Dislocation dynamics: short time existence and uniqueness of the solution.

O. Alvarez [*], P. Hoch [†], Y. Le Bouar [‡], R. Monneau [§]

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Abstract

We study a mathematical model describing dislocation dynamics in crystals. We consider a single dislocation line moving in its slip plane. The normal velocity is given by the Peach-Koehler force created by the dislocation line itself. The mathematical model is an eikonal equation whose velocity is a non-local quantity depending on the whole shape of the dislocation line. We study the special cases where the dislocation line is assumed to be a graph or a closed loop. In the framework of discontinuous viscosity solutions for Hamilton-Jacobi equations, we prove the existence and uniqueness of a solution for small time. We also give physical explanations and a formal derivation of the mathematical model. Finally, we present numerical results based on a level-sets formulation of the problem. These results illustrate in particular the fact that there is no general inclusion principle for this model.

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1 Introduction

1.1 Physical motivation

Plastic deformation of crystals is mainly due to the movement of linear defects called dislocations. This idea was put forward in the XX-th century, in the 30's, by Taylor [64], Orowan [50] and Polanyi [54] and was confirmed in 1956 by the first direct observations of dislocations, based on electron microscopy, by Hirsch, Horne, Whelan [34] and Bollmann [11]. We find a description of static equilibrium of dislocations in classical Physics books already written in the late 60's (see for instance Cottrell [17], Read [56], Friedel [29], Nabarro [49],

[*]Lab. Math. R. Salem, Site Colbert, Université de Rouen, 76821 Mont-Saint-Aignan Cedex, France
[†]CEA/DAM Ile de France, Service DCSA/SSEL, BP 12, 91680 Bruyères Le Chatel, France
[‡]Lab. d'Etude des Microstructures, CNRS-ONERA, 29, av. de la division Leclerc, BP 72, 92322, Châtillon, France
[§]CERMICS, Ecole nationale des Ponts et Chaussées, 6 et 8 avenue Blaise Pascal, Cité Descartes, Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2, France

and Hirth and Lothe [35] for a recent presentation). See also Hull, Bacon [37] for a physical introduction, François, Pineau, Zaoui [28] or Lemaitre, Chaboche [44] for a mechanical presentation and Lardner [41] for a more mathematical presentation.

The theoretical study of dislocations in crystals, together with the development of experimental observations using electron microscopy have largely contributed to better understand the plastic properties of materials. However, even if the elementary mechanisms at the origin of the deformation of monocrystals are rather well understood, many questions are still open concerning the plastic behavior of materials containing a high density of defects. The difficulty lies in the necessity of taking into account, in a 3D geometry, a large number of mechanisms and interactions.

Since the beginning of the 90's, the research field of dislocations is enjoying a new boom, based on the increasing power of computers, partly because it is now possible to directly simulate the collective behavior of a 3D set of dislocations. In the first 3D models, dislocations are described by the interconnection of straight dislocation segments, with only a few given orientations (see Canova, Kubin [14], Devincre, Condat [24], Kubin et al. [40], Devincre [23], Devincre, Kubin [25], Verdier, Fivel, Groma [65]) or with an arbitrary orientation (e.g. Schwarz [60], Politano, Salazar [55], Shenoy, Kukta, Phillips [62]).

Recently, a new approach, called the *phase field model of dislocations* has emerged, where dislocations are described by the variation of continuous fields (see Rodney, Le Bouar and Finel [58], Rodney, Finel [57], Xiang, Cheng, Srolovitz, E [68], Wang, Jin, Cuitino, Khachaturyan [66], Khachaturyan [39], Cuitino, Koslowski, Ortiz [21], Garroni, Müller [30], Haataja, Léonard [32]). This approach has the advantage that the possible topological changes during the dislocation movement are automatically taken into account, and that the interaction of dislocations with other defects or phases can be easily incorporated in the model. In the present paper, we study the phase field model of dislocation dynamics which has been recently proposed by Rodney, Le Bouar and Finel [58].

1.2 Setting of the problem

Dislocation lines are linear defects whose typical length in metallic alloys is of the order of $10^{-6}m$, with thickness of the order of $10^{-9}m$. The mobility of these defects is very sensitive to the crystallographic structure of the crystal. In the face centered cubic structure (FCC is observed in many metals and alloys), dislocations move at low temperature in well defined crystallographic planes (the slip planes), at velocities of the order of $10\ ms^{-1}$. An idealization consists in assuming that the thickness of these lines is zero, and in the case of a single line, in assuming that this line is contained and moves in the (x_1, x_2) plane. More precisely, we will study the case where the initial position of the dislocation line is represented by the graph of a function $x_2 = f_0(x_1)$, see fig. 1.

For the sequel, we define the characteristic function of the subgraph of f_0 for $x = (x_1, x_2)$:

$$\rho_{f_0}(x) = \begin{cases} 1 & \text{if} \quad x_2 < f_0(x_1) \\ 0 & \text{if} \quad x_2 = f_0(x_1) \\ -1 & \text{if} \quad x_2 > f_0(x_1) \end{cases} \tag{1.1}$$

The dislocation line generates an elastic field in the crystal. The normal velocity of the dislocation line only depends on this elastic field (more precisely, it is proportional to the resolved Peach-Koehler force calculated from the elastic field). The equation of motion therefore consists in the coupling of an evolutionary eikonal equation with the stationary

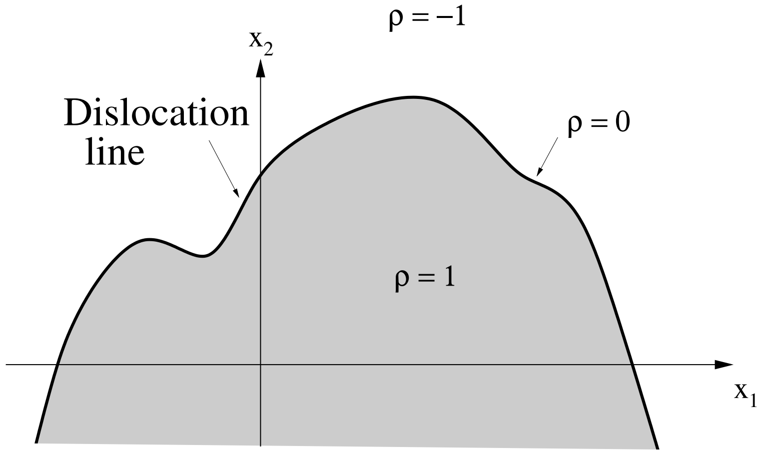

Figure 1: A dislocation line represented by a graph in the plane

equation that gives the elastic field.
Using the Green function of the equations of linearized elasticity, it is possible to reduce the problem (see section 2) to only one non-local eikonal equation on the characteristic function $\rho(x,t)$, namely:

$$\frac{\partial \rho}{\partial t} = (c_0 \star \rho^+(\cdot,t))|\nabla \rho|$$

with the initial condition

$$\rho(0,\cdot) = \rho_{f_0} \tag{1.2}$$

The function c_0 is a kernel associated to the equations of linearized elasticity and we denote $\rho^+ = \max(\rho, 0)$. Here the convolution is done in space for $x \in \mathbb{R}^2$. When we add some external shear stress on the material, or if we want to take into account the inhomogeneities of the crystal (obstacles to the movement of the dislocation like precipitates, other dislocations, ...), we need to modify the equation, and to consider

$$\frac{\partial \rho}{\partial t} = (c_0 \star \rho^+(\cdot,t) + c_1)|\nabla \rho| \tag{1.3}$$

where the function $c_1(x,t)$ describes the external field.

We refer the reader to Elliot, Schätzle, Stoth [26], and Briggs, Claisse, Elliot [12], for similar 1D or 2D non-local equations derived from Hamilton-Jacobi/elliptic coupled systems, for which a comparison principle holds. On the contrary there is no comparison principle for the dislocation dynamics model that we study in the present paper. See also Soravia, Souganidis [63] for an example of a system of equations of an interface moving with a non-local velocity, obtained as the limit of a system of reaction-diffusion equations. Finally, see Amadori [4] and Cont, Tankov, Voltchkova [16] for studies of continuous solutions to monotone non-local equations with applications in finance.

1.3 Main results

In this article we prove an existence and uniqueness result for the solutions ρ for (1.3)-(1.2). Here the functions ρ are by definition discontinuous, and a natural framework is then the theory of discontinuous viscosity solutions (see for instance the monographs of Barles [6], Bardi, Capuzzo-Dolcetta [5] for a presentation of first order equations). The theory of viscosity solutions has been first introduced by Crandall, Lions [19]. We also refer to the article of Barles, Soner, Souganidis [8] and to its references for a detailed presentation of the theory of geometric equations.

1.3.1 The 2D problem for the dislocation Lipschitz graph

In general, there is no reason that the dislocation line stays a graph. This is due to the fact that our non-local equation is not monotone (see the counter-example to the inclusion principle given in subsection 4.4). In section 5, we give a numerical simulation where the dislocation does not stay a graph for all times. Nevertheless we prove, under certain conditions, that if the dislocation line is initially a Lipschitz graph, then its stays a Lipschitz graph on a certain interval of time $[0,T^*)$ and moreover on this interval of time the solution is unique.

To state our main result, we need for a given function $g \in L^1_{\text{loc}}(\mathbb{R}^2)$, to define the quantities

$$||g||_{L^1_{\text{unif}}(\mathbb{R}^2)} = \sup_{x\in\mathbb{R}^2} \int_{Q(x)} |g|\, dy, \qquad ||g||_{L^\infty_{\text{int}}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \|g\|_{L^\infty(Q(x))}\, dx,$$

where $Q(x)$ designates the unit square centered at x

$$Q(x) = \{y \in \mathbb{R}^2 \mid |y_1 - x_1| \le 1/2,\ |y_2 - x_2| \le 1/2\}.$$

We denote respectively by $L^1_{\text{unif}}(\mathbb{R}^2)$ and $L^\infty_{\text{int}}(\mathbb{R}^2)$ the space that consists of the functions for which this quantities are finite.

We recall that for a locally bounded function h defined on $[0,T)\times\mathbb{R}^2$, the function h^* denotes its upper semi-continuous envelope (i.e. the smallest function u.s.c. $\ge h$), and the function h_* its lower semi-continuous envelope.

We refer to section 3 for a precise definition of a discontinuous viscosity solution. We also refer to Definition 3.5 for the definition of discontinuous viscosity solutions to a non-local equation; this is nothing else than the usual definition for the equation where the non-local dependence of the Hamiltonian is frozen. We can now precisely state our first main result:

Theorem 1.1 (Short time existence and uniqueness of a graph dislocation) *Assume that*

$$c_0 \in L^\infty_{int}(\mathbb{R}^2) \cap BV\big(\mathbb{R}^2\big), \quad c_1 \in W^{1,\infty}\big(\mathbb{R}^2 \times [0,+\infty)\big), \quad f_0 \in W^{1,\infty}(\mathbb{R}).$$

Then, there is a time $T^* > 0$ (depending only on the bounds of c_0, c_1 and f_0) for which the following holds.
(Existence) *There exists a function ρ in $C([0,T^*);L^1_{unif}(\mathbb{R}^2))$ which is a discontinuous viscosity solution of the equation*

$$(1.4) \qquad \frac{\partial\rho}{\partial t} = \big(c_0 \star \rho^+(\cdot,t) + c_1\big)\, |\nabla\rho| \quad \textit{in } \mathbb{R}^2\times(0,T^*), \qquad \rho(\cdot,0) = \rho_{f_0} \quad \textit{on } \mathbb{R}^2.$$

Moreover this function $\rho(\cdot,t)$ can be written as the characteristic function of the subgraph of a function $f(\cdot,t)$ as in (1.1): $\rho(\cdot,t) = \rho_{f(\cdot,t)}(\cdot)$, where the function f is a solution of the

graph equation:
(1.5)

$$\frac{\partial f}{\partial t} = \left(c_0 \star \rho^+_{f(\cdot,t)} + c_1\right)(x_1, f(x_1,t))\sqrt{1+\left(\frac{\partial f}{\partial x_1}\right)^2} \quad \textit{in } \mathbb{R}\times(0,T^*), \qquad f(\cdot,0)=f_0 \quad \textit{in } \mathbb{R}$$

(Uniqueness) *If $\rho_1,\ \rho_2 \in C([0,T^*); L^1_{unif}(\mathbb{R}^2))$ are two bounded discontinuous viscosity solutions of (1.4), then $(\rho_1)^* = (\rho_2)^*$, $(\rho_1)_* = (\rho_2)_*$ and, for every $t \in [0,T^*)$, $\rho_1(\cdot,t) = \rho_2(\cdot,t)$ a.e. in $\mathbb{R}^2$.*

The proof of this result relies on an application of the fixed point theorem in the framework of discontinuous viscosity solutions for Hamilton-Jacobi equations. This result has been announced in [3].

The precise expression of the anisotropic kernel c_0 in the model of dislocation dynamics can be quite complicated, as it is explained in the appendix 6. Nevertheless to fix ideas, let us give a rough approximation of an isotropic kernel which satisfies almost all the properties of an admissible kernel:

$$c_0(x) = \begin{cases} -2 & \text{if} \quad |x|<1 \\ \dfrac{1}{|x|^3} & \text{if} \quad |x|\geq 1 \end{cases}$$

In particular, we have $c_0(-x) = c_0(x)$ and $\int_{\mathbb{R}^2} c_0 = 0$. As a consequence of the symmetry of c_0 and the vanishing of its integral, we deduce that

$$c_0 \star 1_\Pi = 0 \quad \text{on} \quad \Gamma = \partial\Pi$$

where Π is any half plane. This result physically means that the self-force of any straight dislocation line Γ is zero, which is a well-known fact in physics. Therefore these properties guarantee that, in the absence of external fields, i.e. $c_1 \equiv 0$, straight dislocation lines stay straight lines which is an expected physical property.

A more physical kernel for isotropic elasticity is the one of Peierls-Nabarro model, given explicitly by its Fourier transform

$$\hat{c}_0(\xi_1,\xi_2) = -\frac{\mu b^2}{2}\left(\frac{\xi_2^2 + \frac{1}{1-\nu}\xi_1^2}{\sqrt{\xi_1^2+\xi_2^2}}\right)e^{-\zeta\sqrt{\xi_1^2+\xi_2^2}}$$

where μ is the shear modulus, ν the Poisson ratio, be_1 the Burgers vector, and ζ the parameter of Peierls-Nabarro which indicates the size of the core of the dislocation.

1.3.2 The 2D problem for the dislocation loop

We now consider the case of a closed dislocation loop. During its evolution the topology of the loop may change (it could for instance split it two curves). Nevertheless, for short time the topology is preserved, and this is in this framework that we will give an existence and uniqueness result, analogous to Theorem 1.1. Our short time existence and uniqueness result for the evolution of a dislocation loop works when the initial dislocation loop is C^3. Nevertheless the statement of our result under slightly more general assumptions (that allow the dislocation loop to be only Lipschitz) is a little bit technical. To this end, we first introduce the following type of open set we shall work with.

Assumption 1.2 (Lipschitz open set Ω_f)

We assume that there is a constant $L > 0$ and a closed curve $\Gamma \in C^3(\mathbb{R}/L\mathbb{Z};\mathbb{R}^2)$ such that Γ is injective and $\tau(s) = \frac{d\Gamma}{ds}(s)$ satisfies $|\tau(s)| = 1$, $n(s) = \tau^{\perp}(s)$, $\frac{d\tau}{ds}(s) = K(s)n(s)$. (Here, $\mathbb{R}/L\mathbb{Z}$ denotes the one-dimensional torus.) We set $K_0 = ||K||_{L^\infty(\mathbb{R}/L\mathbb{Z})}$.

We assume also that there is a function $f \in Lip\,(\mathbb{R}/L\mathbb{Z};\mathbb{R})$ satisfying $||f||_{L^\infty(\mathbb{R}/L\mathbb{Z})} \leq r_0$ for some fixed $r_0 \in (0, \frac{1}{K_0})$, small enough.

Then, the map

$$\Gamma_f(s) = \Gamma(s) + f(s)n(s), \quad s \in \mathbb{R}/L\mathbb{Z}$$

is injective, with $\frac{d\Gamma_f}{ds}(s) \neq 0$ a.e., and $\Gamma_f\,(\mathbb{R}/L\mathbb{Z})$ is the boundary of a bounded, connected, simply connected Lipschitz domain. We call this domain Ω_f. When $f = 0$, we denote this open set by Ω_0.

We define the following characteristic function of Ω_f for $x = (x_1, x_2)$:

$$\tilde{\rho}_f(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega_f \\ 0 & \text{if} \quad x \in \partial\Omega_f \\ -1 & \text{if} \quad x \in \mathbb{R}^2\backslash\Omega_f \end{cases} \tag{1.6}$$

Our second main result is

Theorem 1.3 (Short time existence and uniqueness of the dislocation loop) *Assume that*

$$c_0 \in L^\infty_{int}(\mathbb{R}^2) \cap BV(\mathbb{R}^2), \quad c_1 \in W^{1,\infty}(\mathbb{R}^2 \times [0,+\infty)),$$

and that the initial domain $\Omega_{f_0} \subset \mathbb{R}^2$, given by Assumption 1.2, is a bounded connected and simply connected Lipschitz open set, with $f_0 \in Lip(\mathbb{R}/L\mathbb{Z};\mathbb{R})$ such that $||f_0||_{L^\infty(\mathbb{R}/L\mathbb{Z})} \leq \frac{r_0}{4}$. Then, there is a time $T^* > 0$ (depending only on bounds on c_0, c_1, $\left|\left|\frac{\partial f_0}{\partial s}\right|\right|_{L^\infty(\mathbb{R}/L\mathbb{Z})}$ and on Ω_0) for which the following holds.

(Existence) *There exists a function ρ in $C([0,T^*);L^1_{unif}(\mathbb{R}^2))$ which is a discontinuous viscosity solution of the equation*

$$\frac{\partial\rho}{\partial t} = \left(c_0 \star \rho^+(\cdot,t) + c_1\right)|\nabla\rho| \quad \text{in } \mathbb{R}^2 \times (0,T^*), \qquad \rho(\cdot,0) = \tilde{\rho}_{f_0}(\cdot) \quad \text{on} \quad \mathbb{R}^2 \tag{1.7}$$

Moreover this function $\rho(\cdot,t)$ can be written as the characteristic function of the open set $\Omega_{f(\cdot,t)}$ as in (1.6): $\rho(\cdot,t) = \tilde{\rho}_{f(\cdot,t)}(\cdot)$, where the function f is a solution of the local loop equation:

(1.8)

$$\begin{cases} \frac{\partial f}{\partial t} = \left(c_0 \star \tilde{\rho}^+_{f(\cdot,t)} + c_1\right)\left(\Gamma_{f(\cdot,t)}(s),t\right)\sqrt{1 + \left(\frac{1}{(1-fK)}\frac{\partial f}{\partial s}\right)^2} & \text{in } (\mathbb{R}/L\mathbb{Z}) \times (0,T^*), \\ \\ f(\cdot,0) = f_0 \quad \text{in } \mathbb{R}/L\mathbb{Z} \end{cases}$$

where Γ and K are given in Assumption 1.2.

