those corresponding to $q = 1$.

We next give a simple lemma. All of our main results can be framed as reversals of the implication it establishes (under various auxiliary hypotheses).

Lemma 1.2. *Let* $\mathcal{Y}_j$ *be normed spaces and let* $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ *be linear mappings for* $1 \le j \le d$. *Suppose* $\gamma_j > 0$ *are given. Assume that for some* (p_j) *with* $0 < p_j < \infty$ *we have the condition*

$$\sum_{j=1}^{d} \frac{\gamma_j}{p_j} = 1, \tag{7}$$

[1]We caution that we use the notation $\|g\|_q := \big(\int |g|^q\big)^{1/q}$ and $q' := q/(q-1)$ for $q < 0$ and for $0 < q < 1$, even though in these cases $\|\cdot\|_q$ does not define a norm.

and that there exist nonnegative measurable functions (ϕ_j) *on* X *such that*

$$\prod_{j=1}^{d} \phi_j(x)^{\gamma_j/p_j} \geq 1 \tag{8}$$

almost everywhere on X *and such that*

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\,\mathrm{d}\mu(x)\right)^{1/p_j} \leq A\|f_j\|_{\mathcal{Y}_j} \tag{9}$$

for all $f_j \in \mathcal{Y}_j$. *Then*

$$\int_X \prod_{j=1}^{d} |T_j f_j(x)|^{\gamma_j}\,\mathrm{d}\mu(x) \leq A^{\sum_{j=1}^{d}\gamma_j} \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{\gamma_j} \tag{10}$$

for all $f_j \in \mathcal{Y}_j$.

Proof. Let $\theta_j = \gamma_j/p_j$. Then $\sum_{j=1}^{d}\theta_j = 1$, and, by (8), (9) and Hölder's inequality, we have

$$\begin{aligned}
\int_X \prod_{j=1}^{d} |T_j f_j(x)|^{\gamma_j}\,\mathrm{d}\mu(x) &\leq \int_X \prod_{j=1}^{d} |T_j f_j(x)|^{\gamma_j} \phi_j(x)^{\gamma_j/p_j}\,\mathrm{d}\mu(x)\\
&= \int_X \prod_{j=1}^{d} |T_j f_j(x)|^{p_j\theta_j} \phi_j(x)^{\theta_j}\,\mathrm{d}\mu(x)\\
&\leq \prod_{j=1}^{d}\left(\int_X |T_j f_j(x)|^{p_j}\phi_j(x)\,\mathrm{d}\mu(x)\right)^{\theta_j}\\
&\leq A^{\sum_{j=1}^{d} p_j\theta_j} \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} = A^{\sum_{j=1}^{d}\gamma_j} \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{\gamma_j}. \qquad \square
\end{aligned}$$

Taking $\gamma_j = q\alpha_j$ with q and $\sum_{j=1}^{d}\alpha_j = 1$ as in the preceding discussion makes a point of contact with Theorem 1.1.

Note that Lemma 1.2 has no content in the linear case $d = 1$. Our main concern will therefore be with the converse scenario in the genuinely multilinear case $d \geq 2$. The lemma delineates what we might hope for. More precisely:

Basic Question. *Let* $d \geq 2$. *Suppose* X *is a* σ*-finite measure space*, $\mathcal{Y}_j$ *are normed spaces*, $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ *are saturating linear mappings, and* $\gamma_j > 0$ *for* $1 \leq j \leq d$. *We suppose that* (10) *holds. For which* (p_j) (*if any*) *with* $0 < p_j < \infty$ *satisfying condition* (7) *can we conclude that there exist nonnegative* (ϕ_j) *such that conditions* (8) *and* (9) *hold, perhaps with a loss in the constants*?

Once again we emphasise that we ask this question in the broad context: we seek answers which do not rely upon the precise nature of the operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$, but instead which will hold universally over a wide class of linear operators. We expect that the set of admissible exponents (p_j), in addition to satisfying (7),[2] will reflect whatever geometric structures the normed spaces $\mathcal{Y}_j$ may possess.

[2]For a discussion of why we require this condition, see Proposition A.1 in the Appendix.

We shall give separate answers to this question in the settings of general linear operators and of positive linear operators. It transpires that in order to develop the theory for general linear operators, it first makes sense to consider a related question for positive linear operators: if in Theorem 1.1 we take the lattices $\mathcal{Y}_j$ to be L^{r_j}-spaces, are there stronger, r_j-dependent, conclusions that we can make?

The following result answers our Basic Question for positive linear operators on Lebesgue spaces, with no loss in constants. A corresponding answer in the case of general linear operators on Lebesgue spaces is given in Theorem 1.5.

Theorem 1.3. *Suppose that X and Y_j, for $j = 1, \ldots, d$, are measure spaces and that X is σ-finite. Suppose that the linear operators $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ are positive and that each T_j saturates X. Suppose that $1 \le r_j \le \infty$ for all j. Finally, suppose that for some exponents $\gamma_j > 0$ we have*

$$\int_X \prod_{j=1}^{d} (T_j f_j)(x)^{\gamma_j} \, \mathrm{d}\mu(x) \le A^{\sum_{j=1}^{d} \gamma_j} \prod_{j=1}^{d} \| f_j \|_{L^{r_j}(Y_j)}^{\gamma_j} \tag{11}$$

for all nonnegative simple functions f_j on Y_j, $1 \le j \le d$.

Then for all (p_j) satisfying $0 < p_j < \infty$ for all j, $\sum_{j=1}^{d} \gamma_j / p_j = 1$ and $p_j \le r_j$ for all j, there exist nonnegative (ϕ_j) such that

$$\prod_{j=1}^{d} \phi_j(x)^{\gamma_j / p_j} \ge 1 \tag{12}$$

almost everywhere on X and such that

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x) \, \mathrm{d}\mu(x) \right)^{1/p_j} \le A \| f_j \|_{r_j} \tag{13}$$

for all $f_j \in \mathcal{S}(Y_j)$.

Remark 1. In the Appendix below we give an example of positive linear operators (T_j) satisfying (11) for which the set of (p_j) satisfying $0 < p_j < \infty$ and $\sum_{j=1}^{d} \gamma_j / p_j = 1$, and for which the conclusion of Theorem 1.3 holds, consists *precisely* of those satisfying $p_j \le r_j$ for every j. See Corollary A.7. Thus the condition $p_j \le r_j$ is sharp if we want our result to hold broadly for positive operators without further reference to their individual properties.[3]

Notice that the set

$$\left\{ (p_j) \in (0, \infty)^d : \sum_{j=1}^{d} \frac{\gamma_j}{p_j} = 1 \text{ and } p_j \le r_j \text{ for all } j \right\}$$

is nonempty if and only if $\sum_{j=1}^{d} \gamma_j / r_j \le 1$. In particular, Theorem 1.3 has no content unless $\sum_{j=1}^{d} \gamma_j / r_j \le 1$. In Corollary A.7 we demonstrate, by example, that if $\sum_{j=1}^{d} \gamma_j / r_j > 1$, then the set of (p_j) satisfying the conclusion of Theorem 1.3 may indeed be empty.

[3]For *particular* positive operators (T_j), the result may hold even when $p_j > r_j$ for some j. Indeed, let $X = Y_j = [0, 1]$ with Lebesgue measure, let $r_j = 1$ for all j and let each T_j be given by $T_j f = \int_0^1 f$, so that each $T_j f$ is constant on $[0, 1]$. Then (11) holds for all exponents $\gamma_j > 0$, with $A = 1$. If we take $\phi_j(x) = 1$ for all j, then both (12) and (13) hold for all exponents $0 < p_j < \infty$.

Under hypothesis (11), the disentangled conclusions (13) for $p_j \le \max\{r_j, \gamma_j\}$ alone, with otherwise unspecified but nontrivial (ϕ_j), are more straightforward, and can be established by methods which are not genuinely multilinear.[4] The significant feature of Theorem 1.3 is that under the hypotheses $\sum_{j=1}^d \gamma_j/p_j = 1$ and $p_j \le r_j$ for all j, we can choose (ϕ_j) also satisfying the specific quantitative lower bound (12). Similar remarks apply to our subsequent results.

We point out that the case $p_j = 1$ for all j of Theorem 1.3 directly implies Case I (and therefore Case II) of Theorem 1.1 (in the special case where the spaces $\mathcal{Y}_j$ are taken to be L^{r_j}). The case $p_j = r_j$ of Theorem 1.3 is, however, the crucial one, and in a slightly different notation can be presented as follows:

Theorem 1.4 (disentanglement for positive operators on Lebesgue spaces). *Suppose that X and Y_j, for $j = 1, \dots, d$, are measure spaces and that X is σ-finite. Suppose that the linear operators $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ are positive and that each T_j saturates X. Suppose that $1 \le p_j < \infty$ for all j, and that $\theta_j \ge 0$ are such that $\sum_{j=1}^d \theta_j = 1$. Finally, suppose that*

$$\int_X \prod_{j=1}^d (T_j f_j)(x)^{p_j\theta_j}\, \mathrm{d}\mu(x) \le B \prod_{j=1}^d \|f_j\|_{L^{p_j}(Y_j)}^{p_j\theta_j}$$

for all nonnegative simple functions f_j on Y_j, $1 \le j \le d$. Then there exist nonnegative measurable functions ϕ_j on X such that

$$\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1$$

almost everywhere on X and such that, for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\, \mathrm{d}\mu(x)\right)^{1/p_j} \le B^{1/p_j} \|f_j\|_{L^{p_j}(Y_j)}$$

for all simple functions f_j on Y_j.

In analogy with the Case I of Theorem 1.1, we shall also call this result a disentanglement theorem, and it is an instance of the general disentanglement theorem for positive operators on p_j-convex spaces, which we shall present as Theorem 3.2.

As the reader will have noticed, by homogeneity we may take $B = 1$ (and $A = 1$ in earlier results) without loss. (And by playing with homogeneities the constant B^{1/p_j} can be replaced with $B^{\left(\sum_{j=1}^d p_j\theta_j\right)^{-1}}$).

In order to address our main concern in the paper — the extension of the theory to include general linear operators which are not necessarily positive — we shall consider the analogous situation under hypotheses of Rademacher-type in place of p-convexity. Our use of p-convexity and Rademacher-type proceeds in parallel with their deployment in the development of the Maurey theory; see [García-Cuerva and Rubio de Francia 1985; Albiac and Kalton 2006]. For now we state a sample theorem, which, in the case that the normed spaces $\mathcal{Y}_j$ are L^{r_j}-spaces, answers the Basic Question. We shall significantly generalise this result later; see Theorem 4.3.

[4] The range $p_j \le \max\{r_j, \gamma_j\}$ for this simpler problem is also known to be sharp, as the arguments in the Appendix confirm.

Theorem 1.5. *Suppose that X and Y_j, for $j = 1, \dots, d$, are measure spaces and that X is σ-finite. Suppose that $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ are linear (not necessarily positive) operators and that each T_j saturates X. Suppose that*[5] *$1 \le r_j < \infty$ for all j. Finally, suppose that for some exponents $\gamma_j > 0$ we have*

$$\int_X \prod_{j=1}^d |T_j f_j(x)|^{\gamma_j} \, \mathrm{d}\mu(x) \le A^{\sum_{j=1}^d \gamma_j} \prod_{j=1}^d \|f_j\|_{L^{r_j}(Y_j)}^{\gamma_j} \tag{14}$$

for all simple functions f_j on Y_j, $1 \le j \le d$.

Then for all (p_j) such that $\sum_{j=1}^d \gamma_j / p_j = 1$ and

$$\begin{aligned} &0 < p_j < r_j \quad \text{for those } j \text{ for which } 1 \le r_j < 2, \\ &0 < p_j \le 2 \quad \text{for those } j \text{ for which } 2 \le r_j < \infty, \end{aligned} \tag{15}$$

there exist nonnegative ϕ_j such that

$$\prod_{j=1}^d \phi_j(x)^{\gamma_j/p_j} \ge 1$$

almost everywhere on X and such that

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x) \, \mathrm{d}\mu(x) \right)^{1/p_j} \lesssim_{\{\gamma_j, r_j, p_j\}} A \|f_j\|_{L^{r_j}(Y_j)}$$

for all $f_j \in \mathcal{S}(Y_j)$.

