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DISJOINT PAIRS OF ANNULI AND DISKS FOR HEEGAARD SPLITTINGS

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Abstract. We consider interesting conditions, one of which will be called the *disjoint* (A^2, D^2)*-pair property*, on genus $g \geq 2$ Heegaard splittings of compact orientable 3-manifolds. Here a Heegaard splitting $(C_1, C_2; F)$ admits the *disjoint* (A^2, D^2)*-pair property* if there are an essential annulus A_i normally embedded in C_i and an essential disk D_j in C_j ($(i,j) = (1,2)$ or $(2,1)$) such that ∂A_i is disjoint from ∂D_j. It is proved that all genus $g \geq 2$ Heegaard splittings of toroidal manifolds and Seifert fibered spaces admit the disjoint (A^2, D^2)-pair property.

1. Introduction

Let M denote a compact orientable 3-manifold and $(C_1, C_2; F)$ a genus $g \geq 2$ Heegaard splitting of M. In the 1960s, Haken[4] introduced a condition of Heegaard splittings which is now said to be *reducible.* Here, $(C_1, C_2; F)$ is said to be *reducible* if there are essential disks $D_i \subset C_i$ ($i = 1, 2$) with $\partial D_1 = \partial D_2$. Otherwise, $(C_1, C_2; F)$ is said to be *irreducible.* It is proved that if M is reducible, then any Heegaard splitting of M is reducible. The concept of weak reducibility was introduced by Casson and Gordon[3]. Here, $(C_1, C_2; F)$ is said to be *weakly reducible* if there are essential disks $D_i \subset C_i$ ($i = 1, 2$) with $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, $(C_1, C_2; F)$ is said to be *strongly irreducible.* They proved in [3] that if a Heegaard splitting of M is weakly reducible, then either the splitting is reducible or M contains an orientable incompressible surface. In this direction, Thompson[11] introduced a condition called the *disjoint curve property.* Here, $(C_1, C_2; F)$ admits the *disjoint curve property* if there are essential disks $D_i \subset C_i$ ($i = 1, 2$) and an

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essential loop $z \subset F$ with $(\partial D_1 \cup \partial D_2) \cap z = \emptyset$. In [11], she studied genus 2 closed orientable manifolds with Heegaard splittings satisfying the disjoint curve property. Moreover, Hempel[5] introduced complexity of genus $g \geq 2$ Heegaard splittings of closed orientable 3-manifolds. It is called the 'distance' and is determined by a non-negative integer. The 'distance' is defined by using the curve complex of a Heegaard surface and is extension of the above conditions. In fact, Heegaard splittings with 'distance$= 0$' are reducible splittings and vise versa. A Heegaard splitting has 'distance≤ 1' if and only if the splitting is weakly reducible. A Heegaard splitting has 'distance≤ 2' if and only if the splitting admits the disjoint curve property. He proved that if a closed orientable 3-manifold M is reducible, toroidal or Seifert fibered, then any splitting of M has 'distance≤ 2'. He also showed that for any integer n, there is a closed 3-manifold with a Heegaard splittings of 'distance$> n$'. Note that Schleimer showed in [8] that for a given 3-manifold, the numbers of Heegaard splittings of 'distance≥ 3' is finite.

In this paper, we consider some conditions for Heegaard splittings (see Definition 2.2). One of them is the following: a Heegaard splitting $(C_1, C_2; F)$ admits the *disjoint (A^2, D^2)-pair property* if there are an essential annulus A_i normally embedded in C_i and an essential disk D_j in C_j ($(i,j) = (1,2)$ or $(2,1)$) such that ∂A_i is disjoint from ∂D_j.

We remark that the conditions (in Definition 2.2) are essentially defined by Kobayashi[6] and Schleimer[8]. If a Heegaard splitting is weakly reducible, then it admits the disjoint (A^2, D^2)-pair property and if a Heegaard splitting admits the disjoint (A^2, D^2)-pair property, then it admits the disjoint curve property (see Lemma 3.1).

Our main result is the following.

THEOREM 1.1. *Let M be a compact orientable 3-manifold. If M is reducible, Seifert fibered or toroidal, then any genus $g \geq 2$ Heegaard splitting of M admits the disjoint (A^2, D^2)-pair property.*

We also give an example of a Heegaard splitting such that it does not admit the disjoint (A^2, D^2)-pair property but admits the disjoint curve property. To this end, we will use the concept of *the strong rectangle condition* defined by Kobayashi[6].

2. Preliminaries

Throughout this paper, we work in the piecewise linear category. Let B be a sub-manifold of a manifold A. The notation $N(B; A)$ denotes a regular neighborhood of B in A. The notation $|\cdot|$ denotes the number of connected components. A *surface* means a connected 2-manifold.

A simple loop/arc properly embedded in a surface is said to be *inessential* if the loop/arc cuts off a disk from the surface. A simple loop/arc properly embedded in a surface is *essential* if the loop/arc is not inessential. A disk D^2 properly embedded in a 3-manifold M is *inessential* in M if ∂D^2 is inessential in ∂M. A disk D^2 properly embedded in a 3-manifold M is *essential* in M if D^2 is not inessential in M. A 2-manifold $S(\neq D^2)$ properly embedded in a 3-manifold M is said to be *compressible in M* if there is a disk $D \subset M$ such that $D \cap S = \partial D$ and ∂D is essential in S. The disk D is called a *compression disk of S*. We say that $S(\neq D^2)$ is *incompressible in M* if S is not compressible in M. The surface $S(\neq D^2)$ is *∂-parallel in M* if $\partial M \neq \emptyset$ and S is isotopic into ∂M relative ∂S. In particular, a ∂-parallel annulus A in a 3-manifold cuts off the solid torus $A \times [0,1]$ from M. We say that $S(\neq D^2)$ is *essential in M* if S is incompressible in M and is not ∂-parallel in M. The surface $S(\neq D^2)$ is said to be *∂-compressible* in M if $\partial M \neq \emptyset$, $\partial S \neq \emptyset$ and there is a disk $\delta \subset M$ such that $\delta \cap S = \partial\delta \cap S =: \alpha$ is an essential arc in S and that $\mathrm{cl}(\partial\delta \backslash \alpha)$ is an arc in ∂M. The disk δ is called a *∂-compression disk of S*. We say that $S(\neq D^2)$ is *∂-incompressible in M* if S is not ∂-compressible in M.

A 3-manifold C is a *compression body* if there is a compact connected closed surface F such that C is obtained from $F \times [0,1]$ by attaching 2-handles along mutually disjoint simple loops in $F \times \{1\}$ and capping off the resulting 2-sphere boundary components by 3-handles. Then $\partial_+ C$ denotes the component of ∂C corresponding to $F \times \{0\}$, and $\partial_- C$ denotes $\partial C \backslash \partial_+ C$. If $\partial_- C = \emptyset$, then C is called a *handlebody.* We say that a surface S properly embedded in C is *normally embedded* in C if $S \cap \partial_+ C = \partial S$. It is known that a ∂-compressible essential surface normally embedded in a compression body is a disk.