(Uniqueness) *If ρ_1, $\rho_2 \in C([0,T^*);L^1_{unif}(\mathbb{R}^2))$ are two bounded discontinuous viscosity solutions of (1.7), then $(\rho_1)^* = (\rho_2)^*$, $(\rho_1)_* = (\rho_2)_*$ and, for every $t \in [0,T^*)$, $\rho_1(\cdot,t) = \rho_2(\cdot,t)$ a.e. in $\mathbb{R}^2$.*

1.4 Contribution of this paper and the main open problem

The main achievement of this paper is to bring to the attention of the mathematical community a non-local equation modelling dislocation dynamics, which presents new mathematical difficulties.

In this paper, we show that this equation is well-posed in the framework of viscosity solutions. This can be considered as a preliminary and expected result. Those short time existence and uniqueness results could also probably be studied using classical solutions or other frameworks. But, it turns out that the framework of viscosity solutions is powerful to study the question of the existence (and uniqueness) of a solution for all time: this kind of result has been proved recently in the case where the velocity is nonnegative (see Alvarez, Cardaliaguet, Monneau [2] and Barles, Ley [7]).

In its full generality, the long time existence of a solution to this non-monotone non-local equation is still an open problem. We can even say that this is the essential remaining mathematical question.

Finally let us underline that one contribution of the present paper is to propose a coherent approach to the evolution of dislocations, which includes in some sense a variety of different physical theories describing dislocation dynamics (see section 6).

1.5 Organization of the article

In section 2, we present the physical framework and the formal derivation of the model. In section 3, we recall the definition of discontinuous viscosity solutions, give some fundamental results of existence, uniqueness and stability on Lipschitz solutions of Hamilton-Jacobi equations. We also give an existence and uniqueness result for Lipschitz solutions to non-local Hamilton-Jacobi equations. Finally we state precisely the relationship between geometric equations and graph or loop equations. In section 4, we prove theorems 1.1 and 1.3. In section 5, we present our numerical results. We use in particular the Lax-Friedrichs scheme for Hamilton-Jacobi equations and compute the non-local velocity using fast Fourier transform. These results illustrate in particular the fact that there is no general inclusion principle for this model. In the appendix (section 6), we give more information on the kernel c_0.

2 Physical presentation and formal derivation of the model

Let us consider an orthonormal basis (e_1, e_2, e_3) of $\mathbb{R}^3$. In this section we denote the coordinates by $x = (x_1, x_2, x_3)$, and $x' = (x_1, x_2)$.

2.1 A formulation of linear elasticity without dislocations

We define the operator of incompatibility inc as the curl of the columns of the curl of the rows of a field $e = (e_{ij}) \in (\mathcal{D}'(\mathbb{R}^3))^{3\times 3}_{sym}$ of symmetric 3×3 matrices:

$$(\text{inc}(e))_{ij} = \sum_{i_1, j_1 = 1}^{3} \varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\partial_{i_1}\partial_{j_1}e_{i_2j_2}, \quad i,j = 1,2,3$$

where we denote as usual

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if} \quad (ijk) \quad \text{is a positive permutation of} \quad (123) \\ -1 & \text{if} \quad (ijk) \quad \text{is a negative permutation of} \quad (123) \\ 0 & \text{if} \quad \text{two indices are the same} \end{cases}$$

and where ∂_k stands for $\dfrac{\partial}{\partial x_k}$, for $k = 1, 2, 3$. Then it is known (see for instance [41], Lemma 5 page 277) that

$$\text{inc}(e) = 0 \quad \text{in} \quad \left(\mathcal{D}'(\mathbb{R}^3)\right)^{3\times 3}_{sym} \tag{2.9}$$

if and only if there exists a displacement $u = (u_1, u_2, u_3) \in \left(\mathcal{D}'(\mathbb{R}^3)\right)^3$ such that the strain can be written

$$e = e(u) \quad \text{where} \quad e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

We now consider a linearly elastic material represented by the whole space and whose constant elastic coefficients are given by $\Lambda = (\Lambda_{ijkl}) \in \mathbb{R}^{3\times 3\times 3\times 3}$, and satisfy the following symmetry property

$$\Lambda_{ijkl} = \Lambda_{jikl} = \Lambda_{ijlk} = \Lambda_{klij}, \quad \text{for} \quad i, j, k, l = 1, 2, 3 \tag{2.10}$$

and the following coercivity assumption for some $m > 0$

$$\sum_{i,j,k,l=1}^{3} \Lambda_{ijkl} e_{ij} e_{kl} \quad \geq \quad m \sum_{i,j=1}^{3} (e_{ij})^2 \tag{2.11}$$

for all constant matrices $e = (e_{ij}) \in \mathbb{R}^{3\times 3}_{sym}$, i.e. such that $e_{ij} = e_{ji}$ for $i, j = 1, 2, 3$.

Then the strain e needs to satisfy the equation of linear elasticity (here with zero body forces)

$$\text{div}(\Lambda e) = 0 \tag{2.12}$$

where

$$\left(\text{div}(\Lambda e)\right)_j = \sum_{i=1}^{3} \partial_i \left(\sum_{k,l=1}^{3} \Lambda_{ijkl} e_{kl} \right)$$

Moreover in the absence of dislocations, equation (2.9) is satisfied.

2.2 Description of the dislocation line

Let us assume that the dislocation line is represented by the boundary Γ of a smooth bounded domain $\Omega_0 \subset \mathbb{R}^2$. The plane $\mathbb{R}^2$ is assumed to be the plane where the dislocation can move. This plane is naturally imbedded in the three-dimensional material which is assumed to be the whole space $\mathbb{R}^3$ (see fig. 2).

We then identify Γ and $\Gamma \times \{0\}$. The classical theory of dislocations asserts that the dislocation line Γ creates a distorsion in the strain e such that the operator of incompatibility is non-zero on the dislocation line Γ, and is given by:

$$\text{inc}(e) = -\text{inc}\left(\rho \delta_0(x_3) e^0\right) \quad \text{where} \quad e^0 = \frac{1}{2}\left(b \otimes n + n \otimes b\right) \tag{2.13}$$

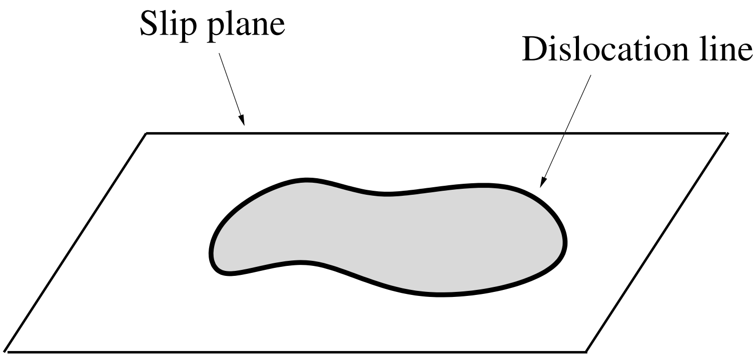

Figure 2: Example of a bounded dislocation line in a slip plane

Here $\delta_0(x_3)$ is the Dirac mass only in the x_3 component, $n = e_3$ is the unit vector perpendicular to the plane which contains the dislocation, $\rho(x) = \rho(x')$ for $x' = (x_1, x_2)$ and is defined by

$$\rho(x') = \begin{cases} 1 & \text{if} \quad x' \in \Omega_0 \\ 0 & \text{if} \quad x' \in \mathbb{R}^2 \backslash \Omega_0 \end{cases}$$

and $b \in \mathbb{R}^3$ is a constant vector associated to the dislocation line, which is called the Burgers vector. It is easily seen that the support of the distribution in the right hand side of (2.13) is the dislocation line Γ. Moreover there exists a unique solution $e \in (\mathcal{D}'(\mathbb{R}^3))^{3\times 3}_{sym}$ satisfying (2.13), (2.12) and

$$e(x) \longrightarrow 0, \quad \text{as} \quad |x| \longrightarrow +\infty \tag{2.14}$$

Here the strain e is a physical quantity, and can not be written as the symmetric part of the gradient of a displacement, because $\text{inc}(e) \not\equiv 0$. If we apply an exterior stress corresponding to a strain e^{ext} (constant or not, if we want to describe the action of body forces), the total strain is then given by

$$e + e^{ext}$$

The total energy (up to a constant $\int \frac{1}{2}\Lambda(e^{ext})^2$, which may be infinite) is then formally given by

$$\int \frac{1}{2}\Lambda e^2 + \Lambda e e^{ext}$$

where we denote

$$\Lambda e^2 = \sum_{i,j,k,l=1}^{3} \Lambda_{ijkl} e_{ij} e_{kl}, \quad \Lambda e e^{ext} = \sum_{i,j,k,l=1}^{3} \Lambda_{ijkl} e_{ij} e^{ext}_{kl}$$

2.3 Mollification by the core tensor

When Ω_0 is bounded, it can be seen (see proposition 6.2 in the appendix) that we have

$$|e(x)| \leq \frac{C}{r^3} \quad \text{for} \quad r = \text{dist}(x, \Gamma) \quad \text{large enough}$$

But one difficulty of the classical theory of dislocations is that the elastic self-energy $\int \frac{1}{2}\Lambda e^2$ has no meaning on the core of the dislocation and is infinite because the strain behaves like

$$e(x) \simeq \frac{1}{r} \quad \text{for} \quad r = \text{dist}(x, \Gamma) \quad \text{small enough}$$

The divergence of the strain near the dislocation line is of course not physical. Indeed, near the dislocation line, the atomic positions can not be described by a small homogeneous deformation of the reference crystal, and so the linear continuum elasticity theory is no longer valid. In other words, the dislocation line is surrounded by a region, known as the dislocation core, within which the linear continuum elasticity theory ceases to be a good approximation (see the volume of Chapman et al [15] for similar descriptions of line singularities in partial differential equations).

To overcome this difficulty within the framework of dislocations in a continuous medium, we need to regularize the strain on the dislocation line. The simplest way to do it consists in introducing a core tensor $\chi = (\chi_{ijkl})$, which satisfies the following

Assumptions

a) χ satisfies the symmetry property (2.10).

b) $\chi_{ijkl} \in M(\mathbb{R}^3)$ for $i,j,k,l = 1,2,3$, where $M(\mathbb{R}^3)$ is the set of Radon measures on $\mathbb{R}^3$ satisfying $|\chi_{ijkl}|(\mathbb{R}^3) < +\infty$.

c) $\chi_{ijkl}(\mathbb{R}^3) = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})$.

d) χ is smooth enough.

Here assumptions b) and d) are necessary to get well defined Peach-Koehler force or energy. Assumptions a) and c) are compatible with the symmetry properties of Λ, and c) guarantees that the convolution of χ with a constant strain tensor is the strain tensor itself.

We now assume that the "true physical strain" e_χ satisfies (2.12),(2.14) and involves a convolution by the core tensor

$$\mathrm{inc}(e_\chi) = -\mathrm{inc}\left(\chi \star \rho\delta_0(x_3)e^0\right)$$

which replaces (2.13), where we define

$$(\chi \star g)_{ij} = \sum_{k,l=1}^{3} \chi_{ijkl} \star g_{kl}$$

for a general tensor g.

If χ is smooth enough, this ensures that, as physically expected, the strain is finite even on the dislocation line, which is experimentally observed. Then the physical energy is now given by

$$E(\Gamma) = \int_{\mathbb{R}^3} \frac{1}{2}\Lambda e_\chi^2 + \int_{\mathbb{R}^3} \Lambda e_\chi e^{ext}$$

where the first term is finite. We will precise later (in subsection 2.6) the meaning of the last term, in particular for constant e^{ext}.

This kind of model has been used in the physical literature: see Rodney, Le Bouar, Finel [58] for a recent reference, and the appendix for more information on the core tensor.

2.4 First variation of the energy

We consider the case where the line Γ is constrained to move in the (x_1, x_2)-plane (case without cross-slip). We denote by s the curvilinear abscissa on Γ and by $\Gamma(s)$ the current point on Γ. Let $n_\Gamma(s)$ be the outer unit normal vector to the open set Ω_0 in the plane $\mathbb{R}^2$. Then for a given function $h(s)$, we define the planar perturbation Γ_δ of Γ by

$$\Gamma_\delta(s) = \Gamma(s) + \delta h(s) n_\Gamma(s)$$

A computation gives

$$-\frac{d}{d\delta}E(\Gamma_\delta)_{|\delta=0} = \int_\Gamma ds\ h(s)c(\Gamma(s))$$

where the function c is called the generalized resolved Peach-Koehler force, and will be computed in subsection 2.6.

2.5 The dislocation dynamics

The simplest dislocation dynamics of a curve $t \longmapsto \Gamma_t$ consists to assume that the normal velocity to the curve is proportional to the resolved Peach-Koehler force c:

$$\frac{d\Gamma}{dt}(s) = \frac{1}{B}\ c(\Gamma(s))n_\Gamma(s) \tag{2.15}$$

where B is a constant called the viscous drag coefficient (see Hirth, Lothe [35], page 208). Physically this constant depends on the properties of the material. Nevertheless, up to rescaling the time, we can remove B in the equation, and we will assume from now on that $B = 1$ in (2.15). In particular, we have formally (if $e^{ext} = e^{ext}(x)$ does not depend on time)

$$\frac{d}{dt}E(\Gamma_t) = -\int_\Gamma c^2$$

which shows that the total energy is non-increasing.

2.6 Computation of the resolved Peach-Koehler force and reformulation of the dynamics

Usually, it is not very convenient to work with the strain e. Because the incompatibility operator inc applied to $e + \rho\delta_0(x_3)e^0$ is zero, we can write

$$e + \rho\delta_0(x_3)e^0 = e(u)$$

for some displacement u. Because of (2.12), we deduce

$$\text{div}(\Lambda e(u)) = \text{div}\left(\Lambda\rho\delta_0(x_3)e^0\right) \tag{2.16}$$

and we can look for such u satisfying moreover

$$u(x) \longrightarrow 0, \quad \text{as} \quad |x| \longrightarrow +\infty \tag{2.17}$$

Remark 2.1 *In the physical literature, the authors usually write* $e - \rho\delta_0(x_3)e^0 = e(u)$*, which is equivalent to our formulation if we change* ρ *in* $-\rho$ *or the Burgers vector* b *in* $-b$*.*

Computation of the self-Peach-Koehler force

We now compute the generalized resolved Peach-Koehler force c in the special case where $e^{ext} = 0$. To this end we remark that, using for instance Fourier transform and the explicit computation of the Green function for linear elasticity, it is possible to see that there exists a forth order tensor R_0 such that $e(u) = (R_0e^0) \star \rho\delta_0(x_3)$ and then we have with $R = -\delta_0(x)Id + R_0$ (for $(Id)_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$):

$$e = R \star \left(\rho\delta_0(x_3)e^0\right)$$

and

$$e_\chi = R \star \chi \star \left(\rho\delta_0(x_3)e^0\right)$$

Therefore we get

$$\begin{aligned}
&\frac{d}{d\delta}E(\Gamma_\delta)_{|\delta=0}\\
&= \int_{\mathbb{R}^3} \Lambda e_\chi (R \star \chi e^0 \star (h\delta_\Gamma))\\
&= \int_{\mathbb{R}^3} \Lambda((R \star \chi e^0) \star (\rho\delta_0(x_3)))((R \star \chi e^0) \star (h\delta_\Gamma))\\
&= \int_{\mathbb{R}^3} \left[\left\{{}^t(\check{R} \star \check{\chi} e^0) \star \Lambda(R \star \chi e^0)\right\} \star (\rho\delta_0(x_3)))\right] (h\delta_\Gamma)\\
&= \int_\Gamma -(c_0 \star \rho)h
\end{aligned}$$

where $\check{\chi}(x) = \chi(-x)$, and we have used a transposition of the tensor χ. Here the last convolution $c_0 \star \rho$ is taken on $\mathbb{R}^2$, and

$$c_0(x') = C_0(x', 0), \quad \text{with} \quad C_0(x) = -{}^t(\check{R} \star \check{\chi} e^0) \star \Lambda(R \star \chi e^0) \tag{2.18}$$

Meaning of the energy for constant exterior forces
We now consider the case of constant e^{ext}. Then we can give the following meaning to the second term in the energy (with $e_\chi + \chi \star (\rho\delta_0(x_3)e^0) = e(u_\chi)$ for some associated displacement u_χ):

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Lambda e_\chi e^{ext}\\
&= \int_{\mathbb{R}^3} \Lambda \left(e(u_\chi) - \chi \star (\rho\delta_0(x_3)e^0)\right) e^{ext}\\
&= -\int_{\mathbb{R}^3} \Lambda \left(\chi \star (\rho\delta_0(x_3)e^0)\right) e^{ext}\\
&= -\int_{\mathbb{R}^3} \left(\check{\chi} \star \Lambda e^{ext}\right) (\rho\delta_0(x_3)e^0)\\
&= -\int_{\mathbb{R}^3} \Lambda e^{ext} (\rho\delta_0(x_3)e^0)\\
&= -\left(\int_{\mathbb{R}^2} \rho\right) \Lambda e^0 e^{ext}
\end{aligned}$$

where we have used an integration by parts on $e(u_\chi)$ and the assumptions on the core tensor χ.

The additional force due to the exterior field

For general (possibly non-constant) e^{ext}, and for a given δ we consider the strain $e_\chi = e_\chi[\Gamma_\delta]$ created by the dislocation line Γ_δ. Then we have

$$\frac{d}{d\delta}\left\{\int_{\mathbb{R}^3} \Lambda e^{ext} e_\chi\right\}_{|\delta=0}$$

$$= \int_{\mathbb{R}^3} \Lambda e^{ext}(R \star \chi e^0 \star (h\delta_\Gamma))$$

$$= \int_\Gamma -c_1 h$$

with

$$c_1 = {}^t(\check{R} \star \check{\chi} e^0) \star \Lambda e^{ext}$$

In particular when e^{ext} is constant we get $c_1 = \Lambda e^0 e^{ext}$ which is a constant force.

The full dynamics

In the general case we have both the self-force and the additional force due to the exterior field. This means that the total resolved force is given by

$$c = c_0 \star \rho + c_1$$

Finally, we see that equation (2.15) with $B = 1$ means that ρ formally satisfies the following equation

$$\frac{\partial \rho}{\partial t} = (c_0 \star \rho + c_1)|\nabla \rho| \tag{2.19}$$

This ends the formal derivation of (2.19), which is the model that we study in this paper.

3 Viscosity solutions of non-local Hamilton-Jacobi equations

3.1 Discontinuous viscosity solutions

We recall in this section a few classical results from the theory of viscosity solutions for the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} = H(x, t, \nabla u) \quad \text{in } \mathbb{R}^N \times (0, T), \qquad u(\cdot, 0) = u_0 \quad \text{on } \mathbb{R}^N \tag{HJ}$$

for a possibly infinite horizon T and for a continuous Hamiltonian H. It is well known that the equation has a unique continuous viscosity solution with an appropriate behavior at infinity under mild assumptions on the Hamiltonian and the initial data u_0.

Let us introduce the functional spaces we shall work with. For $T < +\infty$, we denote by $C(\mathbb{R}^N \times [0, T])$ the set of continuous functions, $C_b(\mathbb{R}^N \times [0, T])$ the subset of continuous bounded functions. We denote by $W^{1,\infty}(\mathbb{R}^N \times [0, T])$ the set of the functions that are Lipschitz continuous and bounded in (x, t).