Remark 2. In the Appendix below we give an example of linear operators (T_j) satisfying (14) for which the set of (p_j) satisfying $0 < p_j < \infty$ and $\sum_{j=1}^d \gamma_j / p_j = 1$, and for which the conclusion of Theorem 1.5 holds, consists *precisely* of those satisfying (15). See Corollary A.8. Thus the condition (15) is sharp if we want our result to hold broadly for linear operators without further reference to their individual properties. For specific operators T_j the conclusion may nevertheless hold even if (15) is violated.

Note that the set of (p_j) satisfying $\sum_{j=1}^d \gamma_j / p_j = 1$ together with (15) will be nonempty if and only if

$$\begin{aligned} \textstyle\sum_{j=1}^d \gamma_j / \min\{r_j, 2\} < 1 &\quad \text{when at least one } r_j < 2, \\ \textstyle\sum_{j=1}^d \gamma_j \le 2 &\quad \text{when all } 2 \le r_j < \infty. \end{aligned}$$

In Corollary A.8 we demonstrate, by example, that if this condition is violated, the set of (p_j) satisfying the conclusion of Theorem 1.5 may indeed be empty.

The special case of this result corresponding to $p_j = 2$ for all j is singled out:

Theorem 1.6 (disentanglement for general linear operators on Lebesgue spaces). *Suppose that X and Y_j, for $j = 1, \dots, d$, are measure spaces and that X is σ-finite. Suppose that the linear operators $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ saturate X. Suppose that $\theta_j > 0$ and $\sum_{j=1}^d \theta_j = 1$. Finally, suppose that for some*

[5]The proof will reveal that the result remains valid under the weaker assumption $0 < r_j < \infty$, provided that we accordingly modify (15) to $0 < p_j < r_j$ for those j for which $0 < r_j < 2$.

exponents $2 \le r_j < \infty$ *we have*

$$\int_X \prod_{j=1}^d |T_j f_j(x)|^{2\theta_j} \, \mathrm{d}\mu(x) \le B \prod_{j=1}^d \|f_j\|_{L^{r_j}(Y_j)}^{2\theta_j}$$

for all simple functions f_j *on* Y_j, $1 \le j \le d$. *Then there exist nonnegative measurable functions* ϕ_j *on* X *such that*

$$\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1$$

almost everywhere on X *and such that, for each* j,

$$\left(\int_X |T_j f_j(x)|^2 \phi_j(x) \, \mathrm{d}\mu(x)\right)^{1/2} \lesssim B^{1/2} \|f_j\|_{L^{r_j}(Y_j)}$$

for all simple functions f_j *on* Y_j.

Theorem 1.6 readily upgrades to the following result (see Section 5), whose formulation can be compared to Case II of Theorem 1.1:

Theorem 1.7 (multilinear duality for general operators on Lebesgue spaces). *Suppose that* X *and* Y_j, *for* $j = 1, \dots, d$, *are measure spaces and that* X *is* σ*-finite. Suppose that the linear operators* $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ *saturate* X. *Suppose that* $\alpha_j > 0$ *and* $\sum_{j=1}^d \alpha_j = 1$. *Finally, suppose that, for some exponents* $q \ge 2$ *and* $2 \le r_j < \infty$, *we have*

$$\left\| \prod_{j=1}^d |T_j f_j|^{\alpha_j} \right\|_q \le B \prod_{j=1}^d \|f_j\|_{L^{r_j}(Y_j)}^{\alpha_j}$$

for all simple functions f_j *on* Y_j, $1 \le j \le d$. *Then for every nonnegative* $G \in L^{(q/2)'}$ *there exist nonnegative measurable functions* g_j *on* X *such that*

$$\prod_{j=1}^d g_j(x)^{\alpha_j} \ge G(x)$$

almost everywhere on X *and such that, for each* j,

$$\left(\int_X |T_j f_j(x)|^2 g_j(x) \, \mathrm{d}\mu(x)\right)^{1/2} \lesssim B \|G\|_{(q/2)'} \|f_j\|_{L^{r_j}(Y_j)}$$

for all simple functions f_j *on* Y_j.

The converse statements to these three results are once again also true, and are easy to verify.

Note that in these last three results we do not assert "$\le$" but only "$\lesssim$" in the conclusions, and moreover the case $r_j = \infty$ is excluded from Theorems 1.5 and 1.7. This is ultimately because we shall need to apply Khintchine's inequality. Note also the numerology familiar from harmonic analysis, in which L^p-boundedness of a positive operator for $p > 1$ (such as a maximal operator) often corresponds to $L^{2p'}$-boundedness of a corresponding nonpositive operator (such as a singular integral operator). Even in the linear case $d = 1$, the duality statement is along the lines that $T : L^r \to L^q$ with $q, r \ge 2$ if and only if $\||T^* g|^2\|_{q'/2} \lesssim \||g|^2\|_{r'/2}$ (rather than $\|T^* g\|_{q'} \lesssim \|g\|_{r'}$).

1.1. *Multilinear restriction and the Mizohata–Takeuchi conjecture.* As an indication of the scope of Theorem 1.7, we consider the so-called multilinear restriction problem for the Fourier transform. For $1 \le j \le n$, let $\Gamma_j : U_j \to \mathbb{R}^n$ (with $U_j \subseteq \mathbb{R}^{n-1}$) be smooth parametrisations of compact hypersurfaces S_j in $\mathbb{R}^n$ with nonvanishing gaussian curvature. We assume that the hypersurfaces are transversal in the sense that if $\omega_j(x)$ denotes a unit normal to S_j at $x \in S_j$, then $|\omega_1(x_1) \wedge \cdots \wedge \omega_n(x_n)| \ge c > 0$ for all $x_j \in S_j$. The Fourier extension (or dual restriction) operator $\mathcal{E}_j$ for S_j is given by

$$\mathcal{E}_j f_j(x) = \int_{U_j} e^{2\pi i x \cdot \Gamma(t_j)} f_j(t_j)\, \mathrm{d}t_j.$$

It is conjectured (see [Bennett et al. 2006]) that these operators satisfy the multilinear bound

$$\int_{\mathbb{R}^n} \prod_{j=1}^n |\mathcal{E}_j f_j(x)|^{2/(n-1)}\, \mathrm{d}x \lesssim \prod_{j=1}^n \|f_j\|_{L^2(U_j)}^{2/(n-1)} \tag{16}$$

or equivalently

$$\Big\| \prod_{j=1}^n |\mathcal{E}_j f_j(x)|^{1/n} \Big\|_{2n/(n-1)} \lesssim \prod_{j=1}^n \|f_j\|_{L^2(U_j)}^{1/n}. \tag{17}$$

This is known up to endpoints (see [Bennett et al. 2006; Tao 2020]) but is as yet unresolved in the form stated here.

These considerations clearly fit into the framework which we were discussing above, in particular Theorem 1.7, and we therefore have the following:

Theorem 1.8 (factorisation for multilinear restriction). *The multilinear restriction bound (17) holds if and only if, for all nonnegative $G \in L^n(\mathbb{R}^n)$, there exist nonnegative $g_1, \dots, g_n$ such that*

$$\prod_{j=1}^n g_j(x)^{1/n} \ge G(x)$$

almost everywhere and, for all j,

$$\Big(\int_{\mathbb{R}^n} |\mathcal{E}_j f_j(x)|^2 g_j(x)\, \mathrm{d}x \Big)^{1/2} \lesssim \|G\|_n \|f_j\|_2.$$

On the other hand, the corresponding endpoint multilinear Kakeya theorem is due to Guth [2010] (see also [Carbery and Valdimarsson 2013]). He proved it by directly establishing the following fundamental factorisation result:

Theorem 1.9 [Guth 2010]. *For $1 \le j \le n$, let $\mathcal{T}_j$ be families of doubly infinite tubes of unit cross-section with transversal directions. For all nonnegative $G \in L^n(\mathbb{R}^n)$, there exist nonnegative $g_1, \dots, g_n$ such that*

$$\prod_{j=1}^n g_j(x)^{1/n} \ge G(x)$$

almost everywhere and, for all j and $T \in \mathcal{T}_j$,

$$\int_T g_j(x)\, \mathrm{d}x \lesssim \|G\|_n.$$

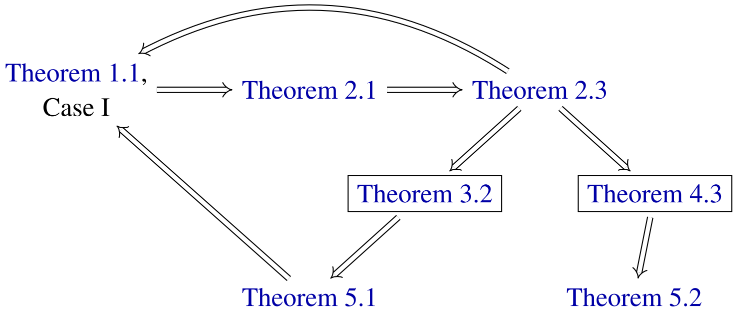

Figure 1. Taxonomy of main theorems.

Moreover, coming from entirely different considerations, there is a conjecture, often attributed to Mizohata and Takeuchi, which states:

Conjecture 1 (Mizohata–Takeuchi conjecture). *Let S be a compact hypersurface of nonvanishing gaussian curvature, with corresponding Fourier extension operator $\mathcal{E}$. Then, for any nonnegative weight w, we have*

$$\int_{\mathbb{R}^n} |\mathcal{E} f(x)|^2 w(x)\,\mathrm{d}x \lesssim \sup_T w(T) \int |f(t)|^2\,\mathrm{d}t,$$

where the sup *is taken over all doubly infinite tubes of unit cross-section with direction normal to S.*

Combining these last two statements we obtain:

Proposition 1.10. *Conditional on the Mizohata–Takeuchi conjecture, the multilinear restriction bound (16) holds.*

Proof. In order to establish (16), we integrate the function $\prod_{j=1}^n |\mathcal{E}_j f_j(x)|^{2/n}$ against a test function G in the unit ball of L^n. We let $\mathcal{T}_j$ consist of tubes with directions normal to S_j. We apply Guth's theorem to G obtain g_j as in Theorem 1.9. Then

$$\int_{\mathbb{R}^n} \prod_{j=1}^n |\mathcal{E}_j f_j(x)|^{2/n} G(x)\,\mathrm{d}x \le \int_{\mathbb{R}^n} \prod_{j=1}^n |\mathcal{E}_j f_j(x)|^{2/n} g_j(x)^{1/n}\,\mathrm{d}x \le \prod_{j=1}^n \Big(\int_{\mathbb{R}^n} |\mathcal{E}_j f_j(x)|^2 g_j(x)\,\mathrm{d}x \Big)^{1/n}$$

by Hölder's inequality. For each j we have

$$\int_{\mathbb{R}^n} |\mathcal{E}_j f_j(x)|^2 g_j(x)\,\mathrm{d}x \lesssim \Big(\sup_{T \in \mathcal{T}_j} \int_T g_j \Big) \int |f_j(t)|^2\,\mathrm{d}t \lesssim \|f_j\|_2^2$$

by the Mizohata–Takeuchi conjecture and the second conclusion of Theorem 1.9. Combining these estimates yields (16). $\square$

1.2. ***Structure of the paper.*** In Section 2 we first state and prove two results, Theorems 2.1 and 2.3, both equivalent to Case I of Theorem 1.1, and then we indicate how we shall use vector-valued techniques to obtain our main theorems. In Section 3 we discuss refinements of Theorem 1.1 for positive operators to the case of p-convex lattices; the main result here is Theorem 3.2. The case of general linear operators is taken up in Section 4, and here we impose conditions of Rademacher-type; the main result in this

$$\begin{tikzcd}
 & \boxed{\text{Theorem 3.2}} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Theorem 1.3} \arrow[r, Rightarrow] & \text{Theorem 1.4} & \text{Theorem 5.1}
\end{tikzcd}$$

Figure 2. Positive operators.

$$\begin{tikzcd}
 & \boxed{\text{Theorem 4.3}} \arrow[ddl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
 & \text{Theorem 4.1} & \text{Theorem 5.2} \arrow[d, Rightarrow] \\
\text{Theorem 1.5} \arrow[r, Rightarrow] & \text{Theorem 1.6} \arrow[r, Rightarrow] & \text{Theorem 1.7}
\end{tikzcd}$$

Figure 3. General linear operators.

setting is Theorem 4.3. In Section 5 we establish sharp multilinear duality and Maurey-type factorisation theorems for both positive and general linear operators, in Theorems 5.1 and 5.2 respectively. The logical connections between these main results are summarised in Figure 1.

