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DISJOINT INVARIANT SUBSPACES

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Let $H^2_{\mathscr{H}}$ denote the (separable) Hilbert space of all functions $F(e^{i\theta})$ defined on the unit circle with values in the separable (usually infinite dimensional) Hilbert space $\mathscr{H}$, and which are weakly in the Hardy class H^2. For a closed subspace of $H^2_{\mathscr{H}}$ "invariant" means invariant under the right shift operator. Such an invariant subspace is said to be of full range if it is of the form $\mathscr{U}H^2_{\mathscr{H}}$, where $\mathscr{U}(e^{i\theta})$ is a.e. a unitary operator on $\mathscr{H}$; i.e., an inner function. We show that if $\mathscr{H}$ is infinite dimensional there exists an uncountable family $\{\mathscr{M}_\alpha\}$ of invariant subspaces of $H^2_{\mathscr{H}}$ of full range such that $\mathscr{M}_\alpha \cap \mathscr{M}_\beta = (0)$ if $\alpha \neq \beta$.

This extends a theorem in the author's paper [2, p. 169] asserting the existence of *two* invariant subspaces $\mathscr{M}, \mathscr{N}$ of full range such that $\mathscr{M} \cap \mathscr{N} = (0)$. For basic definitions and notations consult [1], particularly Chapter VI.

For a bounded operator T on $\mathscr{H}$, $\|T\| < 1$, define the Rota subspace $\mathscr{M}_T$ of T to be all $F \in H^2_{\mathscr{H}}$ with Fourier series $F = \sum_{k=0}^{\infty} \varphi_k e^{kix}$ such that $\sum_{k=0}^{\infty} T^k\varphi_k = 0$. It is known [2, p. 161] that $\mathscr{M}_T$ is of full range. It was shown in [2, p. 169] that if T, U are-one-to one operators on $\mathscr{H}$ with disjoint ranges, then $\mathscr{M}_T \cap \mathscr{M}_U = (0)$. It suffices then to prove the existence in a separable infinite dimensional Hilbert space of an uncountable family of bounded one-to-one operators with disjoint ranges. To do this it suffices to exhibit an uncountable family of disjoint *closed* infinite dimensional subspaces of a separable Hilbert space, since the subspaces are then unitarily equivalent to the original space and the operators can be taken to be of the form $U/2$, where U is unitary as a mapping onto its range. It is convenient to describe such an example in $\mathscr{H}$ realized as $L^2_{\mathscr{K}}$, where $\mathscr{K}$ is some other Hilbert space. Let $\{e_\alpha\}$ be an uncountable family of pairwise linearly independent vectors in $\mathscr{K}$ (which exists if $\mathscr{K}$ is at least two-dimensional) and for the subspaces let

$$\mathscr{N}_\alpha = \{F \in L^2_{\mathscr{K}}\colon\ F(e^{ix}) = f(e^{ix})e_\alpha, \quad \text{for some } f \in L^2\} .$$

The situation when $\mathscr{H}$ is infinite dimensional thus contrasts strongly with the finite dimensional situation [1, p. 70] where the intersection of two invariant subspaces of full range also has full range, and the implication is that only when $\mathscr{H}$ is infinite dimensional can invariant subspaces of full range be pretty small. On the other hand, if nontrivial maximal invariant subspaces of $H^2_{\mathscr{H}}$ exist (or, equivalently, if there exist bounded operators on $\mathscr{H}$ without nontriviant subspaces [1, p. 103]), the existence of an uncountable family of disjoint *maximal* invariant subspaces is conceivable. For if there exists an operator T on $\mathscr{H}$ without an invariant subspace, it may also happen that T is not invertible and the codimension of the range of T is uncountably infinite in the linear space sense. It is then almost certain that one can find an uncountable family of such operators whose ranges are disjoint.

References

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