DISLOCATION SHIELDING OF A CRACK IN A QUASI CONTINUUM APPROXIMATION*

R. M. Thomson and E. R. Fuller, Jr.
National Bureau of Standards
Washington, D.C. 20234

Earlier predictions of toughness by Thomson and Weertman are reviewed. The differing predictions of the two authors are shown to be due to different interpretations of the functional dependence of the size of the elastic region on intrinsic surface energy. An analysis of the quasi continuum model in terms of dislocation rearrangements is given, and the nature of the boundary condition on the elastic enclave boundary is discussed.

*Partially supported by the Army Research Office.

I. Introduction

Thomson[1] and Weertman[2] have recently published expressions (1) and (2), respectively, for the stress intensity factors for equilibrium cracks based on an elastic enclave model of fracture

$$K^2 = 2\pi \left(\frac{2\Gamma\mu}{\pi}\right)^{(1+n)/2n} \left(R_e\sigma_o^{\,2}\right)^{(n-1)/2n} \tag{1}$$

$$K^2 = \frac{4c^{-2}\mu\Gamma}{(1-\nu)} \tag{2}$$

(We have modified Eqn. (1) from that given in Ref. 1 by a numerical factor of $2^{(1+n)/2n}$ to conform to the analysis carried out below.)

K is the stress intensity factor as measured at the external grips for a crack in small scale yielding of a given length, σ_o the yield strength of the material, μ the shear modulus, Γ the intrinsic surface energy, n the strain hardening parameter, R_e the radius of the elastic enclave, ν Poisson's ratio, and c a derived quantity, c<<1. They differ most strikingly in that Eqn. (1) predicts a power law dependence on Γ with a power in the neighborhood of 2-3, while Eqn. (2) apparently yields the linear dependence of pure brittle fracture, except for the multiplying factor, c^{-2}. We will show that one may find a range of dependence on Γ from one of these expressions to the other, depending on how one interprets the variation of the inner radius R_e with external stress. The two expressions were derived using different boundary conditions on R_e, however, and a second aim of the paper will be to explore qualitatively the question of what the boundary conditions should be in terms of dislocation models.

We will work exclusively in Mode III anti-plane strain for the reason that the mathematical analysis in this case is simple, and the consequences for a specific physical model can be clearly worked out. We assume, of course, that predictions for Mode III should have qualitative significance for the physically important Mode I case. Also, we shall work with a time independent, static, deformation law for the reason again that the crucial issue is to decide how to model the crack physically with an elastic enclave, and not to make detailed predictions regarding crack growth.

When we think of embedding a sharp crack in a material containing external dislocation sources distributed randomly around the crack tip, the tip of the crack, on probabilistic grounds, is most likely not to experience the blunting catastrophe which would occur if a slip plane should intersect the exact position of the tip over a significant front of the crack and destroy its sharp character. Rather, slip taking place ahead of the crack will pass it by without changing the geometric configuration of the cohesive zone, and slip taking place behind it will cut the cleavage surface and create typical slip steps on the cleavage surface. The crack will then be supposed to take the general shape depicted in Figure 1. It is our object to work out the predicted toughness of a material possessing such a crack.

II. The Elastic Enclave Model in Mode III

To derive any quantitative results, it is necessary to model the plastically deformed region surrounding the crack by a continuum theory. Further, in the usual fashion, we assume small scale yielding, and assert that outside some finite boundary surrounding the crack the material is stressed below its yield stress, and is linear elastic. We label these regions in Figure 2, II

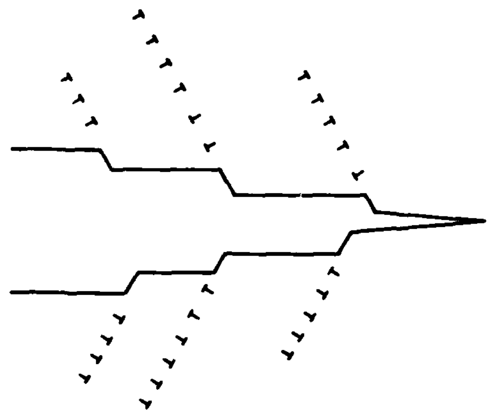

Figure 1 - A sharp crack which has been blunted by impingement of discrete slip lines on its cleavage surface.