We shall also work with functions that are defined on $[0,T)$ for T possibly infinite. Such a function will belong to a certain functional space if its restriction to $[0,S]$ for all $S<T$ is in the functional space defined above. As an example, we shall write

$$W^{1,\infty}(\mathbb{R}^N \times [0,T)) = \bigcup_{S<T} W^{1,\infty}(\mathbb{R}^N \times [0,S])$$

with a straightforward abuse in the notation, which, we hope, will raise no confusion.

For geometric equations, it is most convenient to work with discontinuous viscosity solutions. This corresponds to the case where the initial condition may be discontinuous. We briefly recall the definition. Given a locally bounded function u defined on $\mathbb{R}^N \times [0,T)$, the function u^* designates its upper-semicontinuous envelope (i.e. the smallest u.s.c. function $\geq u$) and the function u_* its lower-semicontinuous envelope.

Definition 3.1 (Discontinuous viscosity solution)
Assume that $H \in C(\mathbb{R}^N \times [0,T) \times \mathbb{R}^N)$ and that the initial data u_0 is locally bounded.

- *A locally bounded function $u : \mathbb{R}^N \times [0,T) \to \mathbb{R}$ is a* viscosity subsolution *of* (HJ) *if it satisfies the following.*
  - *Initial condition.* $u^*(\cdot,0) \leq u_0^*$
  - *Equation in $(0,T)$. For every point $(\overline{x},\overline{t}) \in \mathbb{R}^N \times (0,T)$ and every test function $\phi \in C^1(\mathbb{R}^N \times (0,T))$ that is tangent from above to u^* at $(\overline{x},\overline{t})$ (i.e. satisfying $u^* \leq \phi$ in $\mathbb{R}^N \times (0,T)$ and $u^*(\overline{x},\overline{t}) = \phi(\overline{x},\overline{t})$), we have*

$$\frac{\partial \phi}{\partial t}(\overline{x},\overline{t}) \leq H\big(\overline{x},\overline{t},\nabla\phi(\overline{x},\overline{t})\big).$$

- *A locally bounded function $u : \mathbb{R}^N \times [0,T) \to \mathbb{R}$ is a* viscosity supersolution *of* (HJ) *if it satisfies the following.*
  - *Initial condition.* $u_*(\cdot,0) \geq (u_0)_*$
  - *Equation in $(0,T)$. For every point $(\overline{x},\overline{t}) \in \mathbb{R}^N \times (0,T)$ and every test function $\phi \in C^1(\mathbb{R}^N \times (0,T))$ that is tangent from below to u_* at $(\overline{x},\overline{t})$, we have*

$$\frac{\partial \phi}{\partial t}(\overline{x},\overline{t}) \geq H\big(\overline{x},\overline{t},\nabla\phi(\overline{x},\overline{t})\big).$$

- *A locally bounded function $u : \mathbb{R}^N \times [0,T) \to \mathbb{R}$ is a* discontinuous viscosity solution *if its u.s.c. envelope u^* is a subsolution and its l.s.c. envelope u_* is a supersolution.*

Remark 3.2 *See for instance Giga [31] for a different and new notion of viscosity solution with shocks.*

3.2 Solvability of local Hamilton-Jacobi equations

For further reference, we gather in the next proposition the main properties of viscosity solutions for the Hamilton-Jacobi equation (HJ). The result is quite classical and we refer to the standard works for most proofs and for a detailed presentation of the theory (see the review paper by Crandall, Ishii, Lions [18] and the books by Barles [6] and by Bardi, Capuzzo-Dolcetta [5]).

We shall make assumptions on the Hamiltonian that are well adapted to our dislocation model. They are not the most general and could be improved. An important assumption is the continuity of the Hamiltonian with respect to all the variables, in particular time. Some special attention will be needed later to check that all the Hamiltonians we consider enjoy this property.

A delicate issue for discontinuous viscosity solutions of Hamilton-Jacobi equation is the question of uniqueness. When the initial data is discontinuous, the natural notion of uniqueness is that any two discontinuous solutions must have the same semicontinuous envelopes because the definition of viscosity solutions only view these envelopes. But, as is well-known, uniqueness in this sense may be untrue for non-stationary Hamilton-Jacobi equations (see Barles, Soner, Souganidis [8]). This issue will be discussed in a more detailed manner later in the context of geometric equations. Nonetheless, when the initial data is continuous and the Hamiltonian enjoys certain smoothness properties, there is a unique (discontinuous) viscosity solution and it is continuous. When the initial data is Lipschitz continuous, then so is the solution. We shall need later a sharp estimate of $\|u(\cdot,t)\|_{W^{1,\infty}(\mathbb{R}^N)}$ with respect to time. The one we propose seems to be new.

Proposition 3.3 (Viscosity solution: existence, uniqueness and regularity) *Consider a Hamiltonian $H \in C(\mathbb{R}^N \times [0,T) \times \mathbb{R}^N)$ with the following properties.*

- Local boundedness. *There is a constant C_1 such that $|H(x,t,0)| \le C_1$ for all (x,t).*
- Local Lipschitz regularity in space. *There are nonnegative functions $C_2, C_3 \in L^\infty_{loc}([0,T))$ so that*

$$|H(x,t,p) - H(y,t,p)| \le C_2(t)|p|\,|x-y| + C_3(t)|x-y| \tag{3.20}$$

  for all (x,y,t,p).
- Lipschitz regularity in the gradient. *There is a constant C_4 such that $|H(x,t,p) - H(x,t,q)| \le C_4|p-q|$ for all (x,t,p,q).*

Then, the following holds.

- Comparison principle. *Let u and v are respectively a discontinuous viscosity subsolution and a discontinuous viscosity supersolution of the equation*

$$\frac{\partial u}{\partial t} = H(x,t,\nabla u) \quad \text{in } \mathbb{R}^N \times (0,T).$$

  Assume that u is bounded from above, that v is bounded from below and that $u^*(\cdot,0) \le v_*(\cdot,0)$. Then, $u \le v$ in $\mathbb{R}^N \times [0,T)$.
- Existence. *If u_0 is bounded, then there is a discontinuous bounded viscosity solution of* (HJ)*.*
- Lipschitz regularity. *If $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, then there is a unique viscosity solution of* (HJ)*. It is in $W^{1,\infty}(\mathbb{R}^N \times [0,T))$ and we have the estimates*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le L(t), \quad \|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le M(t), \quad \left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} \le P(t) \tag{3.21}$$

where

$$L(t) = \|u_0\|_{L^\infty(\mathbb{R}^N)} + C_1 t, \qquad P(t) = C_1 + C_4 M(t)$$

and where M is the solution of the differential equation

$$M'(t) = C_2^* M + C_3^* \quad \text{in } (0,T), \qquad M(0) = \|\nabla u_0\|_{L^\infty(\mathbb{R}^N)}. \tag{3.22}$$

where C_2^* and C_3^* are the upper-semicontinuous envelope of C_2 and C_3.

Proof of proposition 3.3 The comparison principle and the construction of a discontinuous solution by Perron's method is classical in the theory of viscosity solutions (see for instance the books Barles [6] or Bardi, Capuzzo-Dolcetta [5]). It follows that when $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, (HJ) has a unique bounded viscosity solution and that it is continuous. The bound for $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}$ follows classically from the comparison principle and from the observation that L and $-L$ are respectively a supersolution and a subsolution of the equation (HJ). The fact that one can control the modulus of continuity of the solution by a solution of a differential equation was established by Crandall Lions [20] and Ishii [38]. As the ODE we need differs from theirs, we briefly explain how to adapt their argument.

We want to estimate the bound on $\|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}$ (see Ley [46] for an interesting estimate from below on the gradient, when the Hamiltonian is convex in the gradient). We first assume that the functions C_2 and C_3 in (3.20) are continuous. The function $w(x,y,t) = u(x,t) - u(y,t)$ is a viscosity subsolution of the equation

$$\frac{\partial w}{\partial t} = H(x,t,\nabla_x w) - H(y,t,-\nabla_y w) \tag{3.23}$$

with initial data $w_0(x,y) = u_0(x) - u_0(y)$. The claim is obvious if u is C^1 and we refer to Crandall Lions [20] when u is a continuous viscosity solution.

For $\varepsilon > 0$, we consider the C^1 function

$$\phi^\varepsilon(x,y,t) = M(t)(|x-y|^2 + \varepsilon^2)^{1/2}.$$

We claim that ϕ^ε is a supersolution of the equation (3.23) with initial condition w_0. The validity of the initial condition follows at once from the definition of M_0 as the Lipschitz constant of u_0. To check the equation, we note that $\nabla_x \phi^\varepsilon = -\nabla_y \phi^\varepsilon$. Therefore, using assumption (3.20) and the differential equation (3.22) solved by M, we compute for every $t > 0$

$$\begin{aligned}
&\frac{\partial \phi^\varepsilon}{\partial t} - H(x,t,\nabla_x \phi^\varepsilon) + H(y,t,-\nabla_y \phi^\varepsilon) \\
&\quad \geq \frac{\partial \phi^\varepsilon}{\partial t} - C_2(t)|\nabla_x \phi^\varepsilon|\,|x-y| - C_3(t)|x-y| \\
&\quad = M'(t)(|x-y|^2+\varepsilon^2)^{1/2} - C_2(t)\frac{|x-y|^2}{(|x-y|^2+\varepsilon^2)^{1/2}} - C_3(t)|x-y| \\
&\quad \geq (|x-y|^2+\varepsilon^2)^{1/2}\big(M'(t) - C_2(t)M(t) - C_3(t)\big) \\
&\quad \geq 0.
\end{aligned}$$

This means that ϕ^ε is a supersolution. As ϕ^ε is bounded from below and w is bounded, we deduce from the comparison principle for (3.23) that $w \leq \phi^\varepsilon$. Sending $\varepsilon \to 0$, we obtain that $u(x,t) - u(y,t) \leq M(t)|x-y|$ for all t, x and y. Exchanging x and y yields $|u(x,t) - u(y,t)| \leq M(t)|x-y|$. Consequently,

$$\|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \sup_{x \neq y} \frac{|u(x,t) - u(y,t)|}{|x-y|} \leq M(t)$$

for every $t \geq 0$.

When the functions C_2 and C_3 are upper-semicontinuous, they can be approximated by a non-increasing sequence of continuous functions C_2^k and C_3^k. Applying the dominated convergence theorem to the explicit formula for P^k and M^k, we see that the sequences converge pointwise to the functions P and M of the proposition (where M is the unique Lipschitz continuous function that solves (3.22) a.e.). Therefore, the a priori estimates still holds when the functions C_2 and C_3 are upper-semicontinuous. This implies of course that, when the functions are locally bounded, the estimates are true provided the functions are replaced by their upper-semicontinuous envelopes.

The equation yields the bound

$$|\frac{\partial u}{\partial t}| = |H(x,t,\nabla u)| \leq |H(x,t,0)| + C_4|\nabla u(\cdot,t)| \leq C_1 + C_4 M(t) = P(t)$$

in the viscosity sense. As it is well-known, this implies that u is Lipschitz continuous in time with Lipschitz constant equal to $P(t)$. □

Proposition 3.4 (Stability with respect to the Hamiltonian)
Let two Hamiltonian H^i, $i = 1$, 2 satisfy the assumptions of Proposition 3.3 with the same constants. Let $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. Let u^i be the solution of (HJ) *with H^i instead of H. Then, for all $t \in [0,T)$, we have the uniform bound*

$$\|u^1(\cdot,t) - u^2(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \sup_{\mathbb{R}^N \times [0,t] \times \mathbb{R}^N} |\frac{H^1(x,s,p) - H^2(x,s,p)}{1+|p|}| \int_0^t \big(1 + M(s)\big)\, ds,$$

where M is the solution of the differential equation (3.22).

Proof of proposition 3.4 The solution u^1 of (HJ) with Hamiltonian H^1 satisfies the equation with H^2 up to the quantity

$$\begin{aligned}\Big|\frac{\partial u^1}{\partial t} - H^2(x,s,\nabla u^1)\Big| &\leq \Big|H^1(x,s,\nabla u^1) - H^2(x,s,\nabla u^1)\Big| \\ &\leq \sup_{\mathbb{R}^N \times [0,t] \times \mathbb{R}^N} |\frac{H^1(x,s,p) - H^2(x,s,p)}{1+|p|}| \big(1 + M(s)\big).\end{aligned}$$

It is a routine exercise to check that the differential inequality actually holds in the viscosity sense.

But, the quantity evaluated for u^2 is 0. Therefore, one deduces from the comparison principle that

$$\|u^1(\cdot,t) - u^2(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \sup_{\mathbb{R}^N \times [0,t] \times \mathbb{R}^N} |\frac{H^1(x,s,p) - H^2(x,s,p)}{1+|p|}| \int_0^t \big(1 + M(s)\big)\, ds.$$

This is the estimate we want. □

3.3 Small time solvability of non-local Hamilton-Jacobi equations

In this section we study the short time solvability of the non-local Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} = H([u], x, t, \nabla u) \quad \text{in } \mathbb{R}^N \times (0,T), \qquad u(\cdot,0) = u_0 \quad \text{on } \mathbb{R}^N \tag{3.24}$$

with initial condition $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. In the equation, the notation $[u]$ or $[u(\cdot,t)]$ refers to the function $u(\cdot,t)$ to distinguish it with its value at the current point (x,t). The solution is required to be Lipschitz continuous in space so that the Hamiltonian only acts on Lipschitz functions.

Under our assumptions, we shall check that, for every Lipschitz function $u \in W^{1,\infty}(\mathbb{R}^N \times [0,T))$, the Hamiltonian

$$H_u(x,t,p) = H([u(\cdot,t)], x, t, p) \tag{3.25}$$

is continuous with respect to all the variables.

More generally we can introduce the following definition of viscosity solutions to non-local equations, which is not restricted to Lipschitz solutions, but is given in general for possibly discontinuous solutions:

Definition 3.5 (Discontinuous viscosity solutions to non-local equations) *We shall say that a locally bounded function $u : \mathbb{R}^N \times [0,T) \to \mathbb{R}$ is a viscosity solution of* (3.24) *if it is a viscosity solution (in the sense of Definition 3.1) of equation* (HJ) *for the Hamiltonian H_u defined in (3.25), when $H_u(x,t,p)$ is continuous in (x,t,p).*

Remark 3.6 *Let us remark that when the Hamiltonian has a local dependence in u, i.e. for $H(u,x,t,p)$, we can in particular set $H([u],x,t,p) = H(u,x,t,p)$, and consider this Hamiltonian as having a non-local dependence in u. When moreover the considered solutions u are continuous, Definition 3.5 gives a sense to solutions u of local equations of the type:*

$$\frac{\partial u}{\partial t} = H(u, x, t, \nabla u) \quad \textit{in } \mathbb{R}^N \times (0,T), \qquad u(\cdot,0) = u_0 \quad \textit{on } \mathbb{R}^N$$

Here, we do not assume that H is monotone with respect to the non-local term $[u]$. Therefore, the comparison principle is not expected. An elementary example is the equation

$$u_t = \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \wedge 1$$

For the initial condition $u_0 = 0$, the solution is $u = 0$, while for $v_0(x) = \sin x - 1$, the solution is $v(x,t) = \sin x - 1 + t$. Though $u_0 \geq v_0$, the solutions are not comparable for $0 < t < 2$ and we have $u(\cdot,t) \leq v(\cdot,t)$ for $t \geq 2$. A geometric example in the context of our dislocation model will be given at the end of the next section. The non validity of the comparison principle renders delicate the question of defining for large time a notion of weak solution to (3.24).

Theorem 3.7 (Short time solvability of non-local Hamilton-Jacobi equations) *Consider a non-local Hamiltonian $H : W^{1,\infty}(\mathbb{R}^N) \times \mathbb{R}^N \times [0,T) \times \mathbb{R}^N \to \mathbb{R}$ with the following properties.*

– *For every fixed $h \in W^{1,\infty}(\mathbb{R}^N)$, the Hamiltonian $H([h],\cdot,\cdot,\cdot)$ satisfies the assumptions of Proposition 3.3. Moreover, there are constants K_1 and K_2 that are independent of h such that*

$$\big|H([h],x,t,0)\big| \le K_1,$$

$$\big|H([h],x,t,p) - H([h],y,t,p)\big| \le K_2\big(1+|p|\big)\big(1+\|\nabla h\|_{L^\infty(\mathbb{R}^N)}\big)|x-y| \tag{3.26}$$

$$\big|H([h],x,t,p) - H([h],x,t,q)\big| \le K_3|p-q|.$$

– *There is a constant K_4 such that*

$$\big|H([h^1],x,t,p) - H([h^2],x,t,p)\big| \le K_4\big(1+|p|\big)\|h^1-h^2\|_{L^\infty(\mathbb{R}^N)} \tag{3.27}$$

Then, for every $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, there is a time $T^* > 0$ (depending only on K_2 and $\|\nabla u_0\|_{L^\infty(\mathbb{R}^N)}$) so that the non-local Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} = H([u(\cdot,t)],x,t,\nabla u) \quad \text{in } \mathbb{R}^N\times(0,T^*\wedge T), \qquad u(\cdot,0)=u_0 \quad \text{on } \mathbb{R}^N. \tag{NL-HJ}$$

has a unique viscosity solution $u \in W^{1,\infty}(\mathbb{R}^N \times [0,T^*\wedge T))$.

Moreover this solution satisfies

$$\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} \le K_1 + \frac{K_3}{K_2}\frac{1}{(T^*-t)} \tag{3.28}$$

with $T^* = \dfrac{1}{K_2\Big(1+||\nabla u_0||_{L^\infty(\mathbb{R}^N)}\Big)}$.

A similar result was proved by N. Alibaud [1]. Within our framework, the main difference is the assumption that estimate (3.26) is uniform in h. The strengthening of the assumption permits to construct a solution for all time to the non-local Hamilton-Jacobi equation (NL-HJ). But, of course, this excludes graph-like equations.

Proof of theorem 3.7 Put

$$M_0 = \|\nabla u_0\|_{L^\infty(\mathbb{R}^N)}.$$

The solution N of the differential equation

$$N' = K_2(1+N)^2$$

with initial condition M_0 blows up at time

$$T^* = \frac{1}{K_2(1+M_0)}.$$

To prove the theorem, we shall apply the fixed point theorem for contraction mappings to a set of functions that are Lipschitz continuous up to a certain time $S < T^*\wedge T$.

We choose $0 < S < T^*\wedge T$ so that

$$K_4\int_0^S \big(1+N(t)\big)\,dt \le \frac{1}{2}$$

and we consider the space

$$E = \Big\{ u \in W^{1,\infty}(\mathbb{R} \times [0,S]) \mid \|u(\cdot,t)\|_{L^\infty} \le L(t),\ \|\nabla u(\cdot,t)\|_{L^\infty} \le N(t), \\ \Big\|\frac{\partial u}{\partial t}(\cdot,t)\Big\|_{L^\infty} \le P(t) \text{ on } [0,S] \Big\}$$

where

$$L(t) = \|u_0\|_{L^\infty(\mathbb{R}^N)} + K_1 t, \qquad P(t) = K_1 + K_3 N(t).$$

It is a nonempty complete metric space for the uniform convergence. Given $u \in E$, we denote by $v = \Phi(u)$ the solution of the equation

$$\frac{\partial v}{\partial t} = H([u], x, t, \nabla v) \quad \text{in } \mathbb{R}^N \times (0,S), \qquad u(\cdot,0) = u_0 \quad \text{on } \mathbb{R}^N. \tag{3.29}$$

We claim that $\Phi : E \to E$ is well-defined and is a contraction mapping.