The implications between the main result for positive operators on p-convex lattices, Theorem 3.2, and its more basic manifestations Theorems 1.3 and 1.4 for L^r-spaces, are given in Figure 2.

For general linear operators on normed spaces of (nontrivial) Rademacher-type, the corresponding logical implications between the main result, Theorem 4.3 and the more basic manifestations Theorems 1.5, 1.6 and 1.7 for L^r-spaces, are given by Figure 3. Finally, in the Appendix, we consider the necessity of the conditions we have imposed on the exponents (p_j) in the Basic Question and in Theorems 1.3 and 1.5, and we show that they cannot in general be dispensed with. We also show that one cannot avoid the hypothesis of (p_j)-convexity in Theorem 3.2.

2. Vector-valued disentanglement

In this section we state and prove two results, both of which are equivalent to the disentanglement result given by Case I of Theorem 1.1. These will be crucial in the development of both the positive theory stated in terms of p-convexity and of the general linear theory using Rademacher-type. At the end of this section we describe the strategy that we will adopt in order to achieve these aims in the succeeding sections.

2.1. ***Functional form.*** We first derive an equivalent, arguably more primordial, form of Case I of Theorem 1.1, which makes no reference to saturating positive linear operators, nor to normed lattices, but instead is couched in terms of saturating families of nonnegative measurable functions on a σ-finite measure space X.

Let $(X, \mathrm{d}\mu)$ be a σ-finite measure space. Suppose that for each $1 \leq j \leq d$ we have an indexing set $\mathcal{K}_j$ and a family $\{g_{k_j}\}_{k_j \in \mathcal{K}_j}$ of nonnegative measurable functions on X. We assume that, for each j, the family

$\{g_{k_j}\}_{k_j\in\mathcal{K}_j}$ *saturates* X in the sense that, for every $E \subseteq X$ with $\mu(X) > 0$, there is a subset $E' \subseteq E$ with $\mu(E') > 0$ and a $k_j \in \mathcal{K}_j$ such that $g_{k_j} > 0$ on E'.

Theorem 2.1 (disentanglement of functions). *With* $(X, \mathrm{d}\mu)$ *and* $\{g_{k_j}\}_{k_j\in\mathcal{K}_j}$ *as above, and* $\alpha_j > 0$ *such that* $\sum_{j=1}^d \alpha_j = 1$, *assume that*

$$\int_X \prod_{j=1}^d \Big(\sum_{k_j\in\mathcal{K}_j} \beta_{k_j} g_{k_j} \Big)^{\alpha_j} \mathrm{d}\mu \le A \prod_{j=1}^d \Big(\sum_{k_j\in\mathcal{K}_j} \beta_{k_j} \Big)^{\alpha_j} \tag{18}$$

for all (*finitely supported*) *nonnegative* $\{\beta_{k_j}\}$. *Then there exist nonnegative* ϕ_j *such that*

$$\prod_{j=1}^d \phi_j(x)^{\alpha_j} \ge 1 \tag{19}$$

almost everywhere on X, *and such that*, *for all* j,

$$\int_X g_{k_j}(x)\phi_j(x)\,\mathrm{d}\mu(x) \le A \tag{20}$$

for all $k_j \in \mathcal{K}_j$.

Proof. Let $\mathcal{Y}_j$ be the normed lattice $l^1(\mathcal{K}_j)$ with counting measure on $\mathcal{K}_j$, whose members are denoted by $\boldsymbol{\beta}_j = \{\beta_{k_j}\}_{k_j\in\mathcal{K}_j}$. (There is no requirement on $\mathcal{K}_j$ to be countable.) Define $T_j : l^1(\mathcal{K}_j) \to \mathcal{M}(X)$ by

$$T_j(\boldsymbol{\beta}_j) := \sum_{k_j\in\mathcal{K}_j} \beta_{k_j} g_{k_j}.$$

Note that T_j are saturating positive linear operators. Then (18) becomes

$$\int_X \prod_{j=1}^d (T_j\boldsymbol{\beta}_j)^{\alpha_j}\,\mathrm{d}\mu \le A \prod_{j=1}^d \|\boldsymbol{\beta}\|_{\mathcal{Y}_j}^{\alpha_j}.$$

By Case I of Theorem 1.1, there exist ϕ_j such that (19) holds and such that

$$\int_X (T_j\boldsymbol{\beta}_j)\phi_j\,\mathrm{d}\mu \le A\|\boldsymbol{\beta}_j\|_{\mathcal{Y}_j},$$

which is the same as

$$\int_X \Big(\sum_{k_j\in\mathcal{K}_j} \beta_{k_j} g_{k_j} \Big) \phi_j \mathrm{d}\mu \le A \sum_{k_j\in\mathcal{K}_j} \beta_{k_j},$$

or, equivalently, (20). □

Theorem 2.1 can be equivalently rephrased in terms of convex families of functions as follows:

Theorem 2.2 (disentanglement of convex families of functions). *Let* $(X, \mathrm{d}\mu)$ *be a* σ*-finite measure space. Suppose that* $\sum_{j=1}^d \alpha_j = 1$ *and that each* $\alpha_j > 0$. *For each* $j \in \{1, \dots, d\}$ *let* $\mathcal{G}_j$ *be a saturating convex set of nonnegative measurable functions. Assume that*

$$\int_X \prod_{j=1}^d g_j(x)^{\alpha_j}\,\mathrm{d}\mu(x) \le A \quad \textit{for all } g_j \in \mathcal{G}_j.$$

Then there exist nonnegative ϕ_j such that

$$\prod_{j=1}^{d} \phi_j(x)^{\alpha_j} \geq 1$$

almost everywhere on X, and such that, for all j,

$$\int_X g_j(x)\phi_j(x)\,\mathrm{d}\mu(x) \leq A \quad \textit{for all } g_j \in \mathcal{G}_j.$$

Proof. The equivalence of Theorems 2.1 and 2.2 is clear from the following observation: writing $\gamma_{k_j} := \beta_{k_j}/\big(\sum_{k_j\in\mathcal{K}_j} \beta_{k_j}\big)$ and using homogeneity, assumption (18) of Theorem 2.1 can be rephrased as

$$\int_X \prod_{j=1}^{d} g_j^{\alpha_j}\,\mathrm{d}\mu \leq A \quad \text{for all } g_j \in \operatorname{conv} \mathcal{G}_j,$$

where $\operatorname{conv}\mathcal{G}_j$ is the convex hull of $\mathcal{G}_j$. □

2.2. *Vector-valued form.* The viewpoint of Theorem 2.1 lends itself more readily to applications which are far from obvious from the viewpoint of the formulation of Theorem 1.1. For some of these applications we shall need to work with quasinormed spaces rather than normed spaces $\mathcal{Y}_j$. We recall that a quasinormed space $\mathcal{Y}$ is one in which we have the quasitriangle inequality $\|x+y\|_{\mathcal{Y}} \leq K(\|x\|_{\mathcal{Y}} + \|y\|_{\mathcal{Y}})$ for some $K \geq 1$ in place of the usual triangle inequality.[6]

For example, we have:

Theorem 2.3. *Suppose that $(X, \mathrm{d}\mu)$ is a σ-finite measure space, $\mathcal{Y}_j$ are quasinormed spaces and $0 < p_j < \infty$. Suppose $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are homogeneous of degree 1 — that is, $T_j(\lambda f_j) = \lambda T_j f_j$ for all $f_j \in \mathcal{Y}_j$ and all scalars λ. Assume that, for all j, the functions $\{|T_j f_j| : f_j \in \mathcal{Y}_j\}$ saturate X. Let $\theta_j > 0$ satisfy $\sum_{j=1}^{d} \theta_j = 1$ and suppose that we have the (p_j)-vector-valued inequality*

$$\int_X \prod_{j=1}^{d} \Big(\sum_{k=1}^{N} |T_j f_{jk}(x)|^{p_j}\Big)^{\theta_j} \mathrm{d}\mu(x) \leq A \prod_{j=1}^{d} \Big(\sum_{k=1}^{N} \|f_{jk}\|_{\mathcal{Y}_j}^{p_j}\Big)^{\theta_j} \tag{21}$$

uniformly in N. Then there exist nonnegative ϕ_j such that

$$\prod_{j=1}^{d} \phi_j(x)^{\theta_j} \geq 1$$

almost everywhere on X and such that, for each j,

$$\Big(\int_X |T_j f_j(x)|^{p_j}\phi_j(x)\,\mathrm{d}\mu(x)\Big)^{1/p_j} \leq A^{1/p_j}\|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

Notice that we do not need $\mathcal{Y}_j$ to have a lattice structure, nor do we need linearity or positivity of T_j.

[6] We shall not use the quasitriangle inequality, and so the constant K will not appear explicitly in our analysis. In fact, every quasinormed space $\mathcal{Y}$ is r-normable and hence has Rademacher-type r for some $0 < r \leq 1$; see for example [Kalton 2005]. The Rademacher-type constant $R_r(\mathcal{Y})$ will instead feature.

Proof. Consider the saturating families

$$\left\{\left(\frac{|T_j f_j(x)|}{\|f_j\|_{\mathcal{Y}_j}}\right)^{p_j} : f_j \in \mathcal{Y}_j \setminus \{0\}\right\}$$

of nonnegative functions defined on X. Assumption (21) translates into (18) with $\alpha_j = \theta_j$, with the same constant A. So by Theorem 2.1 there are nonnegative ϕ_j such that (19) and (20) hold. And (20) translates into

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\,\mathrm{d}\mu(x)\right)^{1/p_j} \le A^{1/p_j} \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$. □

To complete the assertion that Theorems 1.1 (Case I), 2.1 and 2.3 are all equivalent, we note that Theorem 2.3 implies Case I of Theorem 1.1. Indeed, the scalar-valued inequality (the hypothesis of Theorem 1.1) readily upgrades to the vector-valued inequality (the hypothesis of Theorem 2.3 with $p_j = 1$ for all j) via positivity, as follows: we have

$$\begin{aligned}\int_X \prod_{j=1}^d \Big(\sum_k |T_j f_{jk}(x)|\Big)^{\theta_j} \mathrm{d}\mu(x) &\le \int_X \prod_{j=1}^d \Big|T_j\Big(\sum_k |f_{jk}|\Big)(x)\Big|^{\theta_j} \mathrm{d}\mu(x)\\ &\le A \prod_{j=1}^d \Big\|\sum_k |f_{jk}|\Big\|_{\mathcal{Y}_j}^{\theta_j} \le A \prod_{j=1}^d \Big(\sum_k \|f_{jk}\|_{\mathcal{Y}_j}\Big)^{\theta_j}.\end{aligned}$$

(Note that the use of the triangle inequality for $\mathcal{Y}_j$ here is legitimate since in the implication under consideration the spaces $\mathcal{Y}_j$ are indeed normed spaces.) Summarising, Theorems 1.1 (Case I), 2.1 and 2.3 are all equivalent.

The reader will readily verify using Hölder's inequality that the converse statements to Theorems 2.1 and 2.3 also hold.

2.3. ***Vector-valued approach to disentanglement.*** We now give a preview of how we shall employ Theorem 2.3 to establish the main disentanglement theorems of the following sections. Indeed, thanks to Theorem 2.3 (and its easy converse), given weights (θ_j) with $\sum_{j=1}^d \theta_j = 1$, exponents (p_j) with $p_j > 0$, a measure space (X, μ) and linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ defined on quasinormed spaces $\mathcal{Y}_j$, the following two statements are equivalent:

- (disentanglement of p_j-th powers) The norm inequality

$$\int_X \prod_{j=1}^d |T_j f_j(x)|^{p_j\theta_j}\,\mathrm{d}\mu(x) \le A \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j}$$

implies that there exist nonnegative ϕ_j such that $\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1$ almost everywhere on X and such that, for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\,\mathrm{d}\mu(x)\right)^{1/p_j} \le \tilde{A}^{1/p_j} \|f_j\|_{\mathcal{Y}_j}.$$

- (scalar-valued implies vector-valued inequality) The scalar-valued inequality

$$\int_X \prod_{j=1}^d |T_j f_j(x)|^{p_j\theta_j}\,\mathrm{d}\mu(x) \le A \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j}$$

implies the vector-valued inequality

$$\int_X \prod_{j=1}^d \Big(\sum_k |T_j f_{jk}(x)|^{p_j}\Big)^{\theta_j}\,\mathrm{d}\mu(x) \le \tilde{A} \prod_{j=1}^d \Big(\sum_k \|f_{jk}\|_{\mathcal{Y}_j}^{p_j}\Big)^{\theta_j}.$$

In the following sections, we prove disentanglement theorems via this *vector-valued approach*: subject to geometric properties of the spaces $\mathcal{Y}_j$ (p-convexity for positive linear operators, Rademacher-type for general linear operators), we deduce the vector-valued inequality from the corresponding scalar-valued inequality, and thereby establish our disentanglement theorems via the equivalence we have just set out.

