Disocclusion : a variational approach using level lines

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Abstract— **Object recognition, robot vision, image and film restoration may require the ability to perform disocclusion. We call disocclusion the recovery of occluded areas in a digital image by interpolation from their vicinity. It is shown in this paper how disocclusion can be performed by means of the level-lines structure, which offers a reliable, complete and contrast-invariant representation of images. Level-lines based disocclusion yields a solution that may have strong discontinuities. The proposed method is compatible with Kanizsa's *amodal completion* theory.**

Keywords— **Image interpolation, Disocclusion, Inpainting, BV, Level lines, Kanizsa, Amodal completion.**

I. Introduction

We address in this paper the following interpolation problem: given a digital image corrupted by spots whose shapes and positions are known, how to restore the non-texture information from the vicinity of the spots? We call *disocclusion* this restoration process since spots can obviously be considered as occlusions. We proposed in [14] a disocclusion method based on the continuation of the level lines "broken" by the spots. From a functional viewpoint, our method consists in the minimization of a relaxed, level-lines based formulation of the criterion

$$\int |\nabla u|(\alpha + \beta \big|\mathrm{div}\, \frac{\nabla u}{|\nabla u|}\big|^p), \qquad p \geq 1,\, \alpha > 0,\, \beta \geq 0.$$

We gave in [14] a sketch of the algorithm in the case $p = 1$. Our purpose in this paper is to give a theoretical justification of the method when $p = 1$ and a detailed description of the algorithm.

Our approach is closely related to a natural ability of the human visual system, the so-called *amodal completion* process. In a natural scene, an object is seldom totally visible. It is generally partially hidden by other objects. But our perception is, under certain geometric conditions, able to "reconstruct" the whole object by interpolating the missing parts. This is illustrated in Figure 1 : the first drawing shows four black independent "butterflies"; the second drawing is obtained from the first by adding four white rectangles with black borders. The visual reconstruction is such that we "see" black disks partially hidden by the rectangles. A totally different visual reconstruction is shown in the third drawing where the addition to the "butterflies" of new lines simulating a white cross make us "see" a black rectangle occluded by the cross.

This ability of human vision to reconstruct partially hidden objects has been widely studied by psychophysicists, particularly by Gaetano Kanizsa [11]. It appears that continuation of objects boundaries plays a central role in the disocclusion process. This continuation is performed between T-junctions, which are points where image edges form a "T" as illustrated in the fourth drawing of Figure 1. The *amodal completion* is precisely this extension process of visible edges "behind" occluding objects and between T-junctions. According to psychophysicists, the continuation process is such that restored edges must be as smooth and straight as possible, which explains why our perception restores circles in the second drawing and a rectangle in the third drawing of Figure 1.

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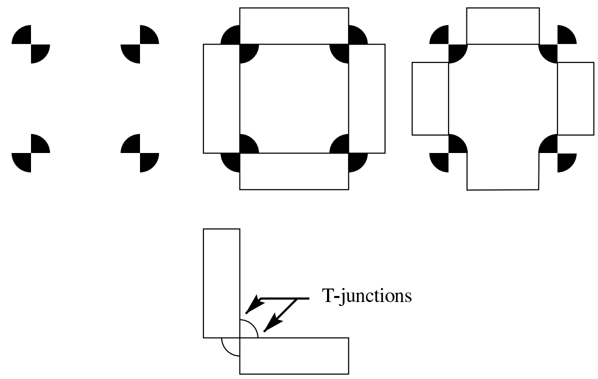

Fig. 1. The amodal completion process is related to T-junctions : image edges are extended "behind" occlusions and between T-junctions.

In [17], Nitzberg, Mumford and Shiota deduced from the amodal completion principles a method for detecting and recovering occluded objects in a still image within the framework of a segmentation and depth computing algorithm. Their method is based on the detection of image edges and T-junctions followed by a variational continuation process, which consists in connecting T-junctions of approximatively the same level by a new edge with minimal length and curvature. This interpolating edge is computed as a spline approximation of the Euler's elastica, which, by definition, minimizes $\int (1+\kappa^2)ds$ where s denotes the arc-length and κ the curvature along the edge. Such energy has a physical justification when the edges to be connected have similar directions at the corresponding T-junctions. However, the Nitzberg-Mumford-Shiota's method can be applied only to highly segmented images with few T-junctions and few possible continuations between them, so that automatic disocclusion of natural images is not directly possible. Indeed, it is a natural requirement that all the image information in the neighborhood of the occlusion be taken into account and not only the partial information issued from a segmentation process. Moreover, the dependence of the Nitzberg-Mumford-Shiota's method on a previous edge detection stage is particularly a major drawback. Indeed, edges are very sensitive to contrast changes and do not offer a complete representation of the image. Now, the independence with respect to contrast changes is crucial. According to the Gestalt school, and particularly M. Wertheimer [20], human vision is essentially sensitive to the only ordering of gray levels in an image. The intensity difference between two pixels is not a reliable characterization of an image since it arbitrarily depends on the sensor used for image capture as well as on the illumination conditions.

