DISJOINTNESS AND WEAK MIXING OF MINIMAL SETS

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In [3, Problem G, p. 34] the following problem is proposed: characterize the class $\mathfrak{D}^{\perp}$ of all flows which are disjoint from every distal flow. We show here that $\mathfrak{D}^{\perp}$ consists precisely of the weakly mixing minimal flows. In arriving at this conclusion we make use of a result (Corollary to Theorem 1) which has been arrived at independently by Keynes and Robertson [4, Theorem 3.4, p. 366]. In the following all transformation groups will be assumed to have compact Hausdorff phase spaces. For any unexplained notation or terminology the reader is referred to [1] and [3]. This research is part of the author's doctoral dissertation prepared at Yale University under the guidance of Professor S. Kakutani, whose assistance the author gratefully acknowledges.

We say that a transformation group (X, T) is *weakly mixing* if given nonempty open subsets A, B, C, D of X there is $t \in T$ such that $At \cap C \neq \emptyset$ and $Bt \cap D \neq \emptyset$.

LEMMA. *Let (X, T) be a transformation group and suppose that T is abelian. Then (X, T) is weakly mixing if and only if given nonempty open subsets A and B of X there is $t \in T$ such that $At \cap A \neq \emptyset$ and $At \cap B \neq \emptyset$.*

PROOF. It is clear that if (X, T) is weakly mixing then the stated condition is satisfied. Suppose then that the condition holds and let nonempty open subsets A, B, C, D of X be given; we need to find $t \in T$ such that $At \cap C \neq \emptyset$ and $Bt \cap D \neq \emptyset$.

We may choose $t_1 \in T$ such that $E = At_1 \cap B \neq \emptyset$, $t_2 \in T$ such that $F = Et_2 \cap Ct_1 \neq \emptyset$, and $t_3 \in T$ such that $Ft_3 \cap F \neq \emptyset$ and $Ft_3 \cap D \neq \emptyset$. Let $t = t_2 t_3$. Then

$$\begin{aligned}(At \cap C)t_1 &= At_1 t \cap Ct_1 \supseteq At_1 t \cap Bt \cap Ct_1 = (At_1 \cap B)t \cap Ct_1 \\ &= E(t_2 t_3) \cap Ct_1 \supseteq Ft_3 \cap Ct_1 \supseteq Ft_3 \cap F \neq \emptyset,\end{aligned}$$

so $At \cap C \neq \emptyset$; and

$$At_1 t \cap Bt \cap D = (At_1 \cap B)t \cap D = E(t_2 t_3) \cap D \supseteq Ft_3 \cap D \neq \emptyset,$$

so $Bt \cap D \neq \emptyset$.

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If (X, T) is a transformation group, we will denote by RP and S_e the regionally proximal and equicontinuous structure relations of X, respectively [**1**, Definition 6, p. 261 and Definition 4, p. 260].

THEOREM 1. *If (X, T) is a weakly mixing transformation group, then $RP = X \times X$. If (X, T) is a minimal abelian transformation group for which $S_e = X \times X$, then (X, T) is weakly mixing.*

PROOF. The first statement in the theorem is easily verified. Suppose then that (X, T) is a minimal abelian transformation group for which $S_e = X \times X$. By a result of Veech [**5**, Theorem 1.1, p. 723], this implies that given $x \in X$ and an open neighborhood U of X, the set $\{xt_1t_2^{-1} : xt_1 \in U, xt_2 \in U\}$ is dense in X. Let nonempty open subsets A and B of X be given, and let $x \in A$. Then there are $t_1, t_2 \in T$ such that $xt_1 \in A$, $xt_2 \in A$, and $xt_1t_2^{-1} \in B$. Thus $A(t_1t_2^{-1}) \cap B \neq \emptyset$. Since $xt_2 \in A$, we have $x \in At_2^{-1}$ and $xt_1 \in At_2^{-1}t_1 = At_1t_2^{-1}$, and hence $xt_1 \in A(t_1t_2^{-1}) \cap A$. The lemma now implies that (X, T) is weakly mixing.