LEMMA 2.1. *Let D be an essential disk in a compression body C and γ an arc in C such that γ joins ∂D to itself and the interior of γ is disjoint from ∂D. Let A be an annulus obtained by pushing the interior of $D \cup N(\gamma; F)$ into the interior of C. Suppose that A is incompressible in C and is ∂-parallel in C. Then D cuts off a solid torus V from C and there is a non-separating essential disk E in $V \subset C$ with $E \cap D = \emptyset$ and $|E \cap \gamma| = 1$.*

Proof. Recall that since a ∂-parallel surface in a 3-manifold is separating, we see that A is separating in C. Hence D is separating in C. If D does not cut off a solid torus from C, then A cannot also cut off a solid torus. This contradicts that A is ∂-parallel. Hence D cuts off a solid torus V from C. Let D' be a copy of D in ∂V. Note that γ is an arc properly embedded in $\mathrm{cl}(\partial V \backslash D')$. Since A is incompressible in C, each component of $\partial A'$ does not bound a disk in V, where A' is a copy of A in ∂V. Let δ be a ∂-compression disk of A such that $\delta \cap N(\gamma; F)$ is an arc intersecting γ in a single point and that $\delta \cap A =: \gamma_\delta$ is an essential arc in A. Recall that A cuts off $A \times [0,1] \subset V$ from C. Set $E = \delta \cup \gamma_\delta \times [0,1]$. Then E is the desired disk. $\square$

We say that $(C_1, C_2; F)$ is a *Heegaard splitting* of a 3-manifold M if each of C_i $(i = 1, 2)$ is a compression body, $M = C_1 \cup C_2$ and $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = F$. The surface F is called a *Heegaard surface* of M and the genus of F is called the genus of the Heegaard splitting. We say that $(C_1, C_2; F)$ is *stabilized* if there are essential disks D_i $(i = 1, 2)$ in C_i with $|\partial D_1 \cap \partial D_2| = 1$. It is well-known that if a genus $g \geq 2$ Heegaard splitting is stabilized, then the splitting is reducible.

Definition 2.2. Let $(C_1, C_2; F)$ be a Heegaard splitting of a compact orientable 3-manifold.

1. The splitting $(C_1, C_2; F)$ admits the *disjoint* (A^2, D^2)*-pair property* if there are an essential annulus A_i normally embedded in C_i and an essential disk D_j in C_j $((i,j) = (1,2)$ or $(2,1))$ such that ∂A_i is disjoint from ∂D_j.
2. The splitting $(C_1, C_2; F)$ admits the *joined* (A^2, A^2)*-pair property* if there are an essential annuli A_i $(i = 1, 2)$ normally embedded in C_i such that one of the components of ∂A_1 is isotopic to one of the components of ∂A_2 and that the other components are mutually disjoint.
3. The splitting $(C_1, C_2; F)$ admits the *disjoint* (A^2, A^2)*-pair property* if there are an essential annuli A_i $(i = 1, 2)$ normally embedded in C_i with $\partial A_i \cap \partial A_j = \emptyset$.

It is an easy observation that if a Heegaard splitting admits the joined (A^2, A^2)-pair property, then the splitting admits the disjoint (A^2, A^2)-pair property. If a Heegaard splitting is weakly reducible, then it admits the disjoint (A^2, D^2)-pair property and if a Heegaard splitting admits the disjoint (A^2, D^2)-pair property, then it admits the disjoint curve property (see Lemma 3.1). Similarly, if a Heegaard splitting is weakly reducible, then it admits the joined (A^2, A^2)-pair property and if a Heegaard splitting admits the joined (A^2, A^2)-pair property, then it admits the disjoint curve property (see Lemma 3.2). See Figure 1.

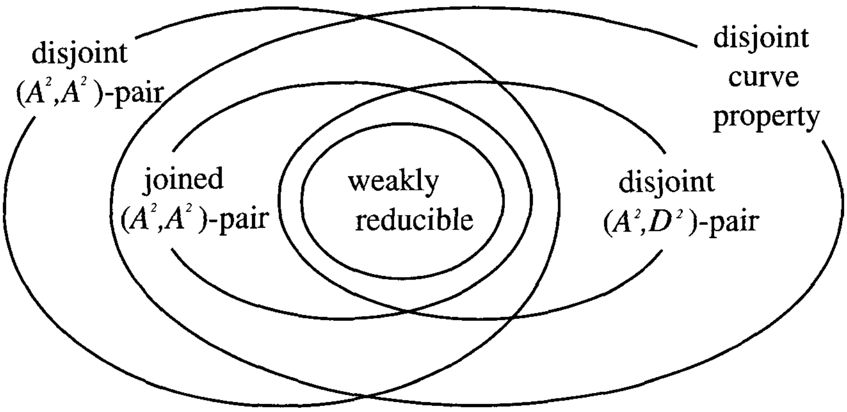

FIGURE 1

If one follows the point of view in Definition 2.2, the irreducibility of Heegaard splittings can be also called the *joined* (D^2, D^2)*-pair property* and the weak reducibility can be also called the *disjoint* (D^2, D^2)*-pair property*. Similarly, one can also define the *joined* (A^2, D^2)*-pair property*. Namely, a Heegaard splitting $(C_1, C_2; F)$ admits the *joined* (A^2, D^2)*-pair property* if there are an essential annulus A_i normally embedded in C_i and an essential disk D_j in C_j ($(i, j) = (1, 2)$ or $(2, 1)$) with $\partial D_j \subset \partial A_i$. However, the following lemma implies that an irreducible Heegaard splitting admits the joined (A^2, D^2)-pair property if and only if the splitting is weakly reducible.

LEMMA 2.3. *Let $(C_1, C_2; F)$ be a genus $g \geq 2$ Heegaard splitting of a compact orientable 3-manifold. If $(C_1, C_2; F)$ admits the joined (A^2, D^2)-pair property, then the splitting is weakly reducible. Conversely, if $(C_1, C_2; F)$ is weakly reducible, then either the splitting is reducible or the splitting admits the joined (A^2, D^2)-pair property.*

Proof. The first conclusion is an easy observation. Let A_i be an essential annulus normally embedded in C_i and D_j an essential disk in C_j ($(i, j) = (1, 2)$ or $(2, 1)$) with $\partial D_j \subset \partial A_i$. By changing the subscripts, if necessary, we may assume $(i, j) = (1, 2)$. Note that A_1 is ∂-compressible in C_1. Hence we obtain an essential disk D_1 in C_1 by ∂-compression of A_1. Since ∂D_1 is disjoint from ∂A_1, we see that ∂D_1 is disjoint from ∂D_2. Hence $(C_1, C_2; F)$ is weakly reducible.