3. Positive operators and p-convexity

In this section we state and prove a more general form of Theorem 1.3 applying to normed lattices which enjoy p-convexity properties.

Definition 3.1 (p-convexity). Let $1 \le p < \infty$. A normed lattice $\mathcal{Y}$ is p-convex if for all finite sequences (f_j) in $\mathcal{Y}$ we have

$$\Big\| \Big(\sum_j |f_j|^p\Big)^{1/p} \Big\|_{\mathcal{Y}} \le C_p(\mathcal{Y}) \Big(\sum_j \|f_j\|_{\mathcal{Y}}^p\Big)^{1/p}.$$

The least such constant is denoted by $C_p(\mathcal{Y})$ and is called the p-convexity constant of $\mathcal{Y}$. Clearly $C_p(\mathcal{Y}) \ge 1$.

Notice that L^p is p-convex with p-convexity constant equal to 1, and that every normed lattice is 1-convex with 1-convexity constant equal to 1. If a lattice $\mathcal{Y}$ is p-convex for some $1 \le p < \infty$, then it is $\tilde{p}$-convex for all $1 \le \tilde{p} \le p$; see, for example, [Lindenstrauss and Tzafriri 1979].

Using the fact that L^r is p-convex for $1 \le p \le r$, with p-convexity constant 1, Theorem 1.3 follows directly from the next, more general result, which is the principal result of this section. This answers our Basic Question for positive linear operators defined on p-convex lattices upon taking $\gamma_j = p_j\theta_j$.

Theorem 3.2 (disentanglement theorem for positive operators on p-convex lattices). *Suppose that X is a σ-finite measure space and that $\mathcal{Y}_j$, for $j = 1, \dots, d$, are p_j-convex normed lattices for some $1 \le p_j < \infty$. Suppose that the linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are positive, and that each T_j saturates X. Suppose that $\theta_j > 0$ and that $\sum_{j=1}^d \theta_j = 1$. Finally, suppose that*

$$\int_X \prod_{j=1}^d (T_j f_j)(x)^{p_j\theta_j}\,\mathrm{d}\mu(x) \le B \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} \tag{22}$$

for all nonnegative f_j in $\mathcal{Y}_j$, $1 \le j \le d$.

Then there exist nonnegative measurable functions ϕ_j on X such that

$$\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1 \tag{23}$$

almost everywhere on X and such that for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j}\phi_j(x)\,\mathrm{d}\mu(x)\right)^{1/p_j} \le B^{1/p_j} C_{p_j}(\mathcal{Y}_j)\|f_j\|_{\mathcal{Y}_j} \tag{24}$$

for all $f_j \in \mathcal{Y}_j$.

Remark 3. The necessity of the geometric assumption that each lattice $\mathcal{Y}_j$ is p_j-convex is addressed in the Appendix — see Proposition A.9.

We establish Theorem 3.2 using the strategy described above in Section 2.3. Indeed, by the discussion there, and some playing with homogeneities, it suffices to show that under the assumptions of the theorem, the scalar-valued inequality

$$\int_X \prod_{j=1}^d |T_j f_j(x)|^{p_j\theta_j}\,\mathrm{d}\mu(x) \le B\prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} \tag{25}$$

implies the (p_j)-vector-valued inequality

$$\int_X \prod_{j=1}^d \left(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\right)^{\theta_j}\mathrm{d}\mu(x) \le B\prod_{j=1}^d C_{p_j}(\mathcal{Y}_j)^{p_j\theta_j}\prod_{j=1}^d\left(\sum_{k=1}^N \|f_{jk}\|_{\mathcal{Y}_j}^{p_j}\right)^{\theta_j}, \tag{26}$$

and this is exactly what we do in the next lemma:

Lemma 3.3 (scalar-valued to vector-valued). *Suppose that $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are positive linear operators and that $\mathcal{Y}_j$ are p_j-convex normed lattices for some $p_j \ge 1$. Then (25) implies (26).*

Note that when each $\mathcal{Y}_j$ is an L^{r_j}-space for $r_j \ge p_j$, the constant in (26) is precisely B since then we have $C_{p_j}(L^{r_j}) = 1$.

Proof. By homogeneity, we may assume that, for each j, $\left(\sum_{k=1}^N \|f_{jk}\|_{\mathcal{Y}_j}^{p_j}\right)^{1/p_j} = 1$.

We are seeking a bound for the left-hand side of (26), and start by linearising the expression $\left(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\right)^{1/p_j}$ in a pointwise manner. We do this by using classical duality for l^p spaces, together with positivity. Indeed, we have, with the sup taken over all (λ_k) with $\sum_k \lambda_k^{p_j'} = 1$,

$$\begin{aligned}\left(\sum_{k=1}^N \left|T_j f_{jk}(x)\right|^{p_j}\right)^{1/p_j} &= \sup_{(\lambda_k)}\left|\sum_{k=1}^N \lambda_k T_j f_{jk}(x)\right| = \sup_{(\lambda_k)}\left|T_j\left(\sum_{k=1}^N \lambda_k f_{jk}\right)(x)\right| \\ &\le \sup_{(\lambda_k)} T_j\left[\left(\sum_{k=1}^N \lambda_k^{p_j'}\right)^{1/p_j'}\left(\sum_{k=1}^N |f_{jk}|^{p_j}\right)^{1/p_j}\right](x) \\ &= T_j\left[\left(\sum_{k=1}^N |f_{jk}|^{p_j}\right)^{1/p_j}\right](x) := T_j F_j(x).\end{aligned}$$

Now we are in a position to apply (25), and we thus have

$$\int_X \prod_{j=1}^d \left(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\right)^{\theta_j}\mathrm{d}\mu(x) \le \int_X \prod_{j=1}^d T_j F_j(x)^{p_j\theta_j}\,\mathrm{d}\mu(x) \le B\prod_{j=1}^d \|F_j\|_{\mathcal{Y}_j}^{p_j\theta_j}.$$

We use the definition of p-convexity to obtain

$$\|F_j\|_{\mathcal{Y}_j} = \left\| \left[\left(\sum_{k=1}^{N} |f_{jk}|^{p_j} \right)^{1/p_j} \right] \right\|_{\mathcal{Y}_j} \leq C_{p_j}(\mathcal{Y}_j) \left(\sum_{k=1}^{N} \|f_{jk}\|_{\mathcal{Y}_j}^{p_j} \right)^{1/p_j} = C_{p_j}(\mathcal{Y}_j).$$

Combining these inequalities establishes the lemma. □

Notice that we really use linearity of T_j in this argument; sublinearity does not suffice for it to work.

Remark 4. The essence of the vector-valued approach to disentanglement lies in upgrading a scalar-valued estimate into the corresponding vector-valued estimate. From the viewpoint of disentanglement of convex families of functions, this amounts to upgrading the estimate

$$\int_X \prod_{j=1}^{d} |g_j(x)|^{\theta_j} \, \mathrm{d}\mu(x) \leq A \quad \text{for all } g_j \in \mathcal{G}_j$$

from the family

$$\mathcal{G}_j := \mathcal{G}(T_j, \mathcal{Y}_j, p_j) := \left\{ \frac{|T_j f_j|^{p_j}}{\|f_j\|_{\mathcal{Y}_j}^{p_j}} \right\}$$

to its convex hull $\operatorname{conv} \mathcal{G}_j$. Now, Lemma 3.3 loosely states that, under its assumptions, the family $\mathcal{G}_j$ is "essentially convex". Indeed, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be sets of nonnegative measurable functions and $C > 0$ be a constant. Let us write $\mathcal{F}_1 \leq C\mathcal{F}_2$ if for each $f_1 \in \mathcal{F}_1$ there is $f_2 \in \mathcal{F}_2$ such that $f_1 \leq C f_2$. Assume that $T : \mathcal{Y} \to \mathcal{M}(X)$ is a positive linear operator on a p-convex normed lattice $\mathcal{Y}$ with p-convexity constant $C_p(\mathcal{Y})$. Then from the definition of p-convexity it follows that

$$\operatorname{conv} \mathcal{G}(T, \mathcal{Y}, p) \leq C_p(\mathcal{Y}) \mathcal{G}(T, \mathcal{Y}, p).$$

4. General linear operators and Rademacher-type

We now consider general linear (not necessarily positive) operators. We will follow the same general lines of argument as in the previous section. The key new ingredient in this setting will be an analogue of the argument of Lemma 3.3 which converts scalar to vector inequalities, but now without a positivity hypothesis. Once again we shall first need to linearise the expression $\left(\sum_{k=1}^{N} |T_j f_{jk}(x)|^{p_j}\right)^{1/p_j}$ in a pointwise manner. We no longer have positivity at our disposal, so we shall instead use the sequence of Rademacher functions, which we denote by (ϵ_k).

Let us first suppose for simplicity that each $p_j = 2$. In this case, we have, for each j,

$$\left(\sum_{k=1}^{N} |T_j f_{jk}(x)|^2 \right)^{1/2} = \left(\mathbb{E} \left| \sum_{k=1}^{N} \epsilon_k T_j f_{jk}(x) \right|^2 \right)^{1/2},$$

$$\sim_{\theta_j} \left(\mathbb{E} \left| \sum_{k=1}^{N} \epsilon_k T_j f_{jk}(x) \right|^{2\theta_j} \right)^{1/2\theta_j} = \left(\mathbb{E} \left| T_j \left(\sum_{k=1}^{N} \epsilon_k f_{jk} \right)(x) \right|^{2\theta_j} \right)^{1/2\theta_j}$$

by Khintchine's inequality, so that

$$\int_X \prod_{j=1}^d \Big(\sum_{k=1}^N |T_j f_{jk}(x)|^2\Big)^{\theta_j} \,\mathrm{d}\mu(x) \lesssim_{\{\theta_j\}} \mathbb{E} \int_X \prod_{j=1}^d \Big| T_j\Big(\sum_{k=1}^N \epsilon_{jk} f_{jk}\Big)(x)\Big|^{2\theta_j} \,\mathrm{d}\mu(x).$$

If we now assume (25) with $p_j = 2$ for all j, we can dominate this last expression by

$$B\,\mathbb{E} \prod_{j=1}^d \Big\| \sum_{k=1}^N \epsilon_{jk} f_{jk} \Big\|_{\mathcal{Y}_j}^{2\theta_j}.$$

If $\mathcal{Y}_j$ is assumed to be of Rademacher-type 2, that is to say

$$\Big(\mathbb{E}\Big\|\sum_{k=1}^N \epsilon_k F_k\Big\|_{\mathcal{Y}_j}^2\Big)^{1/2} \le R_2(\mathcal{Y}_j)\Big(\sum_{k=1}^N \|F_k\|_{\mathcal{Y}_j}^2\Big)^{1/2}$$

for some finite $R_2(\mathcal{Y}_j)$, we will obtain (using Jensen's inequality $\mathbb{E}(X^\theta) \le \mathbb{E}(X)^\theta$ for $0 < \theta < 1$)

$$\int_X \prod_{j=1}^d \Big(\sum_{k=1}^N |T_j f_{jk}(x)|^2\Big)^{\theta_j} \,\mathrm{d}\mu(x) \lesssim_{\{\theta_j\}} B \prod_{j=1}^d R_2(\mathcal{Y}_j)^{2\theta_j} \prod_{j=1}^d \Big(\sum_{k=1}^N \|f_{jk}\|_{\mathcal{Y}_j}^2\Big)^{\theta_j},$$

which is the analogue of (26) in this setting.

(Note that even in the case that each $\mathcal{Y}_j$ is an L^2-space, and so $R_2(\mathcal{Y}_j) = 1$, there is an implicit constant greater than 1 in this last conclusion, due to the use of Khintchine's inequality.)

