Disk Instantons, Mirror Symmetry and the Duality Web

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We apply the methods recently developed for computation of type IIA disk instantons using mirror symmetry to a large class of D-branes wrapped over Lagrangian cycles of non-compact Calabi-Yau 3-folds. Along the way we clarify the notion of "flat coordinates" for the boundary theory. We also discover an integer IR ambiguity needed to define the quantum theory of D-branes wrapped over non-compact Lagrangian submanifolds. In the large N dual Chern-Simons theory, this ambiguity is mapped to the UV choice of the framing of the knot. In a type IIB dual description involving (p, q) 5-branes, disk instantons of type IIA get mapped to (p, q) string instantons. The M-theory lift of these results lead to computation of superpotential terms generated by M2 brane instantons wrapped over 3-cycles of certain manifolds of G_2 holonomy.

Key words: Supersymmetry; Open String Theory; Topological Theories; Mirror Symmetry.

1. Introduction

D-branes wrapped over non-trivial cycles of a Calabi-Yau threefold provide an interesting class of theories with 4 supercharges (such as $N = 1$ supersymmetric theories in $d = 4$). As such, they do allow the generation of a superpotential on their worldvolume. This superpotential depends holomorphically on the chiral fields which parameterize normal deformations of the wrapped D-brane.

On the other hand, F-terms are captured by topological string amplitudes [1], and in particular the superpotential is computed by topological strings at the level of the disk amplitude [1 - 4]. More generally the topological string amplitude at genus g with h holes computes superpotential corrections involving the gaugino superfield W and the $N = 2$ graviphoton multiplet $\mathcal{W}$ given by $h \int d^2\theta (\text{Tr} W^2)^{h-1}(\mathcal{W}^2)^g$ [5]. So the issue of computation of topological string amplitudes becomes very relevant for this class of supersymmetric theories.

In the context of type IIA superstrings such disk amplitudes are given by non-trivial worldsheet instantons, which are holomorphic maps from the disk to the CY with the boundary ending on the D-brane. Such computations are in general rather difficult. The same questions in the context of type IIB strings involve classical considerations of the worldsheet theory. In a recent paper [6] it was shown how one can use mirror symmetry in an effective way to transform the type IIA computation of disk instantons to classical computations in the context of a mirror brane on a mirror CY for type IIB strings. The main goal of this paper is to extend this method to more non-trivial Calabi-Yau geometries.

One important obstacle to overcome in generalizing [6] is a better understanding of "flat coordinates" associated with the boundary theory, which we resolve by identifying it with BPS tension of associated domain walls. We also uncover a generic IR ambiguity given by an integer in defining a quantum Lagrangian D-brane. We relate this ambiguity to the choice of the regularizations of the worldsheet theory associated to the boundaries of moduli space of Riemann surfaces with holes (the simplest one being two disks connected by an infinite strip). In the context of the Large N Chern-Simons dual [7] applied to Wilson Loop observables [4] this ambiguity turns out to be related to the UV choice of the framing of the knot, which is needed for defining the Wilson loop observable by point splitting [8].

Along the way, for gaining further insight, we consider other equivalent dual theories, including the lift to M-theory, involving M-theory in a G_2 holonomy

background. In this context we are able to transform the generation of superpotential by Euclidean M2 branes (with the topology of S^3) to disk instantons of type IIA[1] for M-theory on G_2 holonomy manifolds and use mirror symmetry to compute them! We also relate this theory to another dual type IIB theory in a web of (p, q) 5-branes in the presence of ALF-like geometries.

The organization of this paper is as follows: In Sect. 2 we review the basic setup of [6]. In Sect. 3 we consider the lift of these theories to M-theory in the context of G_2 holonomy manifolds, as well as to type IIB theory with a web of (p, q) 5-branes in an ALF-like background. In Sect. 4 we identify the flat coordinates for boundary fields by computing the BPS tension of D4 brane domain walls ending on D6 branes wrapping Lagrangian submanifolds. In Sect. 5 we discuss the integral ambiguity in the computation of topological string amplitudes and its physical meaning. This is discussed both in the context of Large N Chern-Simons / topological string duality, as well as in the context of the type IIB theory with a web of (p, q) 5-branes. In Sect. 6 we present a large class of examples, involving non-compact CY 3-folds where the D6 brane wraps a non-compact Lagrangian submanifold. In appendix A we perform some of the computations relevant for the framing dependence for the unknot and verify that in the large N dual description this UV choice maps to the integral IR ambiguity we have discovered for the quantum Lagrangian D-brane.

2. Review of Mirror Symmetry for D-branes

In this section we briefly recall the mirror symmetry construction for non-compact toric Calabi-Yau manifolds (specializing to the case of threefolds), including the mirror of some particular class of (special) Lagrangian D-branes on them.

Toric Calabi-Yau threefolds arise as symplectic quotient spaces $X = \mathbf{C}^{3+k}//G$, for $G = U(1)^k$. The quotient is obtained by imposing the k D-term constraints

$$D^a = Q_1^a |X^1|^2 + Q_2^a |X^2|^2 + \ldots Q_{3+k}^a |X^{3+k}|^2 - r^a$$

$$= 0, \qquad (2.1)$$

where $a = 1, \ldots k$, and dividing by G

$$X^i \rightarrow e^{iQ_i^a \epsilon_a} X^i. \qquad (2.2)$$

The $c_1(X) = 0$ condition is equivalent to $\sum_i Q_i^a = 0$. The Kähler structure is encoded in terms of the r^a, and varying them changes the sizes of various 2 and 4 cycles. In the linear sigma model realization [9] this is realized as a (2,2) supersymmetric $U(1)^k$ gauge theory with $3 + k$ matter fields X^i with charges given by Q_i^a, and with k FI terms for the $U(1)^k$ gauge group given by r^a.

The mirror theory is given in terms of $n + k$ dual $\mathbf{C}^*$ fields Y^i [10], where

$$\mathrm{Re}(Y^i) = -|X^i|^2 \qquad (2.3)$$

with the periodicity $Y^i \sim Y^i + 2\pi i$. The D-term equation (2.1) is mirrored by

$$Q_1^a Y^1 + Q_2^a Y^2 + \ldots Q_{3+k}^a Y^{3+k} = -t^a, \qquad (2.4)$$

where $t^a = r^a + i\theta^a$ and θ^a denotes the θ-angles of the $U(1)^a$ gauge group. Note that (2.4) has a three-dimensional family of solutions. One parameter is trivial and is given by $Y^i \rightarrow Y^i + c$. Let us pick a parameterization of the two non-trivial solutions by u, v.

The mirror theory can be represented as a theory of variations of complex structures of a hypersurface Y

$$xz = e^{Y^1(u,v)} + \ldots + e^{Y^{k+3}(u,v)} \equiv P(u, v), (2.5)$$

where

$$Y^i(u, v) = a^i u + b^i v + t^i(t) \qquad (2.6)$$

is a solution to (2.4) (in obtaining this form, roughly speaking the trivial solution of shifting of all the Y^i has been replaced by x, z whose product is given by the above equation). We choose the solutions so that the periodicity condition of the $Y^i \sim Y^i + 2\pi i$ are consistent with those of u, v and that it forms a fundamental domain for the solution. Note that this in particular requires a^i, b^i to be integers. Even after taking these constrains into account there still is an SL(2, $\mathbb{Z}$) group action on the space of solutions via

$$u \rightarrow au + bv,$$

$$v \rightarrow cu + dv.$$

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[1] More generally we can map the generation of superpotential-like terms associated to topological strings at genus g with h boundaries to Euclidean M2 brane instantons on a closed 3-manifold with $b_1 = 2g + h - 1$.

Note that the holomorphic 3-form for CY is given by

$$\Omega = \frac{\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v}{x},$$

and is invariant under the SL(2, $\mathbb{Z}$) action.

2.1. Special Lagrangian Submanifolds and Mirror Branes

In [6] a family of special Lagrangian submanifolds of the A-model geometry was studied, characterized by two charges q_i^α with $i = 1, ..., k + 3$ and $\alpha = 1, 2$, subject to

$$\sum_i q_i^\alpha = 0$$

and in terms of which the Lagrangian submanifold is given by three constrains. Two of them are given by

$$\sum q_i^\alpha |X^i|^2 = c^\alpha \qquad (2.7)$$

and the third is $\sum \theta^i = 0$, where θ^i denotes the phase of X^i. The worldsheet boundary theory for this class of theories has been further studied in [11].

The submanifolds in question project to the one dimensional subspaces of the toric base (taking into account the constrains (2.1), (2.7)),

$$|X^i|^2 = r + b^i \qquad (2.8)$$

for some fixed b^i (depending on c^α, r^a) and $r \in \mathbf{R}^+$. In order to get a smooth Lagrangian submanifold one has to double this space (by including $\sum \theta^i = \pi$). The topology of the Lagrangian submanifold is $R \times S^1 \times S^1$. There is, however, a special choice of c^α which makes the Lagrangian submanifold pass through the intersection line of two faces of the toric base. The topology of the Lagrangian submanifold will be different in this limit. It corresponds to having one of the S^1 cycles pinched at a point of R in the Lagrangian submanifold. This is topologically the same as two copies of $\mathbf{C} \times S^1$ touching at the origin of $\mathbf{C}$. In this limit we view the Lagrangian submanifold as being made of *two* distinct ones intersecting over an S^1. We can now have a deformation, which moves the two Lagrangian submanifolds *independently*, where the end point of each one should be a point (not necessarily the same) on the base of the toric geometry (see the example below).

Under mirror symmetry, the A-brane maps to a holomorphic submanifold of the Y given by

$$x = 0 = P(u, v) = e^{Y^1(u,v)} + \ldots + e^{Y^{3+k}(u,v)}. \qquad (2.9)$$

The mirror brane is one-complex dimensional, and is parameterized by z. Its moduli space is one complex dimensional parameterized by a point on a Riemann surface $P(u, v) = 0$. The choice of the point depends on c^α and the Wilson line around S^1 and it is possible to read it off in the weak coupling limit of large volume of Calabi-Yau and large parameters c^α, as discussed in [6].

2.2. Example

For illustration consider $X = O(-1) \oplus O(-1) \to P^1$, which is also called small resolution of conifold. This sigma model is realized by $U(1)$ gauge theory with 4 chiral fields, with charges $Q = (1, 1, -1, -1)$. The D-term potential vanishes on $|X^1|^2 + |X^2|^2 - |X^3|^2 - |X^4|^2 = r$, and X is a quotient of this by $U(1)$. The D-term equations can be regarded as linear equations by projecting $X^i \to |X^i|^2$, and solved graphically in the positive octant of R^3 (see Fig. 1).

X is fibered over this base with a fiber which is torus of phases of X^i's modulo $U(1)$, $T^3 = T^4/U(1)$. Note that r is the size of a minimal $\mathbf{P^1}$ at $X^3 = 0 = X^4$.

Consider a special Lagrangian D-brane in this background with $q_1 = (1, 0, 0, -1)$, $q_2 = (0, 0, 1, -1)$. This gives the two constrains $|X^1|^2 - |X^4|^2 = c_1$ and $|X^3|^2 - |X^4|^2 = c_2$ in the base which determine a two dimensional family of Lagrangians, but D-branes of
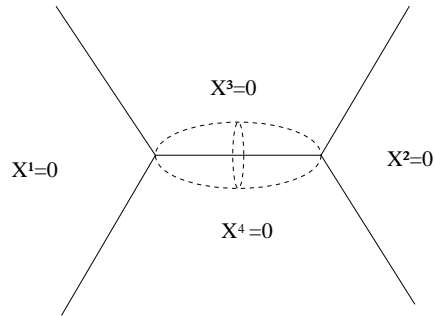

Fig. 1. $X = O(-1) \oplus O(-1) \to \mathbf{P}^1$ viewed as a toric fibration. The base is $(|X^1|^2, |X^3|^2, |X^4|^2)$ as generic solution to the vanishing of the D-term potential, but is bounded by the $|X^2|^2 \geq 0$ hyperplane. Over the faces of the bounding hyperplanes some cycles of the fiber shrink. For example, there is a minimal $\mathbf{P^i}$ in X which lies over the finite edge.
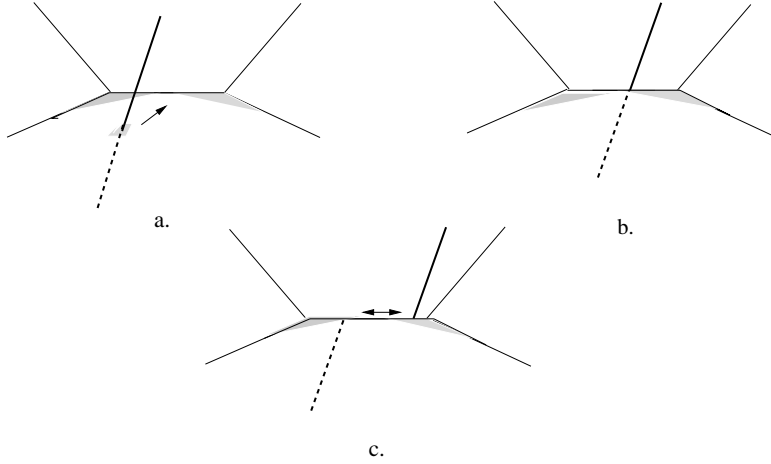
a.

b.

c.

Fig. 2. The special Lagrangian submanifold which has the topology $R \times T^2$ for generic values of c_i (case a) can degenerate (case b) and split (case c) into two Lagrangian submanifolds, when it approaches a one-dimensional edge of the toric base. The two resulting components have the topology $\mathbf{C} \times \mathbf{S}^1$, and can move independently, but only along one-dimensional edges.
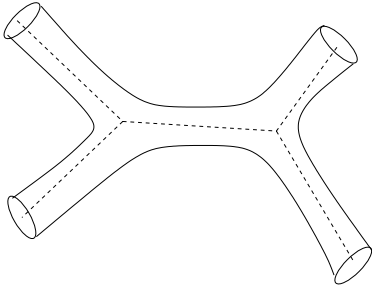

Fig. 3. Riemann surface $\Sigma : P(u,v) = 0$ corresponding to the mirror of $X = O(-1) \oplus O(-1) \to \mathbf{P}^1$. Σ is related to the toric diagram of X by thickening out the one-dimensional edges of the base in Figure 1.

$\longrightarrow$ infinity

Fig. 4. In the limit in which the size t of the $\mathbf{P}^1$ in $X = O(-1) \oplus O(-1) \to \mathbf{P}^1$ goes to infinity, the manifold looks locally like $\mathbf{C}^3$, together with a Lagrangian D-brane.

topology $\mathbf{C} \times S^1$ are further constrained to live on the one dimensional faces of the base. For this we need for example $c_2 = 0$, and c_1 arbitrary but in the $(0, r)$ interval. As discussed above, this can be viewed as coming from the deformation of a Lagrangian submanifold which splits to two when it intersects the edges of the toric geometry and move them independently on the edge. See Figure 2. Typically we would be interested in varying the position of one brane, keeping the other brane fixed (or taken to infinity along an edge).