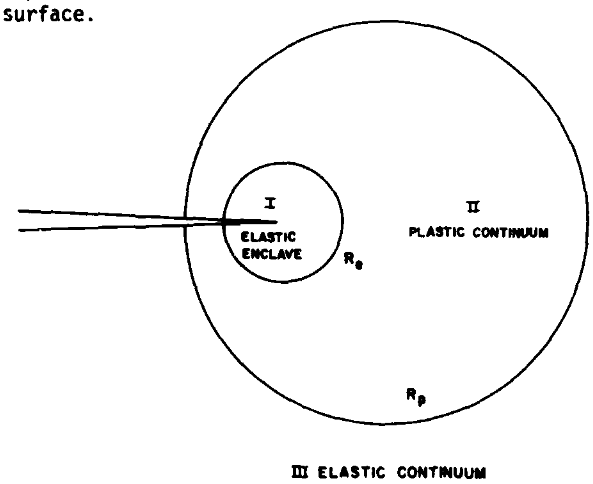

Figure 2 - Schematic of a crack surrounded by elastic and plastic regions. I, elastic enclave surrounding the crack tip of radius R_e; II, plastic continuum region of radius R_p; III, elastic region. The crack is centered in R_e, but R_p is not concentric.

and III, respectively. Finally, if the immediate local vicinity of the crack tip possesses no dislocation sources and is devoid of slip planes, then in this local region, termed by Weertman the elastic enclave, the material is again linearly elastic until the stresses become so high in the cohesive region of the crack that discrete nonlinear effects must be considered. In the following, we think of the cohesive region as characterized by a critical local K in the Barenblatt sense, and consider the crack to be an elastic singularity with this K. In Figure 2 we label the elastic enclave I.

For reasons which will become apparent in the following, we have shown all boundaries as circles. The outer boundary between II and III is characterized as the locus of points where the stress equals the critical shear stress, σ_o. The inner boundary, however, is more difficult to specify. In reference (1) we related it to the maximum dislocation density. However, the meaning of region I is best seen by considering the qualitative form of the crack opening displacement as a function of the distance from the crack tip. In Figure 3 we show in a) the displacement function for the cleavage surface for a fully brittle continuum crack, and in b) that for a continuum plastic crack. In c) we show what a crack displacement function would look like if the deformation is heterogeneous. From these figures, we should identify R_e with the distance between the discrete steps representing actual slip planes near the crack tip. Further, as a plausible procedure for solving the stress distribution problem, we shall interpret R_e as a lower "cut-off" for the plastic solution, and assume that the solution for $R > R_e$ is given by the continuum plastic solution. Within R_e, only the elastic portions of the strain and displacement functions persist, and we match elastic values of the solution on the boundary R_e. Since there are no external sources of stress on R_e, the stress must also be continuous there.

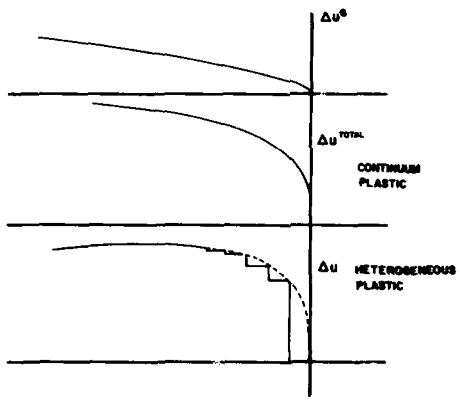

Figure 3 - a. crack opening of elastic crack; b. crack opening of fully plastic crack; c. crack opening when the plastic strain is discretized on slip planes.