We next show the latter conclusion. Suppose that $(C_1, C_2; F)$ is weakly reducible. We may assume that $(C_1, C_2; F)$ is irreducible. Let E_i ($i = 1, 2$) be essential disks in C_i with $\partial E_1 \cap \partial E_2 = \emptyset$. Let γ be an arc in F such that γ joins ∂E_1 to ∂E_2 and that the interior of γ is disjoint from $\partial E_1 \cup \partial E_2$. Let A_1' be an annulus obtained by pushing the interior of $E_1 \cup N(\gamma \cup \partial E_2; F)$ into the interior of C_1. By assumption that $(C_1, C_2; F)$ is irreducible, ∂E_2 does not bound a disk in C_1. This implies that A_1' is incompressible in C_1. If A_1' is ∂-parallel, then it follows from Lemma 2.1 that there is a non-separating essential disk E_1' in C_1 with $|\partial E_1' \cap \partial E_2| = 1$. This implies that $(C_1, C_2; F)$ is stabilized and hence reducible, contradicting the assumption that $(C_1, C_2; F)$ is irreducible. Therefore A_1' is an essential annulus normally embedded in C_1 and $\partial A_1' \supset \partial E_2$. Hence $(C_1, C_2; F)$ admits the joined (A^2, D^2)-pair property. $\square$

3. Fundamental properties

LEMMA 3.1. *Let $(C_1, C_2; F)$ be a genus $g \geq 2$ Heegaard splitting of a compact orientable 3-manifold.*

1. *If $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property, then the splitting admits the disjoint curve property.*
2. *If $(C_1, C_2; F)$ is weakly reducible, then the splitting admits the disjoint (A^2, D^2)-pair property.*

Proof. We first prove the conclusion 1. Let A_i be an essential annulus normally embedded in C_i and D_j an essential disk in C_j ($(i,j) = (1,2)$ or $(2,1)$) such that A_i and D_j give the disjoint (A^2, D^2)-pair property. By changing the subscripts, if necessary, we may assume $(i,j) = (1,2)$, that is, ∂A_1 is disjoint from ∂D_2 in F. Let D_1 be an essential disk in C_1 obtained by ∂-compression of A_1. We isotope D_1 so that $\partial D_1 \cap \partial A_1 = \emptyset$. Then we see that each of D_i ($i = 1, 2$) is disjoint from ∂A_1 and therefore $(C_1, C_2; F)$ admits the disjoint curve property.

We next show the conclusion 2. If $(C_1, C_2; F)$ is irreducible, then it follows from Lemma 2.3 that the splitting admits the joined (A^2, D^2)-pair property (hence the disjoint (A^2, D^2)-pair property). So we further assume that $(C_1, C_2; F)$ is reducible. Let $E_i \subset C_i$ ($i = 1, 2$) be essential disks with $\partial E_1 = \partial E_2$. If E_1 does not cut off a solid torus from C_1, then let α be an essential loop in F such that α does not bound a disk in C_1 and that α is disjoint from $\partial E_1 = \partial E_2$. If E_1 cuts off a solid torus V from C_1, then we also require that the loop α is not a longitude of V. Let γ be an arc in F such that γ joins ∂E_1 to α and that the interior of γ is disjoint from $\partial E_1 \cup \alpha$. Then we obtain an annulus $A_1' \subset C_1$ by pushing the interior of $E_1 \cup N(\gamma \cup \alpha; F)$ into the interior of C_1. Since α does not bound a disk in C_1, we see that A_1' is incompressible in C_1. Recall that even if E_1 cuts off a solid torus V from C_1, α is not a longitude of V. Hence it follows from Lemma 2.1 that A_1' is not ∂-parallel in C_1. Therefore A_1' is an essential annulus normally embedded in C_1 and hence we see that A_1' and E_2 imply that $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property. □

LEMMA 3.2. *Let $(C_1, C_2; F)$ be a genus $g \geq 2$ Heegaard splitting of a compact orientable* 3*-manifold.*

1. *If $(C_1, C_2; F)$ admits the joined (A^2, A^2)-pair property, then the splitting admits the disjoint curve property.*
2. *If $(C_1, C_2; F)$ is weakly reducible, then the splitting admits the joined (A^2, A^2)-pair property.*

Proof. We first prove the conclusion 1. Let $A_i \subset C_i$ $(i = 1, 2)$ be essential annuli which give the joined (A^2, A^2)-pair property. Then $\partial A_1 \cap \partial A_2$ is an essential loop z in F. For each $i = 1$ and 2, let D_i be an essential disk in C_i obtained by ∂-compression of A_i. We isotope D_i $(i = 1, 2)$ so that $\partial D_i \cap z = \emptyset$. Then D_1 and D_2 imply that $(C_1, C_2; F)$ admits the disjoint curve property.

We next show the conclusion 2. Let $E_i \subset C_i$ $(i = 1, 2)$ be essential disks in C_i which give the weak reducibility of $(C_1, C_2; F)$.

Case 1. $(C_1, C_2; F)$ is reducible.

We choose E_1 and E_2 so that $\partial E_1 = \partial E_2$. If each of E_i $(i = 1, 2)$ does not cut off a solid torus from C_i, then let α be an essential loop in F such that α does not bound disks both in C_1 and in C_2, and that α is disjoint from $\partial E_1 = \partial E_2$. If E_i $(i = 1$ or $2)$ cuts off a solid torus V_i from C_i, then we also require that the loop α is not a longitude of V_i. Let γ be an arc in F such that γ joins $\partial E_1 = \partial E_2$ to α and that the interior of γ is disjoint from $\partial E_1 = \partial E_2$. Then for each $i = 1$ and 2, we obtain an annulus $A_i' \subset C_i$ by pushing the interior of $E_i \cup N(\gamma \cup \alpha; F)$ into the interior of C_i. Note that $\partial A_1' = \partial A_2'$. Since α does not bound disks both in C_1 and in C_2, we see that each of A_i' $(i = 1, 2)$ is incompressible in C_i. Recall that even if E_i $(i = 1$ or $2)$ cuts off a solid torus V_i from C_i, α is not a longitude of V_i. Hence it follows from Lemma 2.1 that each of A_i' $(i = 1, 2)$ is not ∂-parallel in C_i. Therefore each of A_i' $(i = 1, 2)$ is an essential annulus normally embedded in C_i and hence we see that A_1' and A_2' imply that $(C_1, C_2; F)$ admits the joined (A^2, A^2)-pair property.

Case 2. $(C_1, C_2; F)$ is irreducible.

Let γ' be an arc in F such that γ' joins ∂E_1 to ∂E_2 and that the interior of γ' is disjoint from $\partial E_1 \cup \partial E_2$. For each $(i,j) = (1,2)$ and $(2,1)$, we obtain an annulus $A''_i \subset C_i$ by pushing the interior of $E_i \cup N(\gamma' \cup \partial E_j; F)$ into the interior of C_i. Since $(C_1, C_2; F)$ is irreducible, we see that each of A''_i $(i = 1,2)$ is incompressible in C_i. If A''_1 is ∂-parallel, then it follows from Lemma 2.1 that there is a non-separating essential disk E'_1 in C_1 which satisfies $|\partial E'_1 \cap \partial E_2| = 1$. This implies that $(C_1, C_2; F)$ is stabilized and hence reducible, contradicting the assumption that $(C_1, C_2; F)$ is irreducible. Therefore A''_1 is an essential annulus normally embedded in C_1. Similarly, we see that A''_2 is an essential annulus normally embedded in C_2. Therefore A''_1 and A''_2 imply that $(C_1, C_2; F)$ admits the joined (A^2, A^2)-pair property. □