The argument now proceeds exactly in accordance with the remarks in Section 2.3, and we arrive at:

Theorem 4.1 (disentanglement theorem for general linear operators on spaces of Rademacher type 2). *Suppose that X is a σ-finite measure space and that $\mathcal{Y}_j$, for $j = 1, \dots, d$, are normed spaces which are of Rademacher-type 2. Suppose that the linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ saturate X, and that $\sum_{j=1}^d \theta_j = 1$. Finally, suppose that*

$$\int \prod_{j=1}^d |T_j f_j(x)|^{2\theta_j} \,\mathrm{d}\mu(x) \le B \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{2\theta_j}$$

for all f_j in $\mathcal{Y}_j$, $1 \le j \le d$.

Then there exist nonnegative measurable functions ϕ_j on X such that

$$\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1$$

almost everywhere on X, and such that, for each j,

$$\Big(\int_X |T_j f_j(x)|^2 \phi_j(x) \,\mathrm{d}\mu(x)\Big)^{1/2} \lesssim_{\{\theta_j\}} B^{1/2} R_2(\mathcal{Y}_j) \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

The special case of this result when each $\mathcal{Y}_j$ is an L^{r_j}-space with $2 \le r_j < \infty$ is Theorem 1.6, which immediately follows from Theorem 4.1 upon using the fact (see below) that the Lebesgue space L^r with $r \ge 2$ has Rademacher-type 2.

We now need to discuss what happens when one or more of the p_j are not equal to 2. We need the notion of Rademacher-type p.

Definition 4.2 (Rademacher-type). Let $0 < p \le 2$. A quasinormed space $\mathcal{Y}$ is of Rademacher-type p if for all finite sequences (F_k) in $\mathcal{Y}$ we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^{N} \epsilon_k F_k\right\|_{\mathcal{Y}}^{p}\right)^{1/p} \le R_p(\mathcal{Y})\left(\sum_{k=1}^{N} \|F_k\|_{\mathcal{Y}}^{p}\right)^{1/p}$$

for some finite constant $R_p(\mathcal{Y})$.

The least such constant is denoted by $R_p(\mathcal{Y})$ and is called the p-Rademacher-type constant of $\mathcal{Y}$. When $0 < r \le 2$, the Lebesgue space L^r has Rademacher-type p for $0 < p \le r$; when $2 < r < \infty$, L^r has Rademacher-type p for $0 < p \le 2$. Every normed space $\mathcal{Y}$ has Rademacher-type 1. Note that by Khintchine's inequality, if a quasinormed space is of Rademacher-type p, then it is also of Rademacher-type $\tilde{p}$ for all $0 < \tilde{p} \le p$. Observe that the one-dimensional normed space $\mathbb{R}$ (and more generally any Hilbert space) has Rademacher-type 2 with corresponding constant 1. When $0 < p < 1$, Rademacher-type p is equivalent to p-normability, i.e., the existence of a constant C such that

$$\left\|\sum_{k=1}^{N} F_k\right\|_{\mathcal{Y}} \le C\left(\sum_{k=1}^{N} \|F_k\|_{\mathcal{Y}}^{p}\right)^{1/p}.$$

Ideally we would hope to have:

Aspiration (general disentanglement aspiration for linear operators). *Suppose that X is a σ-finite measure space and that $\mathcal{Y}_j$, for $j = 1, \ldots, d$, are quasinormed spaces which are of Rademacher-type p_j for certain $0 < p_j \le 2$. Suppose that the linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ saturate X, and that $\sum_{j=1}^{d} \theta_j = 1$. Finally, suppose that*

$$\int \prod_{j=1}^{d} |T_j f_j(x)|^{p_j\theta_j} \, \mathrm{d}\mu(x) \le B \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} \tag{27}$$

for all f_j in $\mathcal{Y}_j$, $1 \le j \le d$.

Then there exist nonnegative measurable functions ϕ_j on X such that

$$\prod_{j=1}^{d} \phi_j(x)^{\theta_j} \ge 1 \tag{28}$$

almost everywhere on X and such that, for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x) \, \mathrm{d}\mu(x)\right)^{1/p_j} \lesssim_{\{\theta_j, p_j\}} B^{1/p_j} R_{p_j}(\mathcal{Y}_j) \|f_j\|_{\mathcal{Y}_j} \tag{29}$$

for all $f_j \in \mathcal{Y}_j$.

We cannot hope for this to be true in general in situations in which some $p_j < 2$; see the Appendix. Nevertheless, we are able to prove something slightly weaker, namely that the aspiration is in fact a theorem under the stronger hypothesis that for those j with $p_j < 2$, the normed spaces $\mathcal{Y}_j$ have Rademacher-type *strictly* larger than p_j.

Theorem 4.3 (disentanglement theorem for general linear operators on spaces of nontrivial Rademacher type). *Let X be a σ-finite measure space and $\mathcal{Y}_j$ quasinormed spaces. Let $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ be linear operators. Suppose that the linear operators T_j saturate X. Let $0 < p_j \le 2$ and $\sum_{j=1}^d \theta_j = 1$. Assume that*

$$\int \prod_{j=1}^d |T_j f_j(x)|^{p_j\theta_j} \, \mathrm{d}\mu(x) \le B \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} \tag{30}$$

for all f_j in $\mathcal{Y}_j$, $1 \le j \le d$.

Suppose moreover that each space $\mathcal{Y}_j$ has Rademacher-type $r_j = 2$ for those j with $p_j = 2$, and has Rademacher-type $r_j > p_j$ for those j with $p_j < 2$.

Then there exist nonnegative measurable functions ϕ_j on X such that

$$\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1$$

almost everywhere on X and such that, for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x) \, \mathrm{d}\mu(x)\right)^{1/p_j} \lesssim_{\{\theta_j, p_j, r_j\}} B^{1/p_j} R_{r_j}(\mathcal{Y}_j) \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

Using the fact that the Lebesgue space L^r (with $0 < r < \infty$) has Rademacher-type $\min\{2, r\}$, and hence also Rademacher-type $\tilde{r}$ for every $0 < \tilde{r} \le \min\{2, r\}$, we immediately obtain Theorem 1.5 (and also the assertion made in the accompanying footnote).

Proof. Once again the key issue is to pass from the scalar-valued inequality (30) to the vector-valued inequality analogous to (26), and this is achieved by linearising the expression

$$\left(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\right)^{1/p_j}$$

for each j. When $p_j = 2$ the Rademacher functions achieve this, but they are unsuited to do so when $0 < p_j < 2$ and instead we use p-stable random variables. (For simplicity of notation, in what follows we shall assume that $p_j < 2$ for all j; the easy modifications when $p_j = 2$ for some j are left to the reader.)

We recall that for $0 < p \le 2$, a real-valued random variable γ on a probability space is called (normalised) *p-stable* if it satisfies $\mathbb{E}(e^{it\gamma}) = e^{-|t|^p}$. Note that the distribution (i.e., the pushforward measure on the real line) of a p-stable random variable is unique because the characteristic function (i.e., the Fourier transform up to a sign) of a random variable determines its distribution. These random variables enjoy the following key property:

Lemma 4.4 (key property of independent p-stable random variables). *Let $0 < q < p \le 2$. Let (γ_k) be a sequence of independent p-stable random variables. Then*

$$\left(\mathbb{E}\left|\sum_k \gamma_k a_k\right|^q\right)^{1/q} \sim_{p,q} \left(\sum_k |a_k|^p\right)^{1/p}$$

for all sequences (a_k) of scalars.

Pisier [1974] proved that this property can be upgraded to the vector-valued setting under an appropriate hypothesis of Rademacher-type:

Lemma 4.5 (Rademacher-type r implies stable-type $p < r$). *Let $0 < q < p < r \leq 2$. Let $\mathcal{Y}$ be a quasinormed space of Rademacher-type r. Let (γ_k) be a sequence of independent p-stable random variables. Then*

$$\left(\mathbb{E}\left\|\sum_k \gamma_k f_k\right\|_{\mathcal{Y}}^q\right)^{1/q} \lesssim_{p,q,r} R_r(\mathcal{Y})\left(\sum_k \|f_k\|_{\mathcal{Y}}^p\right)^{1/p}$$

for all sequences (f_k) of vectors.

Note that we need $q < p$ in the above lemmas because p-stable random variables fail to be p-integrable. For a textbook treatment of Rademacher and p-stable random variables and Rademacher and p-stable types, see for example [Albiac and Kalton 2006, Sections 6.2, 6.4, and 7.1].

Now, for each $j = 1, \ldots, d$, let (γ_{jk}) be a sequence of independent p_j-stable random variables. Then, by Lemma 4.4, we have

$$\left(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\right)^{1/p_j} \sim_{\{\theta_j\}} \left(\mathbb{E}\left|\sum_k \gamma_{jk} T_j f_{jk}(x)\right|^{p_j\theta_j}\right)^{1/p_j\theta_j}.$$

Using this linearisation we can rephrase the left-hand side of the vector-valued inequality in terms of the left-hand side of the scalar-valued inequality,

$$\begin{aligned}\int_X \prod_{j=1}^d \left(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\right)^{\theta_j} \mathrm{d}\mu(x) &\sim_{\{\theta_j\}} \mathbb{E}\int_X \prod_{j=1}^d \left|\sum_k \gamma_{jk} T_j f_{jk}(x)\right|^{p_j\theta_j} \mathrm{d}\mu(x) \\ &= \mathbb{E}\int_X \prod_{j=1}^d \left|T_j\left(\sum_k \gamma_{jk} f_{jk}\right)(x)\right|^{p_j\theta_j} \mathrm{d}\mu(x).\end{aligned}$$

Using the assumed scalar-valued inequality (30), we have the estimate

$$\begin{aligned}\mathbb{E}\int_X \prod_{j=1}^d \left|T_j\left(\sum_k \gamma_{jk} f_{jk}\right)(x)\right|^{p_j\theta_j} \mathrm{d}\mu(x) &\leq B\,\mathbb{E}\prod_{j=1}^d \left\|\sum_k \gamma_{jk} f_{jk}\right\|_{\mathcal{Y}_j}^{p_j\theta_j} \\ &= B\prod_{j=1}^d \mathbb{E}\left(\left\|\sum_k \gamma_{jk} f_{jk}\right\|_{\mathcal{Y}_j}^{p_j\theta_j}\right).\end{aligned}$$

By Lemma 4.5, together with the assumption that each space $\mathcal{Y}_j$ has Rademacher-type $r_j > p_j$, and the fact that $\theta_j < 1$, we obtain

$$\mathbb{E}\left(\left\|\sum_k \gamma_{jk} f_{jk}\right\|_{\mathcal{Y}_j}^{p_j\theta_j}\right) \lesssim_{\theta_j, p_j, r_j} R_{r_j}(\mathcal{Y}_j)^{p_j\theta_j}\left(\sum_k \|f_{jk}\|_{\mathcal{Y}}^{p_j}\right)^{\theta_j}$$

for each j and therefore

$$\mathbb{E}\prod_{j=1}^d \left\|\sum_k \gamma_{jk} f_{jk}\right\|_{\mathcal{Y}_j}^{p_j\theta_j} \lesssim_{\theta_j, p_j, r_j} \prod_{j=1}^d R_{r_j}(\mathcal{Y}_j)^{p_j\theta_j}\left(\sum_k \|f_{jk}\|_{\mathcal{Y}}^{p_j}\right)^{\theta_j}.$$

Summarising, we have proved that if the quasinormed spaces $\mathcal{Y}_j$ have Rademacher-type r_j, then the scalar-valued inequality (30) implies the vector-valued inequality

$$\int_X \prod_{j=1}^d \Big(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j} \Big)^{\theta_j} \,\mathrm{d}\mu(x) \lesssim_{\theta_j, p_j, r_j} \prod_{j=1}^d R_{r_j}(\mathcal{Y}_j)^{p_j\theta_j} \Big(\sum_k \|f_{jk}\|_{\mathcal{Y}}^{p_j} \Big)^{\theta_j}.$$

By the remarks in Section 2.3, this suffices to establish Theorem 4.3. □

Remark 5. Since the linearisation arguments of Theorems 3.2 and 4.3 run componentwise, in the case where some of the operators are positive on p_j-convex lattices and some nonpositive on r_j-Rademacher-type normed spaces, we may obtain a hybrid of these two theorems, whose precise formulation we leave to the interested reader.

5. Multilinear duality and Maurey factorisation extended

In this section we apply the two main disentanglement theorems (Theorem 3.2 for positive linear operators and Theorem 4.3 for general linear operators) to deduce multilinear duality and multilinear Maurey factorisation theorems in the spirit of Theorem 1.1. The treatment we give is very much in parallel to the manner in which Cases II and III of Theorem 1.1 can be deduced from Case I.