The mirror of X is

$$xz = e^u + e^v + e^{-t-u+v} + 1,$$

obtained by solving $Y^1 + Y^2 - Y^3 - Y^4 = -t$ for Y^2, fixing the trivial solution by setting $Y^4 = 0$, and putting $Y^1 = u$ and $Y^3 = v$.
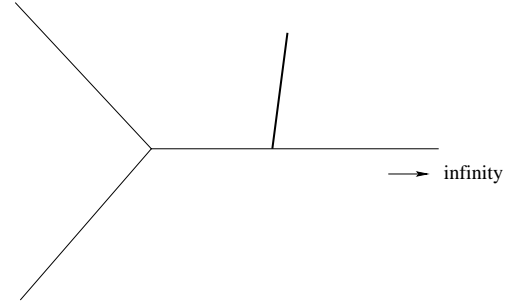
The mirror B-brane propagates on the Riemann surface $0 = P(u,v) = e^u + e^v + e^{-t-u+v} + 1$ shown in Figure 3.

Note that mirror map (2.3) gives the B-brane at $\mathrm{Re}(u) = -c_1$ and $\mathrm{Re}(v) = 0$ which is on the Riemann surface in the large radius limit, $r \gg 0$ and $r/2 > c_1 \gg 0$. In other words, in the large radius limit, classical geometry of the D-brane moduli space is a good approximation to the quantum geometry given by Σ.

We can also construct, as a limit, Lagrangian submanifolds of $\mathbf{C}^3$ by considering the limit $r + i\theta = t \to \infty$ holding c_1 fixed, as shown in Figure 4. In this limit the mirror geometry becomes $xz = e^u + e^v + 1$. This case was studied in detail in [12].

2.3. Disk Amplitude

The disk amplitudes of the topological A-model give rise to an $N = 1$ superpotential in the corre-

sponding type IIA superstring theory [1 - 4] where we view the D6 brane as wrapping the Lagrangian submanifold and filling the spacetime. The corresponding superpotential for the mirror of the Lagrangian submanifolds we have discussed above was computed in [6], and is given in terms of the Abel-Jacobi map

$$W(u) = \int_{u_*}^{u} v(u)\mathrm{d}u, \qquad (2.10)$$

where u_* is some fixed point on the Riemann surface $P(u, v) = 0$ and the line integral is done on this surface. This defines the superpotential up to an addition of a constant. More physically, if we construct the 'splitting' of the Lagrangian brane over the toric edges, we can view u_* as the location of one of the Lagrangian halves, which we consider fixed.

Note that, if we move the point u on the Riemann surface over a closed cycle and come back to the same point, the superpotential (2.10) may change by an overall shift which depends on the choice of the cycle as well as the moduli of the Riemann surface (given by t's). It is natural to ask what is the interpretation of this shift. This shift in superpotential can be explained both from the viewpoint of type IIA and type IIB. In the context of type IIA this corresponds to taking the Lagrangian D6 brane over a path whose internal volume traces a 4-dimensional cycle C_4 of CY (fixing the boundary conditions at infinity). By doing so we have come back to the same Brane configuration, but in the process we have *shifted* the RR 2-form flux. The 4-cycle C_4 is dual to a 2-form which we identify with the shift in the RR 2-form flux. In the type IIA setup this process changes the superpotential by (the quantum corrected) $\int_{C_4} k \wedge k$, as discussed in [13 - 15]. The Type IIB version of this involves varying the D5 brane wrapped over a 2-cycle over a path and bringing it back to the original place. During this process the brane traces a 3-cycle in the internal Calabi-Yau which contributes the integral of the holomorphic 3-form Ω over the 3-cycle to the superpotential. This is interpreted as shifting the RR flux of H along the dual 3-cycle. Note that we can use this idea to generate fluxes by bringing in branes not intersecting the toric edge, to the edges, splitting them on the edge and bringing it back together and then moving it off the toric edge. The process leads to the same CY but with some RR flux shifted.

The superpotential (2.10) is not invariant under different choices of parameterization of the fundamental domain for u, v given by an SL(2, $\mathbb{Z}$) transformation, but transforms as

$$W(u) \rightarrow W(u) + \int \mathrm{d}[acu^2/2 + bdv^2/2 - bcuv]$$

$$= W(u) + acu^2/2 + bdv^2/2 - bcuv,$$

where v is defined implicitly in terms of u by $P(u, v) = 0$. Note that if we added a boundary term it could have canceled this change in superpotential, which can be viewed as a choice of boundary condition at infinity on the non-compact brane [6]. Thus this IR choice is needed for the definition of the brane, and as we see it affects the physics by modifying the superpotential. As discussed in [6] the choice of the splitting to u, v depends on the boundary conditions at infinity on the fields normal to the brane. Each SL(2, $\mathbb{Z}$) action picks a particular choice of boundary conditions on the D-brane. Using the mirror symmetry and what A-model is computing, below we will be able to fix a canonical choice, up to an integer, which we will interpret physically.

As noted above, in terms of the topological A-model, superpotential W is generated by the disk amplitudes. The general structure of these amplitudes has been determined in [4], where it was found that

$$W = \sum_{k,n,\boldsymbol{m}} \frac{1}{n^2} N_{k,\boldsymbol{m}} \exp(n[ku - \boldsymbol{m} \cdot \boldsymbol{t}]). \quad (2.11)$$

Here u parameterizes the size of a non-trivial holomorphic disk, and where $N_{k,\boldsymbol{m}}$ are integers capturing the number of domain wall $D4$ branes ending on the $D6$ brane, which wrap the CY geometry in the 2-cycle class captured by $\boldsymbol{m}$, and k denotes the wrapping number around the boundary.

In the large volume limit (where the area of 2-cycles ending or not ending on the D-brane are large) the A-model picture is accurate enough. In this case we do not expect a classical superpotential as there is a family of special Lagrangian submanifolds. Since $dW/du = v$ and W should be zero for any moduli of the brane, we learn that $v = 0$ on the brane. This in particular chooses a natural choice of parameterization of the curve adapted to where the brane is. In particular the D-brane is attached to the line which is classically specified by $v = 0$ (which can always be done). u should be chosen to correspond to the area of a basic disk instanton. However this can be done in

many ways. In particular, suppose we have one choice of such u. Then

$$u \rightarrow u + nv$$

is an equally good choice, because v vanishes on the Lagrangian submanifold in the classical limit. So even though this ambiguity by an integer is irrelevant in the classical limit, in the quantum theory, since v is non-vanishing due to worldsheet instanton corrections, this dramatically changes the quantum answer. Thus we have been able to fix the u, v coordinates up to an integer choice n for each particular geometry of brane. We will discuss further the meaning of the choice of n in Sect. 3 and 4.

Later we will see that there is a further correction to what u, v are quantum mechanically. In particular, as we will discuss in Sect. 3, this arises because the quantum area of the disk differs from the classical computation which gives u. This is similar to what happens for the closed string theory where the parameter t which measures the area of the basic sphere is replaced by the quantum corrected area T. This is usually referred to as the choice of the "flat coordinates" for the Calabi-Yau moduli.

3. G_2 Holonomy and Type IIB 5-brane Duals

Consider type IIA superstrings on a non-compact Calabi-Yau threefold X with a special Lagrangian submanifold $L \subset X$. Consider wrapping a D6 brane around L and filling R^4. This theory has $N = 1$ supersymmetry on R^4 and we have discussed the superpotential generated for this theory. In this section we would like to relate this to other dual geometries.

3.1. M-theory Perspective

D6 branes are interpreted as KK monopoles of M-theory. This means that in the context of M-theory the theories under consideration should become purely geometric. This in fact was studied in [16 - 20], where it was seen that the M-theory geometry corresponds to a 7 dimensional manifold with G_2 holonomy. In other words we consider a 7-fold which is roughly $Y \sim X \times S^1$ where S^1 is fibered over the CY manifold X and vanishes over the location of the Lagrangian submanifold $L \subset X$. In this context the superpotentials that we have computed must be generated by
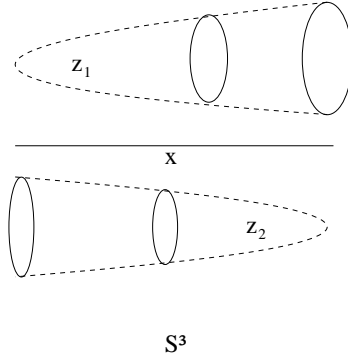

Fig. 5. We can view S^3 as an $S^1 \times S^1$ fibration over an interval. Near the ends of the interval it can be viewed as a complex plane $\mathbf{C} \times S^1$, where the complex plane is z_1 at one end and z_2 at the other. This gives two inequivalent descriptions of S^3 in terms of a circle fibered over a disk.

M2 brane instantons wrapping around non-trivial 3-cycles. Some examples of Euclidean M2 brane instantons for G_2 holonomy manifolds have been studied in [21] In fact there is a direct map from the disks ending on L to a closed 3-cycle with the topology of S^3. In order to explain this we first discuss some topological facts about S^3.

We can view S^3 as

$$|z_1|^2 + |z_2|^2 = 1$$

with z_i complex numbers. Let $x = |z_1|^2$. The range for x varies from 0 to 1. There is an $S^1 \times S^1$ of S^3 which projects to any fixed x with $0 < x < 1$, given by the phases of z_1 and z_2. At $x = 0$, the circle corresponding to the phase of z_1 shrinks and at $x = 1$ the circle corresponding to the phase of z_2 shrinks. So we can view the S^3 as the product of an interval with two S^1's where one S^1 shrinks at one end and the other S^1 shrinks at the other end. See Figure 5.

We can also view S^3 as a disk times a circle where the circle vanishes on one boundary – this can be done in two different ways, as shown in Figure 5.

Now we are ready to return to our case. Consider a disk of type IIA. The M2 brane Euclidean instanton can be viewed as the disk times an S^1, where the S^1 is the '11-th' circle. Note that on the boundary of the disk, which corresponds to the Lagrangian submanifold, the 11-th circle shrinks. Therefore, from our discussion above, this three dimensional space has the topology of S^3.

We have thus seen that using mirror symmetry, by mapping the type IIA geometry with a brane to

an equivalent type IIB with a brane, and computing the superpotential, in effect we have succeeded in transforming the question of computation of super-potentials generated by M2 brane instantons in the context of G_2 holonomy manifolds, to an application of mirror symmetry in the context of D-branes! More generally, one can in principle compute, using mirror symmetry [22], a partition function for higher genus Riemann surfaces with boundaries. This computes F-term corrections to the spacetime theory [5] given by $h \int d^4x d^2\theta F_{g,h}(\mathcal{W}^2)^g \text{Tr}(W^2)^{h-1}$, where $\mathcal{W}$ is the gravi-photon multiplet, and W the $N = 1$ gaugino superfield containing the $U(M)$ field strength on the worldvolume of M coincident KK monopoles (if we wish to get infinitely many such contributions we need $M \rightarrow \infty$). It is easy to see that the topology of the corresponding M2 brane instantons is a closed 3-manifold with $b_1 = 2g + h - 1$. The case of the ordinary superpotential is a special case of this with $g = 0, h = 1$.

3.2. Dual Type IIB Perspective

We have already given one dual type IIB theory related to our type IIA geometry, and that is given by the mirror symmetry we have been considering. However there is another type IIB dual description which is also rather useful.

Consider M-theory on a non-compact Calabi-Yau X compactified to 5 dimensions, which admits a T^2 action, possibly with fixed points. We can use the duality of M-theory on T^2 with type IIB on S^1 [23] to give a dual type IIB description for this class of Calabi-Yau manifolds. Note that the complex structure of the T^2 gets mapped to the coupling constant of type IIB. The non-compact Calabi-Yau manifolds we have been considering do admit T^2 actions and in this way they can be mapped to an equivalent type IIB theory. This in fact has been done in [24] where it was shown that this class of CY gets mapped to type IIB propagating on the web of (p, q) 5-branes considered in [25]. In this picture the 5-branes fill the 5 dimensional space time and extend along one direction in the internal space, identified with various edges of the toric diagram. The choice of (p, q) 5-branes encodes the (p, q) cycle of T^2 shrinking over the corresponding edge. The 5-branes are stretched along straight lines ending on one another and making very specific angles dictated by the supersym-
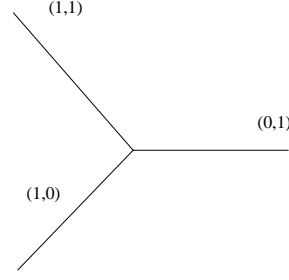

Fig. 6. A (p, q) web of 5-branes which is dual to M-theory on $\mathbf{C}^3$. This web is a junction involving a D5 brane which has $(p, q) = (1, 0)$, an NS5 brane which has $(0, 1)$ and a $(1, 1)$ brane.

metry requirement (balancing of the tensions) depending on the value of the type IIB coupling constant τ. In particular each (p, q) fivebrane is stretched along one dimensional line segments on a 2-plane which is parallel to the complex vector given by $p + q\tau$. An example of a configuration involving a D5 brane, NS 5 brane and a $(1,1)$ 5-brane is depicted in Figure 6.

Now we recall from the previous discussion that to get the G_2 holonomy manifold we need to consider an extra S^1 which is fibered over the corresponding CY. In other words we are now exchanging the '5-th' circle with the '11-th' circle. So we consider going down to 4 dimensions on a circle which is varying in size depending on the point in Calabi-Yau. In particular the circle (i.e. the one corresponding to the 5-th dimension) vanishes over a 2-dimensional subspace of 5-dimensional geometry of type IIB (it vanishes along the radial direction of the Lagrangian submanifold on the base of the toric geometry as well as on the S^1 which is dual to the T^2 of M-theory). Indeed it corresponds to putting the IIB 5-brane web in to a background of ALF geometry dictated by the location of the Lagrangian submanifold in the base times S^1, and varying the geometry and splitting the ALF geometry into two halves, as shown in Figure 2. In this picture the worldsheet disk instantons of type IIA get mapped to (p, q) Euclidean worldsheet instantons, wrapping the 5-th circle and ending on the 5-branes. In particular, if we follow the map of the Euclidean instanton to this geometry, it is the other disk realization of S^3 (see Fig. 7)[2].

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[2] Note that a D6 brane wrapped around S^3 is realized in type IIB as an NS 5-brane in the x direction, a $D5$ brane in the y direction separated in the z direction, and where the 5-th circle vanishes along the interval in the z direction joining the two branes.
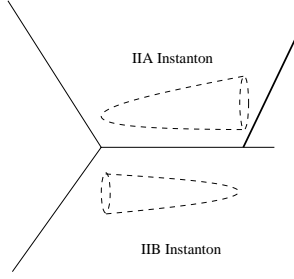
Fig. 7. The M2 brane instanton with a topology of S^3 wrapping over a 3-cycle of a local G_2 manifold gets mapped to two alternative disk projections of S^3, depending on which duality we use. In one case we get the description involving a type IIA string theory on a CY with the D-brane wrapped over a Lagrangian submanifold and in the other we get a web of IIB 5-branes in the presence of ALF like geometries.

4. Choice of Flat Coordinates

In this section we consider the map between the moduli of the brane between the A- and B-model.

As discussed in Sect. 2, the moduli of the D-branes in the A-model are labeled by c which measures the size of the disk instanton ending on the Lagrangian submanifold. In quantum theory c gets complexified by the choice of the Wilson line on the brane, and gets mapped to the choice of a complex point on the mirror type IIB geometry. The choice is characterized by the choice of a point on a Riemann surface $F(u, v) = 0$, which we choose to be our 'u' variable. However it could be that the 'size' of the disk instanton receives quantum corrections, and this, as we will now discuss, is relevant for finding the natural ("flat") coordinates parameterizing the moduli space of Lagrangian D-branes.