LEMMA 3.3. *Let $(C_1, C_2; F)$ be a genus $g \geq 2$ Heegaard splitting of a compact orientable 3-manifold M. Suppose that there are essential disks $E_i \subset C_i$ $(i = 1,2)$ with $|\partial E_1 \cap \partial E_2| \leq 2$. Then $(C_1, C_2; F)$ admits both the disjoint (A^2, D^2)-pair property and the joined (A^2, A^2)-pair property.*

Proof. If $|\partial E_1 \cap \partial E_2| \leq 1$, then we see that $(C_1, C_2; F)$ is weakly reducible. It follows from Lemmas 3.1 and 3.2 that $(C_1, C_2; F)$ admits both the disjoint (A^2, D^2)-pair property and the joined (A^2, A^2)-pair property. Hence we may assume that $(C_1, C_2; F)$ is strongly irreducible, that is, the (minimal) geometric intersection number between ∂E_1 and ∂E_2 is equal to two. Let γ_i and γ'_i $(i = 1,2)$ be the arcs obtained by cutting ∂E_i along the two points $\partial E_1 \cap \partial E_2$.

Claim 1. We may assume $\langle \partial E_1, \partial E_2 \rangle = 0$. Here, $\langle , \rangle$ denotes the algebraic intersection number.

Suppose $\langle \partial E_1, \partial E_2 \rangle \neq 0$. Then each of E_i $(i = 1,2)$ must be a non-separating disk in C_i. Let D_1 be a separating essential disk in C_1 bounded by $\partial N(\partial E_1 \cup \gamma_2; F)$. Recall that the geometric intersection number between ∂D_1 and ∂E_2 is equal to two. Since D_1 is separating in C_1, we see that $\langle \partial D_1, \partial E_2 \rangle = 0$. By replacing E_1 to D_1, we have Claim 1.

By Claim 1, we obtain an annulus A_1 in C_1 by pushing the interior of $E_1 \cup N(\gamma_2; F)$ into the interior of C_1. Similarly, we also obtain an annulus A_2 in C_2 by pushing the interior of $E_2 \cup N(\gamma_1; F)$ into the interior of C_2. Since the (minimal) geometric intersection number between ∂E_1 and ∂E_2 is equal to two, each component of ∂A_i $(i = 1,2)$ is essential in F.

Claim 2. Each of A_i $(i = 1, 2)$ is essential in C_i.

If A_1 is compressible in C_1, then we see that $\gamma_1 \cup \gamma_2$ bounds an essential disk E_1' in C_1. Hence E_1' and E_2 imply that $(C_1, C_2; F)$ is weakly reducible, a contradiction. Therefore A_1 is incompressible in C_1. Suppose that A_1 is ∂-parallel in C_1. Then it follows from Lemma 2.1 that E_1 cuts off a solid torus V_1 from C_1 and that there is an essential disk E_1'' in V_1 with $|\partial E_1'' \cap \gamma_2| = 1$. Since $|\partial E_1 \cap \partial E_2| = 2$, we see that $\partial E_2 \cap \partial V_1 = \gamma_2$. This implies that $|\partial E_1'' \cap \partial E_2| = 1$ and hence $(C_1, C_2; F)$ is stabilized, a contradiction. Hence we see that A_1 is essential in C_1. Similarly, we can show that A_2 is also essential in C_2 and therefore we have Claim 2.

Hence A_1 and E_2 imply that $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property. Note that a component of ∂A_1 is isotopic to $\gamma_1 \cup \gamma_2$ and the other is isotopic to $\gamma_1' \cup \gamma_2$. Note also that a component of ∂A_2 is isotopic to $\gamma_1 \cup \gamma_2$ and the other is isotopic to $\gamma_1 \cup \gamma_2'$. Therefore we see that $(C_1, C_2; F)$ also admits the joined (A^2, A^2)-pair property. $\square$

LEMMA 3.4. *Let $(C_1, C_2; F)$ be a genus $g \geq 2$ Heegaard splitting of a compact orientable 3-manifold M. Suppose that $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property. Let A_i be an essential annulus normally embedded in C_i and D_j an essential disk in C_j $((i, j) = (1, 2)$ or $(2, 1))$ such that A_i and D_j give the disjoint (A^2, D^2)-pair property. Then one of the following holds.*

1. *$(C_1, C_2; F)$ admits the disjoint (A^2, A^2)-pair property.*
2. *$g = 2$, C_j is a genus two handlebody, D_j cuts C_j into two solid tori and each component of ∂A_i is a longitude of one of the solid tori.*

Proof. If a component of ∂A_i bounds a disk in C_j, then $(C_1, C_2; F)$ admits the joined (A^2, D^2)-pair property. Hence it follows from Lemma 2.3 that $(C_1, C_2; F)$ is weakly reducible. By Lemma 3.3, we see that $(C_1, C_2; F)$ admits the joined (A^2, A^2)-pair property. This implies that $(C_1, C_2; F)$ admits the disjoint (A^2, A^2)-pair property. Hence we may assume that ∂A_i does not bound a disk in C_j.

Case 1. D_j is non-separating in C_j.

Let γ be an arc in F such that γ joins ∂D_j to one of the components of ∂A_i and that the interior of γ is disjoint from $\partial A_i \cup \partial D_j$. Then we obtain an annulus A_j by pushing the interior of $D_j \cup N(\alpha \cup \gamma; F)$ into the interior of C_j. Since ∂A_i does not bound a disk in C_j, we see that A_j is incompressible in C_j. Note that A_j is non-separating in

C_j. It follows that A_j is not ∂-parallel in C_j. Hence A_j is essential in C_j and therefore A_i and A_j imply that $(C_1, C_2; F)$ admits the joined (A^2, A^2)-pair property. This implies that $(C_1, C_2; F)$ admits the disjoint (A^2, A^2)-pair property.

Case 2. D_j is separating in C_j.

Subcase 2.1. $g \geq 3$.

Let α be an essential loop in F such that α does not bound disks both in C_1 and in C_2, and that α is disjoint from $\partial A_i \cup \partial D_j$. If D_j cuts off a solid torus V_j from C_j, then we further require that the loop α is not a longitude of V_j. Note that since $g \geq 3$ and $\partial A_i \cup \partial D_j$ consists of at most three components, we can always find such a loop α. Hence by an argument similar to the proof of Lemma 3.2, we can obtain an essential annulus normally embedded in C_j whose boundary is disjoint from ∂A_i. Hence $(C_1, C_2; F)$ admits the disjoint (A^2, A^2)-pair property.

Subcase 2.2. $g = 2$.

Suppose that C_j is not a handlebody. Then D_j cuts C_j into two compression bodies V and V'. We may assume that $V \cong T^2 \times [0, 1]$, where T^2 is a torus. Set $T = \partial V \cap F$. Note that $\partial T = \partial D_j$. If $T \cap \partial A_i \neq \emptyset$, then let α be a component of $T \cap \partial A_i$. Otherwise, let α be an essential loop in T. Let γ be an arc in T such that γ joins α to ∂T and that the interior of γ is disjoint from α. Then we obtain an annulus A'_j in C_j by pushing the interior of $D_j \cup N(\alpha \cup \gamma; F)$ into the interior of C_j. The construction of A'_j assures that A'_j is essential in C_j. Therefore we see that A_i and A'_j imply that $(C_1, C_2; F)$ admits the disjoint (A^2, A^2)-pair property.