For $u \in E$ fixed, we first note that the Hamiltonian $H([u(\cdot,t)],\cdot,\cdot,\cdot)$ fulfils the conditions of Proposition 3.3. Indeed, its regularity with respect to x and p is guaranteed by assumption. Its continuity in time follows from the estimate

$$\begin{aligned} &|H([u(\cdot,t)],x,t,p) - H([u(\cdot,s)],x,s,p)| \\ &\qquad \le |H([u(\cdot,t)],x,t,p) - H([u(\cdot,t)],x,s,p)| + K_4\big(1+|p|\big)\|u(\cdot,t) - u(\cdot,s)\|_{L^\infty(\mathbb{R}^N)} \\ &\qquad \le |H([u(\cdot,t)],x,t,p) - H([u(\cdot,t)],x,s,p)| + K_4\big(1+|p|\big)P\big(\max(t,s)\big)\ |t-s| \end{aligned}$$

and from the continuity of $s \mapsto H([u(\cdot,t)],x,s,p)$ for t fixed. Therefore, by Proposition 3.3, equation (3.29) has a unique viscosity solution v. Moreover, using assumption (3.26), we know that v is Lipschitz continuous in x with the bounds $\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le L(t)$ and $\|\nabla v(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le M(t)$ for the a.e. solution M of the differential equation

$$M'(t) = K_2\big(1 + \|\nabla u(\cdot,t)\|^*_{L^\infty(\mathbb{R}^N)}\big)\ \big(1 + M(t)\big)$$

with initial condition M_0. But, since $u \in E$, we get that $\|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le N(t)$. Taking the u.s.c. envelope, we obtain $\|\nabla u(\cdot,t)\|^*_{L^\infty(\mathbb{R}^N)} \le N(t)$. We deduce that $M' \le K_2(1+M)(1+N) = N'\dfrac{1+M}{1+N}$. As $M(0) = N(0) = M_0$, we conclude upon integrating the differential inequality that

$$M \le N \quad \text{in } [0, T^* \wedge T).$$

Therefore, $\|\nabla v(\cdot,t)\|_{L^\infty} \le N(t)$ in $[0,T^* \wedge T)$. Using Proposition 3.3 again, we also get that

$$\Big\|\frac{\partial v}{\partial t}(\cdot,t)\Big\|_{L^\infty} \le K_1 + K_3 M(t) \le K_1 + K_3 N(t) = P(t). \tag{3.30}$$

This implies that $v \in E$. Therefore, the mapping $\Phi : E \to E$ is well defined.

We now show that Φ is a contraction mapping. Let $u^1, u^2 \in E$ and put $v^i = \Phi(u^i)$. Using the assumption (3.27) and the stability result (proposition 3.4), we deduce that

$$\begin{aligned} \|v^1 - v^2\|_{L^\infty(\mathbb{R}^N \times [0,S])} &\le \sup_{\mathbb{R}^N \times [0,S] \times \mathbb{R}^N} \Big|\frac{H([u^1],x,t,p) - H([u^2],x,t,p)}{1+|p|}\Big|\ \int_0^S \big(1 + N(t)\big)\,dt \\ &\le K_4 \|u^1 - u^2\|_{L^\infty(\mathbb{R}^N \times [0,S])} \int_0^S \big(1 + N(t)\big)\,dt. \end{aligned}$$

By the choice of S, this reads as

$$\|\Phi(u^1) - \Phi(u^2)\|_{L^\infty(\mathbb{R}^N\times[0,S])} \leq \frac{1}{2}\|u^1 - u^2\|_{L^\infty(\mathbb{R}^N\times[0,S])}.$$

Hence, Φ is a contraction mapping for the uniform norm.

The fixed point theorem guarantees that the equation (3.24) in $(0, S)$ has a unique bounded Lipschitz continuous solution in E. We can observe that this implies that the equation has a unique Lipschitz continuous solution without prescribing a priori bounds for the function and its derivatives. Indeed, any Lipschitz continuous viscosity solution u of (3.24) belongs to E. To see this, we know by Proposition 3.3 and (3.26) that $\|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq M(t)$ where M solves

$$M'(t) = K_2\big(1 + \|\nabla u(\cdot,t)\|^*_{L^\infty(\mathbb{R}^N)}\big)\ \big(1 + M(t)\big)$$

with the initial condition $M(0) = M_0$. Therefore,

$$M'(t) \leq K_2\big(1 + M(t)\big)^2.$$

Since N solves the equation with initial condition M_0, we must have $M \leq N$ on $[0, S]$. The bounds for $\|u(\cdot,t)\|_{L^\infty}$ and $\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_{L^\infty}$ are obtained in an elementary manner. Therefore, $u \in E$.

It remains to show the solvability of (3.24) up to time $T^* \wedge T$. We put

$$S = \sup\{0 \leq S' < T^* \wedge T \mid (3.24) \text{ has a unique solution } u \text{ in } (0, S')\}$$

for $u \in W^{1,\infty}(\mathbb{R} \times [0, S'])$. If we had $S < T^* \wedge T$, then we could find $\eta > 0$ so that $0 < S - \eta$ and $S + \eta \leq T^* \wedge T$ with

$$K_4 \int_{S-\eta}^{S+\eta} \big(1 + N(t)\big)\, dt \leq \frac{1}{2}$$

as well as $S' \in (S - \eta, S]$ for which there is a unique solution u of (3.24) in $(0, S')$. The fixed point theorem above, up to a translation in time, yields a unique solution v to

$$\frac{\partial v}{\partial t} = H([v], x, t, \nabla v) \text{ in } \mathbb{R}^N \times (S - \eta, S + \eta), \quad v(\cdot, S - \eta) = u(\cdot, S - \eta) \text{ in } \mathbb{R}^N.$$

By uniqueness, we must have $v = u$ in $\mathbb{R}^N \times [S - \eta, S')$. The function that coincides with u in $\mathbb{R}^N \times [0, S')$ and with v in $\mathbb{R}^N \times [S - \eta, S + \eta)$ is then clearly a Lipschitz continuous viscosity solution of (3.24). It is unique as uniqueness holds on $[0, S')$ and on $(S - \eta, S + \eta)$. Therefore, (3.24) has a unique viscosity solution up to $S + \eta$. This contradicts the definition of S. So, we must have $S = T^* \wedge T$.

Finally the estimate (3.28) on $\dfrac{\partial u}{\partial t}$ is a consequence of (3.30). □

3.4 Geometric viscosity solutions

Consider the following geometrical problem. Given an oriented hypersurface with normal n, we make it evolve by prescribing its normal speed c. In our dislocation model, the speed will depend on the full dislocation line, but, to keep the discussion elementary, we shall restrict ourselves in this section to the case of a local normal speed $c(x,t,n)$. As long as the hypersurface is of class C^1, there is no difficulty to define the motion. However, after a certain time, the hypersurface may loose its smoothness and develop topological singularities (such as the appearance of an interior).

The phase field method, which was introduced by Barles, Soner, Souganidis [8] in the context of Hamilton-Jacobi equations as a limiting case of the level set approach, allows to define an evolution that is global in time by exploiting the theory of discontinuous viscosity solutions. It relies on the extension to an arbitrary closed set of the notion of an oriented hypersurface. A (generalized) front is a partition $(\Omega^-, \Gamma, \Omega^+)$ of $\mathbb{R}^N$ consisting of a closed set Γ and of two open sets Ω^- and Ω^+. When Γ is the oriented boundary of a smooth open set Ω, then we simply choose $\Omega^+ = \Omega$ and $\Gamma = \partial\Omega$. Its outer normal n, which points from Ω^+ to Ω^-, endows Γ with its canonical orientation. In the general case, the partition of the complement of the front Γ into two open sets appears as a very weak way to orientate it. We attach to the generalized front $(\Omega^-, \Gamma, \Omega^+)$ the discontinuous function

$$\rho_\Gamma(x) = +1 \text{ if } x \in \Omega^+, \quad \rho_\Gamma(x) = 0 \text{ if } x \in \Gamma, \quad \rho_\Gamma(x) = -1 \text{ if } x \in \Omega^-.$$

It is called the phase field (as used in the physics literature) or the (signed) characteristic function.

Let us informally explain how one can relate the motion of an hypersurface to a Hamilton-Jacobi equation for its phase field. When the front Γ_t is the boundary of a smooth open set and moves with normal speed $c(x,t,n)$, the derivatives of its phase field $\rho(\cdot,t) = \rho_{\Gamma_t}$ in the sense of distribution are

$$\nabla\rho = -2n\, d\sigma_t, \qquad \frac{\partial\rho}{\partial t} = 2c(x,t,n)\, d\sigma_t$$

where σ_t is the surface area of Γ_t. This can be very informally rewritten as an Hamilton-Jacobi equation

$$\frac{\partial\rho}{\partial t} = H(x,t,\nabla\rho)$$

for the positively homogeneous Hamiltonian

$$H(x,t,p) = c\big(x,t,-\frac{p}{|p|}\big)\ |p| \quad \text{if } p \neq 0, \qquad H(x,t,0) = 0. \tag{3.31}$$

The idea of the phase field method is therefore to solve the Hamilton-Jacobi equation

$$\frac{\partial\rho}{\partial t} = H(x,t,\nabla\rho) \quad \text{in } \mathbb{R}^N \times (0,T), \qquad \rho(\cdot,0) = \rho_0 \quad \text{on } \mathbb{R}^N \tag{3.32}$$

with the Hamiltonian above, for the initial data which is the characteristic function of the initial front $(\Omega_0^-, \Gamma_0, \Omega_0^+)$, i.e. $\rho_0 = \rho_{\Gamma_0}$. Under the assumptions of Proposition 3.3, we know that there is a discontinuous solution ρ to (3.32). The function ρ takes values in $[-1,1]$ by the comparison principle. It defines a generalized front at time t by the formula

$$\Omega_t^+ = \{\rho_*(\cdot,t) > 0\}, \qquad \Omega_t^- = \{\rho^*(\cdot,t) < 0\}.$$

The front may be non unique, but, if all the discontinuous solutions have the same semi-continuous envelopes, then the generalized front will be independent of the solution.

When the initial front is a graph

$$\Gamma_0 = \{x_N = f_0(x') \mid x' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}\},$$

one may look for a generalized front of the graph form, i.e. such that $\Gamma_t = \{x_N = f(x', t)\}$ for some function f. The outward normal to the (hypo)graph is proportional to $(-\nabla_{x'} f, 1)$. The normal speed of the point $x = \big(x', f(x', t)\big)$ is therefore $\dfrac{\partial f}{\partial t}(1 + |\nabla_{x'} f|^2)^{-1/2}$. Thus, when evaluated along the graph, the geometric motion of the hypersurface reduces to the Hamilton-Jacobi equation

$$\frac{\partial f}{\partial t} = H\big(x', f(x', t), t, \nabla_{x'} f, -1\big) \tag{3.33}$$

for the Hamiltonian given by (3.31). Of course, for general Hamiltonians, this graph equation may not have a solution for all time. The solution will blow up when the front curls up.

The next result gives a precise meaning to this informal discussion and establishes the equivalence between the graph equation (3.33) and the geometric equation (3.32), when the initial front is a graph. It is expressed with the help of the oriented characteristic function for the graph

$$\rho_f(x', x_N) = +1 \text{ if } x_N < f(x'),\ \ \rho_f(x', x_N) = 0 \text{ if } x_N = f(x'),\ \ \rho_f(x', x_N) = -1 \text{ if } x_N > f(x').$$

When f also depends on t, we still denote by ρ_f the function $\rho_f(x, t) = \rho_{f(\cdot, t)}(x)$.

Proposition 3.8 (Geometric and graph equations)
Consider a Hamiltonian that is positively homogeneous in the gradient and that satisfies the assumptions of Proposition 3.3. Let the initial condition f_0 be in $W^{1,\infty}(\mathbb{R}^{N-1})$. Assume that there is a time $T > 0$ for which the graph equation

$$\frac{\partial f}{\partial t} = H\big(x', f(x', t), t, \nabla_{x'} f, -1\big) \quad \text{in } \mathbb{R}^{N-1} \times (0, T), \qquad f(\cdot, 0) = f_0 \quad \text{on } \mathbb{R}^{N-1} \tag{3.34}$$

has a viscosity solution $f \in W^{1,\infty}\big(\mathbb{R}^{N-1} \times [0, T)\big)$.

Then, the function ρ_f is the unique bounded discontinuous viscosity solution of the geometric equation (3.32).

Proof of proposition 3.8 We first show that ρ_f is a discontinuous viscosity solution of (3.32). As f is a solution of the graph equation (3.34) in $(0, T)$, the function

$$u(x', x_N, t) = f(x', t) - x_N$$

is a viscosity solution of

$$\frac{\partial u}{\partial t} = H(x', f(x', t), t, \nabla u) \tag{3.35}$$

in $(0, T)$ with initial condition $u_0(x', x_N) = f_0(x') - x_N$. Because of the positive homogeneity of the Hamiltonian in the gradient, for every increasing C^1 function Φ, $\Phi(u)$ is a viscosity solution of (3.35) for the initial condition $\Phi(u_0)$. This claim, which is trivial for C^1 solutions, is classical in the viscosity sense (see for instance [8]).

Consider a decreasing sequence Φ_k of C^1 bounded increasing functions that converges pointwise to the function $\sigma_+ = 1_{[0,+\infty)} - 1_{(-\infty,0)}$. By the stability properties of viscosity solutions, the upper semilimit

$$\overline{u}(x,t) = \limsup_{k\to\infty,\, s\to t,\, y\to x} \Phi_k(u)(s,y)$$

is a subsolution of the equation (3.35). An immediate computation reveals that

$$\overline{u}(x,t) = \inf_k \Phi_k(u)(x,t) = \sigma_+[u(x,t)] = \rho_f^*(x,t).$$

Therefore, ρ_f is a subsolution of (3.35) with initial condition ρ_{f_0}.

This implies that ρ_f is actually a subsolution of (3.32). Indeed, at every point (x,t) such that $f(x',t) \neq x_N$, the equation trivially holds in the viscosity sense because ρ_f is constant in a neighborhood of this point and because $H(x,t,0) = 0$. On the other hand, at every (x,t) such that $f(x',t) = x_N$, the equation (3.32) and (3.35) are the same.

Similarly, by considering an increasing sequence of C^1 bounded increasing functions converging pointwise to $\sigma_- = 1_{(0,+\infty)} - 1_{(-\infty,0]}$, one can show that ρ_f is a supersolution of (3.32).

Next, let ρ be a bounded discontinuous solution of (3.32). Fix $S < T$ arbitrary. For every $\varepsilon > 0$, an elementary computation shows that the function

$$f^\varepsilon = f + \varepsilon e^{Kt}$$

with $K = \|C_2\|_{L^\infty([0,S])}\ (1 + \|\nabla f\|^2_{L^\infty(\mathbb{R}^{N-1}\times[0,S])})^{1/2} + \|C_3\|_{L^\infty([0,S])}$ is a supersolution of the graph equation in $[0,S]$ with initial condition $f_0 + \varepsilon$ (the constant C_2 and C_3 are those in assumption (3.20)). The preceding step implies that the function ρ_{f^ε} is a supersolution of (3.32) with initial condition $(\rho_{f_0+\varepsilon})_* = \sigma_-[f_0+\varepsilon] \geq \sigma_+(f_0) = \rho^*_{f_0}$. Since $\rho^*(\cdot,0) \leq \rho^*_{f_0}$, we deduce from the comparison principle of Proposition 3.3 that $\rho \leq \rho_{f^\varepsilon}$ in $\mathbb{R}^N \times [0,S]$. But, $\rho_{f^\varepsilon} \downarrow \rho_f^*$ pointwise as $\varepsilon \to 0$. Passing to the limit in the inequality, we deduce that $\rho \leq \rho_f^*$ in $\mathbb{R}^N \times [0,S]$. Since $S < T$ is arbitrary, we conclude that $\rho \leq \rho_f^*$ in $\mathbb{R}^N \times [0,T)$.

One shows similarly that $\rho \geq (\rho_f)_*$. Hence, $(\rho_f)_* \leq \rho \leq (\rho_f)^*$ As $\left((\rho_f)_*\right)^* = \rho_f^*$ and $\left(\rho_f^*\right)_* = (\rho_f)_*$, we conclude that ρ and ρ_f must have the same semicontinuous envelopes. □

When the initial front is a Lipschitz compact $(N-1)$-dimensional submanifold of $\mathbb{R}^N$ we can state similar results. For simplicity, let us work in dimension $N = 2$; this is the natural framework for planar dislocation dynamics. We assume that the initial front is a Lipschitz simple closed curve in the plane $\mathbb{R}^2$

$$\Gamma_{f_0} = \{\Gamma_{f_0}(s) := \Gamma(s) + f_0(s)n(s), \quad s \in \mathbb{R}/L\mathbb{Z}\}$$

as given in Assumption 1.2 for some fixed constant $L > 0$. Then one may look for a generalized front of the loop form, i.e. $\Gamma_{f(\cdot,t)}$ for some function f. The unit tangent vector is parallel to $(1 - K(s)f(s,t))\ \tau(s) + \dfrac{\partial f}{\partial s}(s,t)\ n(s)$ where we recall that $\tau(s) = \dfrac{d\Gamma}{ds}(s)$ satisfies $|\tau(s)| = 1$, and $n(s) = \tau^\perp(s)$, $\dfrac{d\tau}{ds}(s) = K(s)n(s)$. Then the normal unit vector is parallel to $n(s) - \dfrac{1}{(1-K(s)f(s,t))}\ \dfrac{\partial f}{\partial s}(s,t)\ \tau(s)$. When evaluated along the curve, the geometric motion of the loop reduces to the Hamilton-Jacobi equation

$$\frac{\partial f}{\partial t}(s,t) = H\left(\Gamma_{f(\cdot,t)}(s), t, -\left(n(s) - \frac{1}{(1-K(s)f(s,t))}\frac{\partial f}{\partial s}(s,t)\ \tau(s)\right)\right) \tag{3.36}$$

for the Hamiltonian given by (3.31).

Then we have the following result which is the analogue of Proposition 3.8.

Proposition 3.9 (Geometric and loop equations)
Consider a Hamiltonian that is positively homogeneous in the gradient and that satisfies the assumptions of Proposition 3.3 in dimension $N = 2$. Let the initial condition f_0 be in $W^{1,\infty}(\mathbb{R}/L\mathbb{Z})$. Assume that there is a time $T > 0$ for which the loop equation

(3.37)
$$\begin{cases} \dfrac{\partial f}{\partial t} = H\left(\Gamma_{f(\cdot,t)}(s), t, -\big(n(s) - \dfrac{1}{(1-K(s)f(s,t))}\dfrac{\partial f}{\partial s}(s,t)\ \tau(s)\big)\right) & \text{in } (\mathbb{R}/L\mathbb{Z}) \times (0,T), \\ f(\cdot,0) = f_0 \quad \text{on } \mathbb{R}/L\mathbb{Z} & \end{cases}$$

has a viscosity solution $f \in W^{1,\infty}\big((\mathbb{R}/L\mathbb{Z}) \times [0,T)\big)$.

Then, the function $\tilde{\rho}_f$ (as given in (1.6)) is the unique bounded discontinuous viscosity solution of the geometric equation (3.32) with the initial condition $\rho_0 = \tilde{\rho}_{f_0}$.

