Discrete Comput Geom 4:287-290 (1989)

Discrete & Computational Geometry

Disjoint Edges in Geometric Graphs

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Abstract. Answering an old question in combinatorial geometry, we show that any configuration consisting of a set V of n points in general position in the plane and a set of $6n-5$ closed straight line segments whose endpoints lie in V, contains three pairwise disjoint line segments.

A *geometric graph* is a pair $G=(V, E)$, where V is a set of points ($=$*vertices*) in general position in the plane, i.e., no three on a line, and E is a set of distinct, closed, straight line segments, called *edges*, whose endpoints lie in V. An old theorem of the second author [Er] (see also [Ku] for another proof), states that any geometric graph with n points and $n+1$ edges contains two disjoint edges, and this is best possible for every $n \geq 3$. For $k \geq 2$, let $f(k\ n)$ denote the maximum number of edges of a geometric graph on n vertices that contains no k pairwise disjoint edges. Thus, the result stated above is simply the fact $f(2, n)=n$ for all $n \geq 3$. Kupitz [Ku] and Perles [Pe] (see also [AA]) raised the problem of determining or estimating $f(k\ n)$ for $k \geq 3$. In particular, they asked if $f(3, n) \leq O(n)$. This specific problem, of determining or estimating $f(3, n)$, was already mentioned in 1966 by Avital and Hanani [AH], and it seems it was a known problem even before that. In this note we answer this question by proving the following.

Theorem 1. *For every $n \geq 1, f(3, n) < 6n-5$, i.e., any geometric graph with n vertices and $6n-5$ edges contains three pairwise disjoint edges.*

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* Research supported in part by an Allon Fellowship and by a Bat Sheva de-Rothschild grant.

Before proving this theorem we note that clearly

$$f(3, n) = \binom{n}{2} \quad \text{for } n \le 5$$

and the best-known lower bound for $n \ge 6$, given by Perles [Pe], is

$$f(3, n) \ge \begin{cases} \frac{5}{2}n - \frac{5}{2} & \text{for odd } n \ge 5, \\ \frac{5}{2}n - 4 & \text{for even } n \ge 2. \end{cases} \tag{1}$$

To prove inequality (1) for odd n consider the geometric graph G_n whose n vertices are the $n-1$ points $v_j = (\cos(2\pi j/(n-1)), \sin(2\pi j/(n-1))), 0 \le j < n-1$, together with the additional point $u = (\varepsilon, \delta)$ where ε and δ are small numbers chosen so that $\{v_0, \ldots, v_{n-2}, u\}$ is in general position. The edges of G_n are the $\frac{5}{2}(n-1)$ line segments

$$\{[u, v_j]: 0 \le j < n-1\}$$
$$\cup \{[v_j, v_{j+(n-3)/2}], [v_j, v_{j+(n-1)/2}], [v_j, v_{j+(n+1)/2}]: 0 \le j < n-1\},$$

where all indices are reduced modulo $n-1$. We can easily check that if ε and δ are sufficiently small then G_n contains no three pairwise disjoint edges. Thus $f(3, n) \ge \frac{5}{2}n - \frac{5}{2}$ for every odd $n \ge 5$. For even n, let G_n be the geometric graph obtained from G_{n+1} by deleting one of its vertices of degree 4. Then G_n has $\frac{5}{2}n - 4$ edges and contains no three pairwise disjoint edges. This establishes (1). On the other hand, Perles [Pe] showed that every geometric graph whose n vertices are the vertices of a convex n-gon in the plane, with more than $(k-1)n$ edges, contains k pairwise disjoint edges. In particular, in the convex case $2n+1$ edges guarantee three pairwise disjoint edges. Comparing this with (1) we conclude that the convex case differs from the general one.

Our final remark before the proof of Theorem 1 is that a special case of one of the results in [AA] implies that, for every $k = o(\log n), f(k, n) = o(n^2)$. It is very likely that, for every fixed $k, f(k, n) = O(n)$, and that, for every $k = o(n)$, $f(k, n) = o(n^2)$, but this remains open.

Proof of Theorem 1. Let G be a geometric graph with n vertices and $6n-5$ edges. We must show that G contains three pairwise disjoint edges. It is first convenient to apply an affine transformation on the plane, in order to make all the edges of G almost parallel to the x-axis. This is done by first choosing the x-axis so that any two distinct points of G have different x-coordinates, and then, by rescaling the y-coordinates so that the difference between the x-coordinates of any two distinct points of G is at least 1000 times bigger than the difference between their y-coordinates. Since any affine transformation maps disjoint segments into disjoint segments we may apply the above transformations, and hence may assume that G satisfies the following:

The small angle between any edge of G and the x-axis is less than $\pi/200$. (2)

We now define the clockwise derivative and the counterclockwise derivative of an arbitrary geometric graph. Let $H=(V, E)$ be a geometric graph and let $e=[u, v]$ be an edge of H. We say that e is *clockwise good at* u if there is another edge $e'=[u, v']$ of H such that the directed line $\overrightarrow{uv'}$ is obtained from $\overrightarrow{uv}$ by rotating it clockwise around u by an angle smaller than $\pi/100$. If e is not clockwise good at u, we say that it is *clockwise bad at* u. The edge $e=[u, v]$ is *clockwise good* if it is clockwise good at both u and v. The *clockwise derivative* of H, denoted by ∂H, is the geometric graph whose set of vertices is the set of all vertices of H, and whose set of edges consists of all clockwise good edges of H. The notions of an edge $e=[u, v]$ which is *counterclockwise good at* u and that of an edge which is *counterclockwise good* are defined analogously. The *counterclockwise derivative* of H, denoted by $H\partial$, is also defined in an analogous manner.