First we have to discuss what we mean by the "natural" choice of coordinate for the A-model. This is motivated by the integrality structure of the A-model expansion parameter. There is a special choice of coordinates [4] on the moduli space of D-branes in terms of which the A-model disk partition function has integer expansion (2.11), and this is the coordinate which measures the tension of the $D4$ brane domain walls. There is no reason to expect this to agree with the classical size of the disk that the B-model coordinate measures, and in general the two are not the same, as we will discuss below. This is what we take as the natural coordinates on the A-model side.

The B-model and the A-model are equivalent theories, and this dictates the corresponding flat coordinate on the B model moduli space. This is the tension of

the domain-wall D-brane which is mirror brane to the D4 brane of the A-model.

The D4 brane wrapping a minimal disk D is magnetically charged under the gauge field on the D6 brane. Consider the domain wall which in the $R^{3,1}$ is at a point in x_3 and fills the rest of the spacetime. The Bianchi identity for the gauge field-strength F on L, modified by the presence of the D4 brane, says that if the B is the cycle Poincaré dual to the boundary of the disk, $\partial D \subset L$. Recall that our brane L has the topology of $\mathbf{C} \times S^1$, so B can be identified with $\mathbf{C}$. Then the charge n of the domain wall is measured by

$$2\pi n = \int_B \mathrm{d}z^2 [F(x_3 = \infty) - F(x_3 = -\infty)]$$

$$= \int_{\partial B} \mathrm{d}s [A(x_3 = \infty) - A(x_3 = -\infty)]$$

Recall that $\mathrm{Im}(u)$ and $\mathrm{Im}(v)$ map to the one forms related to the $S^1 \times S^1$ cycles of the Lagrangian geometry, viewed as a cone over T^2. Thus $\mathrm{Im}(u)$ is the mirror of the Wilson-line $\int_{\partial D} \mathrm{d}s\, A$, and the Wilson-line around the dual $S^1 = \partial B$ is identified with $\mathrm{Im}(v)$. Thus we find that v jumps over the mirror domain wall by

$$v \to v + 2\pi i n.$$

The case of $n = 1$ is depicted in Figure 8.

This allows us to find the tension of the mirror domain wall as discussed in [6]. The BPS tension of the domain wall is given by ΔW, the difference of the superpotentials on the two sides of the domain wall. Since $W = \frac{1}{2\pi i} \int_{u_*}^u v\mathrm{d}u$, the tension of the BPS domain wall is simply the integral $\frac{1}{2\pi i} \int_{C_u} v\mathrm{d}u$, where C_u denotes the appropriate cycle shifting $v \to v + 2\pi i$, beginning and ending on a given u. We thus define the flat coordinate

$$\hat{u}(u, \boldsymbol{t}) = \frac{1}{2\pi i} \int_{C_u} v\mathrm{d}u. \tag{4.1}$$

To summarize, we predict that the disk partition function (2.10), expanded in terms of $\hat{u} = \hat{u}(u, \boldsymbol{t})$, and the corresponding closed string counterpart $\hat{t}(t)$– has the integral expansion (2.11), the coefficients of which count the "net number" of $D4$ brane domain walls ending on the Lagrangian submanifold L (for a more precise definition see [4, 26]).

It is not hard to see that u and $\hat{u}$ as defined above agree at the classical level and differ by instanton
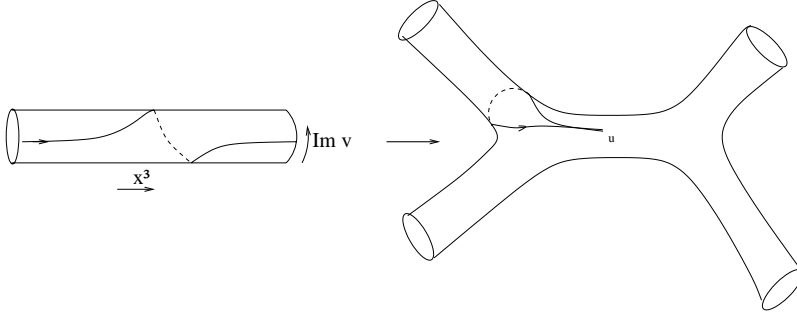
Fig. 8. The D4 brane ending on the D6 brane is mirrored to a domain wall in type IIB where v shifts by $2\pi i$ across it. This projects to a closed cycle on the Riemann surface $P(u, v) = 0$.

generated corrections. In the large radius limit, the local A-model geometry in the neighborhood of the disk D is just $\mathbf{C}^3$ – all toric vertices other than the one supporting D go away to infinity. In this limit, the equation of the mirror simply becomes $P(u, v) \rightarrow e^u + e^v + 1$, so

$$\hat{u} = \frac{1}{2\pi i} \int_{C_u} v(u)\mathrm{d}u \rightarrow \frac{1}{2\pi i} \int_{C_u} [\log(1 + e^u) + i\pi] \, \mathrm{d}u.$$

This has a branch point in the $u-$plane around which v has the monodromy $v \rightarrow v + 2\pi i$. The contour C_u receives a contribution only from the difference of values of v on the two sides of the cut, and thus for a single domain-wall $\hat{u} = u + i\pi$.

Away from the classical limit we can have subleading corrections to the above relation that can in principle be subleading in e^{-t} and in e^{-u}. But we will now argue that it is of the form

$$\hat{u} = u + \mathrm{const} + O(e^{-t}).$$

In other words, we show that $\delta \hat{u}/\delta u = 1$ is exact even away from the classical limit. This in fact is obvious from the definition of (4.1) because as we change u, the change in (4.1) can be computed from the beginning and the end of the path. But the integrands are the same except for the shift of v by $2\pi i$, and therefore the difference is given by $\delta \hat{u} = \delta \int \mathrm{d}u = \delta u$. We have thus shown that $\hat{u}$ differs from u by closed string instanton corrections only. Another way to see this is to note that

$$\hat{u} = \frac{1}{2\pi i} \int_{C_u} v du = \frac{1}{2\pi i}\Delta(uv) - \frac{1}{2\pi i} \int_{C_u} udv \, .$$

Noting that $\Delta(uv) = 2\pi iu$ due to the shift in v, and using the fact that u is not shifting and that dv is well defined, we deduce that the $-\frac{1}{2\pi i} \int_{C_u} udv = \Delta$

is independent of u, by the deformation of the contour C_u, and only depends on the class of the contour C. It thus depends only on the bulk moduli.

A cautionary remark is in order. We have talked about the cycle C_u on the B-model side as a cycle on the Riemann surface $F(u, v) = 0$. In general the cycles on the Riemann surface $F(u, v) = 0$ can be divided in those that lift to cycles where u and v come back to the original values, or those that shift by an integer multiple of $2\pi i$. The cycles that come back to themselves without any shifts in u and v correspond to closed 3-cycles in the underlying CY. Integration of vdu over those cycles correspond to computation of electric and magnetic BPS masses for the underlying $N = 2$ theory in 4 dimensions (and are relevant for the computation of the "flat" coordinate for the bulk field $\hat{t}(t)$). However, the cycles whose u or v values shift by an integer multiple of $2\pi i$ do not give rise to closed 3-cycles in the CY (as the CY in question does have non-trivial cycles corresponding to shifting u or v by integer multiples of $2\pi i$). Nevertheless, as we discussed above such cycles are important for finding the natural coordinates in the context of D-branes.

Note that closed string periods which determine $\hat{t}$ can also be expressed in terms of linear combinations of periods where the u's and v's shift. Thus computing periods where u and v shift are the fundamental quantities to compute. We will discuss these in the context of examples in Section 6.

Just as we have defined $\hat{u}$ as the quantum corrected tension of a domain wall, we can define $\hat{v}$ aa the quantum corrected tension of the domain wall associated with shifting $u \rightarrow u + 2\pi i$. Note that in the derivation of the superpotential [6] u, v are conjugate fields of the holomorphic Chern-Simons field. Thus replacing u by $\hat{u}$ will require[3] replacing v by the quantum cor-

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[3]To see this from the target space viewpoint it is natural to consider the 1+1 realization of this theory as D4 brane wrapped

rected conjugate field $\hat{v}$, and so the equation satisfied by the superpotential changes to

$$\frac{\partial W}{\partial u} = v \rightarrow \frac{\partial W}{\partial \hat{u}} = \hat{v}$$

which is the equation we will use in Sect. 6 to compute W.

5. Quantum Ambiguity for Lagrangian Submanifold

We have seen that the choice of flat coordinates naturally adapted to the A-model Lagrangian D-branes are fixed up to an integer choice. In particular we found that if u, v are complex coordinates satisfying $P(u, v) = 0$, and if the brane is denoted in the classical limit by $v = 0$ and u classically measures the size of the disk instanton, then we can consider a new u given by

$$u \rightarrow u + nv$$

for any n, which classically still corresponds to the disk instanton action. In this section we explain why fixing the arbitrary choice is indeed needed for a *quantum* definition of the A-model Lagrangian D-brane. In particular specifying the A-model Lagrangian D-brane just by specifying it as a classical subspace of the CY *does not uniquely fix the quantum theory*, given by string perturbation theory. The choice of n reflects choices to be made in the quantum theory, which has no classical counterpart. In this section we show how this works in two different ways: First we map this ambiguity to an UV Chern-Simons ambiguity related to framing of the Wilson Loop observables. Secondly we relate it to the choice of the Calabi-Yau geometry at infinity, and for this we use the type IIB 5-brane web dual, discussed in Section 3.

5.1. Framing Choices for the Knot

To see how this works it is simplest to consider the case where the D-brane topological amplitudes were

computed using the observables of the Chern-Simons theory [4, 26, 27] These were obtained by considering expectation values for Wilson loop observables in the large N Chern-Simons theory, in the context of the large N duality of Chern-Simons / closed topological strings proposed in [28]. Let us briefly recall this setup.

Consider the $SU(N)$ Chern-Simons theory on S^3. As was shown in [29], if we consider the topological A-model on the conifold, which has the same symplectic structure as T^*S^3, and consider wrapping N D3 branes on S^3, the open string field theory living on the D3 brane is $SU(N)$ Chern-Simons theory where the level of the Chern-Simons theory (up to a shift by N, i.e., $g_s = 2\pi i/(k + N)$) is identified with the inverse of the string coupling constant. The large N duality proposed in [28] states that this topological string theory is equivalent to topological strings propagating on the non-compact CY 3-fold $O(-1) \oplus O(-1) \rightarrow \mathbf{P}^1$, which is the resolution of the conifold, where the complexified Kähler class on $\mathbf{P}^1$ has the size $t = N g_s$. In [4] it was shown how to use this duality to compute Wilson loop observables. The idea is that for every knot $\gamma \subset S^3$ one considers a non-compact Lagrangian submanifold $L_\gamma \subset T^*S^3$ such that $L_\gamma \cap S^3 = \gamma$. We wrap M D3 branes over L_γ, which gives rise to an $SU(M)$ Chern-Simons gauge theory on L_γ. In addition, bi-fundamental fields on γ transforming as $(N, \overline{M})$ arise from open strings with one end on the D-branes wrapped over S^3 and with the other end on D-branes wrapped over L_γ. Integrating out these fields give rise to the insertion of

$$\left\langle \exp\left(\sum_n \frac{\mathrm{tr}U^n \mathrm{tr}V^n}{n} \right) \right\rangle,$$

where U and V denote the holonomies of the $SU(N)$ and $SU(M)$ gauge groups around γ respectively. Considering the $SU(M)$ gauge theory as a spectator, we can compute the correlations of the $SU(N)$ Chern-Simons theory and obtain

$$\left\langle \exp\left(\sum_n \frac{\mathrm{tr}U^n \mathrm{tr}V^n}{n} \right) \right\rangle = \exp(-F(V, t, g_s)) \; (5.1)$$

It was shown in [4] that the right-hand side can be interpreted as the topological string amplitude in the large N gravitational dual, where the N D-branes have disappeared and are replaced by S^2. In this dual

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over the Lagrangian submanifold. Then, as discussed in [4] the disk amplitude computes $S = \int d^2x d^2\theta (dW/d\Sigma)\Sigma$ for the $U(1)$ gauge theory in $1 + 1$ dimension, where Σ is the twisted chiral gauge field strength multiplet whose bottom component is $\hat{u}$. In this formulation the domain wall associated with shifting of Σ to $\Sigma + 2\pi i$ is realized by $u \rightarrow u + 2\pi i$, whose BPS mass we have denoted by $\hat{v}$. From S, the change in the value of the superpotential under shifting Σ is given by $dW/d\hat{u}$ which leads to the statement that $dW/d\hat{u} = \hat{v}$. This provides an alternative, and more physical derivation of the main formula we use for the computation of W.
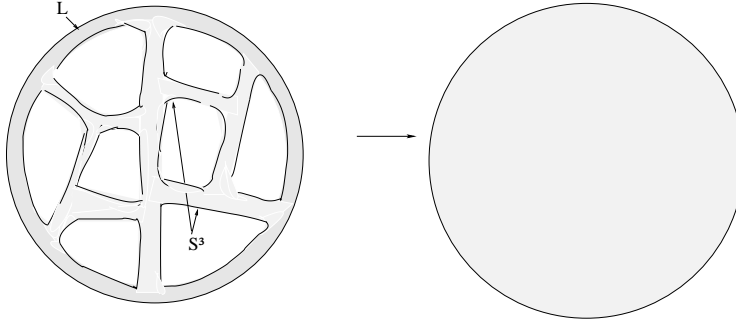
Fig. 9. The Wilson loop observables arise from worldsheet diagrams where some boundaries end on L_γ and N branes wrapping S^3. In the large N limit the holes ending on S^3 get "filled" and we end up with a Riemann surface which has only the boundaries associated with L_γ. In the above figure the outer hole is the only one ending on L_γ. All of the interior holes end on S^3 and disappear in the large N limit, leaving us with a disk.

geometry the M non-compact D-branes are left-over and wrapped over some Lagrangian submanifold in $O(-1) \oplus O(-1) \to \mathbf{P}^1$. This Lagrangian submanifold was constructed for the case of the unknot explicitly in [4] and extended to algebraic knots in [26] (this latter construction has been recently generalized to all knots [30]). Moreover $F(V, t, g_s)$ denotes the topological string amplitude in the presence of the M D-branes wrapped over some particular Lagrangian submanifold in $O(-1) \oplus O(-1) \to \mathbf{P}^1$ with a nontrivial S^1 cycle. Note that a term in $F(V, t, g_s)$ of the form $\prod_{i=1}^b \mathrm{tr} V^{k_i}$ comes from a worldsheet with b boundaries, where the i-th boundary of the worldsheet wraps the S^1 of the Lagrangian submanifold k_i times.

The left-hand side of (5.1) is computable by the methods initiated in [8], and in this way gives us a way to compute the open string topological amplitudes for this class of D-branes. Note in particular that the disk amplitude corresponds to the $1/g_s$ term in $F(V, t, g_s)$. A particular case of the brane we have considered in $O(-1) \oplus O(-1) \to \mathbf{P}^1$ corresponds to the unknot. This is depicted in the toric Figure 2.