Hence we assume that C_j is a genus two handlebody. Let V'_j and V''_j be solid tori obtained by cutting C_j along D_j. If a component of ∂A_i is a longitude neither of V'_j nor of V''_j, then by an argument similar to that in Case 1, we see that $(C_1, C_2; F)$ admits the joined (A^2, A^2)-pair property. The other cases are included in the conclusion 2 of Lemma 3.4. □

REMARK 3.5. The conclusion 2 of Lemma 3.4 can be happened in case of the exceptional splittings of orientable Seifert fibered spaces. For details, see Section 6.

4. Essential tori and Klein bottles

We divide a proof of Theorem 1.1 into three sections. In this section, we consider Heegaard splittings of compact orientable 3-manifolds containing essential tori or Klein bottles.

LEMMA 4.1. *Let $(C_1, C_2; F)$ be a genus $g \geq 2$ strongly irreducible Heegaard splitting of a compact orientable 3-manifold M. Suppose that M contains an essential torus or Klein bottle. Then $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property.*

Proof. Let T be an essential torus or Klein bottle in M. Then by an argument similar to the proof of Lemma 3.6 in [5], we see that each component of $T_i = T \cap C_i$ $(i = 1, 2)$ is an essential annulus or Möbius band in C_i.

Case 1. T is an essential torus.

Recall that an essential annulus normally embedded in a compression body is ∂-compressible. Let A_2 be an annulus component of T_2 such that a ∂-compression disk δ of A_2 is disjoint from the other components of T_2. Then we obtain an essential disk D_2 in C_2 by ∂-compression of A_2 along δ. Moreover, we can isotope D_2 so that ∂D_2 is disjoint from $T \cap F$. Hence D_2 and a component of T_1 imply that $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property.

Case 2. T is a Klein bottle.

If each component both of T_1 and of T_2 is an annulus, then we obtain the desired conclusion by an argument similar to that in Case 1. Hence by changing the subscript, if necessary, we may assume that T_1 contains a Möbius band component A_1^m.

Subcase 2.1. There is an annulus component A_2 of T_2 with $\partial A_2 \supset \partial A_1^m$.

Let D_1^m be an essential disk of C_1 by ∂-compression of A_1^m. Then A_1^m is re-constructed from D_1^m by attaching a band along an appropriate arc γ^m. Let D_1 be a disk obtained by joining two parallel copies of D_1^m with a band along γ^m. Note that D_1 is essential and separating in C_1. Let A_1 be an annulus obtained by joining D_1 to itself with a band along γ^m (*cf.* Figure 2). Since each component of ∂A_1 is isotopic to ∂A_1^m, we see that A_1 is essential in C_1.

Let D_2 be an essential disk in C_2 obtained by ∂-compression of A_2. Then we can isotope D_2 so that ∂D_2 is disjoint from ∂A_2 (hence ∂A_1).

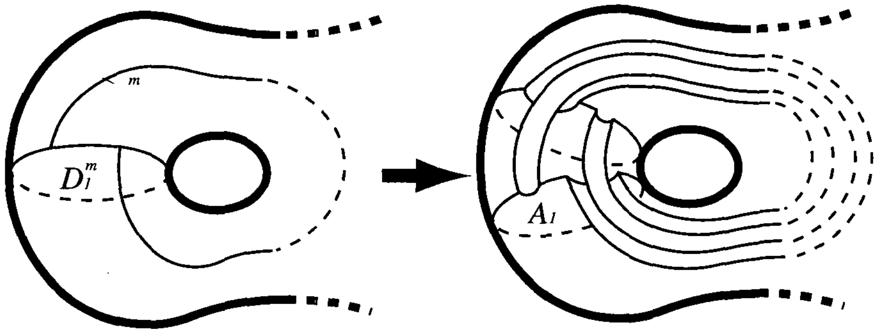

FIGURE 2

Hence A_1 and D_2 imply that $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property.

Subcase 2.2. ∂A_1^m is the boundary of a Möbius band component A_2^m of T_2.

Then for each $i = 1$ and 2, T_i consists of a Möbius band in C_i. As we found the annulus A_1 in Subcase 2.1, we can find a separating essential annulus A_1' normally embedded in C_1. On the other hand, as we found the disk D_1 in Subcase 2.1, we can find a separating essential disk D_2' in C_2. Note that each component of $\partial A_1'$ is isotopic to ∂A_1^m and that $\partial D_2'$ is disjoint from $\partial A_2^m (= \partial A_1^m)$. This implies that $(C_1, C_2; F)$ admits the disjoint (A^2, D_2)-property. □

LEMMA 4.2. *Let $(C_1, C_2; F)$ be a genus $g \geq 2$ strongly irreducible Heegaard splitting of a compact orientable 3-manifold M. Suppose that M contains an essential torus or Klein bottle. Then $(C_1, C_2; F)$ admits the joined (A^2, A^2)-pair property.*

Proof. Let T be an essential torus or Klein bottle. Then we see that each component of $T_i := T \cap C_i$ $(i = 1, 2)$ is an essential annulus or Möbius band in C_i. If T is a torus, we immediately obtain the desired conclusion. Hence we may assume that T is a Klein bottle.

Case 1. For $i = 1$ or 2, T_i contains an annulus component.

We may assume that T_1 contains an annulus component A_1. If a component of ∂A_1 is a boundary component of an annulus component of T_2, then we are done. Otherwise, there is a Möbius band component A_2^m of T_2 with $\partial A_2^m \subset \partial A_1$. Let D_2^m be an essential disk of C_2 by ∂-compression of A_2^m. Then A_2^m is re-constructed from D_2^m by attaching a band along an appropriate arc γ^m. Let D_2 be a disk obtained by joining two parallel copies of D_2^m with a band along γ^m. Note that D_2 is essential and separating in C_2. Let A_2 be an annulus obtained by joining D_2 to itself with a band along γ^m. Since each component of ∂A_2 is isotopic to ∂A_2^m, we see that A_2 is essential in C_2. Therefore A_1 and A_2 imply that $(C_1, C_2; F)$ admits the joined (A^2, A^2)-pair property.

Case 2. For $i = 1$ and 2, T_i does not contain an annulus component.

Then for each $i = 1$ and 2, T_i consists of a Möbius band in C_i. As we found the annulus A_2 in Case 1, we can find a separating essential annulus A_i $(i = 1, 2)$ in C_i with $\partial A_1 = \partial A_2$. This implies that $(C_1, C_2; F)$ admits the joined (A^2, A^2)-pair property. □

We remark that the orientable Seifert fibered spaces over non-orientable base spaces must contain essential tori or Klein bottles. Hence we obtain the following.