Note that multilinear Maurey factorisation theorems below (Cases III of Theorems 5.1 and 5.2) in the linear case $d = 1$ recover the Maurey factorisation theorems [1974] for linear operators. We emphasise, however, that our main theorems (Theorems 3.2 and 4.3) have no linear counterparts since in the case $d = 1$ they are vacuous.

5.1. ***Positive operators.*** We begin with the setting of positive operators.

Theorem 5.1. *Suppose that X is a σ-finite measure space and that $\mathcal{Y}_j$, for $j = 1, \dots, d$, are p_j-convex normed lattices for some $1 \le p_j < \infty$. Suppose that the linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are positive and that each T_j saturates X. Suppose that $\theta_j > 0$ and that $\sum_{j=1}^d \theta_j = 1$. Finally, suppose that for some $0 < q \le \infty$ we have*

$$\Big\| \prod_{j=1}^d (T_j f_j)^{p_j\theta_j} \Big\|_{L^q(\mathrm{d}\mu)} \le B \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} \tag{31}$$

for all nonnegative f_j in $\mathcal{Y}_j$, $1 \le j \le d$.

Case I: (*disentanglement*). $q = 1$. *See Theorem 3.2.*

Case II: (*multilinear duality*) *If $q > 1$, then for every nonnegative $G \in L^{q'}(X)$ there exist nonnegative measurable functions g_j on X such that*

$$G(x) \le \prod_{j=1}^d g_j(x)^{\theta_j}$$

almost everywhere, and such that

$$\Big(\int_X |T_j f_j(x)|^{p_j} g_j(x) \,\mathrm{d}\mu(x) \Big)^{1/p_j} \le B^{1/p_j} C_{p_j}(\mathcal{Y}_j) \|G\|_{q'} \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

Case III: (*multilinear Maurey factorisation*) *If* $0 < q < 1$ *then there exist nonnegative measurable functions* g_j *on* X *such that*

$$\left\| \prod_{j=1}^{d} g_j(x)^{\theta_j} \right\|_{q'} = 1$$

and such that

$$\left(\int_X |T_j f_j(x)|^{p_j} g_j(x)\, \mathrm{d}\mu(x) \right)^{1/p_j} \leq B^{1/p_j} C_{p_j}(\mathcal{Y}_j) \| f_j \|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

Note that Theorem 5.1 in the special case $p_j = 1$ for all j is precisely Theorem 1.1.

Proof. We begin with Case II. Suppose that

$$\left\| \prod_{j=1}^{d} (T_j f_j)^{p_j \theta_j} \right\|_{L^q(X)} \leq B \prod_{j=1}^{d} \| f_j \|_{\mathcal{Y}_j}^{p_j \theta_j}$$

for all nonnegative $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d$. Then, for all nonnegative $G \in L^{q'}(X)$ with $\|G\|_{L^{q'}} = 1$, we have

$$\int_X \prod_{j=1}^{d} (T_j f_j(x))^{p_j \theta_j} G\, \mathrm{d}\mu(x) \leq \left\| \prod_{j=1}^{d} (T_j f_j)^{p_j \theta_j} \right\|_q \leq B \prod_{j=1}^{d} \| f_j \|_{\mathcal{Y}_j}^{p_j \theta_j}.$$

It is easy to see that if T_j saturates X with respect to the measure $\mathrm{d}\mu$, then it also does so with respect to $G\,\mathrm{d}\mu$. Moreover, the measure $G\,\mathrm{d}\mu$ is σ-finite. Therefore, by Theorem 3.2 applied with the measure $G\,\mathrm{d}\mu$ in place of $\mathrm{d}\mu$, there are nonnegative measurable functions γ_j such that

$$1 \leq \prod_{j=1}^{d} \gamma_j(x)^{\theta_j} \quad G\,\mathrm{d}\mu\text{-a.e. on } X,$$

and such that, for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j} \gamma_j(x) G(x)\, \mathrm{d}\mu(x) \right)^{1/p_j} \leq B^{1/p_j} C_{p_j}(\mathcal{Y}_j) \| f_j \|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$. Setting $g_j = \gamma_j G$ gives the desired conclusion.

Now we turn to Case III. The main hypothesis (31) is that

$$\int_X \prod_{j=1}^{d} (T_j f_j)^{p_j \theta_j q}\, \mathrm{d}\mu \leq B^q \prod_{j=1}^{d} \| f_j \|_{\mathcal{Y}_j}^{p_j \theta_j q}$$

for all nonnegative $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d$.

We introduce a new one-dimensional normed lattice $\mathcal{Y}_{d+1}$ with a nonnegative element y of unit norm. Let $T_{d+1} : \mathcal{Y}_{d+1} \to \mathcal{M}(X)$ be given by $\lambda y \to \lambda \mathbf{1}$, where $\mathbf{1}$ denotes the constant function taking the value 1 on X.

Then we have

$$\int_X \prod_{j=1}^{d+1} (T_j f_j)^{p_j \theta_j q}\, \mathrm{d}\mu \leq B^q \prod_{j=1}^{d+1} \| f_j \|_{\mathcal{Y}_j}^{p_j \theta_j q}$$

for all $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d+1$, where the exponents $\theta_{d+1} > 0$ and $p_{d+1} > 0$ are at our disposal.

We want to impose the condition $\theta_{d+1} = 1/q - 1 > 0$ because, with $\tilde{\theta}_j := \theta_j q$, we then have $\sum_{j=1}^{d+1} \tilde{\theta}_j = 1$ and

$$\int_X \prod_{j=1}^{d+1} (T_j f_j)^{p_j \tilde{\theta}_j} \, \mathrm{d}\mu \le B^q \prod_{j=1}^{d+1} \| f_j \|_{\mathcal{Y}_j}^{p_j \tilde{\theta}_j}$$

for all $f_j \in \mathcal{Y}_j$, $1 \le j \le d+1$.

By Theorem 3.2 we therefore have that there exist ψ_j, $1 \le j \le d+1$, such that

$$\prod_{j=1}^{d+1} \psi_j(x)^{\tilde{\theta}_j} = 1$$

almost everywhere, and

$$\left(\int_X |T_j f_j(x)|^{p_j} \psi_j(x) \, \mathrm{d}\mu(x) \right)^{1/p_j} \le B^{q/p_j} C_{p_j}(\mathcal{Y}_j) \| f_j \|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$, $1 \le j \le d+1$.

The case $j = d+1$ of this last inequality tells us that (if we choose $p_{d+1} = 1$)

$$\int_X \psi_{d+1}(x) \, \mathrm{d}\mu(x) \le B^q$$

and, since by the previous equality we have

$$\psi_{d+1}(x) = \prod_{j=1}^{d} \psi_j(x)^{-\tilde{\theta}_j / \tilde{\theta}_{d+1}} = \prod_{j=1}^{d} \psi_j(x)^{-\theta_j / \theta_{d+1}} = \prod_{j=1}^{d} \psi_j(x)^{\theta_j q'},$$

it gives

$$\left\| \prod_{j=1}^{d} \psi_j(x)^{\theta_j} \right\|_{q'} \ge B^{q/q'}.$$

If we now set $g_j = B^{-q/q'} \psi_j$ for $1 \le j \le d$ we obtain

$$\left\| \prod_{j=1}^{d} g_j(x)^{\theta_j} \right\|_{q'} \ge 1$$

and

$$\left(\int_X |T_j f_j(x)|^{p_j} g_j(x) \, \mathrm{d}\mu(x) \right)^{1/p_j} \le B^{1/p_j} C_{p_j}(\mathcal{Y}_j) \| f_j \|_{\mathcal{Y}_j}$$

for all $1 \le j \le d$, and for all $f_j \in \mathcal{Y}_j$. □

5.2. *General linear operators.* Next we turn to general linear operators and state a result which in particular contains Theorem 1.7. The proof follows exactly the same arguments as in Theorem 5.1, with the exception that the application of Theorem 3.2 there is now replaced by that of Theorem 4.3. (We also need for Case III to observe that the one-dimensional normed space $\mathcal{Y}_{d+1}$ which we introduce has Rademacher-type strictly greater than 1 — indeed it has Rademacher-type 2 with constant 1 as we noted earlier.) We leave the remaining details to the reader.

Theorem 5.2. *Let X be a σ-finite measure space and $\mathcal{Y}_j$ quasinormed spaces. Let $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ be linear operators. Suppose that the linear operators T_j saturate X. Let $0 < p_j \le 2$ and $\sum_{j=1}^d \theta_j = 1$. Assume that for some $0 < q \le \infty$ we have*

$$\Big\| \prod_{j=1}^d |T_j f_j|^{p_j \theta_j} \Big\|_{L^q(\mathrm{d}\mu)} \le B \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j \theta_j}$$

for all f_j in $\mathcal{Y}_j$, $1 \le j \le d$.

Suppose moreover that each space $\mathcal{Y}_j$ has Rademacher-type $r_j = 2$ for those j with $p_j = 2$, and has Rademacher-type $r_j > p_j$ for those j with $p_j < 2$.

Case I: (*disentanglement*) $q = 1$. *See Theorem 4.3.*

Case II: (*multilinear duality*) *If $q > 1$, then for every nonnegative $G \in L^{q'}(X)$ there exist nonnegative measurable functions g_j on X such that*

$$G(x) \le \prod_{j=1}^d g_j(x)^{\theta_j}$$

almost everywhere, and such that

$$\left(\int_X |T_j f_j(x)|^{p_j} g_j(x) \, \mathrm{d}\mu(x) \right)^{1/p_j} \lesssim_{\{\theta_j, p_j, r_j\}} B^{1/p_j} R_{r_j}(\mathcal{Y}_j) \|G\|_{q'} \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

Case III: (*multilinear Maurey factorisation*) *If $0 < q < 1$ then there exist nonnegative measurable functions g_j on X such that*

$$\Big\| \prod_{j=1}^d g_j(x)^{\theta_j} \Big\|_{q'} = 1$$

and such that

$$\left(\int_X |T_j f_j(x)|^{p_j} g_j(x) \, \mathrm{d}\mu(x) \right)^{1/p_j} \lesssim_{\{\theta_j, p_j, r_j\}} B^{1/p_j} R_{r_j}(\mathcal{Y}_j) \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

There are further extensions to Case II in both Theorems 5.1 and 5.2 when we replace the role of L^q for $q > 1$ by Köthe function spaces as in [Carbery et al. 2022]. We leave the details to the interested reader.

Appendix: Why certain conditions are needed

At various points in the development of our results we have imposed conditions whose necessity might not be immediately obvious. For example, in the Basic Question we imposed the homogeneity condition (7), in Theorems 1.3 and 1.5 we imposed upper bounds on the exponents p_j, and in Theorem 3.2 we imposed p_j-convexity on the lattices $\mathcal{Y}_j$. In this final section we establish that, in all these cases, the conditions we impose are indeed needed in order for our results to have a sufficiently broad scope so as to include certain natural examples.

A.1. *Condition (7) in the Basic Question.* We first want to clarify to what extent condition (7) is needed in the formulation of the Basic Question.

Proposition A.1. *Fix $r_j \geq 1$ and $\gamma_j > 0$ for $1 \leq j \leq d$. Suppose that (p_j) is such that whenever $T_j : L^{r_j}(\mathbb{R}) \to \mathcal{M}(\mathbb{R}^d)$ are positive linear operators such that*

$$\int_{\mathbb{R}^d} \prod_{j=1}^{d} |T_j f_j(x)|^{\gamma_j} \, \mathrm{d}x \lesssim \prod_{j=1}^{d} \|f_j\|_{L^{r_j}(\mathbb{R})}^{\gamma_j} \tag{32}$$

holds, then there exists (ϕ_j) such that

$$\prod_{j=1}^{d} \phi_j(x)^{\gamma_j/p_j} \geq 1 \tag{33}$$

and

$$\left(\int_{\mathbb{R}^d} |T_j f_j(x)|^{p_j} \phi_j(x) \, \mathrm{d}x \right)^{1/p_j} \lesssim \|f_j\|_{L^{r_j}(\mathbb{R})} \tag{34}$$

hold. Then (p_j) must necessarily satisfy

$$\sum_{j=1}^{d} \frac{\gamma_j}{p_j} = 1.$$

Proof. Let $\Phi_j \in L^{\gamma_j}(\mathbb{R}) \setminus \bigcup_{\beta_j \neq \gamma_j} L^{\beta_j}(\mathbb{R})$ and $g_j \in L^{r_j'}(\mathbb{R})$ be nonzero and strictly positive. Let $T_j : L^{r_j}(\mathbb{R}) \to L^{\gamma_j}(\mathbb{R})$ be given by

$$T_j f(s) = \left(\int_{\mathbb{R}} f g_j \right) \Phi_j(s).$$

Extend T_j to $T_j : L^{r_j}(\mathbb{R}) \to \mathcal{M}(\mathbb{R}^d)$ by defining

$$(T_j f)(x_1, \ldots, x_d) := T_j f(x_j).$$

Then (32) holds with exponents (γ_j), but if we replace any γ_j by any other exponent, its left-hand side becomes infinite for all nontrivial nonnegative $f_j \in L^{r_j}(\mathbb{R})$.