The match between the computation in this case using mirror symmetry, and the result expected from the Chern-Simons theory was demonstrated in [6]. However as we have discussed here the disk amplitudes have an integer ambiguity when we use mirror symmetry for their computation. Thus apparently the right hand side of (5.1) is defined once we pick an integer related to the boundary conditions at infinity on the B-brane in the type IIB mirror. If the right hand side of (5.1) is ambiguous, then so should the left hand side. In fact the computation of Wilson loop observables also has an ambiguity given by an integer! In particular we have to choose a framing on the knot γ to make the computation well defined in the quantum theory [8]. A framing is the choice of a normal vector field on the knot γ, which is non-vanishing everywhere on the knot. Note that if we are given a framing of the knot, any other topologically distinct framing is parameterized by an integer, given by the class of the map $S^1 \to S^1$, where the domain S^1 parameterizes γ and the range denotes the relative choice of the framing which is classified by the direction of the vector field on the normal plane to the direction along the knot. The framing of the knot enters the gauge theory computation by resolving UV divergencies of the Chern-Simons theory in the presence of Wilson loops. It arises when we take the Greens function for the gauge field coming from the same point on the knot. The framing of the knot allows a point splitting definition of the Greens function.

We have thus seen that both sides of (5.1) have a quantum ambiguity that can be resolved by a choice of an integer. On the left hand side the ambiguity arises from the UV. On the right hand side the ambiguity arises from the IR (i.e. boundary conditions on the brane at infinity). We have checked that the two ambiguities match for the case of the unknot, by comparing the disk amplitudes on both sides (using CS computation of the framing dependence of the knot on the left and comparing it with the mirror symmetry computation of the knot on the right). Some aspects of this computation are presented in the Appendix A. The computation of the disk amplitude for this case, using mirror symmetry, is presented in Section 6.

Here let us discuss further how this match arises. Consider the disk amplitude at large N corresponding to a given knot. In the gauge theory side the computation arises from open string diagrams of a planar diagram with the outer hole on the Lagrangian submanifold L_γ, and the rest of the holes ending on D-branes wrapping the S^3, as shown in Figure 9. In the large N limit, the interior holes get "filled" and we get the topology of the disk.

Short Propagator
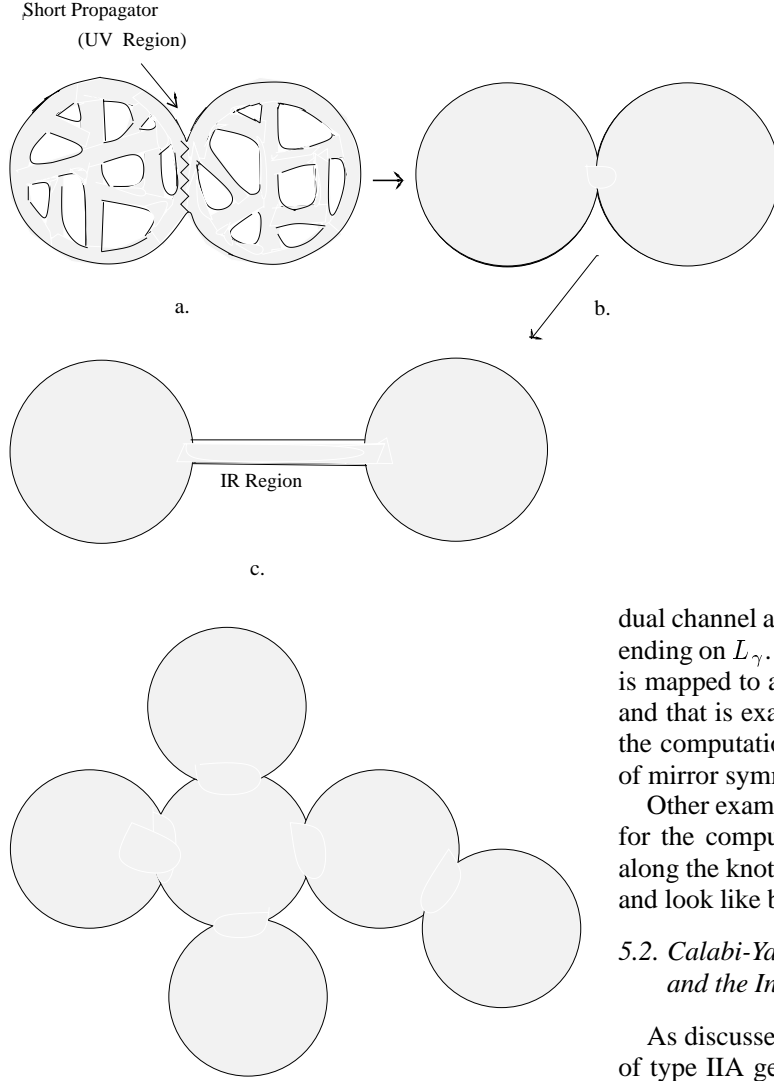(UV Region)

a.

b.

IR Region

c.

Fig. 10. The Wilson loop observable has UV divergencies which need to be regulated, coming from points along the knot where the gauge boson propagator is of zero size (a). In the large N limit these map to disks touching at a point (b), which can be viewed via a conformal transformation as a long propagator (c). Note that the boundaries of the long propagator are on the Lagrangian submanifold L_γ (or more precisely its large N dual) and correspond to an open string propagating on it. Thus the UV framing choice of CS gets mapped to the choice of large distance (IR) physics of modes on the brane.
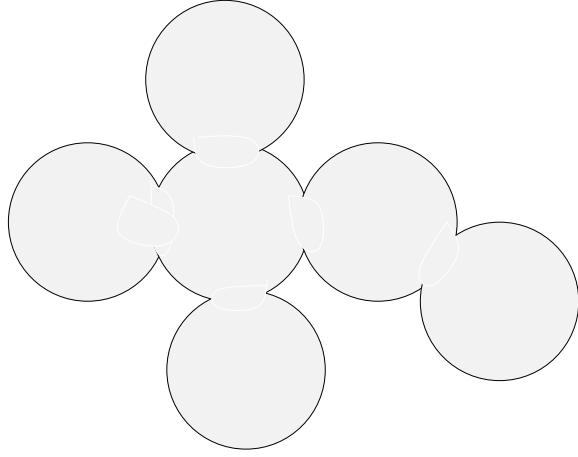
Fig. 11. Other examples of the large N limit of UV divergencies of Wilson loop observables.

Now consider where the UV divergencies of the gauge theory would arise. They would arise from Feynman diagrams where the Schwinger parameter for the gauge field goes to zero–an example of this is depicted in Figure 10a. Note that the two end points of the short propagator will be mapped to the same point on the knot γ in the limit of zero length propagator. In the large N limit, where the disk gets filled, these get mapped to configurations such as that shown in Figure 10b. In this dual description by a conformal transformation the worldsheet can be viewed as that depicted in Figure 10c. In other words, we have in the

dual channel a long Schwinger time of an open string ending on L_γ. This means that the issue of ambiguity is mapped to an IR behaviour of fields living on L_γ, and that is exactly where we found the ambiguity in the computation of the superpotential in the context of mirror symmetry.

Other examples of the large N limit of UV regions for the computation of the Wilson loop observable along the knot get mapped to disks shown in Fig. 11, and look like branched trees of disks.

5.2. Calabi-Yau Geometry, 5-brane Perspective and the Integral Ambiguity

As discussed in Sect. 3, we have dual descriptions of type IIA geometry with D6 branes wrapped over Lagrangian submanifolds in terms of M-theory on G_2 holonomy manifolds (viewed as circle fibration over CY manifolds) or in terms of type IIB web of 5 branes in an ALF-like background in R^6. To be precise, the M-theory on T^2/type IIB on S^1 duality relates the coupling constant of type IIB to the complex structure of the T^2. Fibering this duality gives rise to the duality just mentioned. However, in the type IIB picture we typically fix the type IIB coupling at infinity. This means that, by this chain of duality, the G_2 holonomy manifold is a circle fibration over the CY where the complex structure of the T^2 fibration of CY is fixed at infinity. Turning this around, this duality predicts the existence of a particular class of CY and G_2 holonomy metrics with a particular behaviour of the metric

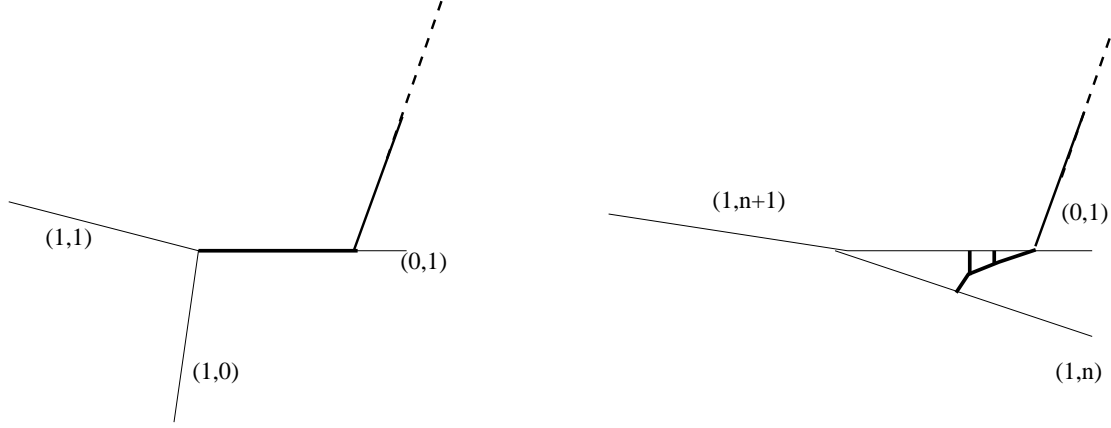Fig. 12. In the weak coupling limit, we consider an NS $(0, 1)$ 5-brane with an ALF like space ending on it. If this is connected to $(1, 0)$ and $(1, 1)$ 5-branes in a junction, the only instantons allowed involve the worldsheet instanton stretched along the interval shown on the left-hand side of the figure above, times the 5-th circle which shrinks at the attachment point of the ALF space. There is no other instanton allowed in this geometry. However if an NS $(0, 1)$ 5-brane is attached to $(1, n)$ and $(1, n + 1)$ 5-branes, the geometry of the intersection dramatically changes (as long as $n \neq 0, -1$) and now we can have many more allowed configurations for the worldsheet (p, q) instantons. This is depicted on the right hand side of the above figure.

at infinity (we can also fix the area of T^2 at infinity as that gets mapped to the inverse of the radius of type IIB S^1). We now argue that the choice of the complex structure of T^2, or equivalently the type IIB coupling constant at infinity, *changes* the number of Euclidean M2 branes, exactly as is expected by the ambiguity. Instead of being general we simply illustrate this idea in the context of a simple example, namely the $\mathbf{C}^3$ geometry discussed before and represented by Fig.6, consisting of NS 5-brane $(0, 1)$ and D5-brane $(1, 0)$ and the $(1, 1)$ 5-brane. We already did show that M2 brane instantons get mapped to worldsheet disk instantons consisting of the (p, q) string shown in Fig. 7 times an extra circle (the '5-th' circle) which vanishes at the position of the ALF space. However now we consider changing the coupling constant. Then the angles between the 5-branes will be changed. Beyond some critical angles we *can* get new worldsheet instantons. Basically we can consider the web of strings, and the only data that we need to take into account is that a BPS (p, q) string can end only on a (p, q) 5-brane (and perpendicularly). This web of strings, however, can end on the projection of the ALF space on the web at any angle, since at that point now the '5-th' circle is shrinking (again in a perpendicular fashion). To be concrete, let us consider the limiting choices of the type IIB coupling constant given by $\tau \to -1/n$. This can be obtained from $\tau \to i\infty$ by the modular trans-

formation $\frac{-1}{\tau} \to \frac{-1}{\tau} + n$. Said differently, we can take the weak coupling limit $\tau \to i\infty$ by considering the inverse modular transformation where we consider the 5-brane web given by the NS fivebrane $(0, 1)$ and the fivebranes $(1, n)$ and $(1, n+1)$. It is easy to see that this geometry of webs has an $n \to -(n + 1)$ symmetry, when we reverse the direction of the handedness on the D5 brane. Thus, the counting of the worldsheet instantons here will exhibit this symmetry. This is exactly the symmetry we will find for the ambiguity of the mirror to $\mathbf{C}^3$, as discussed in Section 6. Moreover it is easy to see that except for $n = 0, -1$, where there is exactly one choice of disk instanton, for all other n's we get a large number of disk instantons, allowed by the geometry of the 5-branes, as shown in Figure 12. This is also in line with the result we find from mirror symmetry in Section 6. Thus as the geometry of Calabi-Yau changes, we get jumps in the number of instantons as predicted by this picture.

It is natural to ask what IR aspect of the type IIA CY geometry with the brane, this choice of τ is reflected in. One's natural guess is that the normalizability of the 1-form on the brane represented by the Wilson line vev, which represents the mode corresponding to moving the brane, is the relevant issue. If this is the case the possible choices of metrics for which a particular 1-form is normalizable will have to be classified by the integer ambiguity n, which in turn is

related to the choice of the CY metric with fixed τ at infinity, labeled by n. Similarly, in the M-theory lift this would be related to the possible choices of normalizable deformations of the G_2 holonomy metric. It would be interesting to study these issues further.

6. Examples

In this section we will consider a number of examples which illustrate the general discussion we have presented for the computation of disk amplitudes for D-branes wrapping a Lagrangian submanifold of non-compact CY 3-folds. The examples include considerations of Lagrangian branes in $\mathbf{C}^3$, $\mathbf{P}^1 \times \mathbf{P}^1$, $\mathbf{P}^2$ and its blow up at a point $\mathbf{F}_1$. In these examples we consider inequivalent configurations of Lagrangian branes. For the case of $\mathbf{C}^3$ we exhibit the dependence of the topological string computations on the integer ambiguity we discussed earlier. For all of these cases we find the predicted integrality properties of the disk amplitudes[4]. We also consider different Lagrangian branes (for example ending on the "internal" or "external" edges of the toric diagrams).

As discussed in Sect. 4, a non-trivial part of the story involves finding the flat coordinates for the open string variables. As discussed there, we find the flat coordinates to be given by

$$\hat{u} = u + \Delta(t_i),$$

where $\Delta(t_i)$ (up to an addition of a constant) is an exponentially suppressed function of Kähler moduli of the Calabi-Yau ($O(e^{-t_i})$), and Δ depends on the choice of the brane. We compute Δ using the methods discussed in Sect. 4 for all the examples.

Note that using $\partial_{\hat{u}} W = \hat{v}(\hat{u})$ and (2.11), we have

$$\partial_{\hat{u}} W = \hat{v}(\hat{u}) = -\sum_k k N_{k,\boldsymbol{m}} \log(1 - \exp[k\hat{u} - \boldsymbol{m} \cdot \hat{\boldsymbol{t}}]).$$

So by solving for $\hat{v}$ in terms of $\hat{u}$ (from $P(u, v) = 0$ and the correction to $u, v \to \hat{u}, \hat{v}$ due to the mirror map) we find a prediction of an expansion in terms of integers $N_{k,\boldsymbol{m}}$. We verify these highly non-trivial integrality predictions in the examples below which we now turn to.