The method we describe in this paper can be viewed as a generalization of the Nitzberg-Mumford-Shiota's variational continuation framework to the level-lines structure, which is more precise and reliable than edges. From a theoretical viewpoint, we will consider an image as a function of bounded variation (BV) in the plane, that is a function in $L^1(\mathbb{R}^2)$ whose distributional derivative is a vector-valued Radon measure on $\mathbb{R}^2$. Readers may refer to [1], [8] for a detailed study of the BV space. This model is well adapted to our problem for it seems reasonable to view an image as an integrable function which may have discontinuities, but concentrated on rectifiable curves.

II. Level lines continuation

Recent works [9] have emphasized the importance of level lines for image understanding and representation. Let $u(x)$ denote the gray level of an image u at a point x. We define level lines as the boundaries of upper level sets, given at each gray level t by $X_t u = \{x,\ u(x) \geq t\}$. In contrast to the edge representation, the family of level lines is a complete representation of u, from which u can be reconstructed thanks to the equality $u(x) = \sup\{t : x \in X_t u\}$. In addition, it is easily seen that the level lines family is globally invariant with respect to any increasing contrast change (up to the gray level quantization effects), which again is not true for an edge representation.

The method we describe here allows to recover functions with strong discontinuities, like BV functions. Let us denote by Ω the occlusion and $\partial\Omega$ its boundary. We shall assume here that Ω has no hole. The result stated in Theorem 1 could actually be extended to the more general case of a finite number of holes but the design of an algorithm performing the disocclusion in such situation, in the framework of level lines connection, remains an open problem.

Given a function u outside Ω, our aim is to interpolate u inside Ω by performing a continuation of its level lines. Let us consider as T-junctions the intersection points between $\partial\Omega$ and the level lines of u outside Ω - this point will be clarified in the next section. Following Kanizsa and Nitzberg *et al*, we wish to interpolate the level lines between the T-junctions by means of short and not too oscillating curves. This can be formulated as a variational problem. Let us first remark that each level line intersecting $\partial\Omega$ can be associated with a positive or negative orientation, according to the orientation of ∇u along the line. Now, for each level t represented on $\partial\Omega$, denote by $(L_i^t)_{i\in I(t)}$ the corresponding set of level lines and remark that we can assume $I(t)$ to be finite with no loss of generality (see the next section). Since a level line arriving at $\partial\Omega$ must necessarily leave it, it is worth noticing that there is an even number of T-junctions associated with the family $(L_i^t)_{i\in I(t)}$ and more precisely, as many T-junctions with positive and negative orientation. Then our problem consists in finding an optimal set of curves $(\Gamma_j^t)_{j\in J(t)}$, with $\operatorname{card} J(t) = \operatorname{card} I(t)/2$, pairwise connecting the T-junctions with same level and same orientation, and minimizing the following energy :

$$E = \int_{-\infty}^{+\infty} \sum_{j\in J(t)} \left(\int_{\Gamma_j^t} (\alpha + \beta|\kappa|^p) ds + \text{angles} \right) \tag{1}$$

where α, β are positive constants and "angles" denotes the sum at both endpoints of Γ_j^t of the angles between the direction of Γ_j^t and the direction of the associated level lines. The parameter p can be seen as a generalization of the curvature exponent appearing in the Euler's elastica energy. The method we describe and theoretically justify in this paper gives the optimal solution in the case $p = 1$, which corresponds to a continuation of level lines by polygonal paths. Similar theoretical arguments can be used to prove the existence of a minimal solution whenever $p > 1$, still within the framework of level lines continuation in a simply connected domain. However, unlike for $p = 1$, it is still an open problem how to find an optimal set of curves pairwise connecting the T-junctions and minimizing the criterion (1). Briefly speaking, though it is known how to compute a minimal curve with respect to the single energy $\int(1 + |\kappa|^p)ds$, one actually needs to minimize a global energy corresponding to a *set* of curves that do not cross each other. And, in contrast with the case $p = 1$, it is no more equivalent to minimize individually the energy of each curve.