Proof of proposition 3.9 We proceed similarly as in the proof of Proposition 3.8. The only change is in checking that $\tilde{\rho}_f$ is a discontinuous solution of (3.32). To this end, we introduce the map

$$\begin{array}{rlcl} \Psi: & (\mathbb{R}/L\mathbb{Z}) \times (-r_0/2, r_0/2) & \longrightarrow & U \subset \mathbb{R}^2 \\ & (s,h) & \longmapsto & \Psi(s,h) = x = (x_1,x_2) = \Gamma(s) + h\ n(s) \end{array}$$

which is a C^1-diffeomorphism, for r_0 given in Assumption 1.2. Here U is a neighborhood of Γ (where Γ is introduced in Assumption 1.2). For $x \in U$, we denote $(s(x), h(x)) = \Psi^{-1}(x)$, and easily check that the function $h(x)$ is the signed distance function to $\partial\Omega_0$ which is negative on Ω_0 (where Ω_0 is given in Assumption 1.2). Then we consider the function

$$u(x,t) = f(s(x),t) - h(x).$$

We easily check that $\nabla u(x,t) = -n(s(x)) + \dfrac{1}{(1-K(s(x))h(x))}\dfrac{\partial f}{\partial s}(s(x),t)\ \tau(s(x))$, and then u is a viscosity solution on $U \times (0,T)$ of

$$\frac{\partial u}{\partial t}(x,t) = H(\Gamma_{f(\cdot,t)}(s(x)), t, M(x,t)\cdot \nabla u(x,t)) \tag{3.38}$$

in $(0,T)$ with initial condition $u_0(x) = f_0(s(x)) - h(x)$. Here the matrix M is given by

$$M(x,t) = n(s(x)) \otimes n(s(x)) + \left(\frac{1-K(s(x))h(x)}{1-K(s(x))f(s(x),t)}\ \tau(s(x)) \otimes \tau(s(x))\right).$$

Note that $M(x,t)$ is the identity matrix when $x \in \Gamma_{f(\cdot,t)}$.

Because of the positive homogeneity of the Hamiltonian in the gradient, we can use the same argument as in the proof of Proposition 3.8 to conclude that $\tilde{\rho}_f$ is a discontinuous solution of (3.32) on $U \times (0,T)$, and then on $\mathbb{R}^2 \times (0,T)$. □

4 Small-time solvability of the dislocation dynamics

4.1 Preliminaries

Given a function $f \in L^1_{\rm loc}(\mathbb{R}^2)$, we define the quantities

$$||f||_{L^1_{\rm unif}(\mathbb{R}^2)} = \sup_{x\in\mathbb{R}^2} \int_{Q(x)} |f|\, dy, \qquad ||f||_{L^\infty_{\rm int}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \|f\|_{L^\infty(Q(x))}\, dx,$$

where $Q(x)$ designates the unit square centered at x

$$Q(x) = \{y \in \mathbb{R}^2 \mid |y_1 - x_1| \le 1/2,\ |y_2 - x_2| \le 1/2\}.$$

We denote respectively by $L^1_{\rm unif}(\mathbb{R}^2)$ and $L^\infty_{\rm int}(\mathbb{R}^2)$ the space that consists of the functions for which these quantities are finite. The relevancy of $L^1_{\rm unif}$ is that it locally coincides with L^1 and that it contains the phase field functions.

Lemma 4.1 (Convolution inequality)
For every $f \in L^1_{unif}(\mathbb{R}^2)$ and $g \in L^\infty_{int}(\mathbb{R}^2)$, the convolution product $f \star g$ is bounded and satisfies

$$\|f \star g\|_{L^\infty(\mathbb{R}^2)} \le \|f\|_{L^1_{unif}(\mathbb{R}^2)} \|g\|_{L^\infty_{int}(\mathbb{R}^2)}. \tag{4.39}$$

Proof of lemma 4.1 For every $x \in \mathbb{R}^2$, we can compute

$$|f \star g(x)| \le \int_{\mathbb{R}^2} |f(x-y)g(y)|\, dy = \sum_{k\in\mathbb{Z}^2} \int_{Q(k)} |f(x-y)g(y)|\, dy \le \|f\|_{L^1_{\rm unif}} \sum_{\mathbb{Z}^2} \|g\|_{L^\infty(Q(k))}.$$

Applying the above estimate to the translate $\tau_z g = g(\cdot - z)$ for z fixed, we obtain

$$|f \star g(x)| = |f \star \tau_z g(x+z)| \le \|f\|_{L^1_{\rm unif}} \sum_{\mathbb{Z}^2} \|\tau_z g\|_{L^\infty(Q(k))} \le \|f\|_{L^1_{\rm unif}} \sum_{\mathbb{Z}^2} \|g\|_{L^\infty(Q(k-z))}.$$

We now integrate with respect to $z \in Q(0)$ to get

$$\begin{aligned} |f \star g(x)| &\le \|f\|_{L^1_{\rm unif}} \sum_{\mathbb{Z}^2} \int_{Q(0)} \|g\|_{L^\infty(Q(k-z))}\, dz \\ &= \|f\|_{L^1_{\rm unif}} \sum_{\mathbb{Z}^2} \int_{Q(k)} \|g\|_{L^\infty(Q(y))}\, dy = \|f\|_{L^1_{\rm unif}} \int_{\mathbb{R}^2} \|g\|_{L^\infty(Q(y))}\, dy = \|f\|_{L^1_{\rm unif}} \|g\|_{L^\infty_{\rm int}}. \end{aligned}$$

We conclude that $f \star g$ is bounded with the bound given by (4.39). □

For further references, we denote by $M(\mathbb{R}^2)$ the set of the signed Radon measures endowed with the total variation norm $\|\mu\|_{M(\mathbb{R}^2)} = |\mu|(\mathbb{R}^2)$. We denote by $BV(\mathbb{R}^2)$ the set of the functions with bounded variation, i.e. the functions $f \in L^1(\mathbb{R}^2)$ with $\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2} \in M(\mathbb{R}^2)$.

4.2 The 1D problem for the graph equation

In this section, we are interested in the graph equation associated to our dislocation model equation

$$\frac{\partial f}{\partial t}(x_1,t) = \left(c_0 \star \rho_f^+ + c_1\right)(x_1, f(x_1,t), t)\sqrt{1 + \left(\frac{\partial f}{\partial x_1}(x_1,t)\right)^2}$$

where the initial condition $f_0 \in W^{1,\infty}(\mathbb{R})$ is the function whose graph is the initial dislocation line. We recall that the convolution acts on the space variable $x \in \mathbb{R}^2$.

Theorem 4.2 (Short time existence and uniqueness of a solution of the graph equation)
Assume that

$$c_0 \in L^\infty_{int}(\mathbb{R}^2) \cap BV\left(\mathbb{R}^2\right), \quad c_1 \in W^{1,\infty}\left(\mathbb{R}^2 \times [0,+\infty)\right), \quad f_0 \in W^{1,\infty}(\mathbb{R}).$$

Then, there is a time $T^* > 0$ (depending only on the bounds of c_0, c_1 and f_0) so that the graph equation
(4.40)

$$\frac{\partial f}{\partial t} = \left(c_0 \star \rho_f^+ + c_1\right)(x_1, f(x_1,t), t)\sqrt{1 + \left(\frac{\partial f}{\partial x_1}\right)^2} \quad \text{in } \mathbb{R}\times(0,T^*), \qquad f(\cdot,0) = f_0 \quad \text{in } \mathbb{R}$$

has a unique viscosity solution $f \in W^{1,\infty}(\mathbb{R}\times[0,T^*))$.

Proof of theorem 4.2
The theorem results from the solvability of non-local Hamilton-Jacobi equation (Theorem 3.7) for the Hamiltonian

$$H([h], x_1, t, p) = c[h]\left(x_1, h(x_1), t\right)\sqrt{1+p^2}$$

where the non-local velocity of the graph $c[h]$ is given by

$$c[h](x_1,x_2,t) = c_0 \star \rho_h^+(x_1,x_2) + c_1(x_1,x_2,t).$$

First, we note that $c[h]$ is bounded with

$$\|c[h]\|_{L^\infty} \le \|c_0\|_{L^1} + \|c_1\|_{L^\infty}$$

because $|\rho_h^+| \le 1$. It is continuous in time. Moreover, it is clearly Lipschitz continuous in space with the bound

$$\|\nabla c[h]\|_{L^\infty} \le \|\nabla c_0\|_M + \|\nabla c_1\|_{L^\infty}.$$

The function $c[h]\left(x_1, h(x_1), t\right)$ is therefore Lipschitz continuous in x_1 with

$$|c[h]\left(x_1, h(x_1), t\right) - c[h]\left(y_1, h(y_1), t\right)| \le \|\nabla c[h]\|_{L^\infty}\left(1 + \|\frac{\partial h}{\partial x_1}\|_{L^\infty(\mathbb{R})}\right)|x_1 - y_1|.$$

Therefore, the Hamiltonian $H([h], x_1, t, p)$ satisfies the first set of assumptions of Theorem 3.7 for the constants

$$K_1 = K_3 = \|c_0\|_{L^1} + \|c_1\|_{L^\infty}, \qquad K_2 = \|\nabla c_0\|_M + \|\nabla c_1\|_{L^\infty}. \tag{4.41}$$

As concerns the proof of the Lipschitz regularity of Hamiltonian in h, we note that, for every $x \in \mathbb{R}^2$, we have

$$\begin{aligned}
\|\rho_{h^1} - \rho_{h^2}\|_{L^1(Q(x))} &\le \|\rho_{h^1} - \rho_{h^2}\|_{L^1(\mathbb{R}\times[x_1-\frac{1}{2},x_1+\frac{1}{2}])} \\
&\le \int_{x_1-\frac{1}{2}}^{x_1+\frac{1}{2}} \int |1_{\{y_2<h^1(y_1)\}} - 1_{\{y_2<h^2(y_1)\}}| + |1_{\{y_2>h^1(y_1)\}} - 1_{\{y_2>h^2(y_1)\}}| \, dy_2\, dy_1 \\
&= 2\int_{x_1-\frac{1}{2}}^{x_1+\frac{1}{2}} |h^1(y_1) - h^2(y_1)|\, dy_1 \\
&\le 2\|h^1 - h^2\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

Taking the supremum over x, we get

$$\|\rho^+_{h^1} - \rho^+_{h^2}\|_{L^1_{\text{unif}}} \le \|\rho_{h^1} - \rho_{h^2}\|_{L^1_{\text{unif}}} \le 2\|h^1 - h^2\|_{L^\infty(\mathbb{R})}. \tag{4.42}$$

Therefore, recalling the convolution inequality (Lemma 4.1), we deduce that

$$\begin{aligned}
&\left|c[h^1]\big(x_1, h^1(x_1), t\big) - c[h^2]\big(x_1, h^2(x_1), t\big)\right| \\
&\quad \le \left|c[h^1]\big(x_1, h^1(x_1), t\big) - c[h^2]\big(x_1, h^1(x_1), t\big)\right| + \left|c[h^2]\big(x_1, h^1(x_1), t\big) - c[h^2]\big(x_1, h^2(x_1), t\big)\right| \\
&\quad \le \|c[h^1] - c[h^2]\|_{L^\infty} + \|\nabla c[h^2]\|_{L^\infty}\|h^1 - h^2\|_{L^\infty(\mathbb{R})} \\
&\quad \le \|c_0\|_{L^\infty_{\text{int}}}\|\rho^+_{h^1} - \rho^+_{h^2}\|_{L^1_{\text{unif}}} + K_2\|h^1 - h^2\|_{L^\infty(\mathbb{R})}. \\
&\quad \le (2\|c_0\|_{L^\infty_{\text{int}}} + K_2)\|h^1 - h^2\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

This clearly implies that the Hamiltonian is Lipschitz continuous in h and that it satisfies the inequality (3.27) for the constant $K_4 = 2\|c_0\|_{L^\infty_{\text{int}}} + K_2$. □

4.3 The 1D problem for the loop equation

In this section, we are interested in the loop equation associated to our dislocation model equation

$$\frac{\partial f}{\partial t} = \left(c_0 \star \tilde{\rho}^+_{f(\cdot,t)} + c_1\right)\left(\Gamma_{f(\cdot,t)}(s), t\right)\sqrt{1 + \left(\frac{1}{(1 - fK)}\frac{\partial f}{\partial s}\right)^2}$$

where the initial condition $f_0 \in W^{1,\infty}(\mathbb{R}/L\mathbb{Z})$ is a function such that the boundary of Ω_{f_0} is the initial dislocation line. To avoid any problems when $fK = 1$, we set $r_1 = r_0/2$ where r_0 is given in Assumption 1.2 and define the truncation function

$$T_{r_1}(f) = \begin{cases} r_1 & \text{if } \ f \ge r_1 \\ f & \text{if } \ |f| \le r_1 \\ -r_1 & \text{if } \ f \le -r_1 \end{cases}$$

Then we consider the modified equation

$$\frac{\partial f}{\partial t} = \left(c_0 \star \tilde{\rho}^+_{T_{r_1}(f(\cdot,t))} + c_1\right)\left(\Gamma_{T_{r_1}(f(\cdot,t))}(s), t\right)\sqrt{1 + \left(\frac{1}{(1 - T_{r_1}(f)\ K)}\frac{\partial f}{\partial s}\right)^2}$$

Theorem 4.3 (Short time existence and uniqueness of a solution of the truncated loop equation)
Assume that

$$c_0 \in L^\infty_{int}(\mathbb{R}^2) \cap BV\big(\mathbb{R}^2\big), \quad c_1 \in W^{1,\infty}\big(\mathbb{R}^2 \times [0,+\infty)\big), \quad f_0 \in W^{1,\infty}(\mathbb{R}/L\mathbb{Z}).$$

Then, there is a time $T^{*'} > 0$ *(depending only on the bounds of* c_0, c_1 *and* Ω_{f_0}*) so that the truncated loop equation*

$$\begin{cases} \dfrac{\partial f}{\partial t} = \Big(c_0 \star \tilde{\rho}^+_{T_{r_1}(f(\cdot,t))} + c_1\Big)\big(\Gamma_{T_{r_1}(f(\cdot,t))}(s), t\big)\sqrt{1 + \left(\dfrac{1}{(1 - T_{r_1}(f)\ K)}\dfrac{\partial f}{\partial s}\right)^2} \\ \qquad\quad in \ \ (\mathbb{R}/L\mathbb{Z}) \times (0, T^{*'}), \\ f(\cdot, 0) = f_0 \quad in \ \mathbb{R}/L\mathbb{Z} \end{cases}$$

has a unique viscosity solution $f \in W^{1,\infty}((\mathbb{R}/L\mathbb{Z}) \times [0, T^{*'}))$.

Moreover there exists a constant $C_0 > 0$ *(depending only on bounds on* c_0, c_1 *and on* Ω_0 *given in Assumption 1.2), such that*

$$\left|\left|\frac{\partial f}{\partial t}(\cdot, t)\right|\right|_{L^\infty(\mathbb{R}/L\mathbb{Z})} \le C_0\left(1 + \frac{1}{T^{*'} - t}\right), \tag{4.43}$$

with $T^{*'} = \dfrac{1}{C_0\left(1 + ||\frac{\partial f_0}{\partial s}||_{L^\infty(\mathbb{R}/L\mathbb{Z})}\right)}$.

Proof of theorem 4.3
The theorem results from the solvability of non-local Hamilton-Jacobi equation (Theorem 3.7) for the Hamiltonian

$$H([h], s, t, p) = \tilde{c}[T_{r_1}(h)]\big(\Gamma_{T_{r_1}(h)}(s), t\big)\sqrt{1 + \left(\frac{p}{1 - K(s)T_{r_1}(h)(s)}\right)^2}$$

where the non-local velocity of the loop $\tilde{c}[g]$ is given by

$$\tilde{c}[g](x_1, x_2, t) = c_0 \star \tilde{\rho}^+_g(x_1, x_2) + c_1(x_1, x_2, t).$$

First, we note that $\tilde{c}[g]$ is bounded with

$$\|\tilde{c}[g]\|_{L^\infty} \le \|c_0\|_{L^1} + \|c_1\|_{L^\infty},$$

is continuous in time, and is Lipschitz continuous in space with the bound

$$\|\nabla\tilde{c}[g]\|_{L^\infty} \le \|\nabla c_0\|_M + \|\nabla c_1\|_{L^\infty}.$$

Let us now remark that for $||g||_{L^\infty(\mathbb{R}/L\mathbb{Z})} \le r_1 \le 1/(2K_0)$, we have

$$|\Gamma_g(s_2) - \Gamma_g(s_1)| \le |s_2 - s_1|\left(1 + \left|\left|\frac{\partial g}{\partial s}\right|\right|_{L^\infty(\mathbb{R}/L\mathbb{Z})} + K_0 r_1\right).$$

The function $\tilde{c}[g]\big(\Gamma_g(s),t\big)$ is therefore Lipschitz continuous in s with

$$|\tilde{c}[g]\big(\Gamma_g(s_2),t\big) - \tilde{c}[g]\big(\Gamma_g(s_1),t\big)| \leq \|\nabla\tilde{c}[g]\|_{L^\infty}\Big(\frac{3}{2} + \|\frac{\partial g}{\partial s}\|_{L^\infty(\mathbb{R}/L\mathbb{Z})}\Big)\ |s_2 - s_1|.$$

We also remark that for $||g||_{L^\infty(\mathbb{R}/L\mathbb{Z})} \leq r_1$, we have

$$\left|\sqrt{1+\Big(\frac{p}{1-K(s_2)g(s_2)}\Big)^2} - \sqrt{1+\Big(\frac{p}{1-K(s_1)g(s_1)}\Big)^2}\right|$$

$$\leq 4|p|\left(r_1\ \left|\left|\frac{\partial K}{\partial s}\right|\right|_{L^\infty(\mathbb{R}/L\mathbb{Z})} + K_0\ \left|\left|\frac{\partial g}{\partial s}\right|\right|_{L^\infty(\mathbb{R}/L\mathbb{Z})}\right)|s_2 - s_1|$$

Applying these results to $g = T_{r_1}(h)$, we get that the Hamiltonian $H([h],s,t,p)$ satisfies the first set of assumptions of Theorem 3.7 for the constants

$$K_1 = \frac{1}{2}K_3 = \|c_0\|_{L^1} + \|c_1\|_{L^\infty},$$

$$K_2 = 4(1+K_0)\left(1 + \frac{1}{2K_0}\left|\left|\frac{\partial K}{\partial s}\right|\right|_{L^\infty(\mathbb{R}/L\mathbb{Z})}\right)(\|c_0\|_{L^1} + \|c_1\|_{L^\infty} + \|\nabla c_0\|_M + \|\nabla c_1\|_{L^\infty}).$$