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[4] The integrality requirement is highly non-trivial. Originally, we experimentally found the highly non-trivial flat coordinates only by requiring the integrality of the domain wall degeneracies. Later we showed that they agree with the BPS tension of domain walls, which is the definition used in this section.

6.1. Almost $\mathbf{C}^3$

We consider the simplest Calabi-Yau $X = \mathbf{C}^3$. It is described by just three chiral fields X^i and no gauge group. The special Lagrangian D-brane L we are interested in is given by $q_1 = (1, 0, -1)$ and $q_2 = (0, 1, -1)$:

$$|X^1|^2 - |X^3|^2 = c_1 \quad |X^2|^2 - |X^3|^2 = 0.$$

This geometry can be regarded as a local approximation, in the limit that all radii are large, to a more involved geometry it is embedded in, for example the one discussed in Section 2. The value of c_1 on the brane and the Wilson-line around the one finite circle on L form one complex modulus u of the A-brane, which measures the classical BPS tension of the D4 brane domain wall.

The mirror manifold can be written as

$$xz = P(u, v) = e^u + e^v + 1, \tag{6.1}$$

where we have set Y^3 to zero, and $Y^1 = u$, $Y^2 = v$.

Equations (2.3) fix the classical limit of the brane

$$c^1 = \mathrm{Re}(u),$$

so the transverse coordinate to the B-brane is u. In this limit, v is zero on the brane corresponding to the vanishing of the classical superpotential.

The SL(2, $\mathbb{Z}$) group of reparametrizations acts on (6.1) by linear transformations $\begin{pmatrix} \hat{u}' \\ \hat{v}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$ that leave the holomorphic $(3, 0)$ form $\Omega = \frac{\mathrm{d}z}{z} \mathrm{d}u \mathrm{d}v$ invariant. The subgroup of SL(2, $\mathbb{Z}$) that preserves the classical limit above consists of transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$. These transformations result in a family of parameterizations of the mirror geometry

$$xz = P_p(u, v) = e^{u+pv} + e^v + 1,$$

and a family of superpotentials obtained by solving the order p polynomial in e^v,

$$W_p(u) = \int v_p(u) du.$$

To compute the numbers of disk domain walls we need to know the flat coordinates.

For $p = 0$ we have computed in Sect. 4 the BPS tension of the D-brane in the u-phase and also the v−phase, by trivial relabeling, to be $\hat{u}_{p=0} = u + i\pi$, $\hat{v}_{p=0} = v + i\pi$. The knowledge of these suffices to find the BPS tensions for a different choice of framing p.

The SL(2, $\mathbb{Z}$) transformations $u \rightarrow u + pv$, $v \rightarrow v$ act on the basis of one-cycles on the Riemann surface that are associated to periodicity of Im(u) and Im(v). Since Δ_u and Δ_v which provide the quantum corrections to u and v

$$\hat{u} = u + \Delta_u, \ \hat{v} = v + \Delta_v,$$

are obtained by evaluating periods of the one-form $\lambda = vdu$ over these, changing the basis of one-cycles by an SL(2, $\mathbb{Z}$) transformation to $u' = au + bv$ and $v' = cu + dv$ acts in an analogous way on the periods, i. e.

$$\Delta'_u = a\Delta_u + b\Delta_v.$$

Thus, we find that

$$\hat{u}_p = u + (p + 1)i\pi, \ \hat{v}_p = v + i\pi.$$

The equation

$$P_p(\hat{u}, \hat{v}) = 0 = 1 - e^{\hat{u}+p\hat{v}} - e^{\hat{v}} \qquad (6.2)$$

is solved iteratively, using the ansatz $e^{\hat{v}} = \sum_{k=0}^{\infty} a_k e^{k\hat{u}}$. Using $(\sum_{k=0}^{\infty} a_k x^k)^p = \sum_{m=0}^{\infty} c_m x^m$, where $a_0 = c_0 = 1$ and $c_m = \frac{1}{m}\sum_{k=1}^{m}(kp - m + k)a_k c_{m-k}$ we get immediately a recursive formula $a_m = \frac{1}{m-1}\sum_{m=1}^{m-1}(kp - m + k + 1)a_k a_{m-k}$ for the coefficients. This can be summed using Stirlings coefficients of the first kind (see e. g. [31]), which are defined by $x(x - 1)(x - 2) \cdot \ldots \cdot (x - n + 1) = \sum_{m=0}^{n} S_n^{(m)} x^m$. Using the relation $\binom{m}{r} S_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} S_{n-k}^{(r)} S_k^{(m-r)}$ the result of the summation is

$$a_m = \frac{(-1)^m}{m!} \prod_{j=0}^{m-2}(mp - j). \qquad (6.3)$$

Solving now for $\hat{v} = \partial_{\hat{u}}W$ and integrating we get, up to trivial integration constants, the superpotential

$$W = \sum_{m=1}^{\infty} \frac{1}{mm!} \prod_{j=1}^{m-1}(mp - j)e^{mu}. \qquad (6.4)$$

We can write this in the general form (2.11). The case at hand, that is given by $W = \sum_{m,k} \frac{N_m}{k^2}e^{km\hat{u}}$, yields the following integers for N_m:

$$N_1 = (-1)^p,$$

$$N_2 = -\frac{p}{2} + \frac{1 - (-1)^p}{4} = -\left[\frac{p}{2}\right],$$

$$N_3 = \frac{(-1)^p}{2}p(p - 1),$$

$$N_4 = -\frac{1}{3}(2p - 1)(p - 1)p,$$

$$N_5 = \frac{5}{24}(-1)^p p(p - 1)(5p^2 - 5p + 2),$$

$$N_6 = \frac{1}{20}(p - 1)p(36p^3 - 54p^2 + 31p - 9),$$

$$N_7 = \frac{7}{720}(-1)^p p(p - 1)$$

$$\cdot (343p^4 - 686p^3 + 539p^2 - 196p + 36),$$

$$\vdots$$

To show integrality for a given expression $N_m(p)$ for all $p \in \mathbb{Z}$, one may factorize the denominator into $\prod_i p_i^{n_i}$ with p_i prime and check that one can factor $p_i^{n_i}$ from the numerator for all $p = p_i^{n_i}n - k_i$ with $k_i = 1, \ldots, p_i^{n_i}$. That has been checked for $m \leq 50$.

Note that N_m is a polynomial in p of degree $m - 1$. As discussed in Sect. 5, the p dependence of the superpotential for this particular Lagrangian brane (viewed as a large t limit of the $O(-1) \oplus O(-1) \rightarrow \mathbf{P}^1$ geometry) can be mapped to the framing ambiguity of the unknot for the Chern-Simons theory. In Appendix A we discuss this computation and verify that the large N Chern-Simons computation leads to a polynomial of degree $m - 1$ in p for N_m. Moreover we have verified that for N_i for $i = 1, 2, 3$ the p dependence of the framing choice agrees with the above result. We also have shown from the computation of the framing

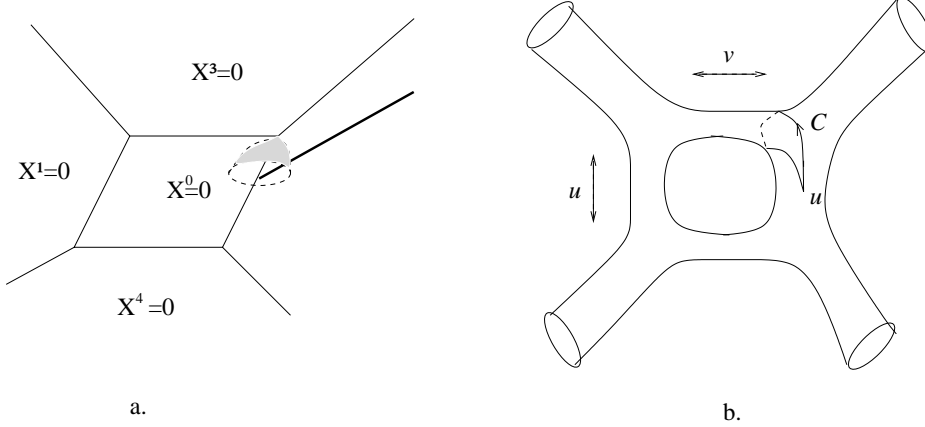a.   b.

Fig. 13. The Fig. a. depicts the base of the toric fibration for $X = O(K) \to \mathbf{P}^1 \times \mathbf{P}^1$ together with a special Lagrangian brane in X. The minimal disk which ends on the A-brane in the figure wraps partly the $\mathbf{P}^1$ over which the brane is. Figure b. depicts the Riemann surface Σ of the mirror geometry, and the mirror B-brane as a choice of a point on Σ. The D4 brane wrapping the disk in Fig. a. is mirror to a domain-wall which projects to path $\mathcal{C}$ in Figure b.

dependence of the CS theory that for all m the coefficient of the leading term p^{m-1} is given by $m^{m-2}/m!$ in agreement with the above result.

Results of the prediction of [4] for the topological string amplitudes for the case of the unknot have recently been verified mathematically using localization methods in [32, 33]. In verifying the predictions of [4], there were some choices made for the toric action in these works. The authors of [32] have recently checked and verified that the integer choice of ambiguity changes the topological string amplitudes exactly as predicted by the above results[5]. Moreover this ambiguity arises from the boundary of moduli space of Riemann surfaces with holes, as we already discussed.

6.2. $\mathcal{O}(K) \to \mathbf{P}^1 \times \mathbf{P}^1$

This manifold involves a compact $\mathbf{P}^1 \times \mathbf{P}^1$ geometry inside a CY 3-fold with a non-compact direction given by the canonical line bundle. It can be described by a linear sigma model with $G = U(1)^2$ and five chiral fields X^i, for $i = 0, \ldots 4$ with charges $Q^1 = (-2, 1, 1, 0, 0)$ and $Q^2 = (-2, 0, 0, 1, 1)$ which leads to the D-terms

$$|X^1|^2 + |X^2|^2 - 2|X^0|^2 = r_t,$$

[5]We are grateful to S. Katz and C.-C. Liu for performing these computations upon our request. The form we have presented our answer (6.4) was chosen to simplify the comparison with their formula.

$$|X^3|^2 + |X^4|^2 - 2|X^0|^2 = r_s.$$

The solutions to the D-term equation, projected to the toric base are given in Figure 13. The $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ corresponds to the divisor class represented by $X^0 = 0$ and is visible in Fig. 13 as the minimal parallelogram in the toric base. The fiber $O(K)$ corresponds to the normal direction. The special Lagrangian D-branes of the A-model for this Calabi-Yau have inequivalent phases, depending on where one puts the brane, together with a $\mathbf{Z}$-family of choices of framing in each. The D-brane charge can be taken to be $q_1 = (-1, 0, 1, 0, 0)$, $q_2 = (-1, 0, 0, 1, 0)$ and so

$$|X^2|^2 - |X^0|^2 = c^1, \ |X^3|^2 - |X^0|^2 = c^2.$$

The constants c_i are required to be chosen so that the Lagrangian D-brane lies on one dimensional toric edges, and different toric legs give rise to generally different phases of the theory.

The mirror variables Y^i satisfy

$$Y^1 + Y^2 - 2Y^0 = -t, \quad Y^3 + Y^4 - 2Y^0 = -s,$$

where the real parts of the complex structure parameters t, s measure the classical sizes of the two minimal P^1's in the toric base, $\mathrm{Re}(t) = r_t, \mathrm{Re}(s) = r_s$. The mirror manifold is given by

$$xz = e^u + e^{-t-u} + e^v + e^{-s-v} + 1,$$
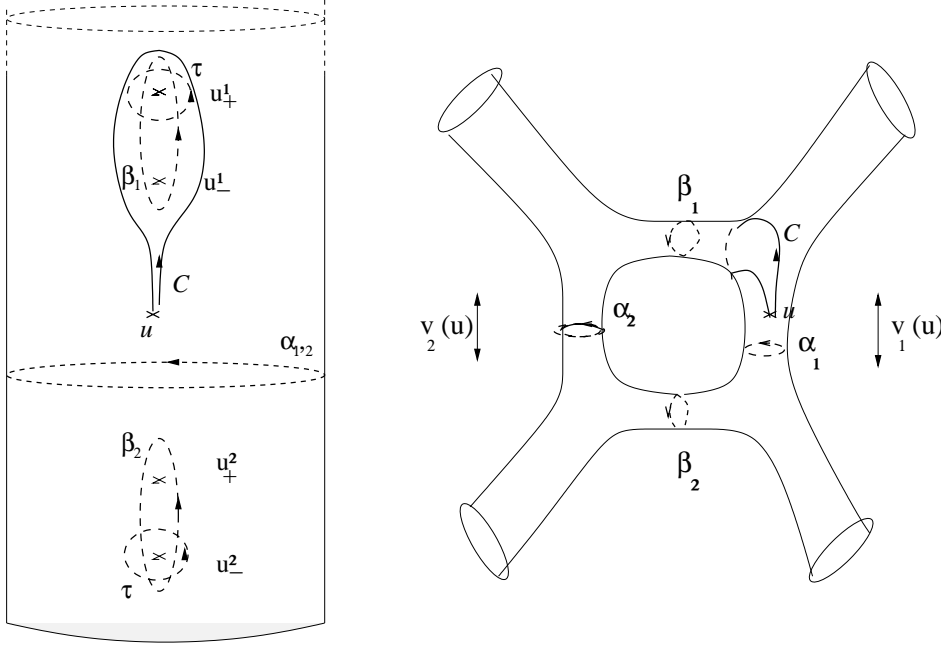
Fig. 14. The Riemann surface $\Sigma : P(u,v) = 0$ can be viewed as a branched cover of the u-cylinder by two solutions $v_{1,2}(u)$. The solutions are branched over 4 points $u_{1,2}^{\pm}$, with monodromies $\tau_{1,2}$ which exchanges $v_1 \leftrightarrow v_2$, and $\beta_{1,2}$ which take $v_{1,2} \to v_{1,2} + 2\pi i$ corresponding to the v-cylinders which glue the two roots on the figure to the right. Monodromies $\alpha_{1,2}$ arise from the periodicity of u itself.

where $Y^2 = u$, $Y^3 = v$, $Y^0 = 0$ and the two holomorphic constrains are solved in terms of these. The Riemann surface Σ which is the configuration space of the mirror B-brane is given by $0 = P(u,v) = e^u + e^{-t-u} + e^v + e^{-s-v} + 1$. When viewed in terms of the single valued variables e^u and e^v, Σ is related to the A-model geometry by "thickening" of the one-dimensional edges of the toric diagram (or more precisely, their projection onto the $X^0 = 0$ plane). In the large radius limit $0 \ll t$, s the A-model geometry becomes classical and the legs of the Riemann surface of the B-model become long and thin, so the A- and the B-model can be related already at the classical level.

For example, the A-brane on Fig. 13 is on the internal leg which is given by $c_2 = 0$, and c_1 in the $[0, r_t]$ interval. The large radius limit is, in addition to r_t and r_s being large, the limit of large disk size. For example in the regime $c_1 < r_t/2$ the basic disk is the one on Fig.13a, and the classical limit corresponds to c_1 being of order of $r_t/2$. The size parameter c_1, together with the Wilson line $u = c_1 + i \int A$, is the classical tension of the D4 brane domain wall wrapping this disk and ending on the D6 brane.