COROLLARY. *For a minimal abelian transformation group (X, T), with X compact metric, the following statements are equivalent:*

(1) *(X, T) is weakly mixing.*

(2) *$RP = X \times X$.*

(3) *$S_e = X \times X$.*

(4) *(X, T) has no nontrivial distal homomorphic image.*

PROOF. That (1), (2), and (3) are equivalent is direct from the theorem, remembering that $RP \subseteq S_e$. That (3) is equivalent to (4) is proved with the aid of [**2**, Theorem 2.4, p. 483].

By a *flow* (X, ϕ) we will mean a compact metric space X together with a homeomorphism $\phi : X \to X$. We denote by $\mathfrak{D}$ the class of all distal flows, by $\mathfrak{M}$ the class of all minimal flows, and by $\mathfrak{W}$ the class of all weakly mixing flows. If $\mathfrak{C}$ is a class of flows, $\mathfrak{C}^{\perp}$ will denote the class of all flows which are disjoint [**3**, Definition II.1, p. 24] from every flow in $\mathfrak{C}$.

THEOREM 2. $\mathfrak{D}^{\perp} = \mathfrak{W} \cap \mathfrak{M}$.

PROOF. It is proved in [**3**, Theorem II.3, p. 33] that $(\mathfrak{D} \cap \mathfrak{M})^{\perp} \supseteq \mathfrak{W}$, so we have $(\mathfrak{D} \cap \mathfrak{M})^{\perp} \cap \mathfrak{M} \supseteq \mathfrak{W} \cap \mathfrak{M}$. If $(X, \phi) \in (\mathfrak{D} \cap \mathfrak{M})^{\perp} \cap \mathfrak{M}$, then (X, ϕ) can have no nontrivial distal homomorphic image, since it would not be disjoint from such an image. By the above Corollary, then, $(X, \phi) \in \mathfrak{W} \cap \mathfrak{M}$. Therefore $(\mathfrak{D} \cap \mathfrak{M})^{\perp} \cap \mathfrak{M} = \mathfrak{W} \cap \mathfrak{M}$.

It is a result of Furstenberg's [**3**, Theorem II.1, p. 32] that if two flows are disjoint then one of them must be minimal. Since $\mathfrak{D}$ contains flows which are not minimal, we have $\mathfrak{D}^{\perp}\cap\mathfrak{M}=\mathfrak{D}^{\perp}$. Since $\mathfrak{D}\cap\mathfrak{M}\subseteq\mathfrak{D}$, we have $(\mathfrak{D}\cap\mathfrak{M})^{\perp}\supseteq\mathfrak{D}^{\perp}$, and hence $(\mathfrak{D}\cap\mathfrak{M})^{\perp}\cap\mathfrak{M}\supseteq\mathfrak{D}^{\perp}\cap\mathfrak{M}=\mathfrak{D}^{\perp}$. Thus $\mathfrak{W}\cap\mathfrak{M}\supseteq\mathfrak{D}^{\perp}$.

Now let $(X,\phi)\in\mathfrak{W}\cap\mathfrak{M}$ and $(Y,\psi)\in\mathfrak{D}$; we need to prove that (X,ϕ) and (Y,ψ) are disjoint. This amounts to showing [**3**, Lemma II.1, p. 24] that if V is a closed invariant subset of the flow $(X\times Y, \phi\times\psi)$ which projects onto X and onto Y, then $V=X\times Y$. Suppose that V is such a set. It is a well-known result of Ellis that, (Y,ψ) being distal, Y is the disjoint union of the minimal subsets Y_α of (Y,ψ). Thus it suffices to prove that $V\supseteq X\times Y_\alpha$ for each α. Now for each α, $(Y_\alpha,\psi)\in\mathfrak{D}\cap\mathfrak{M}$, so by the first part of the proof (X,ϕ) and (Y_α,ψ) are disjoint; therefore $(X\times Y_\alpha, \phi\times\psi)$ is minimal. Since $V\cap(X\times Y_\alpha)$ is a nonempty closed invariant subset of $(X\times Y_\alpha, \phi\times\psi)$, we must have $V\cap(X\times Y_\alpha)=X\times Y_\alpha$. Therefore $V=X\times Y$ and the proof is complete.

We remark that as an application of these results it can be proved that a minimal flow which is prime in the sense that it has no proper homomorphic images must be either a translation on a cyclic group of prime order or else weakly mixing. We conjecture that the corollary to Theorem 1 remains valid if the requirement that X be metrizable is dropped.

References

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