Recapitulating, our procedure amounts to a recognition that the displacement is composed of two components, elastic and plastic, which normally must be coupled to determine the overall solution, but in our case must be decoupled because the plastic component undergoes a cut-off before the elastic component does. Within the confines of a continuum theory, there is no rigorous way to accomplish this decomposition, but clearly an approximate solution will be achieved if we adopt the full continuum solution of Figure 3b., separate off the elastic portion, and match an elastic crack in I to the elastic component of II at R_e.

Analytically, we follow Rice's[3] solution of the work hardening Mode III problem. We assume a constitutive work hardening law of the form

$$\sigma = \sigma_0 \left(\frac{\gamma}{\gamma_0}\right)^n \tag{3}$$

where σ and γ are principal stress and strain, respectively, and σ_o, and γ_o are the corresponding critical yield values. We assume the crack lies along the negative x-axis with its tip at the origin. In Rice's problem $R_e = 0$. He introduces a potential function $\Psi(\gamma,\phi)$, where ϕ is the angle between γ and the y axis, with the defining equations.

$$x = -\sin\phi \frac{\partial\Psi}{\partial\gamma} - \frac{\cos\phi}{\gamma} \frac{\partial\Psi}{\partial\phi}$$

$$y = \cos\phi \frac{\partial\Psi}{\partial\gamma} - \frac{\sin\phi}{\gamma} \frac{\partial\Psi}{\partial\phi} \tag{4}$$

which satisfies the differential equation

$$\gamma^2 \frac{\partial^2\Psi}{\partial\gamma^2} + n\gamma \frac{\partial\Psi}{\partial\gamma} + n\frac{\partial^2\Psi}{\partial\phi^2} = 0 \tag{5}$$

The boundary conditions are that

$$\left.\frac{\partial\Psi}{\partial\phi}\right]_{\phi = \pm\pi/2} = 0 \tag{6}$$

If we solve Eqn. (5) by separation of variables then Ψ must have the form

$$\Psi = \Psi(\gamma) \sin m\phi \qquad m = \pm 1, \pm 2--- \tag{7}$$

Then Eqn. (5) is solved by setting

$$\Psi_m = \gamma^p \tag{8}$$

where p satisfies the quadratic indicial equation

$$p(p-1) - np - nm^2 = 0 \tag{9}$$

For each value of m, p has two solutions, p_+ and p_-.

The general solution for Eqn. (4) is the series

$$\Psi = \sum_{m=1}^{\infty} (a_m\gamma^{p_+(m)} + b_m\gamma^{p_-(m)}) \sin m\phi \tag{10}$$

Rice was able to satisfy the boundary conditions for $\Delta u = 0$ at the crack tip and the boundary conditions on the elastic-plastic boundary by retaining only one term of the series in Eqn. (10),

$$\Psi = a_1 \frac{\sin\phi}{\gamma n} \tag{11}$$

From this, he shows

$$x = X(\gamma) + R(\gamma)\cos 2\phi$$

$$y = R(\gamma)\sin 2\phi$$

$$R(\gamma) = \frac{K^2}{2\pi\sigma_o^2}\left(\frac{\gamma_o}{\gamma}\right)^{n+1} = R_p\left(\frac{\gamma_o}{\gamma}\right)^{n+1}$$

$$X(\gamma) = \frac{1-n}{1+n} R(\gamma)$$

$$2\phi = \theta \tag{12}$$

This solution is represented by a set of nested circles. On the circumference of each circle, γ is a vector of constant magnitude, whose polar components are given by the equations

$$\gamma_R = \gamma \sin \theta/2 \qquad \sigma_R = \sigma \sin \theta/2$$
$$\gamma_\theta = \gamma \cos \theta/2 \qquad \sigma_\theta = \sigma \cos \theta/2$$
$$\theta = 2\phi \tag{13}$$