In the limit the A-model geometry is classical, it follows from the mirror map (2.3) that the mirror B-brane is located in the region on Σ, where v is constant, $v \sim 0$, and u is a parameter which is large, $u \sim t/2$. Away from the large radius limit, this deforms to a root of the equation $P(u,v) = 0$. This has two solutions for v at every value of u,

$$v = v_{1,2}(u) = \log \left[\frac{1 + e^u + e^{-t-u}}{2} \right.$$
$$\left. \pm \frac{\sqrt{(1 + e^u + e^{-t-u})^2 - 4e^{-s}}}{2} \right] + i\pi$$

and the mirror B-brane propagates along a region of the root $v = v_1(u)$. As we discussed above, the superpotential on this leg is given by

$$W(u) = \int v_1(u) \mathrm{d}u.$$

This vanishes in the classical limit, since $\partial_u W = v_1(u) \to 0$.

The flat coordinate computes the tension $\hat{u}$ of the D4 brane domain wall wrapped on the basic disk as in Fig. 13a, and we will compute this by computing the tension of the mirror B-brane domain wall. As explained in Sect. 4. the mirror domain wall projects to the loop $\mathcal{C}$ on the Riemann surface which starts at the location of the B-brane at a fixed u, winds around $v \to v+2\pi i$ before "attaching" again (see Fig.13b). In this configuration, the tension is given by a classical integral on the Riemann surface

$$\hat{u} = \frac{1}{2\pi i} \int_{\mathcal{C}} v_1(u)\mathrm{d}u.$$

The period integral $\frac{1}{2\pi i} \int_{\mathcal{C}} v_1(u)\mathrm{d}u$ is, as explained in Sect. 4, the sum of two pieces. There is the classical contribution to the domain wall tension which is equal to $\frac{1}{2\pi i}\Delta(v_1 u) = u$, as the initial and final end-point of $\mathcal{C}$ differ by $v \to v + 2\pi i$. The quantum correction to the BPS mass comes from the "small period", the contour integral around the β_1 cycle (see Fig. 14) of $\frac{1}{2\pi i} \int_{\beta_1} u\mathrm{d}v$.

The small periods can be found as follows. Notice that, since $v_1 + v_2 = -s + 2\pi i$, the sum of two periods around $u \to u + 2\pi i$ is

$$\frac{1}{2\pi i} \int_{\alpha_1} v_1\mathrm{d}u + \frac{1}{2\pi i} \int_{\alpha_2} v_2\mathrm{d}u = s + 2\pi i.$$

On the other hand, the closed string period $\hat{s}$, that measures the mass of the D4 brane wrapping a $\mathbf{P}^1$ of size r_s can also be expressed in terms of small periods as it is computed along the contour $\alpha_s = \alpha_1 - \alpha_2$, (orientations are fixed up to an over-all sign by the requiring that *both* u and v have trivial monodromy around β_s),

$$\frac{1}{2\pi i} \int_{\alpha_1} v_1\mathrm{d}u - \frac{1}{2\pi i} \int_{\alpha_2} v_2\mathrm{d}u = -\hat{s}.$$

From this we can find for example the small period along α_1 as[6] $\frac{s-\hat{s}}{2} + i\pi = \frac{1}{2\pi i} \int_{\alpha_1} v_1\mathrm{d}u$.

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[6]This method of calculating the domain-wall tension gives the result up to factors of $i\pi$. The direct evaluation of the period around $\mathcal{C}$ can be done, and it determines the answer to be the one we presented above.
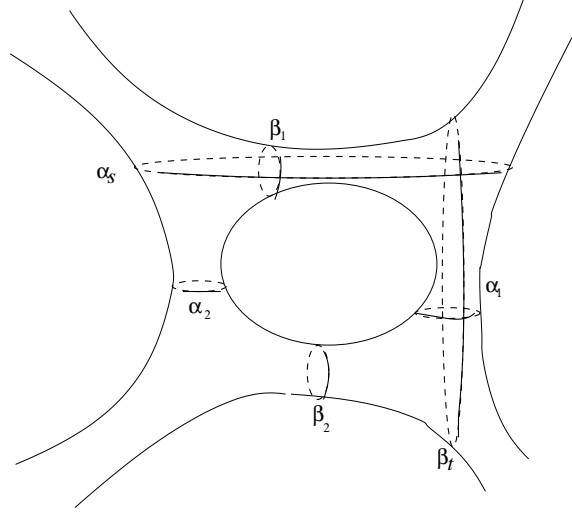
Fig. 15. The closed string periods β_t, β_s are linear combinations of small periods α_i, β_i for $i = 1, 2$ which come from the periodicity of u and v variables.
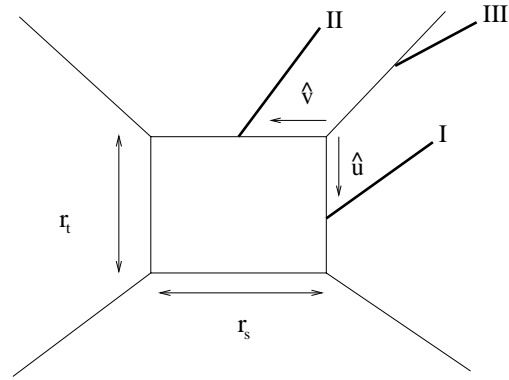
Fig. 16. Three phases of the $A-$brane on $O(K) \to \mathbf{P}^1 \times \mathbf{P}^1$. The phases I and II are related by the exchange of the two $\mathbf{P}^1$'s in the base, and this is reflected in the disc domain wall numbers.

For the B-brane at hand we need the small period around the β_1 cycle on the v-leg (Fig. 15), and by a computation analogous to the one we just did,

$$\hat{u} = u + \frac{t - \hat{t}}{2} + \pi i.$$

In fact the small period of α_1 is the correction for the B-brane on the other leg of the Riemann surface – the leg parameterized by v which corresponds to the

phase II (see Fig. 16). In this phase we have therefore

$$\hat{v} = v + \frac{s - \hat{s}}{2} + i\pi.$$

According to the discussion in the $\mathbf{C}^3$ case, which carries over here as well, the correction terms we computed

$$\Delta_u = \frac{t - \hat{t}}{2} + i\pi, \ \Delta_v = \frac{s - \hat{s}}{2} + i\pi,$$

form a doublet under the SL(2, $\mathbb{Z}$) transformations, just as u, v do. Consider now an SL(2, $\mathbb{Z}$) group element which takes

$$u \to u' = u - v, \ v \to v' = v.$$

The new equation of the curve becomes

$$e^{u' - v'} + e^{-t - u' + v'} + e^{v'} + e^{-s - v'} + 1 = 0.$$

This change of coordinates does not change the classical value of the superpotential in phase I (the phase originally parameterized by u) since $v' = v \sim 0$, but corresponds to a different choice of framing with $n = -1$, which changes the fluctuating field on the brane from u to $u' = u - v$. The quantum corrected domain wall tension is now[7]

$$\Delta_{u'} = \Delta_u - \Delta_v$$

For the brane in phase II it is no longer true that the classical superpotential is zero, since $u' = u - v$ is not zero after the SL(2, $\mathbb{Z}$) transformation.

With this choice of parameterization, consider the superpotential

$$W = \int u'(v') \mathrm{d}v'$$

with $u'(v)$ obtained by solving $P(u', v) = 0$. This superpotential has no classical piece, as $v' \to \infty$, $u' \sim 0$ is on the curve. In fact, $u' \sim 0$ is the equation of the outer leg of Σ (see Fig. 16), so this computes the superpotential of the brane in phase III. The domain wall tension in the phase III (the outer leg) is given by

$$\Delta_{v'} = \Delta_v = \frac{s - \hat{s}}{2} + i\pi.$$

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[7]As we will see when we discuss the closed string flat coordinates in more detail below, $\Delta_{u'}$ turns out to be zero.

To compute disk numbers in these various phases, we need to write the superpotential in each case in terms of the open and closed string flat coordinates.

The closed string flat coordinates have a complicated dependence on the classical, linear-sigma model, coordinates t. Quite analogously to what we found for the open string, the A-model closed string amplitudes have integrality properties when expanded in terms of flat coordinates which measure the BPS mass of D2-branes wrapped on rational curves in X. The corrected quantities are related transcendentally by the mirror map $\hat{t}_i = \hat{t}_i(t_j)$ to the complex structure variable $z_i = e^{-\hat{t}_i}$. In fact, as discussed above, $\hat{t}$ and $\hat{s}$ are among the periods of the Riemann surface Σ. Alternatively, they can be obtained from the solutions to Picard-Fuch's equations $\mathcal{L}_i f = 0$,

$$\mathcal{L}_i = \prod_{Q_i > 0} (\partial_i)^{Q_i} - \prod_{Q_i < 0} (\partial_i)^{-Q_i}, \qquad (6.5)$$

where $\partial_i = \partial / \partial z^i$. For the $O(K) \to P^1 \times P^1$ the linear differential operators $\mathcal{L}_{t,s}$ are

$$\mathcal{L}_t = \theta_t{}^2 - 2z_t(\theta_t + \theta_s)(2\theta_s + 2\theta_t + 1)$$
$$\mathcal{L}_s = \theta_s{}^2 - 2z_s(\theta_t + \theta_s)(2\theta_s + 2\theta_t + 1), \qquad (6.6)$$

where $\theta_{t,s} = z_{t,s} \partial / \partial z_{t,s}$. There is a constant solution $f_0 = 1$, and near $z_t = z_s = 0$ there are two logarithmic solutions f_1, f_2. The flat coordinates $\hat{t}, \hat{s}$ are given by linear combinations of ratios of the periods $f_i/f_0, i = 1, 2$ picked by the correct classical behavior

$$\hat{t} = t - (2z_t + 2z_s + 3z_t^2 + 12z_t z_s + 3z_s^2 + O(z^3)),$$

$$\hat{s} = s - (2z_t + 2z_s + 3z_t^2 + 12z_t z_s + 3z_s^2 + O(z^3)).$$
$$(6.7)$$

Note that the solutions are symmetric in $z_{t,s}$ except for the logarithmic term. To expand the disk amplitude in terms of the flat coordinates we need the inverse relations $z_{t,s}(q_{t,s})$ for $q_t = e^{-\hat{t}}$ and $q_s = e^{-\hat{s}}$. The first few terms of the expansion are

$$z_s = q_s - 2q_s^2 - 2q_s q_2 + 3q_s^3 + 3q_s q_t^2$$
$$\quad - 4q_s^4 - 4q_s^3 q_t - 4q_s^2 q_t^2 - 4q_s q_t^3 + O(q^5),$$
$$z_t = q_t - 2q_s q_t - 2q_t^2 + 3q_s^2 q_t + 3q_t^3$$
$$\quad - 4q_s^3 q_t - 4q_s^2 q_t^2 - 4q_s q_t^3 - 4q_t^4 + O(q^5),$$
$$(6.8)$$

where $q_t = e^{-\hat{t}}$ and $q_s = e^{-\hat{s}}$.

Table 1. The disk domain wall degeneracies for brane I in the $O(K) \to \mathbf{P}^1 \times \mathbf{P}^1$ geometry for $m > 0$. Exchanging s with t yields the result for brane II.

k_s			— k_t —			
	0	1	2	3	4	5
$m = 1$:						
0	1	0	0	0	0	0
1	1	2	3	4	5	6
2	1	10	45	140	350	756
3	1	30	300	1776	7650	26532
4	1	70	1332	13400	91070	472368
5	1	140	4590	72856	736270	5437530
$m = 2$:						
0	0	0	0	0	0	0
1	1	2	3	4	5	6
2	2	16	62	180	428	896
3	4	70	552	2856	11280	36828
4	6	224	3130	26336	159078	759200
5	9	588	13420	171720	1503135	10016490
$m = 3$:						
0	0	0	0	0	0	0
1	1	2	3	4	5	6
2	4	24	85	230	525	1064
3	11	146	977	4542	16644	51420
4	25	618	6975	50912	278134	1231230
5	49	2070	36637	395818	3068331	18655290
$m = 4$:						
0	0	0	0	0	0	0
1	1	2	3	4	5	6
2	6	34	112	290	638	1260
3	25	276	1645	7040	24246	71400
4	76	1498	14496	94830	476900	1979098
5	196	6248	91935	870220	6103867	34309080
$m = 5$:						
0	0	0	0	0	0	0
1	1	2	3	4	5	6
2	9	46	145	360	770	1484
3	49	482	2640	10592	34674	98028
4	196	3270	28240	169402	795998	3126928
5	635	16642	213083	1816038	11729677	61675880
$m = 6$:						
0	0	0	0	0	0	0
1	1	2	3	4	5	6
2	12	60	182	440	918	1736
3	87	790	4060	15478	48600	132732
4	440	6560	51906	290600	1290870	4840248
5	1764	40050	460355	3604656	21737688	107979508

Table 2. Disk degeneracies for brane I in the $O(K) \to \mathbf{P}^1 \times \mathbf{P}^1$ geometry for $m = -1$.

k_s			— k_t —			
	0	1	2	3	4	5
0	0	−1	0	0	0	0
1	0	−1	−2	−3	−4	−5
2	0	−1	−10	−45	−140	−350
3	0	−1	−30	−300	−1776	−7650
4	0	−1	−70	−1332	−13400	−91070
5	0	−1	−140	−4590	−72856	736270

In phase I we get the values for numbers of primitive disks $N_{k_s, k_t, m}$ given in Table 1.

There is a symmetry which relates the numbers of disk instantons with negative m to those above. We have

$$N_{k_s, k_t, -m} = \begin{cases} 0 & \text{if } k_t - m < 0, \\ -N_{k_s, k_t - m, m} & \text{if } k_t - m \geq 0. \end{cases} \quad (6.9)$$

Examples are given in Table 2.

For the numbers of primitive disks for the brane III see Table 3.

Up to the invariant $N_{0,0,1} = 2$ we have $N_{k_s, k_t, m} = 0$ for $k_t < m$ and for $k_t \geq m$ there is a symmetry

$$N_{k_s, k_t, m} = N_{k_t - m, k_s + m, m}, \quad (6.10)$$

which reflects the exchange symmetry of s and t for the brane in phase III. The instantons with negative m are absent for $k_s \leq |m|$ and their numbers are related to those with positive m by

$$N_{k_s, k_t, -m} = -N_{k_s + m, k_t - m, m}. \quad (6.11)$$

Examples are given in Table 4.

6.3. $\mathcal{O}(-3) \to \mathbf{P}^2$

The linear sigma model for this geometry has $G = U(1)$ and four matter fields with charges $Q = (-3, 1, 1, 1)$

$$|X^1|^2 + |X^2|^2 + |X^3|^2 - 3|X^0|^2 = r. \quad (6.12)$$

The base of the toric fibration is shown in the Figure 17.