Claim 1. *Let $G=(V, E)$ be a geometric graph with $n \geq 2$ vertices and m edges satisfying* (2). *Then the number of edges of ∂G is at least $m-(2n-2)$. Similarly, the number of edges of $G\partial$ is at least $m-(2n-2)$.*

Proof. We prove the assertion for ∂G. The proof for $G\partial$ is analogous. Let $v \in V$ be an arbitrary vertex of G. We claim that the number of edges of the form $[v, u]$ of G which are clockwise bad at v does not exceed 2. Indeed, assume this is false and let $[v, u_1], [v, u_2], [v, u_3]$ be three such edges. Without loss of generality, assume that the x-coordinates of u_1 and u_2 lie in the same side of the x-coordinate of v. By (2), the angle between $[v, u_1]$ and $[v, u_2]$ is smaller than $\pi/100$, and hence at least one of these two edges is clockwise good at v. This contradiction shows that indeed at most two edges of the form $[v, u]$ are clockwise bad at v. The same argument shows that if u is a vertex of G whose x-coordinate is maximum or minimum, then there is at most one edge incident with u which is clockwise bad at u. Altogether, the total number of clockwise bad edges is bounded by $2+2\cdot(n-2)=2n-2$, completing the proof of Claim 1. □

Returning to our graph G with n edges and $6n-5$ edges, which satisfies (2), define $G_1=G\partial$, $G_2=\partial G_1$, $G_3=G_2\partial$. Clearly, all the graphs G_1, G_2, and G_3 satisfy (2) and hence, by applying Claim 1 three times, we conclude that the number of edges of G_3 is at least $6n-5-3(2n-2)=1$. Let $e=[u_1, u_2]$ be an edge of G_3. Since $G_3=G_2\partial, [u_1, u_2]$ is a counterclockwise good edge of G_2. Consequently, there is an edge $[u_1, v_1]$ of G_2 such that the directed line $\overrightarrow{u_1v_1}$ is obtained from $\overrightarrow{u_1u_2}$ by rotating it counterclockwise around u_1 by an angle smaller than $\pi/100$ (see Fig. 1). Similarly, there is an edge $[u_2, v_2]$ of G_2 with $\measuredangle u_1u_2v_2 < \pi/100$, as in Fig. 1. Since $G_2=\partial G_1$ there are edges $[v_1, w_1]$ and $[v_2, w_2]$ of G_1 with $\measuredangle u_1v_1w_1 < \pi/100$ and $\measuredangle u_2v_2w_2 < \pi/100$, as in Fig. 1. (It is worth noting that it may be, for example, that $[v_1, w_1]$ intersects both $[v_2, u_2]$ and $[v_2, w_2]$, or even that $w_1=v_2$.) Finally, as $G_1=G\partial$ there are edges $[w_1, x_1]$ and $[w_2, x_2]$ of G, with $\measuredangle v_1w_1x_1 < \pi/100$ and $\measuredangle v_2w_2x_2 < \pi/100$, as in Fig. 1. All seven edges $[x_2, w_2]$, $[w_2, v_2], [v_2, u_2], [u_2, u_1], [u_1, v_1], [v_1, w_1]$, and $[w_1, x_1]$, depicted in Fig. 1, belong to G. To complete the proof we show that they must contain three pairwise disjoint edges. Without loss of generality we may assume that $\measuredangle u_2u_1v_1 \geq \measuredangle u_1u_2v_2$.

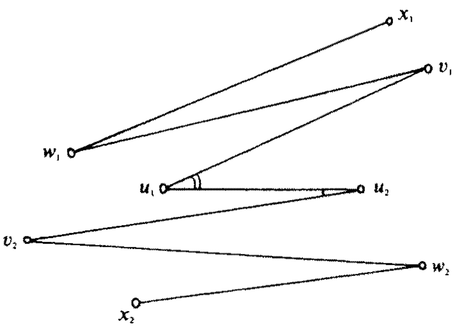

Fig. 1

If the length $l[v_2, u_2]$ of the segment $[v_2, u_2]$ satisfies $l[v_2, u_2] \geq l[u_1, u_2]$ (as is the case in Fig. 1), then we can easily check that $[x_2, w_2]$, $[v_2, u_2]$, and $[u_1, v_1]$ are three pairwise disjoint edges. Otherwise, $l[v_2, u_2] < l[u_1, u_2]$ and then it is easy to check that $[v_2, w_2]$, $[u_1, u_2]$, and $[w_1, v_1]$ are three pairwise disjoint edges. Therefore, in any case, G contains three pairwise disjoint edges, completing the proof of Theorem 1. □

Acknowledgment

We would like to thank Y. Kupitz, I. Krasikov, and M. A. Perles for helpful discussions.

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Received May 19, 1988.