where R and θ are measured in a system of polar coordinates with the origin at the center of each circle. For different values of γ, the centers of the circles shift, being situated at $x = X(\gamma)$, relative to the crack tip. See Figure 4. The stress intensity factor, K, defines the radius of the elastic plastic boundary, R_p

$$K^2 = 2\pi\sigma_o^{\,2}R_p \tag{14}$$

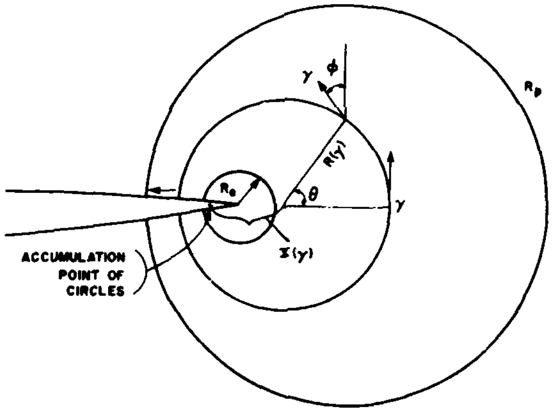

Figure 4 - Schematic drawing of crack embedded in three regions as in Figure 2. The accumulation point for the plastic circles, $R(\gamma)$, is to the left of the center of the elastic enclave. $X(\gamma)$ is the distance from the accumulation point to the center of the circle, $R(\gamma)$. An arbitrary R $(\gamma)$is shown with the principal strain γ on the circumference, and coordinates, R, θ.

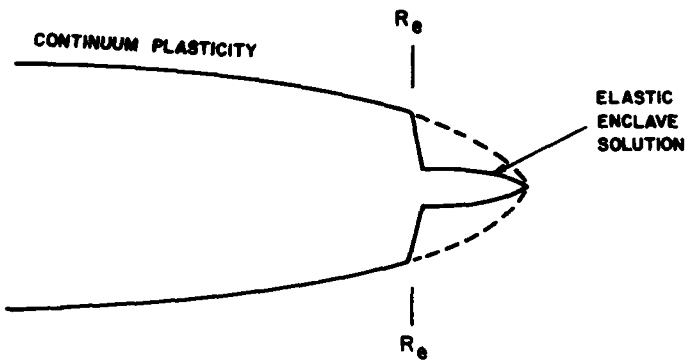

Figure 5 - The continuum plastic solution in our model is replaced by an elastic crack opening at the tip with a discontinuous displacement ar R_e corresponding to concentrating all the continuum dislocations within R_e onto its circumference. The slope of the displacement at R_e is continuous.

Equations (4) through (14) describe a fully continuum plastic crack. In our problem, we assume at a value of R given by R_e, the plastic solution is cut off, and the solution is continued within R_e by a purely elastic solution. (See Figure 5.) We now write the solution for a purely elastic crack situated at the geometric center of R_e. With the usual boundary conditions $\Delta u = 0$ at the tip, and zero stress on the crack surfaces, the solution within R_e is

$$\sigma(Z) = \frac{K_e}{\sqrt{2\pi z}} + \sum_{c=2}^{\infty} \frac{C_n}{i} (-z)^{n/2}$$

$$z = x - iy$$

$$\sigma(z) = \sigma_{32}(x,y) + i\,\sigma_{31}(x,y) \tag{15}$$

where C_n is real, and K_e is the local K for the crack tip enclosed in the elastic conclave. In polar coordinates centered at the center of R_e, the first term has the form

$$\sigma_R = \frac{K_e}{\sqrt{2\pi R}} \sin\theta/2 = \mu\gamma_R$$

$$\sigma_\theta = \frac{K_e}{\sqrt{2\pi R}} \cos\theta/2 = \mu\gamma_\theta \qquad R < R_e$$

$$u = \frac{K_e}{\mu}\sqrt{\frac{2R}{\pi}} \sin\theta/2 \tag{16}$$

where u is the component of the displacement normal to the x-y plane.