COROLLARY 4.3. *Let $(C_1, C_2; F)$ be a genus $g \geq 2$ Heegaard splitting of a compact orientable 3-manifold M. Suppose that M is toroidal or a Seifert fibered space over a non-orientable base space. Then $(C_1, C_2; F)$ admits both the disjoint (A^2, D^2)-pair property and the joined (A^2, A^2)-pair property.*

5. Vertical splittings of Seifert fibered spaces

Moriah and Schultens[7] proved that every irreducible Heegaard splitting of an orientable Seifert fibered space over an orientable base space is either *vertical* or *horizontal*. In particular, Schultens[9] showed that if the 3-manifold has non-empty boundary, then any irreducible splitting is a vertical splitting.

We briefly recall the definition of a vertical splitting(for a horizontal splitting, see the next section). For convenience of our argument, we shall refer to the definition described as in [5]. Let M be an orientable Seifert fibered space over an orientable base space B and $f_1, f_2, \ldots, f_m$ the singular fibers of M. Set $x_i = p(f_i)$ $(i = 1, 2, \ldots, m)$, where $p : M \to B$ is a projection. Let $\partial_1, \partial_2, \ldots, \partial_n$ be the boundary components of B. Then we have a decomposition $B = D_1 \cup D_2 \cup R$ such that each of D_1, D_2 and R is a disjoint union of disks or annuli satisfying the following.

1. Each disk component of D_1 (D_2 resp.) contains at most one point of x_i $(i = 1, 2, \ldots, m)$.

2. Each annulus component of D_1 (D_2 resp.) contains no points of x_i ($i = 1, 2, \dots, m$) and has a single component of ∂_j ($j = 1, 2, \dots, n$) as one of the boundary components.
3. Each component of R is a rectangle containing no points of x_i ($i = 1, 2, \dots, m$) such that one pair of opposite edges attaches to D_1 and the other pair attaches to D_2.
4. The interiors of D_1, D_2 and R are mutually disjoint, and $D_1 \cup R$ and $D_2 \cup R$ is connected.

Set $C_1 = D_1 \times S^1 \cup R \times [0, 1/2]$, $C_2 = D_2 \times S^1 \cup R \times [1/2, 1]$ and $F = C_1 \cap C_2$, where $S^1 = [0, 1]/0 \sim 1$. Then $(C_1, C_2; F)$ is a Heegaard splitting of M. Such a Heegaard splitting is called a *vertical splitting.*

LEMMA 5.1. *Let $(C_1, C_2; F)$ be a genus $g \geq 2$ vertical splitting of an orientable Seifert fibered space M. Then $(C_1, C_2; F)$ admits both the disjoint (A^2, D^2)-pair property and the joined (A^2, A^2)-property.*

Proof. We use the same notations as above. Let e_i ($i = 1, 2$) be arcs in a rectangle component R_0 of R such that each of e_i joins the opposite edges of R_0 containing in D_i and that e_1 meets e_2 in a single point. Set $E_1 = e_1 \times [0, 1/2]$ and $E_2 = e_2 \times [1/2, 1]$. Then we see that E_i ($i = 1, 2$) are essential disks in C_i with $|\partial E_1 \cap \partial E_2| = 2$. Hence we will obtain the desired result by Lemma 3.3. □

6. Horizontal splittings of Seifert fibered spaces

We first recall the definition of a horizontal splitting. Let M be a closed orientable Seifert fibered space and f a fiber in M. Suppose that there is a surface $S \neq B^2$ in a fibration of $M_0 := M \backslash N(f; M)$ over S^1. Let $\phi : S \to S$ be the orientation preserving periodic homeomorphism such that $M_0 = S \times [0, 1]/(x, 0) \sim (\phi(x), 1)$. Note that M is obtained from M_0 by attaching a solid torus V so that a longitude of V is identified with ∂F. Set $C_1 = S \times [0, 1/2]$, $C_2 = V \cup (S \times [1/2, 1])$ and $F = C_1 \cap C_2$. Then $(C_1, C_2; F)$ is a Heegaard splitting of M. Such a splitting is called a *horizontal splitting.* Note that the genus of F is equal to twice the genus of S.

Note that lens spaces have no irreducible Heegaard splittings of genus $g \geq 2$. If the base space of M has positive genus or M has more than three singular fibers, then M contains an essential torus. Hence it follows from Lemma 4.1 that any strongly irreducible Heegaard splitting of M

admits both the disjoint (A^2, D^2)-pair property and the joined (A^2, A^2)-property. Therefore in the remainder, we assume that the base space of M is S^2 and M has exactly three singular fibers.

LEMMA 6.1. *Let M be an orientable Seifert fibered space satisfying the above assumption. Let $(C_1, C_2; F)$ be a horizontal splitting of M. Let S be as above. Suppose that the genus of S is greater than one. Then $(C_1, C_2; F)$ admits both the disjoint (A^2, D^2)-pair property and the joined (A^2, A^2)-pair property.*

Proof. We use the same notations as in the definition of horizontal splittings. We may assume that $(C_1, C_2; F)$ is irreducible by Lemma 3.1. Then by Theorem 3.5 of [5], there is an essential loop α in S with $|\alpha \cap \phi(\alpha)| \leq 1$. Let S' be the 2-manifold obtained by cutting S along $\alpha \cup \phi(\alpha)$. Set $\partial_0 S' = \partial S' \cap \partial S$.

Since the genus of S is greater than one, there is an arc γ in S' such that γ joins $\partial_0 S$ to itself and that γ is essential in S. Set $D_1 = \gamma \times [0, 1/2]$ and $A_2 = \alpha \times [1/2, 1]$. Then D_1 is an essential disk in C_1 and A_2 is an essential annulus in C_2. Note that $\gamma \times \{1/2\} \cap \alpha \times \{1/2\} = \emptyset$ and $\gamma \times \{0\} \cap \alpha \times \{1\} = \emptyset$. Hence we see that $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property.

Note that D_1 is non-separating in C_1. Hence by an argument similar to that in Case 1 of the proof of Lemma 3.2, $(C_1, C_2; F)$ admits the joined (A^2, A^2)-pair property. □

In the remainder, we suppose that S is of genus one. Then $(C_1, C_2; F)$ is a genus two Heegaard splitting of M. It is proved in [1] that any genus two Heegaard splitting of an orientable Seifert fibered space M is isotopic to a vertical splitting except for the following cases (cf. 5.7 of [2]).

1. $M = V(2, 3, a)$ is a 2-fold covering of S^3 branched along the torus knot K_0 of type $(3, a)$ $(a \geq 7)$.
2. $M = W(2, 4, b)$ is a 2-fold covering of S^3 branched along the link $K_1 \cup K_2$, where K_1 is the torus knot of type $(2, b)$ $(b \geq 5)$ and K_2 is a core loop of the standard solid torus in S^3 whose boundary includes K_1.