The D-brane charge can be taken to be $q_1 = (1, 0, -1, 0)$, $q_2 = (0, 1, -0, -1)$ (the choice is unique in all local models), and so

$$|X^1|^2 - |X^0|^2 = c^1, \quad |X^2|^2 - |X^0|^2 = c^2.$$

To get the disk numbers the only remaining task is to expand $W(u)$ in terms of open and closed string flat coordinates. The integrality of the disk amplitude (2.11) implies that we write $\partial_{\hat{u}} W(\hat{u}) = \hat{v}$ as

$$\partial_{\hat{u}} W(\hat{u}) = -\sum_{\substack{k_s = 0, k_t = 0 \\ m = -k_t}}^{\infty} m N_{k_s, k_t, m} \log\left(1 - q_s^{k_s} q_t^{k_t} e^{m\hat{u}}\right).$$

Table 3. The disk degeneracies for brane III in the $O(K) \to$ $\mathbf{P}^1 \times \mathbf{P}^1$ geometry for $m > 0$.

k_s			— k_t —			
	0	1	2	3	4	5
$m = 1$:						
0	2	2	2	2	2	2
1	0	2	12	40	100	210
2	0	2	40	310	1520	5628
3	0	2	100	1520	12908	76488
4	0	2	210	5628	91070	680940
5	0	2	392	17184	353316	4515558
$m = 2$:						
0	0	0	–2	–4	–8	–12
1	0	0	–4	–32	–140	–448
2	0	0	–8	–140	–1188	–6580
3	0	0	–12	–448	–6580	–58240
4	0	0	–28	–1176	–27840	–370428
5	0	0	–24	–2688	–97020	–1859648
$m = 3$:						
0	0	0	0	2	10	28
1	0	0	0	10	100	540
2	0	0	0	28	540	5012
3	0	0	0	62	2100	317072
4	0	0	0	120	6600	147420
5	0	0	0	210	17820	576212

k_s			— k_t —			
	0	...	4	5	6	7
$m = 4$:						
0	0	...	0	–4	–28	–104
1	0	...	0	–28	–336	–2156
2	0	...	0	–104	–2156	–21888
3	0	...	0	–300	–9856	–149940
4	0	...	0	–720	–36036	–787640
5	0	...	0	–1540	–112112	–3406480
$m = 5$:						
0	0	...	0	10	84	396
1	0	...	0	84	1176	8736
2	0	...	0	396	8736	96660
3	0	...	0	1386	45864	724800
4	0	...	0	4004	191100	4273840
5	0	...	0	10090	672672	–
$m = 6$:						
0	0	...	0	–26	–264	–1504
1	0	...	0	–264	–4224	–35640
2	0	...	0	–1504	–35640	–427540
3	0	...	0	–6228	–211200	–3484800
4	0	...	0	–21028	–987360	–
5	0	...	0	–61152	–	–

Consider three phases of the A-brane that are visible classically (see Fig. 17).

$$\text{Phase I}: \ r_t > c^1 > 0, \ c^2 = 0,$$

Table 4. The disk degeneracies for brane III in the $O(K) \to$ $\mathbf{P}^1 \times \mathbf{P}^1$ geometry for $m = -1$.

k_s			— k_t —			
	0	1	2	3	4	5
0	0	0	0	0	0	0
1	–2	–2	–2	–2	–2	–2
2	–2	–12	–40	–100	–210	–392
3	–2	–40	–310	–1520	–5628	–17184
4	–2	–100	–1520	–12908	–76488	–353316
5	–2	–210	–5628	–76488	–680940	–4515558
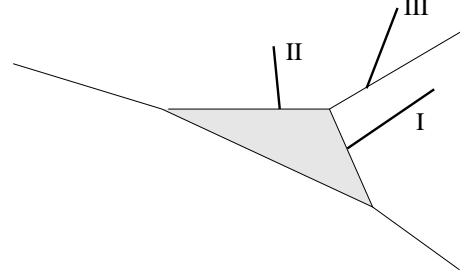
Fig. 17. Three phases of the A–brane on $O(-3) \to \mathbf{P}^2$. Phases I and II are related by $\mathbf{Z}_3$ symmetry of the $\mathbf{P}^2$.

$$\text{Phase II}: \ c^1 = 0, r_t > c^2 > 0,$$

$$\text{Phase III}: \ c^1 = c^2, \ 0 < c^1. \tag{6.13}$$

As the branes in phases I and II are related by the $\mathbf{Z}_3$ symmetry of $\mathbf{P}^2$, the special Lagrangian D-branes of the A-model for this Calabi-Yau have two inequivalent phases, together with a $\mathbf{Z}$-family of choices of framing in each.

The mirror B-model geometry (see e. g. [34 - 36]) can be written as

$$xz = e^u + e^v + e^{-t-u-v} + 1, \tag{6.14}$$

where we have "solved" $Y^1 + Y^2 + Y^3 = -t + 3Y^0$ by $Y^0 = 0$, $Y^1 = u$, $Y^2 = v$ and $Y_3 = -t - u - v$.

In terms of these coordinates, the brane in phase I propagates on the internal leg of the Riemann surface where $v \sim 0$ and u large of order $-t/2$, the brane in phase II is on $u \sim 0$, and $v \sim t/2$, and the brane on external leg has $u \sim v$, both being large.

In the variables of (6.14) the classical superpotential vanishes in phase I and $W(u) = \int v(u) \mathrm{d}u$ computes the disk instanton generated superpotential.

To compute disk numbers we need the flat coordinate. According to the discussion in Sect. 5, this is given by the difference of superpotentials on the two
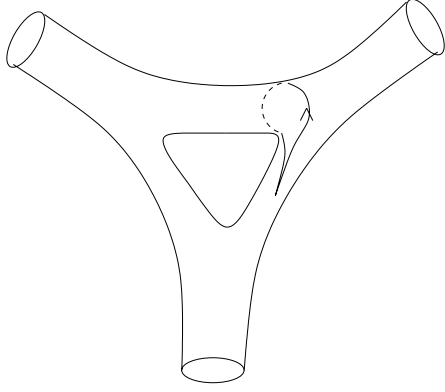
Fig. 18. The curve $P(u, v) = 0$ associated to the mirror of $O(-3) \to \mathbf{P}^2$.

sides of the domain wall as computed along the contour C on Figure 18. The non-trivial contribution to $\hat{u}$ is the exponentially suppressed shift which comes from the small period on Fig. 18:

$$\hat{u} = \int_C v(u)\mathrm{d}u = u + \Delta_u.$$

We find, using the same kind of methods discussed for the $\mathbf{P}^1 \times \mathbf{P}^1$ example, that

$$\Delta_u = \frac{t - \hat{t}}{3} + i\pi,$$

where $\hat{t}$ is the closed string period on the contour α_1 in Figure 18.

The closed string period can be computed either directly by integration on Σ, or from the Picard-Fuchs equation with $\mathcal{L} = \theta^3 + 3z\theta(3\theta + 1)(3\theta + 2)$, where $\theta = z\partial_z$ and $z = e^{-t}$. The solutions to the Picard-Fuchs equation can be expressed, e. g., as Mejer G-functions [37] and

$$\hat{t} = t - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{(3n)!}{(n!)^3} z^n.$$

The inverse $z(\hat{t})$ relation is

$$z = q + 6q^2 + 9q^3 + 56q^4 - 300q^5 + 3942q^6 + \ldots, \quad (6.15)$$

where $q = e^{-\hat{t}}$.

After rewriting $W(u, t)$ in terms of the flat coordinates $\hat{u}, \hat{t}$, and expanding as in (2.11) we obtain the integer invariants of Table 5.

Table 5. Disk degeneracies for brane I or II in the $O(K) \to \mathbf{P}^2$ geometry.

m						— k —			
	0	1	2	3	4	5	6	7	8
–5	0	0	0	0	0	5	–84	1200	–16854
–4	0	0	0	0	–2	28	–344	4360	–57760
–3	0	0	0	1	–10	102	–1160	14274	–185988
–2	0	0	–1	4	–32	326	–3708	45722	–598088
–1	0	1	–2	12	–104	1085	–12660	159208	–2112456
1	1	–1	5	–40	399	–4524	55771	–729256	9961800
2	0	–1	7	–61	648	–7661	97024	–1293185	17921632
3	0	–1	9	–93	1070	–13257	173601	–2371088	33470172
4	0	–1	12	–140	1750	–22955	312704	–4396779	63460184
5	0	–1	15	–206	2821	–39315	559787	–8136830	120497011

Table 6. Disk degeneracies for brane III in the $O(K) \to \mathbf{P}^2$ geometry.

m						— k —			
	0	1	2	3	4	5	6	7	8
1	–1	2	–5	32	–286	3038	–35870	454880	–6073311
2	0	1	–4	21	–180	1885	–21952	275481	–3650196
3	0	1	–3	18	–153	1560	–17910	222588	–2926959
4	0	1	–4	20	–160	1595	–17976	220371	–2869120
5	0	1	–5	26	–196	1875	–20644	249120	–3205528
6	0	1	–7	36	–260	2403	–25812	306095	–3889116
7	0	1	–9	52	–365	3254	–34089	397194	–4981102
8	0	1	–12	76	–528	4578	–46812	535639	–6627840

For the phase III, to compute the integer invariants we need to change the parameterization of the curve. Consider the SL(2, $\mathbb{Z}$) transformation

$$u \to u' = u - v,$$

$$v \to v' = v,$$

which will allow us to compute the superpotential in phase III since in these variables the equation of the phase III leg is $u' = 0$. The equation for Σ written in the new variables becomes

$$e^{u'} + e^{v'} + 1 + e^{-u' + 3v' - t} = 0, \quad (6.16)$$

upon trivial multiplication by $e^{v'}$. The good flat coordinate for this phase is $\hat{v}' = v' + \Delta_v = v' + \frac{t - \hat{t}}{3}$. For the integer invariants in phase III we get that $d_{k,m} = 0$ for $m < 0$ and, for positive m, we obtain the values of Table 6.

From (6.16) we obtain another description of phase I, which is at the classical level equivalent to the one already given, but differes in the quantum theory by

Table 7. Disk degeneracies for brane I in the $O(K) \to \mathbf{P}^2$ geometry for various choice of the ambiguity $n \in Z$.

m					— k —				
	0	1	2	3	4	5	6	7	8
$n = -1$:									
–5	0	0	0	0	0	40	–1274	27885	–528934
–4	0	0	0	0	10	–253	4604	–76068	1214324
–3	0	0	0	3	–54	783	–11058	157347	–2274642
–2	0	0	1	–13	142	–1657	20785	–274473	3769424
–1	0	1	–4	29	–274	3002	–36144	464522	–6262370
1	1	–1	4	–29	274	–3002	36144	–464522	6262370
2	0	0	–1	13	–142	1657	–20785	274473	–3769424
3	0	0	0	–3	54	–783	11058	–157347	2274642
4	0	0	0	0	–10	253	–4604	76068	–1214324
5	0	0	0	0	0	–40	1274	–27885	528934

m		— k —			
	0	1	2	3	4
$n = 1$:					
–4	0	0	0	0	0
–3	0	0	0	0	–1
–2	0	0	0	–1	7
–1	0	1	–1	5	–40
1	1	–2	12	–104	1085
2	–1	4	–32	326	–3708
3	1	–10	102	–1160	14274
4	–2	28	–344	4360	–57760
$n = 2$:					
–3	0	0	0	0	0
–2	0	0	0	–1	13
–1	1	–1	4	–29	274
1	1	–4	29	–274	3002
2	1	–13	142	–1657	20785
3	3	–54	758	–11058	157347
4	10	–253	4608	–76068	1214324
5	40	–1274	27885	–528934	9380474

the relative framing $n = -1$. The flat coordinate in the phase $I_{n=-1}$ is

$$\hat{u}' = u',$$

since under the SL(2, $\mathbb{Z}$) transformation Δ_u and Δ_v cancel off. We have also considered other values of n, i.e the D-branes with shifted $u \to u' = u + nv$ as the dynamical field on the brane. First, note that I_n and $I_{-(n+1)}$ (where I_n denotes the brane in phase I and with framing n) are related by $v \to -v + t$, with u fixed. Thus, we expect

$$N^n_{k,m} = \pm N^{-(n+1)}_{k+m,-m} \,,$$

where m denotes the is boundary class of the disk. The integer invariants for $n = -1, 1, 2$, given in Table 7, clearly respect this.

Integrality of the Bulk Mirror Map

The integrality of mirror map in the bulk, even though it has been proven in some cases, has not been physically explained. Here we will connect this to the integrality of the number of domain walls $N_{k,m}$. In the case $I_{n=-1}$ we can explicitly show that all coefficients of the form $N_{k,1}$ are directly related to the coefficients of the bulk mirror map and using this relation the integrality of one follows from the other.

Define the numbers a_i by

$$e^{\frac{t-\hat{t}}{2}} = 1 + \sum_{i=1}^{\infty} a_i q^i.$$

One can show the integrality of a_i using the integrality of the mirror map (6.15). Furthermore one can show, by explicitly investigating the Taylor series of $v(\hat{u}, q)$, that

$$N_{k,1} = P_k(a_1, \ldots, a_{k-1}) - a_k,$$

where P_k is a polynomial with integer coefficients in the a_i. E. g. we get

$$P_1 = -3,$$

$$P_2 = 30 + 12a_1 + a_1^2,$$

$$P_3 = -420 - 210a_1 - 30a_1^2 - a_1^3 + 12a_2 + 2a_1a_2,$$

$$P_4 = 6930 + 4200a_1 + 840a_1^2 + 60a_1^3 + a_1^4 - 210a_2$$
$$- 60a_1a_2 - 3a_1^2a_2 + a_2^2 + 12a_3 + 2a_1a_3,$$

etc.. The proof that all P_k are integer polynomials is tedious and relies on some formulas for the derivatives of $v(\hat{u}, q)$ for this special example. From this one sees that the integrality of $N_{k,1}$ and a_k are equivalent, and the integrality of a_k follows from the integrality of the mirror map in the bulk (6.15), and vice-versa. The integrality of the mirror map in the bulk had not been physically explained before. Here, by relating it to the integrality of numbers of domain walls $N_{k,1}$ we have found a physical explanation for it. The integrality of $N_{k,m}$ $m \neq 1$ requires special properties of the a_i and seems much more involved.

6.4. $\mathcal{O}(K) \to \mathbf{F}_1$

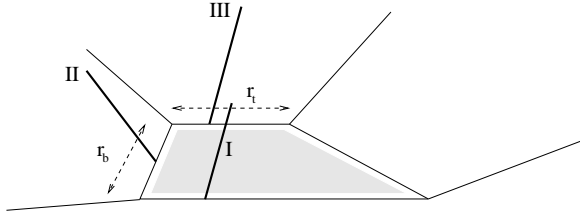Consider a CY geometry containing the blowup of $\mathbf{P}^2$ at one point, which we denote by $\mathbf{F}_1$. The

Fig. 19. Toric base of $O(K) \to \mathbf{F}_1$ and the A-brane in three phases. $O(K) \to \mathbf{F}_1$ can be viewed as a blowup of $O(-3) \to \mathbf{P}^2$ at a point by a $\mathbf{P}^1$ of size r_t.

charges describing the gauged linear σ-model on the non-compact Calabi-Yau are $Q_b = (-2, 0, 0, 1, 1)$ and $Q_f = (-1, 1, 1, -1, 0)$, so the corresponding D-terms are

$$|X^3|^2 + |X^4|^2 - 2|X^0|^2 = r_b,$$
$$|X^1|^2 + |X^2|^2 - |X^0|^2 - |X^3|^2 = r_f, \qquad (6.17)$$

where r_b and r_f are the areas of the base and the fiber of the Hirzebruch surface $\mathbf{F}_1$. The equations are "solved" in Figure 19. The threefold has two classes of divisors: H coming from $\mathbf{P}^2$ itself and the exceptional divisor E with $H^2 = 1$, $E^2 = -1$ and $EH = 0$. The charge vectors (or generators of the mori-cone) correspond to the fiber $F = H - E$ and the base, which is E. Note that $Q_b + Q_f = Q_{P^2}$ with Q_{P^2} is the charge for the $O(-3) \to P^1$, and correspondingly $E + F = H$.