By hypothesis, (p_j) is such that there exists (ϕ_j) satisfying (33) and (34) for this particular (T_j). Let $\lambda = \sum_{j=1}^{d} \gamma_j/p_j$. Then (33) gives

$$\prod_{j=1}^{d} \phi_j(x)^{\gamma_j/\lambda p_j} \geq 1,$$

and so by Lemma 1.2 we can conclude that

$$\int \prod_{j=1}^{d} |T_j f_j(x)|^{\gamma_j/\lambda} \, \mathrm{d}\mu(x) \lesssim \prod_{j=1}^{d} \|f_j\|_{L^{r_j}}^{\gamma_j/\lambda};$$

that is, (32) holds also with exponents (γ_j/λ) in place of (γ_j) for this (T_j). This is a contradiction to what we observed above, unless $\lambda = 1$. $\square$

A.2. ***Sharpness of the exponents in Theorems 1.3 and 1.5.*** As a preliminary observation, we note that the next two lemmas can be used to demonstrate the sharpness of the exponents arising in the classical Maurey–Nikishin–Stein theory of factorisation of linear operators.

Lemma A.2. *For each $1 \le r \le \infty$ and $0 < \gamma < \infty$ we can construct a positive translation-invariant bounded linear operator $T : L^r(\mathbb{G}) \to L^\gamma(\mathbb{G})$ (where $\mathbb{G} = \mathbb{T}$ or $\mathbb{R}$ with Haar measure) such that*

$$\{0 < p < \infty : \textit{for some nontrivial } \phi,\ T : L^r \to L^p(\phi) \textit{ boundedly}\} = I_{r,\gamma} := (0, \max\{\gamma, r\}].$$

This is well known. When $\gamma \le r$, we take $T = I$, and when $\gamma > r$, we take T to be a fractional integral operator (or slight variant thereof when $r = 1$).

We next consider general operators.

Lemma A.3. *For each $1 \le r < \infty$ and $0 < \gamma < \infty$ we can construct a translation-invariant bounded linear operator $T : L^r(\mathbb{G}) \to L^\gamma(\mathbb{G})$ (where $\mathbb{G} = \mathbb{T}$ or $\mathbb{R}$ with Haar measure) such that*

$$\{0 < p < \infty : \textit{for some nontrivial } \phi,\ T : L^r \to L^p(\phi) \textit{ boundedly}\} \\ = J_{r,\gamma} := \begin{cases} (0,\gamma] & \textit{when } 2 \le \gamma < r \textit{ or } \gamma \ge r, \\ (0,2] & \textit{when } \gamma < 2 \le r, \\ (0,r) & \textit{when } \gamma < r < 2. \end{cases}$$

This is also mostly well known. The exponents $\gamma \ge r$ are covered by Lemma A.2 (in which case we can take $\mathbb{G} = \mathbb{T}$ or $\mathbb{R}$ with Haar measure), so it remains to consider the exponents $\gamma < r$ (in which case we shall take $\mathbb{G} = \mathbb{T}$). Note that, by an averaging argument, for a translation-invariant operator on a compact abelian group, $T : L^r \to L^p(\phi)$ boundedly for a nontrivial weight ϕ if and only if $T : L^r \to L^p(\phi)$ boundedly for the weight $\phi = 1$. Thus,

$$\{0 < p < \infty : \text{for some nontrivial } \phi,\ T : L^r(\mathbb{T}) \to L^p(\mathbb{T}, \phi) \text{ boundedly}\} \\ = \{0 < p < \infty : T : L^r(\mathbb{T}) \to L^p(\mathbb{T}) \text{ boundedly}\}.$$

When $r > 2$ we shall also need the following result to assist us in establishing Lemma A.3:

Lemma A.4. *Let $2 \le \gamma < \infty$. Then there is a bounded translation-invariant linear operator $T : L^2(\mathbb{T}) \to L^\gamma(\mathbb{T})$ such that for no $p > \gamma$ is T bounded from $L^\infty(\mathbb{T})$ to $L^p(\mathbb{T})$.*

For the case $\gamma = 2$ of Lemma A.4, an argument based on Rademacher functions can be found in [García-Cuerva and Rubio de Francia 1985, Chapter VI, Example 2.10(e)]. The case $\gamma > 2$ follows readily from Bourgain's solution [1989] of the $\Lambda(p)$-set problem. This result states that for each $2 < \gamma < \infty$ there is a set $E \subseteq \mathbb{Z}$ which is a $\Lambda(\gamma)$-set, but which is not a $\Lambda(\tilde{p})$-set for any $\tilde{p} > \gamma$. If T is the Fourier multiplier operator with multiplier χ_E, then T is bounded from $L^2(\mathbb{T})$ to $L^\gamma(\mathbb{T})$ (since E is a $\Lambda(\gamma)$-set) but unbounded from $L^\infty(\mathbb{T})$ to $L^p(\mathbb{T})$ for every $p > \gamma$ (since if $T : L^\infty \to L^p$ boundedly for some $p > \gamma$, then interpolating between this bound and the bound $T : L^2 \to L^\gamma$ with $\gamma > 2$ gives the bound $T : L^q \to L^{\tilde{p}}$ for some $q < \tilde{p}$ and $\tilde{p} > \gamma$, which would imply that E is a $\Lambda(\tilde{p})$-set, a contradiction). (We thank an anonymous referee for pointing out this connection to us.) Bourgain's argument gives the stronger conclusion that the operator T can also be chosen to satisfy $T^2 = T$. On the other hand, his argument is not constructive, and so we give a simple constructive proof of Lemma A.4 — which is perhaps of independent interest — in Section A.4 below.

We return to the detailed discussion of Lemma A.3.

- When $2 \le \gamma < r$ we appeal to Lemma A.4, and we take T to be a translation-invariant bounded linear operator $T : L^2 \to L^\gamma$ (and hence $T : L^r \to L^\gamma$) that is not bounded from L^∞ to L^p for any $p > \gamma$.
- When $\gamma < 2 < r$ we appeal to Lemma A.4, and we take T to be a translation-invariant bounded linear operator $T : L^2 \to L^2$ (and hence $T : L^r \to L^\gamma$) that is not bounded from L^∞ to L^p for any $p > 2$.
- When $\gamma < r$ and $r = 2$ we take T to be the identity operator.
- When $\gamma < r < 2$ we appeal to a theorem of [Zafran 1975] which states that for each $r < 2$ there is a translation-invariant bounded linear operator $T : L^r(\mathbb{T}) \to L^{r,\infty}(\mathbb{T})$ (and thus $T : L^r(\mathbb{T}) \to L^\gamma(\mathbb{T})$ for all $\gamma < r$) such that T is not bounded on L^r.

By taking tensor products we obtain corresponding multilinear examples. Indeed, by choosing operators $T_j : L^{r_j}(\mathbb{G}_j) \to L^{\gamma_j}(\mathbb{G}_j)$ as in Lemmas A.2 and A.3, and letting the measure space $(X, \mathrm{d}\mu)$ be the product $X = \mathbb{G}_1 \times \cdots \times \mathbb{G}_d$, with $\mathrm{d}\mu$ as product measure, we obtain:

Proposition A.5. *For each $1 \le r_j \le \infty$ and $0 < \gamma_j < \infty$ there is a σ-finite measure space X and there are positive linear operators $T_j : L^{r_j}(\mathbb{G}_j) \to \mathcal{M}(X)$ such that*

$$\int_X \prod_{j=1}^d |T_j f_j|^{\gamma_j} \lesssim \prod_{j=1}^d \|f_j\|_{r_j}^{\gamma_j}$$

and such that

$$\{(p_j) \in (0,\infty)^d : \textit{for each } j,\ T_j : L^{r_j} \to L^{p_j}(\phi_j) \textit{ boundedly for some nontrivial } \phi_j\} = \prod_{j=1}^d I_{r_j,\gamma_j} = \prod_{j=1}^d (0, \max\{\gamma_j, r_j\}].$$

Proposition A.6. *For each $1 \le r_j < \infty$ and $0 < \gamma_j < \infty$ there is a σ-finite measure space X and there are linear operators $T_j : L^{r_j}(\mathbb{G}_j) \to \mathcal{M}(X)$ such that*

$$\int_X \prod_{j=1}^d |T_j f_j|^{\gamma_j} \lesssim \prod_{j=1}^d \|f_j\|_{r_j}^{\gamma_j}$$

and such that

$$\{(p_j) \in (0,\infty)^d : \textit{for each } j,\ T_j : L^{r_j} \to L^{p_j}(\phi_j) \textit{ boundedly for some nontrivial } \phi_j\} = \prod_{j=1}^d J_{r_j,\gamma_j}.$$

As immediate corollaries we have:

Corollary A.7. *For each $1 \le r_j \le \infty$ and $0 < \gamma_j < \infty$ there is a σ-finite measure space X and there are positive linear operators $T_j : L^{r_j}(\mathbb{G}_j) \to \mathcal{M}(X)$ such that*

$$\int_X \prod_{j=1}^d |T_j f_j|^{\gamma_j} \lesssim \prod_{j=1}^d \| f_j \|_{r_j}^{\gamma_j}$$

and such that

$$\left\{ (p_j) \in (0, \infty)^d : \sum_{j=1}^d \frac{\gamma_j}{p_j} = 1 \text{ and, for each } j,\ T_j : L^{r_j} \to L^{p_j}(\phi_j) \text{ boundedly for some nontrivial } \phi_j \right\}$$

is nonempty if and only if $\sum_{j=1}^d \gamma_j / r_j \le 1$, and, when this condition holds, equals

$$\left(\prod_{j=1}^d (0, r_j] \right) \cap \left\{ (p_j) \in (0, \infty)^d : \sum_{j=1}^d \frac{\gamma_j}{p_j} = 1 \right\}.$$

Corollary A.8. *For each $1 \le r_j < \infty$ and $0 < \gamma_j < \infty$ there is a σ-finite measure space X and there are linear operators $T_j : L^{r_j}(\mathbb{G}_j) \to \mathcal{M}(X)$ such that*

$$\int_X \prod_{j=1}^d |T_j f_j|^{\gamma_j} \lesssim \prod_{j=1}^d \| f_j \|_{r_j}^{\gamma_j}$$

and such that

$$\left\{ (p_j) \in (0, \infty)^d : \sum_{j=1}^d \frac{\gamma_j}{p_j} = 1 \text{ and, for each } j,\ T_j : L^{r_j} \to L^{p_j}(\phi_j) \text{ boundedly for some nontrivial } \phi_j \right\}$$
$$= \left(\prod_{j=1}^d J_{r_j, \gamma_j} \right) \cap \left\{ (p_j) \in (0, \infty)^d : \sum_{j=1}^d \frac{\gamma_j}{p_j} = 1 \right\}.$$

This set is nonempty if and only if we have $\sum_{j=1}^d \gamma_j / \min\{r_j, 2\} < 1$ when at least one $r_j < 2$, and $\sum_{j=1}^d \gamma_j \le 2$ when all $r_j \ge 2$. When nonempty, this set equals

$$\left(\prod_{j : r_j < 2} (0, r_j) \times \prod_{j : r_j \ge 2} (0, 2] \right) \cap \left\{ (p_j) \in (0, \infty)^d : \sum_{j=1}^d \frac{\gamma_j}{p_j} = 1 \right\}.$$

These two corollaries establish the assertions concerning sharpness of Theorems 1.3 and 1.5 which we made in the Introduction.

A.3. ***Disentanglement implies p-convexity.*** Here we show that the hypotheses of p-convexity are intrinsic to Theorem 3.2, since p-convexity follows from the conclusion of that result, at least in the case when the spaces $\mathcal{Y}_j$ are Köthe spaces whose duals are norming. This class includes Lorentz spaces and Orlicz spaces.