As concerns the proof of the Lipschitz regularity of Hamiltonian in h, we note that if $||g^i||_{L^\infty(\mathbb{R}/L\mathbb{Z})} \leq r_1$ for $i = 1,2$, if $g^- = \min(g^1,g^2)$, $g^+ = \max(g^1,g^2)$, and if $D = \{(s,r) \in (\mathbb{R}/L\mathbb{Z}) \times \mathbb{R},\quad g^-(s) < r < g^+(s)\}$, then we have

$$\begin{aligned}
\|\tilde{\rho}^+_{g^2} - \tilde{\rho}^+_{g^1}\|_{L^1(\mathbb{R}^2)} &\leq \int_{\Psi(D)} 1 \\
&\leq \int_D |\mathrm{Jac}\Psi| \\
&\leq \int_{\mathbb{R}/L\mathbb{Z}} ds\ \int_{g^-(s)}^{g^+(s)} dr\ (1 - rK(s)) \qquad (4.44)\\
&\leq (1 + r_1K_0)\int_{\mathbb{R}/L\mathbb{Z}} ds\ |g^+(s) - g^-(s)| \\
&\leq \frac{3L}{2}||g^2 - g^1||_{L^\infty(\mathbb{R}/L\mathbb{Z})}
\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}
&\big|\tilde{c}[g^2]\big(\Gamma_{g^2}(s),t\big) - \tilde{c}[g^1]\big(\Gamma_{g^1}(s),t\big)\big| \\
&\quad\leq \big|\tilde{c}[g^2]\big(\Gamma_{g^2}(s),t\big) - \tilde{c}[g^1]\big(\Gamma_{g^2}(s),t\big)\big| + \big|\tilde{c}[g^1]\big(\Gamma_{g^2}(s),t\big) - \tilde{c}[g^1]\big(\Gamma_{g^1}(s),t\big)\big| \\
&\quad\leq \big|\big|\tilde{c}[g^2] - \tilde{c}[g^1]\big|\big|_{L^\infty(\mathbb{R}^2)} + \big|\big|\nabla\tilde{c}[g^1]\big|\big|_{L^\infty(\mathbb{R}^2)}\ \big|\big|g^2 - g^1\big|\big|_{L^\infty(\mathbb{R}/L\mathbb{Z})} \\
&\quad\leq \big|\big|c_0\big|\big|_{L^\infty(\mathbb{R}^2)}\ \big|\big|\tilde{\rho}^+_{g^2} - \tilde{\rho}^+_{g^1}\big|\big|_{L^1(\mathbb{R}^2)} + \big|\big|\nabla\tilde{c}[g^1]\big|\big|_{L^\infty(\mathbb{R}^2)}\ \big|\big|g^2 - g^1\big|\big|_{L^\infty(\mathbb{R}/L\mathbb{Z})} \\
&\quad\leq \Big(\frac{3L}{2}\big|\big|c_0\big|\big|_{L^\infty(\mathbb{R}^2)} + \|\nabla c_0\|_M + \|\nabla c_1\|_{L^\infty}\Big)\ \big|\big|g^2 - g^1\big|\big|_{L^\infty(\mathbb{R}/L\mathbb{Z})}.
\end{aligned}$$

This implies that the Hamiltonian is Lipschitz continuous in h and that it satisfies the inequality (3.27) for the constant $K_4 = 3L||c_0||_{L^\infty} + 2\left(\|\nabla c_0\|_M + \|\nabla c_1\|_{L^\infty}\right) + 4K_0\left(\|c_0\|_{L^1} + \|c_1\|_{L^\infty}\right)$.

Finally the estimate (4.43) on $\dfrac{\partial f}{\partial t}$ is a consequence of (3.28). □

A straightforward consequence of Theorem 4.3 is the

Corollary 4.4 (Short time existence and uniqueness of a solution of the loop equation)
Under the assumptions of Theorem 4.3, if $||f_0||_{L^\infty(\mathbb{R}/L\mathbb{Z})} \leq \dfrac{r_0}{4}$ with r_0 given in Assumption 1.2, then there is a time $T^* > 0$ (depending only on the bounds of c_0, c_1, $\left|\left|\dfrac{\partial f_0}{\partial s}\right|\right|_{L^\infty(\mathbb{R}/L\mathbb{Z})}$ and on Ω_0) so that the loop equation

(4.45)
$$\begin{cases} \dfrac{\partial f}{\partial t} = \left(c_0 \star \tilde{\rho}^+_{f(\cdot,t)} + c_1\right)\left(\Gamma_{f(\cdot,t)}(s), t\right)\sqrt{1 + \left(\dfrac{1}{(1-f\ K)}\dfrac{\partial f}{\partial s}\right)^2} \quad \text{in } (\mathbb{R}/L\mathbb{Z}) \times (0, T^*), \\ \\ f(\cdot, 0) = f_0 \quad \text{in } \mathbb{R}/L\mathbb{Z} \end{cases}$$

has a unique viscosity solution $f \in W^{1,\infty}((\mathbb{R}/L\mathbb{Z}) \times [0, T^*))$.

4.4 The 2D problems : proofs of Theorems 1.1 and 1.3

We give in this section 2D versions of Theorem 4.2 and Corollary 4.4, namely Theorems 4.5 and 4.6, which are a reformulation of Theorems 1.1 and 1.3. Our main purpose is to shed light onto the framework in which the geometric equation

$$\frac{\partial \rho}{\partial t} = \left(c_0 \star \rho^+(\cdot, t) + c_1\right)|\nabla \rho| \quad \text{in } \mathbb{R}^2 \times (0, T^*), \qquad \rho(\cdot, 0) = \rho_{f_0} \quad \text{on } \mathbb{R}^2. \tag{4.46}$$

is well posed. As a matter of fact, we expect that a large time solution of the geometric equation will exist. It should not remain a graph for all time however.

The equivalence between the graph and the geometric equations (Proposition 3.8) ensures immediately the existence of a solution to (4.46), namely ρ_f. The delicate question is that of uniqueness. It pinpoints the functional space where to find solutions of (4.46). It turns out that a convenient space to work with is the set of the bounded discontinuous functions (where naturally leaves a phase field) that belong to $C([0, T^*); L^1_{\text{unif}}(\mathbb{R}^2))$. This regularity ensures in particular that the Hamiltonian

$$H_\rho(x, t, p) = \left(c_0 \star \rho^+(\cdot, t) + c_1\right)|p|$$

is continuous with respect to all the variables.

Our uniqueness result, though a bit technical, is well adapted to the space. It states that two solutions must have the same semicontinuous envelopes. This, we recall, is the natural form of the uniqueness assertion for discontinuous viscosity solutions of Hamilton-Jacobi equations because it allows to define without ambiguity a generalized front. The result also states that two solutions must be equal a.e. in space for all time, i.e. in $C([0, T^*); L^1_{\text{unif}}(\mathbb{R}^2))$. This implies that the two solutions will give birth to the same Hamiltonian H_ρ.

Theorem 4.5 (Short time existence and uniqueness of a solution of the 2D graph problem)
We make the assumptions of the Theorem 4.2 and denote by f the unique solution of the graph equation (4.40)*.*

Then, there is a time $T^* > 0$ (depending only on the bounds of c_0, c_1 and f_0) for which the following holds.
(Existence) *The function ρ_f is a solution in $C([0,T^*);L^1_{unif}(\mathbb{R}^2))$ of the equation* (4.46)
(Uniqueness) *If ρ_1, $\rho_2 \in C([0,T^*);L^1_{unif}(\mathbb{R}^2))$ are two bounded discontinuous viscosity solutions of* (4.46)*, they have the same semi-continuous envelopes and, for every $t \in [0,T^*)$, $\rho_1(\cdot,t) = \rho_2(\cdot,t)$ a.e. in $\mathbb{R}^2$.*

Proof of theorem 4.5
Given a bounded function ρ, we define the non-local velocity

$$c[\rho](x_1,x_2,t) = c_0 \star \rho^+(x_1,x_2) + c_1(x_1,x_2,t)$$

and write indifferently $c[\rho]$ or $c[\rho(\cdot,t)]$ when the function depends on t.

We first verify that ρ_f is a discontinuous solution of (4.46) and that it belongs to $C([0,T^*);L^1_{\text{unif}})$. Estimate (4.42) implies that, for every $0 < s,t \le T < T^*$ and every $x \in \mathbb{R}^2$, we have

$$\|\rho_f(\cdot,t) - \rho_f(\cdot,s)\|_{L^1_{\text{unif}}} \le 2\|f(\cdot,t) - f(\cdot,s)\|_{L^\infty(\mathbb{R})} \le 2|t-s|\, \|\frac{\partial f}{\partial t}\|_{L^\infty(\mathbb{R}\times[0,T])}.$$

Since $f \in W^{1,\infty}(\mathbb{R}\times[0,T])$, we deduce that $\rho_f \in C([0,T];L^1_{\text{unif}})$. As $T < T^*$ is arbitrary, $\rho_f \in C([0,T^*);L^1_{\text{unif}})$.

Next, f solves by definition the graph equation

$$\frac{\partial f}{\partial t} = c[\rho_f]\big(x_1, f(x_1,t), t\big)\sqrt{1+\left(\frac{\partial f}{\partial x_1}\right)^2} \quad \text{in } \mathbb{R}\times(0,T^*), \qquad f(\cdot,0) = f_0 \quad \text{in } \mathbb{R}.$$

The associated Hamiltonian $H(x,t,p) = c[\rho_{f(\cdot,t)}](x,t)\,|p|$ satisfies the assumptions of Proposition 3.8 (its continuity in time follows from the fact that $\rho_f \in C([0,T^*);L^1_{\text{unif}})$, while its regularity with respect to x and p is proved as in Theorem 4.2). Therefore, ρ_f solves the equation

$$\frac{\partial \rho_f}{\partial t} = c[\rho_f](x_1,x_2,t)|\nabla \rho_f| \quad \text{in } \mathbb{R}^2\times(0,T^*), \qquad \rho_f(\cdot,0) = \rho_{f_0} \quad \text{on } \mathbb{R}^2.$$

This is (4.46).

For the proof of uniqueness, we fix $\rho \in C([0,T^*);L^1_{\text{unif}})$, a bounded discontinuous viscosity solution of (4.46). Applying Theorem 3.7 (with a Hamiltonian with ρ frozen, and with a proof similar to the one of Theorem 4.2), we see that the graph equation

$$\text{(4.47)} \qquad \frac{\partial g}{\partial t} = c[\rho]\big(x_1, g(x_1,t), t\big)\sqrt{1+\left(\frac{\partial g}{\partial x_1}\right)^2} \quad \text{in } \mathbb{R}\times(0,T^{**}), \qquad g(\cdot,0) = f_0 \quad \text{in } \mathbb{R}$$

has a unique solution up to time $T^{**} = 1/\big(K_2(1+M_0)\big)$. Here, the constant K_2 is such that assumption (3.26) holds true and $M_0 = \|\partial f_0/\partial x_1\|_{L^\infty}$. As

$$\begin{aligned}|c[\rho]\big(x_1,h(x_1),t\big) - c[\rho]\big(y_1,h(y_1),t\big)| &\le \|\nabla c[\rho]\|_{L^\infty}\Big(1+\|\frac{\partial h}{\partial x_1}\|_{L^\infty(\mathbb{R})}\Big)\,|x_1-y_1| \\ &\le \big(\|\nabla c_0\|_M + \|\nabla c_1\|_{L^\infty}\big)\Big(1+\|\frac{\partial h}{\partial x_1}\|_{L^\infty(\mathbb{R})}\Big)\,|x_1-y_1|,\end{aligned}$$

we can choose for K_2 the constant $\|\nabla c_0\|_M + \|\nabla c_1\|_{L^\infty}$. As this is the same as the one for the graph equation (4.40) (see (4.41) in the proof of Theorem 4.2), we conclude that $T^{**} = T^*$. In other words, the functions f and g will not blow up before the same time.

Using Proposition 3.8 for the Hamiltonian $H(x,t,p) = c[\rho(\cdot,t)](x,t)\,|p|$, we deduce that, up to time T^*, ρ and ρ_g have the same semi-continuous envelopes. As $|\rho - \rho_g| \le \max(\rho^* - (\rho_g)_*, \rho_g^* - \rho_*)$, we get

$$|\rho - \rho_g| \le \rho_g^* - (\rho_g)_* = 2 \cdot 1_{\{x_2 = g(x_1,t)\}}.$$

For every t, the right-hand term has zero Lebesgue integral in $\mathbb{R}^2$. Therefore, $\rho(\cdot,t) = \rho_g(\cdot,t)$ a.e. This implies that $c[\rho] = c[\rho_g]$ everywhere. Consequently, g actually solves the graph equation

$$\frac{\partial g}{\partial t} = c[\rho_g]\big(x_1, g(x_1,t), t\big)\sqrt{1 + \left(\frac{\partial g}{\partial x_1}\right)^2} \quad \text{in } \mathbb{R} \times (0,T^*), \qquad g(\cdot,0) = f_0 \quad \text{in } \mathbb{R}.$$

By uniqueness (Theorem 4.2), we conclude that $f = g$. In view of the relationship between ρ and ρ_g, we see that the functions ρ and ρ_f must have the same semicontinuous envelopes and be equal a.e. in space for all time. □

Similarly we get the following result

Theorem 4.6 (Short time existence and uniqueness of a solution of the 2D loop problem)
We make the assumptions of the Corollary 4.4 and denote by f the unique solution of the loop equation (4.45).

Then, there is a time $T^* > 0$ (depending only on the bounds of c_0, c_1, $\left\|\frac{\partial f_0}{\partial s}\right\|_{L^\infty(\mathbb{R}/L\mathbb{Z})}$ and on Ω_0) for which the following holds.
(Existence) *The function $\tilde{\rho}_f$ is a solution in $C([0,T^*); L^1_{unif}(\mathbb{R}^2))$ of the equation* (4.46) *with initial data $\rho_0 = \tilde{\rho}_{f_0}$.*
(Uniqueness) *If ρ_1, $\rho_2 \in C([0,T^*); L^1_{unif}(\mathbb{R}^2))$ are two bounded discontinuous viscosity solutions of* (4.46), *they have the same semi-continuous envelopes and, for every $t \in [0,T^*)$, $\rho_1(\cdot,t) = \rho_2(\cdot,t)$ a.e. in $\mathbb{R}^2$.*

Sketch of the proof of Theorem 4.6
The proof repeats words by words the proof of Theorem 4.5, replacing estimate (4.42) by (4.44), Proposition 3.8 by Proposition 3.9, equation (4.47) by

$$\frac{\partial g}{\partial t} = c[\rho]\big(\Gamma_{f(\cdot,t)}(s), t\big)\sqrt{1 + \left(\frac{1}{(1-gK)}\frac{\partial g}{\partial s}\right)^2} \text{ in } (\mathbb{R}/L\mathbb{Z}) \times (0,T^{**}), \quad g(\cdot,0) = f_0 \text{ in } \mathbb{R}/L\mathbb{Z}$$

and replacing Theorem 4.2 by Corollary 4.4. □

As explained above, one major difficulty in defining a solution globally in time comes from the non validity of the inclusion principle, which is the analogous for sets of the comparison principle for the solutions of Hamilton-Jacobi equations. This results from the fact that the kernel c_0 will take negative values because $\int c_0\,dx = 0$. We briefly justify this.

The inclusion principle says roughly that if one front encompasses the second one initially, this will continue for all subsequent times. Less formally, the principle says that if two generalized fronts $(\Omega_t^{1-}, \Gamma_t^1, \Omega_t^{1+})$ and $(\Omega_t^{2-}, \Gamma_t^2, \Omega_t^{2+})$ satisfy the inclusions

$$\Omega_t^{1+} \subset \Omega_t^{2+}, \qquad \Omega_t^{1-} \supset \Omega_t^{2-}$$

at time $t = 0$, then the inclusions remain true at every time $t > 0$.

Let us construct an example to show that the inclusion principle is false, with c_0 continuous.

Counter-example to the inclusion principle

Choose a non empty ball B such that $c_0 < 0$ on $-B$ and such that $0 \notin \overline{B}$. Choose for Ω_0^{1+} a smooth set with $\overline{\Omega_0^{1+}} \cap \overline{B} = \emptyset$ and $0 \in \partial\Omega_0^{1+}$. Put $\Omega_0^{2+} = \Omega_0^{1+} \cup B$. The initial fronts are of course $\Gamma_0^1 = \partial\Omega_0^{1+}$ and $\Gamma_0^2 = \partial\Omega_0^{2+} = \Gamma_0^1 \cup \partial B$. At the origin $(x,t) = (0,0)$, the front Γ_0^1 has normal speed $c^1(0) = c_0 \star 1_{\Omega_0^{1+}}(0)$. On the other hand, the front Γ_0^2 has the speed $c^2(0) = c_0 \star 1_{\Omega_0^{2+}}(0) = c_0 \star (1_{\Omega_0^{1+}} + 1_B)(0) = c_0^1(0) - \delta$ for the constant $\delta = -\int_B c_0(-x)\,dx > 0$. Therefore, the set Ω_0^{2+}, though larger than Ω_0^{1+}, will propagate at the origin with a smaller speed; thus, the set Ω_t^{2+} cannot contain Ω_t^{1+} when $t > 0$ is small.

5 Numerical simulations

5.1 A level-sets scheme for dislocation dynamics

We consider discontinuous solutions ρ of the following equation

$$\frac{\partial\rho}{\partial t} = (c_0 \star \rho^+(\cdot,t) + c_1)|\nabla\rho| \tag{5.48}$$

and assume moreover that for some $L_1, L_2 > 0$ the solution is periodic:

$$\rho(x_1 + 2L_1, x_2, t) = \rho(x_1, x_2, t) = \rho(x_1, x_2 + 2L_2, t) \tag{5.49}$$

In order to compute numerically the solution, we first replace equation (5.48) on the discontinuous solution ρ, by a level-sets equation on a continuous function $\tilde\rho$:

$$\begin{cases} \dfrac{\partial\tilde\rho}{\partial t} = c\,|\nabla\tilde\rho| \\[2ex] c = (c_0 \star E(\tilde\rho(\cdot,t)) + c_1) \\[2ex] \tilde\rho(\cdot,0) = \tilde\rho_0 \end{cases} \tag{5.50}$$

where E is the Heavyside function defined by

$$E(\tilde\rho) = \begin{cases} 1 & \text{if} \quad \tilde\rho > 0 \\[2ex] 0 & \text{if} \quad \tilde\rho \le 0 \end{cases}$$

Here, in order to simplify the presentation, we assume that the velocity c_1 is a given constant. We also assume that $\tilde\rho$ satisfies the periodicity conditions (5.49).