The A-brane charges are again such that the brane is special Lagrangian, e.g. $q^1 = (-1, 1, 0, 0, 0)$ and $q^2 = (-1, 0, 0, 0, 1)$ which defines the moduli c_1, c_2:

$$|X^1|^2 - |X^0|^2 = c_1,$$

$$|X^4|^2 - |X^0|^2 = c_2.$$

We will consider the following 3 phases (out of a total of 8 possibilities)[8]

Phase I : $(r_b + r_f)/2 > c_1 > 0 \; c_2 = 0,$

Phase II : $c_1 = 0 \; r_b/2 > c_2 > 0,$ \qquad (6.18)

Phase III : $r_f/2 > c_1 > 0 \quad c_2 = r_b.$

The mirror of $\mathcal{O}(K) \to F_1$ is given by

$$xz = 1 + e^u + e^{-v-u} z_f z_b + e^{-v} z_b + e^v, \quad (6.19)$$

---

[8]We have checked that all other phases also lead to integral expansions for disk amplitudes.

Table 8. Disk degeneracies for brane I and II in the $O(K) \to F_1$ geometry.

k_b	— k_f —					
	0	1	2	3	4	5
$m = -5$:						
Phase II:						
5	0	1	–6	14	–14	5
6	0	2	–30	140	–280	252
Phase I:						
5	0	0	0	0	0	0
6	0	0	0	0	0	42
$m = -4$:						
Phase II:						
4	0	1	–4	5	–2	0
5	0	2	–20	60	–70	28
6	0	3	–68	400	–936	–344
Phase I:						
4	0	0	0	0	–2	0
5	0	0	0	0	–14	28
6	0	0	0	0	–52	308
$m = -3$:						
Phase II:						
3	0	1	–2	1	0	0
4	0	2	–12	20	–10	0
5	0	3	–45	170	–230	102
6	0	4	–130	958	–2612	2940
Phase I:						
3	0	0	0	1	0	0
4	0	0	0	5	–10	0
5	0	0	0	14	–90	102
6	0	0	0	31	–450	1428
$m = -2$:						
Phase II:						
2	0	1	–1	0	0	0
3	0	2	–6	4	0	0
4	0	3	–28	57	–32	0
5	0	4	–90	424	–664	326
6	0	5	–237	2172	–6872	8640
Phase I:						
2	0	0	–1	0	0	0
3	0	0	–2	4	0	0
4	0	0	–4	28	–32	0
5	0	0	–6	112	–390	326
6	0	0	–9	336	–2500	5638

where $z_b = e^{-t_b}$, $z_f = e^{-t_f}$, and we have solved the mirror relations in terms of $Y^1 = u$, $Y^4 = v$. with $Y^0 = 0$. The mirror of the brane in phase I has $u \sim (t_b + t_f)/2$ as a variable with $v \sim 0$, II has $v \sim t_f/2$ as a variable with $u \sim 0$ on the relevant leg of the toric diagram. One finds using the methods discussed for the $\mathbf{P}^1 \times \mathbf{P}^1$ example, that

$$\Delta_{u,v} = (t_f - \hat{t}_f) + i\pi.$$

Table 8 (continued).

k_b	0	1	2	3	4	5
$m = -1$:						
Phase II:						
1	−1	1	0	0	0	0
2	0	2	−2	0	0	0
3	0	3	−15	12	0	0
4	0	4	−60	160	−104	0
5	0	5	−175	1080	−1995	1085
Phase I:						
1	0	1	0	0	0	0
2	0	1	−2	0	0	0
3	0	1	−10	12	0	0
4	0	1	−30	120	−104	0
5	0	1	−70	648	−1596	1085
$m = 1$:						
Phase II:						
0	1	0	0	0	0	0
1	0	−1	0	0	0	0
2	0	−2	5	0	0	0
3	0	−3	30	−40	0	0
4	0	−4	105	−432	399	0
5	0	−5	280	−2520	6370	−4524
Phase I:						
0	1	0	0	0	0	0
1	1	−1	0	0	0	0
2	1	−6	5	0	0	0
3	1	−20	59	−40	0	0
4	1	−50	356	−706	399	0
5	1	−105	1500	−6244	9372	−4524
$m = 2$:						
Phase II:						
0	0	0	0	0	0	0
1	0	−1	0	0	0	0
2	0	−2	7	0	0	0
3	0	−3	38	−61	0	0
4	0	−4	128	−616	648	0
5	0	−5	330	−3420	9744	−7661
Phase I:						
0	0	0	0	0	0	0
1	1	−1	0	0	0	0
2	2	−9	7	0	0	0
3	4	−43	100	−61	0	0
4	6	−147	756	−1263	648	0
5	9	−406	3920	−13122	17260	−7661

Table 8 (continued).

k_b	0	1	2	3	4	5
$m = 3$:						
Phase II:						
0	0	0	0	0	0	0
1	0	−1	0	0	0	0
2	0	−2	9	0	0	0
3	0	−3	48	−93	0	0
4	0	−4	155	−884	1070	0
5	0	−5	390	−4682	15134	−13257
Phase I:						
0	0	0	0	0	0	0
1	1	−1	0	0	0	0
2	4	−13	9	0	0	0
3	11	−85	167	−93	0	0
4	25	−382	1555	−2268	1070	0
5	49	−1344	9813	−27584	32323	−13257
$m = 4$:						
Phase II:						
0	0	0	0	0	0	0
1	0	−1	0	0	0	0
2	0	−2	12	0	0	0
3	0	−3	60	−140	0	0
4	0	−4	188	−1260	1750	0
5	0	−5	460	−6397	23482	−22955
Phase I:						
0	0	0	0	0	0	0
1	1	−1	0	0	0	0
2	6	−18	12	0	0	0
3	25	−155	270	−140	0	0
4	76	−887	3056	−3995	1750	0
5	196	−3873	23040	−56429	60021	−22955
$m = 5$:						
Phase II:						
0	0	0	0	0	0	0
1	0	−1	0	0	0	0
2	0	−2	15	0	0	0
3	0	−3	74	−206	0	0
4	0	−4	225	−1772	2821	0
Phase I:						
0	0	0	0	0	0	0
1	1	−1	0	0	0	0
2	9	−24	15	0	0	0
3	49	−264	421	−206	0	0
4	196	−1876	5700	−6841	2821	0

The numbers of primitive disks in these two phases are listed in Tabel 8.

We consider now the phase III, for which we must change the parameterization of the curve by $u \to u' = -t_b - u$, if we are to have the superpotential which is zero classically. In this phase, v is the transverse coordinate on the brane. The correction to the flat coordinate is again found by requiring integrality of the amplitude, and we find that

$$\hat{v} = v + \Delta_v = v + t_f - \hat{t}_f.$$

The disk numbers follow (Table 9).

Table 9. Disk degeneracies for brane III in the $O(K) \to F_1$.

k_b	0	1	2	— k_f — 3	4	5
$m = 1:$						
0	1	–1	0	0	0	0
1	0	–2	2	0	0	0
2	0	–3	15	–12	0	0
3	0	–4	60	–160	104	0
4	0	–5	175	–1080	1995	–1085
5	0	–6	420	–5040	19110	–27144
6	0	–7	882	–18480	124033	–337617
$m = -1:$						
0	0	0	0	0	0	0
1	–1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	2	–5	0	0	0
4	0	3	–30	40	0	0
5	0	4	–105	432	–399	0
6	0	5	–280	2520	–6370	4524
$m = 2:$						
0	0	–1	1	0	0	0
1	0	–2	6	–4	0	0
2	0	–3	28	–57	32	0
3	0	–4	90	–424	664	–326
4	0	–5	237	–2172	6872	–8640
5	0	–6	532	–8640	48208	–114774
6	0	–7	1072	–28578	258516	–1023679
$m = -2:$						
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	1	0	0	0	0
4	0	2	–7	0	0	0
5	0	3	–38	61	0	0
6	0	4	–128	616	–648	0
$m = 3:$						
0	0	–1	2	–1	0	0
1	0	–2	12	–20	10	0
2	0	–3	45	–170	230	–102
3	0	–4	130	–958	2612	–2940
4	0	–5	315	–4116	19750	–41996
5	0	–6	672	–14520	112970	–398970
6	0	–7	1302	–44073	525031	–2854610
$m = -3:$						
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	1	0	0	0	0
5	0	2	–9	0	0	0
6	0	3	–48	93	0	0

Table 9 (continued).

k_b	0	1	2	— k_f — 3	4	5
$m = 4:$						
0	0	–1	4	–5	2	0
1	0	–2	20	–60	70	–28
2	0	–3	68	–400	936	–945
3	0	–4	180	–1912	7910	–15030
4	0	–5	412	–7290	49096	–155035
5	0	–6	840	–23520	243558	–1185830
6	0	–7	1576	–66660	1015960	–7246659
$m = -4:$						
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	1	0	0	0	0
6	0	2	–12	0	0	0
$m = 5:$						
0	0	–1	6	–14	14	–5
1	0	–2	30	–140	280	–252
2	0	–3	95	–810	2870	–4858
3	0	–4	240	–3472	20150	–56728
4	0	–5	525	–12156	109167	–475047
5	0	–6	1036	–36648	487382	–3116370
6	0	–7	1890	–98340	1869595	–16909871
$m = -5:$						
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	1	0	0	0	0

Appendix. Large *N* Limit of Chern-Simons and Framing of the Knot

As discussed in Sect. 5, Wilson loop observables of Chern-Simons theory at large N are naturally encoded in terms of the expectation values

$$\left\langle \exp \sum_n \frac{1}{n} \mathrm{tr} U^n \mathrm{tr} V^n \right\rangle = \exp(-F(V, t, g_s)),$$

where V is viewed as a source and U is the holonomy of the connection along the loop. If we are interested in extracting the amplitudes with a single hole, we consider terms on the right hand side of the form $\text{tr}V^n$ in the exponent. This will correspond to contributions where the large N worldsheet wraps n times around the loop. From the left-hand side this is equivalent to computation of

$$\left\langle \frac{1}{n}\text{tr}U^n \right\rangle.$$

For a special choice of framing of the unknot this was computed in [4] and it was shown that

$$\left\langle \frac{1}{n}\text{tr}U^n \right\rangle = \frac{1}{n}\text{tr}U_0^n,$$

where U_0 is a particular element of $SU(N)$ given by a diagonal matrix with entries $\exp(\frac{i\pi(N+1-2r)}{N+k})$, as r ranges from 1 to N. This leads, in the leading order in g_s, corresponding to disk amplitude ($g = 0$ and one hole), to

$$\left\langle \frac{1}{n}\text{tr}U^n \right\rangle = \frac{1}{n^2 g_s}[\exp(nt/2) - \exp(-nt/2)],$$

where in the large t limit, and by redefining the V (and absorbing a factor of $\exp(t/2)$ in it) we obtain the $\mathbf{C}^3$ answer for the disk amplitude

$$\frac{1}{g_s}\sum_n \frac{1}{n^2}\text{tr}V^n$$

(where we have identified in this paper $V = e^u$).

Now we ask what happens if we choose a different framing. For the expectation value of the Wilson loops with holonomies in a given representation R of $SU(N)$ the answer is relatively simple:

$$\langle \text{Tr}_R U \rangle_p = \langle \text{Tr}_R U \rangle_0 \exp(2\pi i p \frac{h_R}{k + N}),$$

where p denotes the change in the framing from a given one denoted by 0 (characterized by an element of topological class of winding of an S^1 over an S^1), and h_R denotes the Casimir of the representation. In other words, the result of the change in framing is a multiplication by $\exp(g_s p h_R)$.

However we need to compute the correlation function for $\langle \text{tr}U^n \rangle$ which is the trace in the fundamental representation of U^n. To find the change in this correlation function due to change in framing, we will first have to write it in terms of trace of U in different representations, compute each one, and multiply each one with $\exp(g_s p h_R)$.

There is an identity which is useful for this purpose:

$$\text{tr}U^n = \sum_{s=0}^{n-1}(-1)^s \text{Tr}_{R_{n,s}}U,$$

where $R_{n,s}$ denotes representations of $SU(N)$ with n boxes for the Young Tableau which looks like 'Γ' and which consists of only one non-trivial row and one non-trivial column. $s + 1$ denotes the number of elements in the first column.

Combining all this, we find that the coefficient of $\text{tr}V^n$ for the unknot and with framing number p is given by

$$\left\langle \frac{1}{n}\text{tr}U^n \right\rangle = \frac{1}{n}\sum_{s=0}^{n-1}(-1)^s \exp(p g_s h_{R_{n,s}})\text{Tr}_{R_{n,s}}U_0$$

We are interested in extracting the disk amplitude which is the leading term for this expansion as $g_s \to 0$. Moreover we will be interested in the limit where $t \to \infty$, but where we rescale V by a factor of $e^{t/2}$ so that we would be computing the brane in the $\mathbf{C}^3$ geometry. The most complicated aspect of this computation is finding $\text{Tr}_{R_{n,s}}U_0$. We will reduce this computation to a group theory computation as follows: Let

$$\text{Tr}_{R_{n,s}}U_0 = \sum_{\sum i n_i = n} \alpha_{n_i}^{R_{n,s}} \prod_{i=1}^{n}[\text{tr}U_0^i]^{n_i}.$$

for some group theoretic factor $\alpha_{n_i}^{R_{n,s}}$. Using this and the leading computation at large N for the unknot with the standard framing, we can reduce the above computation to

$$\left\langle \frac{1}{n}\text{tr}U^n \right\rangle = \frac{1}{n}\sum_{s=0}^{n-1}(-1)^s \exp(p g_s h_{R_{n,s}})$$

$$\cdot \sum_{\Sigma\, i n_i = n} \alpha_{n_i}^{R_{n,s}} \prod_{i=1}^{n}(\frac{1}{i g_s})^{n_i}$$

Using this and the fact that up to a further redefinition of V and dropping a subleading term in g_s, $h_{R_{n,s}} = \frac{1}{2}n(n - 1 - 2s)$ we have computed the

above expression for $n = 1, 2, 3$ and found perfect match with the result of the computation done for $\mathbf{C}^3$ when we shift $u \to u + pv$. Moreover we have verified that we obtain for all n a polynomial of degree $n - 1$ in p in agreement with the results of $\mathbf{C}^3$.

Furthermore we have verified $\left[\text{using the identity} \frac{n^{n-2}}{n!2^{n-1}} \sum_{r=1}^{n} (-1)^{r+1} \frac{(n-2r+1)^{n-1}}{(n-r)!(r-1)!} = \frac{n^{n-2}}{n!}\right]$ that the leading power in p agrees between both computations for all n.

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