We therefore assume in what follows that each $\mathcal{Y}_j$ is a Köthe function lattice over the σ-finite measure space $(Y_j, \mathrm{d}\nu_j)$, and that we can realise the norm of any $f \in \mathcal{Y}_j$ as

$$\| f \|_{\mathcal{Y}_j} = \sup_{\|g\|_{\mathcal{Y}_j'} \le 1} \left| \int_{Y_j} f g \, \mathrm{d}\nu_j \right|.$$

We remark that a Köthe dual $\mathcal{Y}'$ is norming if and only if the pointwise convergence $f_n \uparrow f$ implies the norm convergence $\|f_n\|_{\mathcal{Y}} \to \|f\|_{\mathcal{Y}}$ for all pointwise increasing sequences (f_n) (though we shall not need this characterisation here).

Proposition A.9. *Fix $\mathcal{Y}_j$ as above, and fix $1 < p_j < \infty$ for $1 \le j \le d$. Assume that there exists a constant $C_{\{\mathcal{Y}_j\}}$ such that for all weights (θ_j) with $\theta_j > 0$ and $\sum_{j=1}^d \theta_j = 1$, all σ-finite measure spaces $(X, \mathrm{d}\mu)$, and all saturating positive linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ the estimate*

$$\int_X \prod_{j=1}^d |T_j f_j(x)|^{p_j\theta_j}\,\mathrm{d}\mu(x) \le A \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} \quad \textit{for all } f_j \in \mathcal{Y}_j$$

implies the existence of functions ϕ_j such that $\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1$ and such that

$$\left(\int_X |T_j f_j|^{p_j}\phi_j\,\mathrm{d}\mu\right)^{1/p_j} \le C_{\{\mathcal{Y}_j\}} A^{1/p_j} \|f_j\|_{\mathcal{Y}_j}.$$

Then each space $\mathcal{Y}_j$ is p_j-convex.

Proof. Fix j. Let $g_j \in \mathcal{Y}_j'$ be of unit norm. Let $(X, \mathrm{d}\mu) := (Y_j, |g_j|\mathrm{d}\nu_j)$. We define $T_j := I_{\mathcal{Y}_j \to \mathcal{Y}_j}$.

For each $i \ne j$, we choose a nonnegative function F_i on Y_i such that $\|F_i\|_{\mathcal{Y}_i} = 1$. Since $\mathcal{Y}_i'$ is assumed to be norming, for each $\epsilon > 0$ we can choose a nonnegative function G_i on Y_i with $\|G_i\|_{\mathcal{Y}_i'} = 1$ such that $\int_{Y_i} F_i G_i\,\mathrm{d}\nu_i \ge (1-\epsilon)\|F_i\|_{\mathcal{Y}_i} = (1-\epsilon)$. We define $T_i : \mathcal{Y}_i \to \mathcal{M}(X)$ by

$$T_i f(x) = \int_{Y_i} f G_i\,\mathrm{d}\nu_i,$$

so that each $T_i f$ is a constant function on X. Note that $|T_i f_i(x)| \le \|f_i\|_{\mathcal{Y}_i}$ for all $f_i \in \mathcal{Y}_i$ and that $|T_i F_i(x)| \ge (1-\epsilon)$ for all $x \in X$.

Let $\theta_j := 1/p_j \in (0,1)$, and choose the remaining $\theta_i \in (0,1)$ in such a way that $\sum_{i=1}^d \theta_i = 1$.

With these choices, we have

$$\begin{aligned}\int_X \prod_{i=1}^d |T_i f_i(x)|^{p_i\theta_i}\,\mathrm{d}\mu(x) &\le \int_{Y_j} |f_j||g_j|\,\mathrm{d}\mu_j \prod_{i\ne j} \|f_i\|_{\mathcal{Y}_i}^{p_i\theta_i}\\ &\le \|g_j\|_{\mathcal{Y}_j'}\|f_j\|_{\mathcal{Y}_j} \prod_{i\ne j} \|f_i\|_{\mathcal{Y}_i}^{p_i\theta_i} = \prod_{i=1}^d \|f_i\|_{\mathcal{Y}_i}^{p_i\theta_i}.\end{aligned}$$

By assumption, there are (ϕ_i) such that $\prod_{i=1}^d \phi_i(x)^{\theta_i} \ge 1$ and such that, for each i,

$$\left(\int_X |T_i f_i|^{p_i}\phi_i\,\mathrm{d}\mu\right)^{1/p_i} \le C_{\{\mathcal{Y}_j\}}\|f_i\|_{\mathcal{Y}_i}.$$

Hence, by the equivalence set out in Section 2.3, we have the vector-valued inequality

$$\int_X \prod_{i=1}^d \left(\sum_{k=1}^N |T_i f_{i,k}|^{p_i}\right)^{\theta_i} \mathrm{d}\mu \le C_{\{\mathcal{Y}_j\}} \prod_{i=1}^d \left(\sum_{k=1}^N \|f_{i,k}\|_{\mathcal{Y}_i}^{p_i}\right)^{\theta_i}$$

for the same constant $C_{\{\mathcal{Y}_j\}}$.

For $i \neq j$, set $f_{i,k} = F_i$ for $k = 1$ and $f_{i,k} = 0$ for $k = 2, \ldots, N$. We obtain

$$\int_{Y_j} \left(\sum_{k=1}^{N} |f_{j,k}|^{p_j} \right)^{1/p_j} |g_j| \, \mathrm{d}\nu_j \leq C_{\{\mathcal{Y}_j\}} \frac{1}{(1-\epsilon)^{d-1}} \left(\sum_{k=1}^{N} \| f_{j,k} \|_{\mathcal{Y}_j}^{p_j} \right)^{1/p_j}.$$

By assumption, the Köthe dual $\mathcal{Y}_j'$ is norming, and hence taking supremum over g_j in the unit ball of $\mathcal{Y}_j'$ and letting $\epsilon \to 0$ yields

$$\left\| \left(\sum_{k=1}^{N} |f_{j,k}|^{p_j} \right)^{1/p_j} \right\|_{\mathcal{Y}_j} \leq C_{\{\mathcal{Y}_j\}} \left(\sum_{k=1}^{N} \| f_{j,k} \|_{\mathcal{Y}_j}^{p_j} \right)^{1/p_j}.$$

This is the defining inequality of p_j-convexity. □

A.4. ***Constructive proof of Lemma A.4.*** Finally, we turn to our constructive proof of Lemma A.4, which represents a slight strengthening (in the particular case when the underlying group is $\mathbb{T}$) of a result found in [Figà-Talamanca and Price 1973, Theorem 4.4];[7] see the references therein for a full history.

We recall (see for example [Katznelson 2004, p. 33]) the sequence of Rudin–Shapiro polynomials P_m on $\mathbb{T}$. There is a (deterministic) sequence $a_n \in \{\pm 1\}$ such that the sequence of trigonometric polynomials defined for $m \geq 0$ by

$$P_m(x) := \sum_{n=0}^{2^m - 1} a_n e^{2\pi i n x}$$

has the following properties (of which the first and the last are trivial and the second is the interesting one):

- $\| P_m \|_2 = 2^{m/2}$.
- $\| P_m \|_\infty \leq 2^{(m+1)/2}$.
- $2^{(m-1)/2} \leq \| P_m \|_q \leq 2^{(m+1)/2}$ for $1 \leq q \leq \infty$.
- $\| \widehat{P_m} \|_\infty = 1$.

For the third item, the upper bounds are clear from the second item; for the lower bounds it suffices by Hölder's inequality to show that $\| P_m \|_1 \geq 2^{(m-1)/2}$, and this follows from the first two items together with $\| P_m \|_2 \leq \| P_m \|_1^{1/2} \| P_m \|_\infty^{1/2}$.

From the first and fourth of these we deduce by Young's inequality and interpolation that, for $1 \leq r \leq 2$,

$$\| P_m * f \|_2 \leq 2^{m(1/r - 1/2)} \| f \|_r.$$

Let $F_m(x) = \sum_{n=0}^{2^m-1} e^{2\pi i n x}$ so that $\| F_m \|_p \lesssim 2^{m/p'}$ for $1 < p \leq \infty$ and $\| F_m \|_1 \lesssim m$.

Observe that $P_m * F_m = P_m$, so that $\| P_m * F_m \|_q = \| P_m \|_q \gtrsim 2^{m/2}$ for all $1 \leq q \leq \infty$. Let T_m denote convolution with P_m. Using these bounds we can easily see that $\| T_m \|_{L^p \to L^q} \lesssim \| T_m \|_{L^r \to L^2}$ only when $p \geq r$. Indeed, from the upper bounds on $\| F_m \|_p$ we deduce that, for all $1 \leq p, q \leq \infty$, $\| T_m \|_{L^p \to L^q}$ is bounded below by $2^{m(1/2 - 1/p')}$ when $p > 1$ and $m^{-1} 2^{m/2}$ when $p = 1$.

[7] The examples in [Figà-Talamanca and Price 1973] depend in principle also on the exponent p, whereas ours is p-independent.

We now build an explicit example. We first note that $\widetilde{P}_m := e^{2\pi i 2^m x} P_m(x)$ has frequencies in $[2^m, 2^{m+1})$, and similarly for $\widetilde{F}_m(x) := e^{2\pi i 2^m x} F_m(x)$. Performing this modulation does not change any of the estimates on P_m and F_m which we had above, and we have $\widetilde{P}_m * \widetilde{F}_m = \widetilde{P}_m$ and $\widetilde{P}_m * \widetilde{F}_{m'} = 0$ for $m \neq m'$.

Fix an r with $1 \le r \le 2$. Let T (depending on r) be given by convolution with

$$\sum_{m=1}^{\infty} m^{-2} 2^{m/2} 2^{-m/r} \widetilde{P}_m;$$

by the bounds for P_m derived above we see that T is bounded from L^r to L^2.

Fix $p \ge 1$ and let $f_m = m^{-3} 2^{-m/p'} \widetilde{F}_m$ so that

$$\|f_m\|_p \le m^{-3} 2^{-m/p'} \|\widetilde{F}_m\|_p \lesssim 1$$

uniformly in $m \ge 1$.

Moreover, we have

$$T f_m = m^{-5} 2^{m/2} 2^{-m/r} 2^{-m/p'} \widetilde{P}_m * \widetilde{F}_m$$

since $\widetilde{P}_m * \widetilde{F}_{m'} = 0$ for $m \neq m'$. Therefore,

$$\|T f_m\|_1 = m^{-5} 2^{m/2} 2^{-m/r} 2^{-m/p'} \|\widetilde{P}_m * \widetilde{F}_m\|_1 \sim m^{-5} 2^{-m/r} 2^{m/p}$$

for each $m \ge 1$.

Consequently,

$$\|T\|_{L^p \to L^1} \gtrsim \sup_m \|T f_m\|_1 = \infty$$

when $p < r$.

Thus, for each $1 < r \le 2$, we have built an example of an $L^r \to L^2$-bounded translation-invariant operator T on $\mathbb{T}$ such that, for every $1 \le p < r$, we have $\|T\|_{L^p \to L^1} = \infty$.

By duality, for each $2 \le r < \infty$, we have an explicit example of an $L^2 \to L^r$-bounded translation-invariant operator T on $\mathbb{T}$ such that if $q > r$, we have $\|T\|_{L^\infty \to L^q} = \infty$. This establishes the constructive version of Lemma A.4.

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ANTHONY CARBERY: a.carbery@ed.ac.uk
School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Edinburgh, United Kingdom

TIMO S. HÄNNINEN: timo.s.hanninen@helsinki.fi
Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland

and

School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Edinburgh, United Kingdom

STEFÁN INGI VALDIMARSSON: sivaldimarsson@gmail.com
Arion banki, Reykjavík, Iceland

and

Science Institute, Mathematics Division, University of Iceland, Reykjavík, Iceland

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Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de
Vadim Kaloshin	University of Maryland, USA vadim.kaloshin@gmail.com
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca
Anna L. Mazzucato	Penn State University, USA alm24@psu.edu
Richard B. Melrose	Massachussets Inst. of Tech., USA rbm@math.mit.edu
Frank Merle	Université de Cergy-Pontoise, France merle@ihes.fr
William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu
Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de
Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu
Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
András Vasy	Stanford University, USA andras@math.stanford.edu
Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
Jim Wright	University of Edinburgh, UK j.r.wright@ed.ac.uk
Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu

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