In the foregoing, we have used a variety of coordinate origins, and crack tip positions, which we wish to clarify here to avoid possible confusion for the reader. In writing the equations (4) through (14) describing the plastic solution, we had reference to an effective crack tip position shown in Figure 4 to which the continuum solutions were referred. This effective tip position was the accumulation point about which the nested circles converged as $R \to 0$. Also, we described a coordinate system for θ in Eqn. (13) which specified the orientation of the principal strain (and stress) for a particular magnitude of γ and for a given R, where $R_p > R > R_e$. This coordinate system was centered at the center of the given circle R. This center is at a distance $X(\gamma)$ from the effective position of the continuum plastic crack. Finally, we assume that a crack tip, elastic this time, is placed at the center of the circle, R_e, whose circumference defines the boundary of the elastic enclave. We note that by construction, the accumulation point of the nested circles of the plastic solution at the effective plastic crack position is not at the same position at the elastic crack at the Center of R_e. Nevertheless, the actual physical crack is assumed to be positioned at the center of R_e, and coincides with the elastic crack of equations (15) and (16). Its solution within R_e is given by equations (15) and (16), and outside R_e is given by equations (4)-(14) with boundary conditions given on R_e which we now specify.

On the circle, R_e, we set the stress of the elastic solution, Eqn. (16), equal to the stress of the plastic solution, Eqn. (13). We note that since these equations display the same angular dependence, with respect to θ on R_e, this is easily done. With the use of the third equation of Eqn. (12) and the defining relation for K, we have

$$K^2 = K_e^2 \left(\frac{R_p}{R_e}\right)^{(1-n)/(1+n)} \tag{17}$$

If we assume that the elastic crack is in equilibrium, that is, K_e obeys the Griffith relation,

$$K_e^2 = 4\,\Gamma\mu \tag{18}$$

we have

$$K^2 = 4\,\Gamma\mu \left(\frac{R_p}{R_e}\right)^{(1-n)/(1+n)} \tag{19a}$$

$$= 2\pi \left(\frac{2\Gamma\mu}{\pi}\right)^{(1+n)/(2n)} \left(\frac{1}{R_e\sigma_o^2}\right)^{(1-n)/(2n)} \tag{19b}$$

This is the result quoted in Eqn. (1).

We now compare this result with Weertman's results, Eqn. (2), which we take from Eqn. (24) of reference (2). If we rewrite this equation in our notation, we have

$$c = \eta \left(\frac{\sigma_o}{\sigma(R_e)}\right)^{(1-n)/(2n)} \tag{20}$$

Weertman's parameter p can be replaced by $1/(n+1)$, and η is a constant, $\eta<1$. Since in Mode III, $(\sigma_o/\sigma(R_e)) = (R_p/R_e)^{n/n+1}$, we have

$$c = n \left(\frac{R_e}{R_p}\right)^{(1-n)/2(1+n)} \tag{21}$$

Except for the constant, η, this is precisely the result (19a) of this paper.

Thus, a direct comparison of Eqn. (1) with Eqn. (2) shows they are based on precisely the same results. They differ in that in Eqn. (1), we have not taken the ratio, R_p/R_e, to be a constant. There, R_p is written in terms of its dependence on Γ, and R_e is left as an undetermined parameter. Thus the discrepancy in the published results apparently does not revolve around a different use of the boundary conditions, but rather depends upon how the ratio R_p/R_e is interpreted.

In our opinion, there is no physical reason why R_p and R_e will depend on Γ in precisely the same manner. Hence, the dependence of K on Γ should be more complex than a simple linear relation. From the discussion here, however, it is clear that no theory yet exists from which R_e can be related to Γ, and in fact, from Eqn. (19a), experimental measurement of $K(\Gamma)$ would provide evidence on the point.