We see (at least formally) that if $\tilde{\rho}$ satisfies (5.50), then

$$\rho = E(\tilde{\rho})$$

satisfies (5.48).
We work on a cartesian grid where each grid point is defined by

$$(x_1, x_2) = (i\Delta x_1, j\Delta x_2)\,, \quad \text{for} \quad i = -N_1, ..., N_1; \quad j = -N_2, ..., N_2$$

with

$$\Delta x_1 = \frac{L_1}{N_1}, \quad \Delta x_2 = \frac{L_2}{N_2}.$$

Moreover the time is also discretized

$$t = n\Delta t, \quad \text{for} \quad n \in \mathbb{N}$$

We approximate the values of the continuous functions $\tilde{\rho}$ (resp. c) on the grid points $(i\Delta x_1, j\Delta x_2)$ at time $n\Delta t$ by $\tilde{\rho}^n_{i,j}$ (resp. $c^n_{i,j}$), where we assume periodicity in both space directions, i.e. for $g = \tilde{\rho}$ or c:

$$g^n_{-N_1,j} = g^n_{N_1,j}, \quad \text{for} \quad j = -N_2, ..., N_2$$

$$g^n_{i,-N_2} = g^n_{i,N_2}, \quad \text{for} \quad i = -N_1, ..., N_1$$

We choose the Lax-Friedrichs scheme

$$\frac{\tilde{\rho}^{n+1}_{i,j} - \tilde{\rho}^n_{i,j}}{\Delta t} = \left\{ \begin{array}{c} c^n_{i,j}\sqrt{\left(\dfrac{\tilde{\rho}^n_{i+1,j} - \tilde{\rho}^n_{i-1,j}}{2\Delta x_1}\right)^2 + \left(\dfrac{\tilde{\rho}^n_{i,j+1} - \tilde{\rho}^n_{i,j-1}}{2\Delta x_2}\right)^2} \\ +\dfrac{\alpha}{2}\left\{\dfrac{\tilde{\rho}^n_{i+1,j} + \tilde{\rho}^n_{i-1,j} - 2\tilde{\rho}^n_{i,j}}{\Delta x_1} + \dfrac{\tilde{\rho}^n_{i,j+1} + \tilde{\rho}^n_{i,j-1} - 2\tilde{\rho}^n_{i,j}}{\Delta x_2}\right\} \end{array} \right\}$$

It is known (and easy to check) that this scheme is monotone (for a given fixed velocity c^n_{ij}) if the constant α (to be fixed later) satisfies

$$\alpha \geq |c^n_{i,j}| \quad \text{for every} \quad i = -N_1, ..., N_1; \quad j = -N_2, ..., N_2; \quad n \in \mathbb{N}$$

and if we assume the CFL condition

$$\Delta t \leq \frac{1}{\alpha}\left(\frac{1}{\Delta x_1} + \frac{1}{\Delta x_2}\right)^{-1}$$

We define an approximation $c^0_{i,j}$ (which will be specified later) of the kernel c_0. We extend all quantities $g = \tilde{\rho}, c, c^0$ periodically as follows

$$g^n_{i+2N_1,j} = g^n_{i,j} = g^n_{i,j+2N_2} \quad \text{for every} \quad i, j \in \mathbb{Z}; \quad n \in \mathbb{N}$$

Then we approximate the convolution $c_0 \star E(\tilde{\rho})$ by the discrete convolution, and write

$$c^n_{i,j} = c_1 + \sum_{i'=-N_1+1}^{N_1} \sum_{j'=-N_2+1}^{N_2} c^0_{i-i',j-j'} E(\tilde{\rho}^n_{i',j'})$$

where c_1 is the given velocity (assumed constant, to simplify).

The direct computation of the discrete convolution requires a number of operations of the order of $N_1^2 N_2^2$. Here we use the fast Fourier transform (FFT) to compute this convolution, which is simply a product in Fourier space. This reduces the computational cost to a number of operations of the order of $(N_1 \ln N_1)\ (N_2 \ln N_2)$.

We have used the software GO++ in C++ that can be found at the following address:

http://www-rocq.inria.fr/~benamou/gopp/gopp.html.

Here many schemes are implemented, and in particular the Lax-Friedrich scheme.

At every time step we perform a discrete convolution using FFT. We have chosen an algorithm that can be found at the following address: http://www.matpack.de/. We use this package, giving on the one hand $c_0(x)$ in a analytical way and on the other hand the sampling $E(\tilde{\rho}^n_{i,j})$. This algorithm computes the convolution of these two signals.

Remark 5.1 *Let us mention that we can also use a Rouy-Tourin scheme in place of the Lax-Friedrichs scheme. Second order approximations both in space and time can also be considered for first order Hamilton-Jacobi equations. We can for instance consider second order TVD approximations of the gradient with minmod limiters (see Osher, Sethian [51]), and a Runge-Kutta (TVD) time discretization of Osher, Shu [52]. Finally, let us add that the FFT of a signal (given analytically or accurately sampled) can be easily computed using the following package: http://www.fftw.org/. This could be useful when the kernel c_0 is only given in Fourier space.*

5.2 Analysis of the numerical results

Here we propose to study numerically three test cases. In the sequel, we choose the following approximation of the kernel:

$$c^0_{i,j} = \overline{c}^\varepsilon_0(i\Delta x_1, j\Delta x_2)$$

where $\overline{c}^\varepsilon_0(x)$ is defined in each case.

Case 1: collapse of a circle (fig. 3)

Here $L_1 = L_2 = 2$ and the two-dimensional domain is the square $[-2,2]^2$. We chose $N_1 = N_2 = 100$, which leads to $\Delta x_1 = L_1/N_1 = 0.02 = \Delta x_2$. We chose $\Delta t = 0.00018$. Here

$$\overline{c}^\varepsilon_0(x) = c^\varepsilon_0(x)$$

with

$$c^\varepsilon_0(x) = \begin{cases} \dfrac{1}{|x|^3} & \text{if} \quad |x| > \varepsilon \\ \\ -\dfrac{2}{\varepsilon^3} & \text{if} \quad |x| \le \varepsilon \end{cases}$$

Here we took $\varepsilon = 0.1$. We chose periodic conditions in each direction. At time $t = 0$, we chose $\tilde{\rho}_0(x) = 0.5 - |x|$, and the final time is $t \simeq 0.8$. In this simulation the circle collapse isotropically in finite time, which is naturally expected.

Case 2: relaxation of the *sin* curve (fig. 4)

The initial condition is a *sin* curve given by $x_2 = 3\sin(\pi x_1)$. More precisely, at time $t = 0$, we chose $\tilde{\rho}_0(x_1, x_2) = -x_2 + 3\sin(\pi x_1)$. We work with $L_1 = 1$, $L_2 = 10$, i.e. on the rectangle $[-1,1] \times [-10,10]$, discretized with $N_1 = 20$, $N_2 = 200$, which gives $\Delta x_1 = \Delta x_2 = 0.05$.

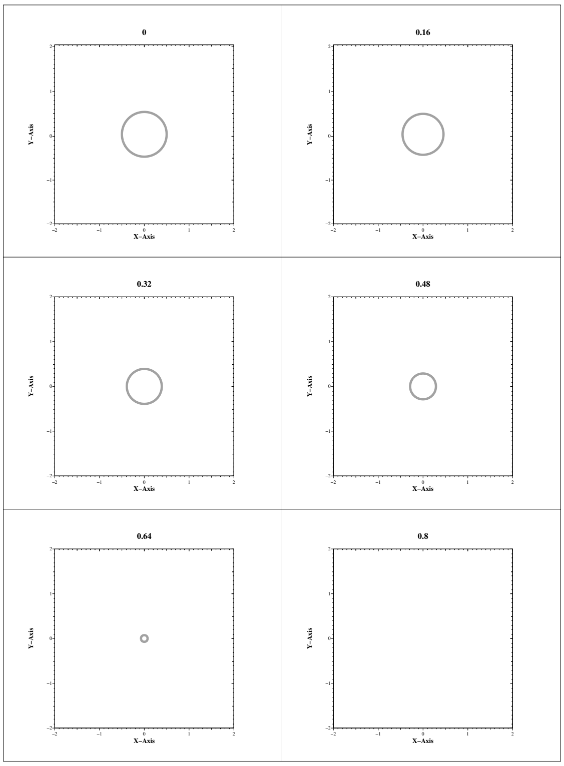

Figure 3: Relaxation of a circle dislocation

In this case we set

$$\overline{c}_0^\varepsilon(x_1, x_2) = \sum_{k=-K}^{k=K} c_0^\varepsilon(x_1 + 2k, x_2) \ .$$

It can be checked that we have to choose K of the order of L_2/L_1. We chose $K = 10$. This periodization of the kernel was done to take into account the periodicity in x_1, and the convolution is done assuming periodic conditions in x_1. On the contrary, for this example, we work with Neumann conditions in x_2. Our choice of Neumann conditions in x_2 is made in order to control the effect of these fixed boundaries on the graph solution inside the domain. This is also why we chose $L_2 >> L_1$. Until our choice of the final time we know that the computation is correct.

Simulation 1 (fig. 4 left)

We chose $\varepsilon = 0.01$, $\Delta t = 0.0014$ and the final time $T \simeq 2$. In this computation the *sin* curve relaxes to a straight line which is physically expected. For this so small value of ε the dynamics of the dislocation line is expected to be very closed to the motion by mean curvature, and then the dislocation line stays smooth.

Simulation 2 (fig. 4 right)

We took $\varepsilon = 5$, $\Delta t = 0.0000185$, and the final time equal to $T \simeq 0.0002$. In this second simulation, the core of the dislocation is much larger than the typical length scale of the dislocation line. This is why we observe short range effects, and the dislocation line does not stay smooth.

Case 3: a graph becomes a non-graph (fig. 5)

In this example we work on the domain $[-2, 2] \times [-2, 10] = (0, 4) + [-2, 2] \times [-6, 6]$, i.e. $L_1 = 2, L_2 = 6$. The initial dislocation is the curve $x_2 = 4e^{(-10x_1^2)}$. More precisely, at time

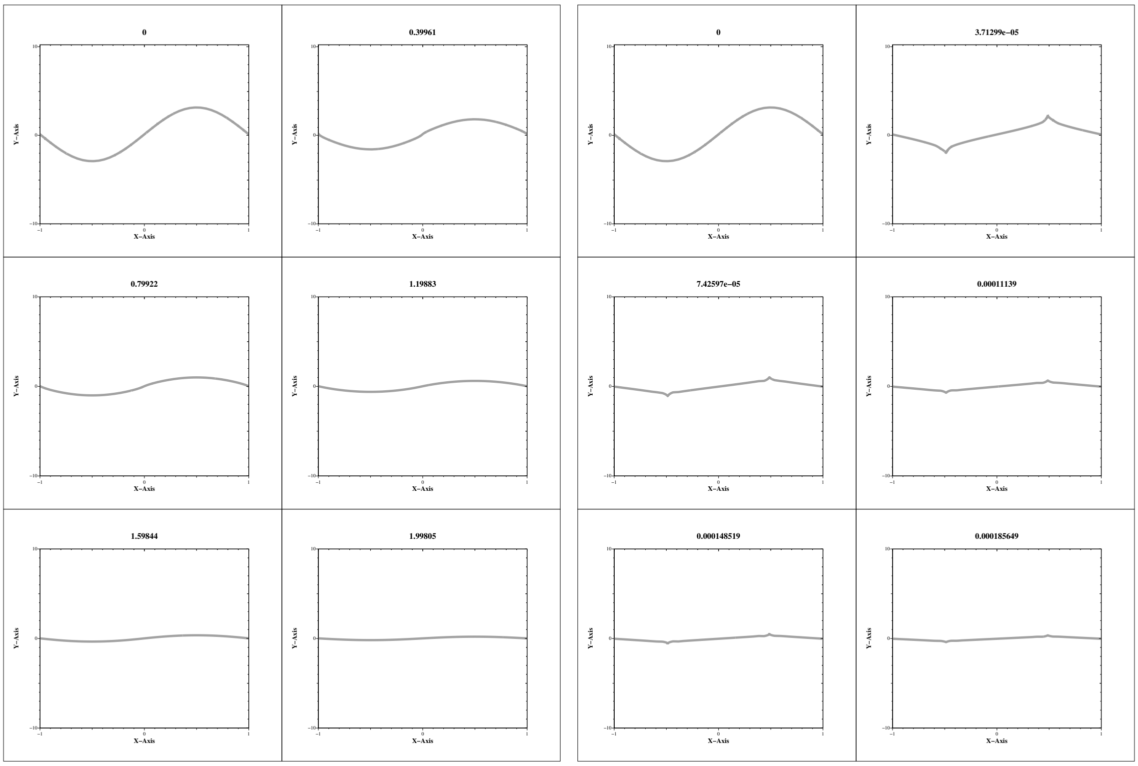

Figure 4: Relaxation of a sinusoidal dislocation line, with $\varepsilon = 0.01$ (left) and $\varepsilon = 5$ (right)

$t = 0$, we chose $\tilde{\rho}_0(x_1, x_2) = -x_2 + 4e^{(-10x_1^2)}$. The discretization is $N_1 = 40$, $N_2 = 100$, which means $\Delta x_1 = 0.05$, $\Delta x_2 = 0.06$. We chose $\Delta t = 0.0000512$, and the final time is $T \simeq 0.005$. Here the kernel is non-physical. We define

$$\Omega = B_1((-1,0)) \cup B_1((1,0)) = \text{supp } \overline{c}_0$$

where $B_1(y) = \{x \in \mathbb{R}^2,\ |x - y| \leq 1\}$. We choose the following function

$$\overline{c}_0^\varepsilon(x) = \overline{c}_0(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega,\ |x_1| > 1 \\ -1 & \text{if} \quad x \in \Omega,\ |x_1| \leq 1 \\ 0 & \text{if} \quad x \notin \Omega \end{cases}$$

In this simulation we see that starting from a graph dislocation, the solution can become a non-graph.

6 Appendix

In this appendix, we use the notations of section 2, with $x = (x_1, x_2, x_3)$ and $x' = (x_1, x_2)$.

6.1 Formal properties of the kernel c_0

We have

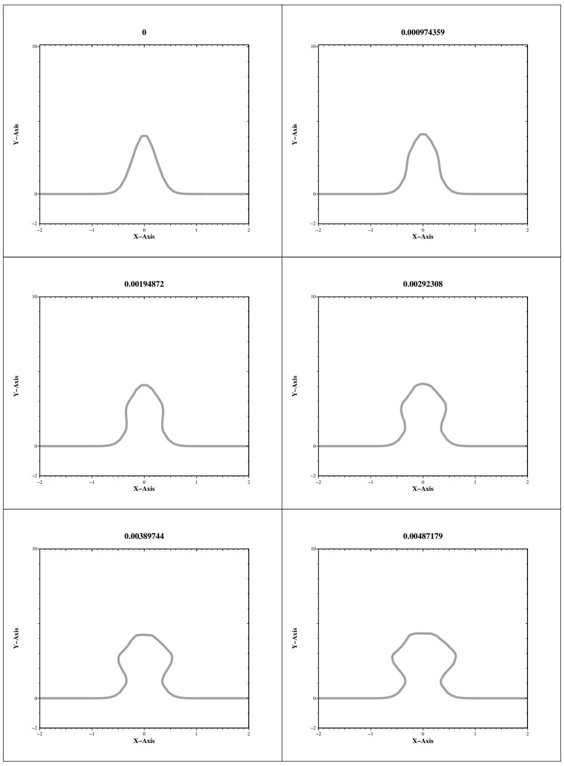

Figure 5: A graph can become a non-graph

Proposition 6.1 (Formal properties of c_0)
The kernel c_0 formally satisfies the following properties:

$$
\begin{array}{ll}
i) & c_0(-x') = c_0(x') \\
ii) & \displaystyle\int_{\mathbb{R}^2} c_0 = 0 \quad \textit{if} \quad R\star\chi = \chi\star R \\
iii) & -\hat{c}_0(\xi') \geq 0 \\
iv) & |c_0(x')| \leq \dfrac{C}{|x'|^3} \quad \textit{for} \quad |x'| \quad \textit{large enough} \\
v) & c_0 \quad \textit{is smooth enough}
\end{array}
$$

Formal proof of proposition 6.1
i)Using Fourier transform on the definition of C_0 (2.18), we get

$$\hat{C}_0(\xi) = -{}^t(\hat{\check{R}}\hat{\check{\chi}}e^0)\Lambda(\hat{R}\hat{\chi}e^0) \tag{6.51}$$

where we recall that $\check{\chi}(x) = \chi(-x)$, and we have $\hat{\check{R}} = \hat{R}$. From the symmetry property (2.10) of Λ, we immediately see that $\hat{C}_0(-\xi) = \hat{C}_0(\xi)$. This implies $C_0(-x) = C_0(x)$ and then

$$c_0(-x') = c_0(x')$$

ii) We have formally

$$\int_{\mathbb{R}^2} -c_0$$

$$= \int_{\mathbb{R}^3} {}^t(\check{R} \star \check{\chi} e^0) \star \Lambda(R \star \chi e^0)\ \delta_0(x_3)$$

$$= \int_{\mathbb{R}^3} \Lambda(R \star \chi e^0)(R \star \chi \star (\delta_0(x_3) e^0))$$

$$= \int_{\mathbb{R}^3} \Lambda(R \star \chi e^0)(\chi \star R \star (\delta_0(x_3) e^0))$$

where we have used in the last line the assumption that $R\star\chi = \chi\star R$. But $\delta_0(x_3)e^0 = e(v)$ for $v = bH(x_3)$ where H is the Heavyside function. As a consequence $R\star(\delta_0(x_3)e^0) = R\star e(v) = 0$ because $\mathrm{inc}(e(v)) = 0$. Therefore we get

$$\int_{\mathbb{R}^2} c_0 = 0$$

iii) We have

$$E(\Gamma)$$

$$= \int_{\mathbb{R}^3} \frac{1}{2} \Lambda e_\chi^2$$

$$= \int_{\mathbb{R}^3} \frac{1}{2} \Lambda(R \star \chi e^0 \star (\rho\delta_0(x_3)))\ (R \star \chi e^0 \star (\rho\delta_0(x_3)))$$

$$= \int_{\mathbb{R}^3} \frac{1}{2} \left[\left\{ {}^t(\check{R} \star \check{\chi} e^0) \star \Lambda(R \star \chi e^0) \right\} \star (\rho\delta_0(x_3)) \right] (\rho\delta_0(x_3))$$

$$= \int_{\mathbb{R}^2} -\frac{1}{2}(c_0 \star \rho)\rho$$

$$= \int_{\mathbb{R}^2} -\frac{1}{2}\hat{c}_0 |\hat{\rho}|^2$$

Because $E(\Gamma) \geq 0$ for every possible function ρ, we deduce formally that

$$-\hat{c}_0(\xi') \geq 0$$

iv) In the expression (6.51) of $\hat{C}_0(\xi)$, the quantity $\hat{R}$ is homogeneous of order one in $\dfrac{\xi \otimes \xi}{|\xi|^2}$. Because by homogeneity, we have formally $\widehat{|x|} = \frac{K}{|\xi|^4}$ for some $K \in \mathbb{R}$, we deduce that C_0 can be written formally:

$$C_0 = D^4(|x|) \star (\text{mollifier})$$

where $D^4(|x|)$ denotes the forth derivatives of the function $x \longmapsto |x|$. This implies that

$$|C_0(x)| \leq \frac{C}{|x|^3} \quad \text{for} \quad |x| \quad \text{large enough}$$

and the corresponding result is true for $c_0(x')$.
v) Finally from the convolution formula (2.18) for C_0, we deduce that c_0 is smooth enough, if χ is smooth enough.
This ends the proof of proposition 6.1.
Scaling properties of the problem
If we replace χ by

$$\chi^\varepsilon(x') = \frac{1}{\varepsilon^2}\chi\left(\frac{x'}{\varepsilon}\right)$$

we see that $c_0(x')$ is replaced by

$$c_0^\varepsilon(x') = \frac{1}{\varepsilon^3}c_0\left(\frac{x'}{\varepsilon}\right)$$

which is the natural scaling for the kernel c_0.
Moreover if ρ satisfies

$$\frac{\partial\rho}{\partial t} = (c_0 \star \rho + c_1)|\nabla\rho|$$

then we still have

$$\frac{\partial\rho^\varepsilon}{\partial t} = (c_0^\varepsilon \star \rho^\varepsilon + c_1^\varepsilon)|\nabla\rho^\varepsilon|$$

with

$$\begin{cases} c_0^\varepsilon(x') = \dfrac{1}{\varepsilon^3}c_0\left(\dfrac{x'}{\varepsilon}\right) \\ \rho^\varepsilon(x',t) = \rho^\varepsilon\left(\dfrac{x'}{\varepsilon},\dfrac{t}{\varepsilon^2}\right) \\ c_1^\varepsilon = \dfrac{c_1}{\varepsilon} \end{cases}$$

6.2 How to compute the kernel c_0 and the core tensor?

In this part of the appendix, we explain formally how to compute explicitly the kernel c_0, leading to quite complicated computations. In a first subsection, we give the general expressions of the kernel c_0 in the special case of a scalar core tensor. And in a second subsection we give the expression of the scalar core tensor for different models, whose the famous Peierls-Nabarro model of dislocations.