In each case, the exceptional splittings are obtained as follows. Let L be the 3-bridge link K_0 or $K_1 \cup K_2$ and $(B_1, \tau_1) \cup (B_2, \tau_2)$ is a 3-bridge decomposition of L. For each $i = 1$ and 2, let C_i be the 2-fold covering of B_i branched along three unknotted arcs τ_i. Then C_i $(i = 1, 2)$ are genus two handlebodies. Set $F = \partial C_1 = \partial C_2$. Then we obtain the desired splitting $(C_1, C_2; F)$. We call such splittings the *exceptional splittings of orientable Seifert fibered spaces.*

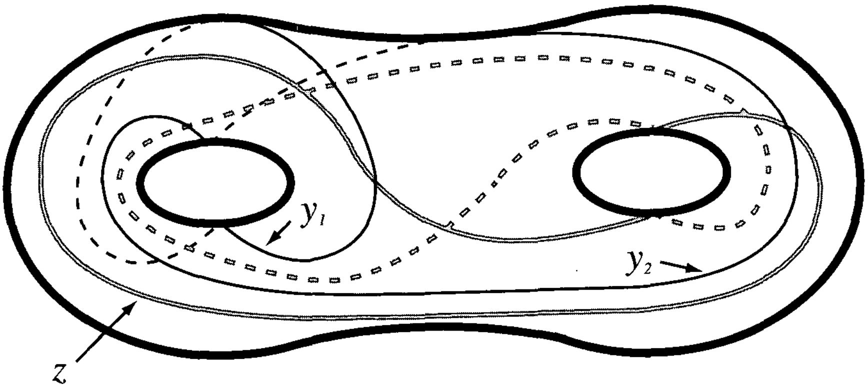

FIGURE 3

LEMMA 6.2. *Set $M = V(2,3,a)$ $(a \geq 7)$ or $W(2,4,b)$ $(b \geq 5)$. Let $(C_1, C_2; F)$ be the exceptional splitting of M. Then $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property.*

Proof. We use the same notations as in the definition of horizontal splittings.

Case 1. $M = V(2,3,3k+1)$, where $k(\geq 2)$ is an integer.

Let y_1, y_2 and z be the loops on ∂C_1 illustrated in Figure 3. Then we see that each of $\tau_z^k(y_i)$ $(i = 1,2)$ bounds a non-separating disk in C_2, where τ_z is a Dehn twist along z. Let A_1 be the essential annulus normally embedded in C_1 illustrated in Figure 4. Note that ∂A_1 is

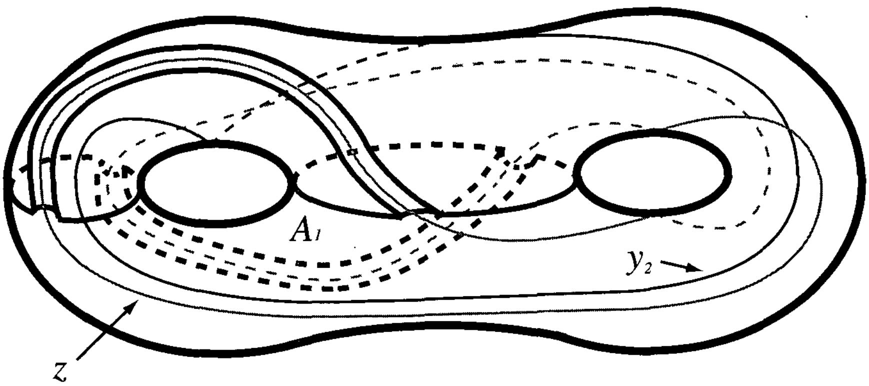

FIGURE 4

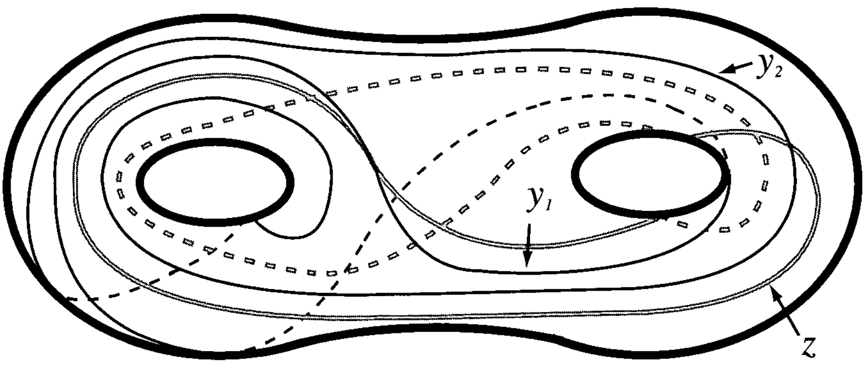

FIGURE 5

disjoint from z and intersects y_2 in a single point. Hence ∂A_1 intersects $\tau_z^k(y_2)$ in a single point. Let α be the component of ∂A_1 intersecting $\tau_z^k(y_2)$ in a single point. Then since $\tau_z^k(y_2)$ bounds a non-separating essential disk in C_2, $\partial N(\alpha \cup \tau_z^k(y_2), F)$ bounds an essential separating disk D_2 in C_2. Therefore A_1 and D_2 imply that $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property.

Case 2. $M = V(2, 3, 3k+2)$, where $k(\geq 2)$ is an integer.

The argument is similar to that of Case 1. Let y_1, y_2 and z be the loops on ∂C_1 illustrated in Figure 5. Then we see that each of $\tau_z^k(y_i)$ $(i = 1, 2)$ bounds a non-separating disk in C_2. Let A_1 be the essential annulus normally embedded in C_1 illustrated in Figure 6. Note that ∂A_1

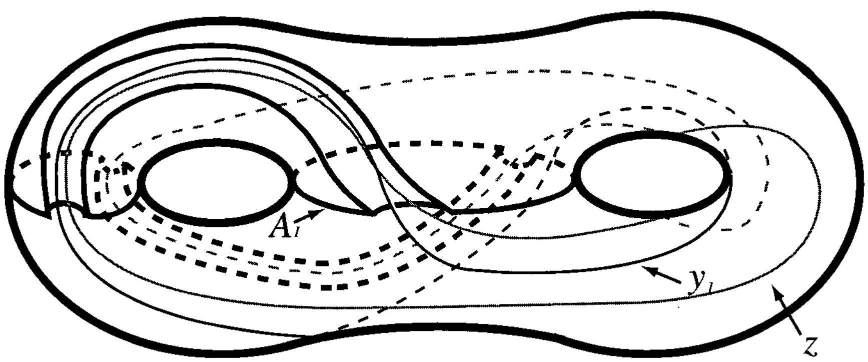

FIGURE 6

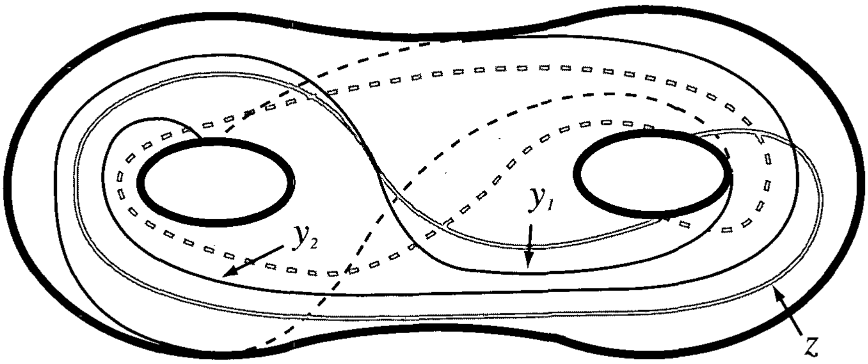

FIGURE 7

is disjoint from z and intersects y_1 in a single point. Hence ∂A_1 intersects $\tau_z^k(y_1)$ in a single point. Let α be the component of ∂A_1 intersecting $\tau_z^k(y_1)$ in a single point. Then $\partial N(\alpha \cup \tau_z^k(y_1), F)$ bounds an essential separating disk D_2 in C_2. Therefore A_1 and D_2 imply that $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property.