III. Alternative Continuum Solutions

In this section, we explore Weertman's suggestion that the total displacement at R_e should be continuous, as well as σ. However, rather than follow Weertman's Mode I development, we believe it is preferable to remain in Mode III, where the analysis is more accessible, and compare Weertman's prescription with the results of Section II of this paper.

Following Weertman, we write a solution in region II of the form $\gamma^{II} \propto R^{-p}$, and then match appropriate power series solutions in regions I and III to this region II plastic solution. We believe that a simplification is possible which is suggested by the fact that since $R_p >> R_e$, the solutions at R_p and the matching process there should be effectively independent of the details of the matching on R_e. That is, microstructural effects are fully damped out when $R \sim R_p$. Thus, in Region III, we follow Rice[3], and take the elastic solution in III to be

$$\sigma_R^{III} = \frac{K}{\sqrt{2\pi R}} \sin \theta/2$$

$$\sigma_\theta^{III} = \frac{K}{\sqrt{2\pi R}} \cos \theta/2$$

$$R > R_p \tag{22}$$

R and θ are the polar coordinates relative to an origin fixed at the center of R_p. That is, in region III, the crack tip has an effective position at the center of the plastic circle, R_p. Also, the plastic solution at R_p should consist only of the expressions for Ψ used by Rice[3], which was the first term m=1, of the expansion, Eqns. (7) and (10).

However, if we are to match the total displacement and stress at R_e, we shall have to consider additional terms in either the expansion, Eqn. (15), for the elastic region I or for the plastic region II, or both. Further, if we include higher order terms in Eqn. (10), they must disappear with R faster than the dominant term, $\gamma \propto R^{-1/(1+n)}$. An inspection of the two expansions, Eqns. (15) and (10), however, shows that in order to match solutions on some boundary corresponding to R_e, <u>all</u> the terms in either or both of these expansions will have to be used, because their angular dependencies are incommensurate. In addition, the matching surface may have a complex shape as well. This was a point noted by Weertman, who in the face of this difficulty, simply ignored the angular dependence, and matched stress and displacement with the smallest possible number of terms in the R parts only. Following this path, we can again, as Weertman did, keep only the dominant solution in region II, and add one term to the elastic expansion in region I. at $R = R_e$ we have

$$\sigma^{II} = \sigma_0 \left(\frac{R_p}{R_e}\right)^{n/(1+n)} \qquad \sigma^{I} = \frac{K_e}{\sqrt{2\pi R_e}} + A\sqrt{R_e}$$

$$u^{II} = 2\gamma_0 \left(\frac{R_p}{R_e}\right)^{1/(1+n)} R_e \qquad u^{I} = \frac{K_e}{\mu}\sqrt{\frac{2R_e}{\pi}} + \frac{2}{3}\frac{A}{\mu} R_e^{3/2} \tag{23}$$

These expressions are obtained from Eqns. (12), (13), (20), and (16), but we have added the term in A to the dominant K_e terms in σ^I and u^I. Writing

$$\sigma_0 \left(\frac{R_p}{R_e}\right)^{\frac{n}{1-n}} = \frac{K_e}{\sqrt{2\pi R_e}} + A\sqrt{R_e}$$

$$2\gamma_0 \left(\frac{R_p}{R_e}\right)^{\frac{1}{1+n}} R = \frac{K_e}{\mu}\sqrt{\frac{2R_e}{\pi}} + \frac{2}{3}\frac{A}{\mu} R^{3/2} \tag{24}$$

for the matching conditions on R_e, we finally obtain

$$\left(\frac{K_e}{K}\right)^2 = \frac{3}{2}\left(\frac{R_p}{R_e}\right)^{\frac{1-n}{1+n}} - \frac{1}{2}\left(\frac{R_e}{R_p}\right)^{\frac{1-n}{1+n}} \tag{25}$$

Since $R_p >> R_e$, Eqn. (25) shows that <u>the elastic crack is not shielded by the deformation field</u>, and $K_e > K$, which is a non-physical result. This anomaly is caused by the large value of the elastic K-field required to match the large continuum displacement at R_e. Also, A has a large negative value, which requires a correspondingly large counterbalancing K-field.