6.2.1 Expressions of the kernel c_0 for a scalar core tensor

In this subsection we are interested in a special case for the core tensor, namely the case of a scalar core tensor, which greatly simplifies the computations:

$$\chi_{ijkl} = \chi_1(x)\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

The core function $\chi_1(x)$ contains certain properties of the physics of the dislocations and drives the physics of interactions between dislocations at short distances. Its explicit expression is difficult to compute and really depends on the assumptions of the model behind. It can be isotropic in the slip plane or anisotropic (see for instance François, Pineau, Zaoui [28] page 213, for an example of an anisotropic core).

General expression of the kernel c_0 for a scalar core tensor. The kernel c_0 is given by

$$c_0(x') = C_0(x', 0)$$

If we define

$$\hat{C}_0(\xi) = \int_{\mathbb{R}^3} d^3x \; e^{-i\xi\cdot x} C_0(x)$$

we have

$$\hat{C}_0(\xi) = -b^2 |\hat{\chi}_1(\xi)|^2 B\left(\frac{\xi}{|\xi|}\right)$$

where

$$\xi = (\xi_1, \xi_2, \xi_3), \quad \text{and} \quad |\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$$

and

$$b^2 B\left(\frac{\xi}{|\xi|}\right) = {}^t(\hat{\bar{R}}e^0)\Lambda(\hat{R}e^0)$$

We see that the computation of the kernel can be split in two parts: first the computation of B (see below), and second the computation of the core function $\chi_1(x)$ (see subsection 6.2.1). We recover $c_0(x') = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} d^2\xi' \; e^{i\xi'\cdot x'} \hat{c}_0(\xi')$, with

$$\hat{c}_0(\xi') = \frac{1}{2\pi}\int_{\mathbb{R}} d\xi_3 \; \hat{C}_0(\xi)$$

General expression of $B(\xi)$ for FCC crystals with cubic elasticity. We consider a direct orthonormal basis (e_1, e_2, e_3) which defines the cell of the cubic crystal. In the special case of cubic crystal, there exist three constants c_{11}, c_{12}, c_{44} (in Voigt notations), such that

$$\Lambda_{ijkl} = \left((c_{11} - c_{12} - 2c_{44})\,\delta_{ik} + c_{12}\right)\delta_{ij}\delta_{kl} + c_{44}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

This means that

$$\Lambda_{iiii} = c_{11}$$

$$\Lambda_{iijj} = c_{12} \quad \text{for} \quad i \neq j$$

$$\Lambda_{ijij} = c_{44} \quad \text{for} \quad i \neq j$$

We define

$$\alpha = \frac{c_{11} - c_{12} - 2c_{44}}{c_{44}}$$

The isotropic case corresponds to the case $\alpha = 0$, where $c_{11} = \lambda + 2\mu, c_{12} = \lambda, c_{44} = \mu$.

If we are interested in dislocations in FCC crystals, we need to consider dislocations moving in a plane $\{111\}$. Let us consider for instance the normal n to the slip plane defined by $n = \frac{1}{\sqrt{3}}(1,1,1)$. Here we can choose as a Burgers vector, the vector $b = |b|e_b$ with $e_b = \frac{1}{\sqrt{2}}(1,-1,0)$. We also define the tensors

$$e_{ij}^{00} = \frac{1}{2}\left((e_b)_i n_j + (e_b)_j n_i\right)$$

$$\sigma_{ij}^{00} = \sum_{k,l} \Lambda_{ijkl} e_{kl}^{00}$$

We introduce $e_c = n \wedge e_b = \frac{1}{\sqrt{6}}(1,1,-2)$, such that (e_b, e_c, n) is a direct orthonormal basis. Every vector ξ can then be written both in the basis (e_b, e_c, n) and (e_1, e_2, e_3):

$$\xi = \xi_b e_b + \xi_c e_c + \xi_n n = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$$

Then we have (see Khachaturyan [39], pages 213 and 244)

$$B(\xi) = \sum_{i,j,k,l \in \{1,2,3\}} \Lambda_{ijkl} e^{00}_{ij} e^{00}_{kl} - \xi_i \sigma^{00}_{ij} \Omega_{jk}(\xi) \sigma^{00}_{kl} \xi_l$$

where we define

$$D(\xi) = c_{11} + \alpha(c_{11} + c_{12}) \left(\xi_1^2\xi_2^2 + \xi_1^2\xi_3^2 + \xi_2^2\xi_3^2\right) + \alpha^2 \left(c_{11} + 2c_{12} + c_{44}\right) \xi_1^2\xi_2^2\xi_3^2$$

and

$$\Omega_{ii}(\xi) = \frac{1}{c_{44} D(\xi)} \left(c_{44} + (c_{11} - c_{44})(\xi_j^2 + \xi_k^2) + \alpha(c_{11} + c_{12})\xi_j^2\xi_k^2\right)$$

$$\Omega_{ij}(\xi) = -\frac{1}{c_{44} D(\xi)} \left((c_{12} + c_{44})(1 + \alpha\xi_k^2)\xi_i\xi_j\right) \quad \text{if} \quad i \neq j$$

Here (ijk) is a circular permutation of the indices (123).

General expression of $B(\xi)$ for crystals with isotropic elasticity. In the case of isotropic elasticity, the previous expression of B can be greatly simplified. If we now choose $\Lambda_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$, $b = |b|e_1$, $n = e_3$, we get

$$\frac{B(\xi)}{2\mu} = \frac{1}{2}\frac{\xi_2^2}{|\xi|^2} + \gamma\frac{\xi_1^2\xi_3^2}{|\xi|^4}$$

where

$$\gamma = \frac{2(\lambda+\mu)}{(\lambda+2\mu)} = \frac{1}{1-\nu} \quad \text{with the Poisson ratio} \quad \nu = \frac{\lambda}{2(\lambda+\mu)}$$

6.2.2 Expressions of the scalar core tensor in the case of isotropic elasticity

Here we present some references to the literature which allow us to compute the core tensor in the special case of a scalar core tensor:

$$\chi_{ijkl} = \chi_1(x)\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

where $\chi_1(x)$ is called the core function, and in the framework of isotropic elasticity $\Lambda_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$, $b = |b|e_1$, $n = e_3$.

General relation between the kernel c_0 and the stress σ_{13}. Let us consider the case of a straight dislocation line $\{x_3 = x_1 = 0\}$ along the direction x_2, moving in the (x_1, x_2) plane, and with Burgers vector parallel to e_1. In this special case we have

$$e^0 = \frac{1}{2}\left(b \otimes n + n \otimes b\right), \quad \text{with} \quad b = |b|e_1, \quad n = e_3$$

Let us recall that the relation between the stress and the kernel c_0 is given by

$$\sigma \cdot e^0 = c_0 \star 1_\Pi \quad \text{with} \quad \Pi = \{x_1 < 0\}$$

i.e.

$$|b|\sigma_{13}(x_1, x_2, 0) = |b|\sigma_{13}(x_1, 0, 0) = \int_{x_1}^{+\infty} dy_1 \int_{\mathbb{R}} dy_2 \ c_0(y_1, y_2) \tag{6.52}$$

We will use expression (6.52) in what follows to compute the core function $\chi_1(x)$ for different models.

The Peierls-Nabarro model. See Peierls [53], Nabarro [48], and Hirth, Lothe [35] or Lardner [41] for a more recent presentation of the Peierls-Nabarro model.

In this model we have in the case of the *edge dislocation* (see Hirth, Lothe [35] page 221, and up to the change of sign in σ_{13}, or Lardner [41], page 121):

$$\sigma_{13}(x_1, 0, 0) = \frac{\mu b}{2\pi(1-\nu)} \frac{x_1}{x_1^2 + \zeta^2} \tag{6.53}$$

where $\zeta > 0$ is a physical parameter (depending on the material) which indicates the size of the core of the dislocation.

Similarly still with a Burgers vector $b = |b|e_1$, but for a *screw dislocation* line $\{x_3 = x_2 = 0\}$ along the direction x_1, moving in the plane (x_1, x_2), we have $|b|\sigma_{13}(x_1, x_2, 0) = |b|\sigma_{13}(0, x_2, 0) = \int_{x_2}^{+\infty} dy_2 \int_{\mathbb{R}} dy_1 \ c_0(y_1, y_2)$, and (see [35] page 225 or [41] page 124)

$$\sigma_{13}(0, x_2, 0) = \frac{\mu b}{2\pi} \frac{x_2}{x_2^2 + \zeta^2} \tag{6.54}$$

After some computations, it can be checked that a general solution for the core function, which matches with expressions (6.53)-(6.54) of the stress, is for $\xi = (\xi_1, \xi_2, \xi_3)$:

$$\hat{\chi}_1(\xi) = e^{-\frac{\zeta}{2}\sqrt{\xi_1^2 + \xi_2^2}}$$

i.e.

$$\chi_1(x) = \chi_0\left(\sqrt{x_1^2 + x_2^2}\right) \delta_0(x_3)$$

for some function χ_0. Here the core function χ_1 is isotropic in (x_1, x_2).

Then we have

$$\hat{c}_0(\xi_1, \xi_2) = -\frac{\mu b^2}{2} \left(\frac{\xi_2^2 + \frac{1}{1-\nu}\xi_1^2}{\sqrt{\xi_1^2 + \xi_2^2}} \right) e^{-\zeta\sqrt{\xi_1^2 + \xi_2^2}}$$

The model of Rodney, Le Bouar, Finel [58]. In this model, we have

$$\hat{\chi}_1(\xi) = 1_{\{|\xi_1| < K, |\xi_2| < K\}} 1_{\{|\xi_3| < K'\}}$$

where $K, K' > 0$ are two parameters to be fixed (where physically $K' > K$), depending on the atomistic properties of the material.

Similar functions χ_1 are also given in [58] for anisotropic elasticity.

The model of Saada and Shi [59]. In this model we have

$$\chi_1(x) = \xi_0(x_1)\delta_0(x_2)\delta_0(x_3)$$

where x_1 is the direction of the Burgers vector and the function ξ_0 formally satisfies (with $\check{\xi}_0(x_1) = \xi_0(-x_1)$)

$$\check{\xi}_0 \star \xi_0(x_1) = \phi(x_1) \tag{6.55}$$

$$\phi(x_1) = c(1 - x_1^2)^2 1_{\{|x_1|<1\}}$$

where the constant $c > 0$ is chosen such that $\int_{\mathbb{R}} \phi = 1$. Here we do not know if ϕ can really be written as in (6.55), i.e. if the Fourier transform of ϕ is non-negative.

Let us remark that in this model the core function is not smooth enough to ensure that C_0 is continuous.

See also Cai et al. [13] for some similar continuous Burgers vectors formulations.

Non-local elasticity models. See Eringen [27], and Lazar (see [42] for screw dislocations and [43] for edge dislocations).

In this kind of models, we have for instance

$$\hat{\chi}_1(\xi) = \frac{1}{(1 + \zeta^2|\xi|^2)^{\frac{1}{2}}}$$

In particular see Eringen [27], page 158 for a picture the regularized stress for a dislocation.

The model of Cuitino, Koslowski, Ortiz [21]. This phase field model is a self-contained model describing the core of a general curved dislocation. Even if it is not completely clear, the core function could be in principle theoretically computed to match with particular solutions of this model, at least for special configurations with straight line dislocations (edge and screw).

More generally a similar estimate of the core function could be made for Ginzburg-Landau-type models that could be imagined for dislocations (see for instance the modelling of Denoual [22]).

From atomic simulations. As explained above, the dislocation line is surrounded by a region, known as the core, within which linear continuum elasticity ceases to be a good approximation. However, we have seen that calculations can still be performed within the linear continuum elasticity, provided that the strains are regularized on the dislocation line. More precisely, the aim of this regularization procedure is to remove the non-physical divergence of the the strains inside the core without modifying (too much) the predictions of the model outside the core. Thus it is clear that the regularization procedure (i.e. the core tensor) is directly linked to the shape and size of the dislocation core.

A way to obtain reliable information on the core tensor, consists in measuring the shape and size of the dislocation core using atomic scale calculations. The latter can be performed using semi-empirical interatomic potentials (see e.g. Schroll, Vitek, Gumbsch [61], Mrovec, Nguyen-Manh, Pettifor, Vitek [47]), or more accurately using 'ab initio' electronic structure calculations (see e.g. Woodward, Rao [67], Blase, Lin, Canning, Louie, Chrzan, [10]). Then the result has to be compared to the linear continuum elasticity solution in order to deduce the shape and size of the core. Finally, the information obtained for different types of dislocations cores (edge, screw ...) together with the symmetries of the atomic crystal can be in principle combined to build a core tensor for the particular crystal under consideration.

6.3 A rough but explicit approximation of the kernel c_0

We start with the following result whose proof is left to the reader:

Proposition 6.2 (Expression of the kernel c_0 for isotropic elasticity and Burgers vector in the slip plane) *Let us choose* $\Lambda_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$, $b = |b|e_1$, *and* $\chi_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\delta_0(x_3)\chi_0(x')$. *Then we have*

$$c_0(x') = \frac{\mu b^2}{4\pi}(\chi_0 \star \check{\chi}_0) \star \left\{\partial_{22}\frac{1}{|x'|} + \gamma\partial_{11}\frac{1}{|x'|}\right\}$$

where

$$\gamma = \frac{2(\lambda+\mu)}{(\lambda+2\mu)} = \frac{1}{1-\nu} \quad \text{with the Poisson ratio} \quad \nu = \frac{\lambda}{2(\lambda+\mu)}$$

and we denote $\partial_{ii} = \dfrac{\partial^2}{\partial x_i^2}$ *for* $i = 1, 2$.

A straightforward computation gives:

$$c_0(x') = \frac{\mu b^2}{4\pi}(\chi_0 \star \check{\chi}_0) \star \left\{\frac{1}{|x'|^5}\left(x_1^2(2\gamma - 1) + x_2^2(2-\gamma)\right)\right\}$$

where the quantity in the bracket has to be taken formally.

Let us recall that Λ satisfies the coercivity condition (2.11) if and only if the Lamé coefficients satisfy

$$\mu > 0, \quad 3\lambda + 2\mu > 0$$

This implies that

$$\gamma \in \left(\frac{1}{2}, 2\right)$$

and then this shows that the kernel c_0 satisfies:

$$c_0(x') > 0 \quad \text{on} \quad \mathbb{R}^2 \backslash B_R(0) \quad \text{for} \quad R \quad \text{large enough}$$

In particular for $\lambda = 0$, we have $\gamma = 1$ and the kernel c_0 is isotropic:

$$c_0(x') = \frac{\mu b^2}{4\pi}(\chi_0 \star \check{\chi}_0) \star \Delta\left(\frac{1}{|x'|}\right)$$

We can use another way to mollify c_0. For instance, when the size of the core is comparable to 1, it seems reasonable to replace $\frac{1}{|x'|}$ by the function G_1 given by

$$G_1(x') = \begin{cases} \dfrac{1}{|x'|} & \text{if} \quad |x'| > 1 \\ \\ \dfrac{1}{2}(3 - |x'|^2) & \text{if} \quad |x'| \leq 1 \end{cases}$$

Here $G_1 \in W^{2,\infty}(\mathbb{R}^2)$. And then we can define:

$$c_0 = \frac{\mu b^2}{4\pi} \left\{ \partial_{22} G_1 + \gamma \partial_{11} G_1 \right\}$$

We get

$$c_0(x') = \frac{\mu b^2}{4\pi} \begin{cases} \frac{1}{|x'|^5} \left(x_1^2(2\gamma - 1) + x_2^2(2-\gamma) \right) & \text{if} \quad |x'| > 1 \\ \\ -(1+\gamma) & \text{if} \quad |x'| \le 1 \end{cases}$$

The main problem of this approximation of the kernel is that this construction does not guarantee that its Fourier transform is non positive everywhere, as it is formally required. In the particular case of isotropic kernels ($\gamma = 1$), we get

$$c_0(x') = \frac{\mu b^2}{4\pi} \begin{cases} \frac{1}{|x'|^3} & \text{if} \quad |x'| > 1 \\ \\ -2 & \text{if} \quad |x'| \le 1 \end{cases}$$

6.4 A "monotone approximation" of the kernel c_0

In this subsection, we mention an interesting "monotone approximation" of the kernel, which has the interest to be much more convenient to study mathematically. This approximation garantees that the evolution equation satisfies the inclusion principle.

Let us define for $\gamma \in (1/2, 2)$

$$\tilde{c}_0(x') = \frac{\mu b^2}{4\pi} \begin{cases} \frac{1}{|x'|^5} \left(x_1^2(2\gamma - 1) + x_2^2(2-\gamma) \right) & \text{if} \quad |x'| > 1 \\ \\ 0 & \text{if} \quad |x'| \le 1 \end{cases}$$

Then we define

$$c_0(x') = \tilde{c}_0(x') - \left(\int_{\mathbb{R}^2} \tilde{c}_0 \right) \delta_0$$

where δ_0 is the Dirac mass. When ρ is the (signed) characteristic function of a smooth open set Ω, we can evaluate formally at $x' \in \partial\Omega$

$$(c_0 \star \rho^+)(x') = (\tilde{c}_0 \star \rho^+)(x') - \frac{1}{2} \left(\int_{\mathbb{R}^2} \tilde{c}_0 \right)$$

The factor 1/2 comes from the formal convolution of the Dirac mass and the Heavyside function. This convention garantees that the solution corresponding to a half plane Ω has a zero velocity on $\partial\Omega$ and then is stationary. Because we are only interested in the velocity on the discontinuity of ρ, we could consider the solutions of

$$\frac{\partial \rho}{\partial t} = \left((\tilde{c}_0 \star \rho^+)(x') - \frac{1}{2} \left(\int_{\mathbb{R}^2} \tilde{c}_0 \right) \right) |\nabla \rho|$$

which is a monotone non-local equation because $\tilde{c}_0 \ge 0$.

Even if the description the self-interaction of the dislocation at short distances (i.e. in the core of the dislocation) is quite rough, this kind of modelling seems acceptable at large distances.

Remark 6.3 *Let us remark that the situation where we consider the simultaneous evolutions of two dislocations in the same plane but with different Burgers vectors, would be modelled by a non-local system of equations which would have no reason to be monotone. This is another motivation to try first to study mathematically the case of a single non-local equation without assuming any kind of monotonicity.*

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P.S.: For any correspondence, please contact:
Régis MONNEAU
CERMICS, Ecole Nationale des Ponts et Chaussées
6 et 8 avenue Blaise Pascal
Cité Descartes, Champs-sur-Marne
77455 Marne-la-Vallée Cedex 2
France
e-mail: monneau@cermics.enpc.fr
http://binky.enpc.fr/ monneau/home.html