Case 3. $M = V(2, 4, 2k+1)$, where $k(\geq 2)$ is an integer.

Let y_1, y_2 and z be the loops on ∂C_1 illustrated in Figure 7. Then we see that each of $\tau_z^k(y_i)$ $(i = 1, 2)$ bounds a non-separating disk in C_2. Let A_1 be the essential annulus normally embedded in C_1 illustrated in Figure 6. Then we obtain the desired conclusion by the same argument as in Case 2. □

By Corollary 4.3, Lemmas 5.1 and 6.1, we have the following.

THEOREM 6.3. *Let M be a compact orientable 3-manifold. If M is reducible, Seifert fibered or toroidal, then any genus $g \geq 2$ Heegaard splitting of M other than the exceptional splittings of orientable Seifert fibered spaces admits the joined (A^2, A^2)-pair property.*

We do not know whether the exceptional splittings of orientable Seifert fibered spaces admit the disjoint/joined (A^2, A^2)-pair property. However, we would like to expect the following.

CONJECTURE 6.4. *The exceptional splittings of orientable Seifert fibered spaces do not admit the disjoint (A^2, A^2)-pair property.*

7. Strong rectangle condition

In this section, we give an example of a Heegaard splitting such that it does not admit the disjoint (A^2, A^2)-pair property, but admits the disjoint curve property. To this end, we will use the concept of *the strong rectangle condition* defined by Kobayashi[6].

Let S be a genus $g(\geq 2)$ closed orientable surface and each of R_i $(i = 1, 2)$ a four holed 2-sphere in S with $\partial R_i = l_1^i \cup l_2^i \cup l_3^i \cup l_4^i$. We suppose that ∂R_1 and ∂R_2 intersect transversely. We say that R_1 and R_2 are *tight* if they satisfy the following.

1. There is not a bigon B in S such that $\partial B = a \cup b$, where a is a subarc of ∂R_1 and b is a subarc of ∂R_2.
2. For each pair (l_s^1, l_t^1) with $s \neq t$ and (l_p^2, l_q^2) with $p \neq q$, there is a rectangle R embedded in R_1 and R_2 such that the interior of R is disjoint from $\partial R_1 \cup \partial R_2$ and that the edges of R are subarcs of l_s^1, l_t^1, l_p^2 and l_q^2.

Let $(C_1, C_2; F)$ be a genus $g(\geq 2)$ Heegaard splitting of a compact orientable 3-manifold M. For each $i = 1$ and 2, let $\{l_1^i, \ldots, l_{3g-3}^i\}$ be a collection of mutually disjoint, non-isotopic, essential loops in F such that each of l_s^i is either the boundary of a disk in C_i or a boundary component of an incompressible, ∂-incompressible annulus properly embedded in C_i. For each $i = 1$ and 2, let $P_1^i, \ldots, P_{2g-2}^i$ be three holed 2-spheres by cutting F along $l_1^i \cup \ldots \cup l_{3g-3}^i$. Then for each $i = 1, 2$ and $j = 1, \ldots, 3g-3$, we obtain a four holed 2-sphere $R_j^i := P_s^i \cup P_t^i$, where P_s^i and P_t^i satisfies that $P_s^i \cap P_t^i = l_j^i$. We say that $(C_1, C_2; F)$ satisfies the *strong rectangle condition* if for each $s = 1, \ldots, 3g-3$ and $p = 1, \ldots, 3g-3$, R_s^1 and R_p^2 are tight.

The following is proved in [6].

LEMMA 7.1. (Theorem 2 of [6]) *Suppose that a Heegaard splitting satisfies that the strong rectangle condition. Then the splitting does not satisfy the disjoint (A^2, A^2)-property.*

Hence we only have to find an example of a Heegaard splitting satisfying the disjoint curve property and admitting the strong rectangle condition. In fact, Figure 8 will give such an example.

Let F be a genus three closed surface illustrated in Figure 8 and x_1, x_2, x_3, y and z be loops in F as in Figure 8. Set $x_i' = \tau_y^n(x_i)$ $(i = 1, 2, 3)$, where τ_y is a Dehn twist along y. We attach 2-handles to $F \times [0, 1]$ along $x_i \times \{0\}$ and $x_i' \times \{1\}$ $(i = 1, 2, 3)$. Moreover, by capping off 3-balls along the boundary, we obtain a closed 3-manifold M. Note that $F \times \{1/2\}$

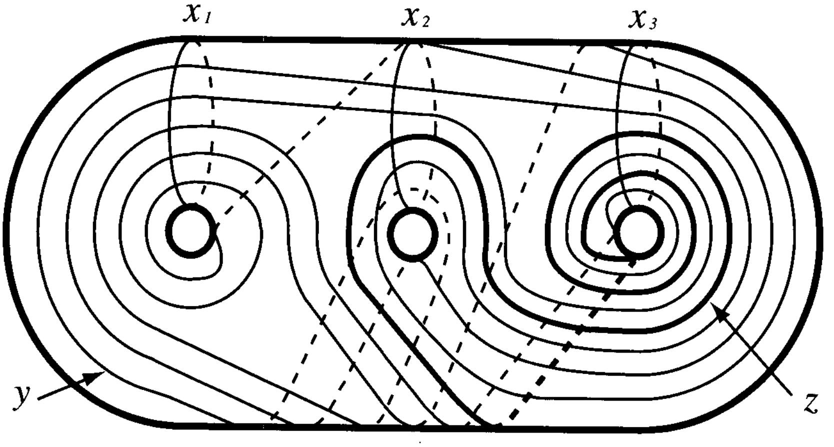

FIGURE 8

decomposes M into two handlebodies C_1 and C_2. Hence $(C_1, C_2; F)$ is a genus three Heegaard splitting of M. Note that $x_1 \cup x_1'$ is disjoint from z. It follows that $(C_1, C_2; F)$ satisfies the disjoint curve property. Moreover, if n is sufficiently large, then we see that $\{x_1, x_2, x_3\}$ and $\{x_1', x_2', x_3'\}$ give the strong rectangle condition (*cf.* Section 7 of [6]). Hence by Lemma 7.1, $(C_1, C_2; F)$ does not admit the disjoint (A^2, A^2)-pair property. Moreover, since $(C_1, C_2; F)$ is a genus three splitting, it does not admit the disjoint (A^2, D^2)-pair property (*cf.* Lemma 3.4).

QUESTION 7.2. *Is there a Heegaard splitting such that the splitting admits the disjoint (A^2, A^2)-pair property and that the splitting does not admit the disjoint curve property?*

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