IV. Conclusion

The published results of Thomson and Weertman are shown to reflect equivalent physical pictures at the crack tip. Their difference is caused by different implications concerning the material parameters which control the size of the elastic enclave. However, as there has yet been no adequate treatment of the size of the elastic enclave, we believe it would be fruitful at this stage to highlight it as a fruitful field for experimental study.

In effect, what our treatment does is to modify the dislocation distribution and their sources in the region of the crack tip from that of the plastic continuum solution. The dislocation distribution implied by the plastic solution is often termed the "geometric" distribution, and can be calculated from the curl of the local stress function, σ. The geometric distribution calculated in this way does not have a strong singularity at the crack tip, and it is therefore possible to modify it by concentrating the distribution totally on the circumference of R_e without markedly changing the overall K. However, modifying the manner in which these dislocations are introduced into the medium from the cleavage surface, for example, by assuming that real dislocations form discrete slip lines on crystallographic planes, will modify the detailed shape of the crack opening near its tip. This is precisely the point of our discussion as shown in Figures 4 and 5, of course. On the other hand, these modifications in crack opening near the tip do not change the total crack opening displacement (COD), provided the total integrated burgers vector of the screening dislocation distribution is unchanged, as shown again in Figures 5 and 6. Finally, such changes within R_e amount to a change in the local stress-strain law, which will modify the stress and strain solutions within R_e markedly. Indeed, in our solution, we have recognized this important change by specifying an elastic stress-strain law within R_e, with the results derived in section II. A more adequate theory, of course, would deal in a unified way with the elastic and plastic regions by means of actual discrete distributions of dislocations. Thus, although we believe we have some of the physics right in Eqns. (19a and 19b), we are less sure of the details, and we have lumped some large questions into our parameter, R_e.

In summary, we do not believe we yet have an adequate theory of the toughness exhibited by sharp cracks in a medium of modest ductility. The problem remaining relates to how one makes a rigorous transition from the plastic continuum to the discrete description of dislocation distributions at the crack tip. The cut-off procedure of this paper is at best ad hoc, and since the cutoff radius, R_e, appears in an essential way in the results (i.e., leaving open the dependence of R_e on Γ, etc.), we believe a proper theoretical treatment still lies in the future.

We have profited from discussions of this problem with E. Hart, I. Lin, and J. Weertman.

References

1. R. Thomson, J. Matls. Sci. 13, 128 (1978).
2. J. Weertman, ibid. 15, 1306.
3. J. Rice, J. Appl. Mech. 34, 287 (1967). See also Fracture, Vol. II, pp. 256, Ed. H. Liebowitz, Academic Press, New York (1968).

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	16810.1-MS
2. GOVT ACCESSION NO.	N/A
3. RECIPIENT'S CATALOG NUMBER	N/A
4. TITLE (and Subtitle)	Dislocation Shielding of a Crack in a Quasi Continuum Approximation
5. TYPE OF REPORT & PERIOD COVERED	Reprint
6. PERFORMING ORG. REPORT NUMBER	N/A
7. AUTHOR(s)	R. M. Thomson E. R. Fuller, Jr.
8. CONTRACT OR GRANT NUMBER(s)	ARO 36-82
9. PERFORMING ORGANIZATION NAME AND ADDRESS	National Bureau of Standards Washington, DC 20234
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	N/A
11. CONTROLLING OFFICE NAME AND ADDRESS	U. S. Army Research Office P. O. Box 12211 Research Triangle Park, NC 27709
12. REPORT DATE	Oct 81
13. NUMBER OF PAGES	11
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	
15. SECURITY CLASS. (of this report)	Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	

16. DISTRIBUTION STATEMENT (of this Report)

Submitted for announcement only.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

82 04 07 066

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)