Our approach, from both a theoretical and practical viewpoint, relies on the continuation of level lines. However, it is a natural question whether disocclusion could be achieved minimizing a criterion depending on functions rather than curves. Actually, using a simple change of variables and omitting the angular term, it is easily seen that (1) rewrites

$$\int_\Omega |\nabla u| (\alpha + \beta \left|\operatorname{div} \frac{\nabla u}{|\nabla u|}\right|^p) dx, \tag{2}$$

which is exactly the criterion we mentioned in the introduction. In a paper in preparation, we prove the existence of a minimal solution with respect to another relaxed formulation of this criterion, depending explicitly on functions rather than level lines. In contrast with the work presented here, occlusions need no more be simply connected domains. This is also the case in [3] where Chan, Kang and Shen give interesting results related to functional (2). In particular, they derive the Euler-Lagrange equation for $p > 1$ and propose numerical schemes and interesting computational examples. In contrast with our work, curvy level lines can be recovered. However, the equation is of fourth order thus the stability and the convergence speed are delicate issues. Another problem is the difficulty to recover sharpness, as pointed out by the authors. Actually, it is still unclear whether the solutions to this equation may present discontinuities, which is a crucial issue in the context of interpolation of BV functions. In contrast, our approach is well adapted to the recovery of discontinuities, from both a theoretical and a practical viewpoint.

Another method for occlusion removal was proposed in [4], consisting in the minimization of the total variation $\int_{\Omega'} |\nabla u| dx$ (Ω' is an open set strictly containing Ω) coupled with a forcing term outside the occlusion. The total variation term can be seen as a simplification of our global variational formulation above, since no second-order term is involved. As a consequence, only straight lines - or polygonal paths - can be recovered within the occlusion. This is also the case for our method in the case $p = 1$ but remark that the curvature term in our energy induces a contribution of the angular variation along each restored line. Thus, it coerces the connection of facing lines which make more or less the same angle with the occlusion boundary and gives a result closer to what we could expect from amodal completion. Actually, the minimizers obtained with our energy really differ from those produced by the only total variation when the occlusion is non convex, which happens very often for occlusions defined on a discrete grid. As to the forcing term, it is a major difference with our method for we did not address at all the issue of noise outside the region to be filled. Actually, our approach is obviously not robust at all when noise is added. Remark however that this is not a really problematic issue since one can always apply a prior denoising filter outside the occlusion before performing disocclusion.

Finally, still in the context of deterministic methods based on a variational formulation, a very interesting and elegant approach for occlusion removal is proposed in [2]. It is closely related to the so-called *inpainting* technique in art restoration. The basic idea consists in filling-in the occlusion by a smooth continuation of information in the direction of level lines. This can be formulated as a variational problem involving a relaxed formulation of (2) where the term $\operatorname{div} \frac{\nabla u}{|\nabla u|}$ is replaced with $\operatorname{div} \theta$, θ being a vector field "approximating" the normal $\frac{\nabla u}{|\nabla u|}$ (see [2] for details). This approach has mainly two advantages : from a theoretical viewpoint, the use of a vector field makes the problem better posed and the relaxed formulation easier to handle. From a practical viewpoint, the corresponding PDE's for the steepest descent method are well-posed, admit solutions with discontinuities and are of order three. Overall, this kind of approach is very promising.

Until now, we did not mention any work based on a non deterministic approach. However, there is a huge literature on the subject, and particularly on texture synthesis and restoration. Let us mention for instance the approach developed in [7], [19]. In these papers, occlusions are iteratively filled-in using a sophisticated "copy and paste" under the assumption that texture may be modelled as a Markov Random Field. This method gives impressive results for texture synthesis but, by nature, does not allow to recover the geometric information in a *reliable* way. In other words, some geometry is recovered but does not issue from an attempt to guess the shape of the occluded objects. In contrast, our approach is more reliable from this viewpoint but is not adapted at all for texture recovery. Recall indeed that our goal is to mimic the amodal completion process, which is essentially a geometric process : it allows the recovery of objects contours using some geometric information - the T-junctions and the conformation of contours outside the occlusion - but does not involve any statistics on the intensity. So is there a choice to make between deterministic and non deterministic methods ? We believe that this question has no sense due to the nature of an image. Indeed, it is now well admitted that an image contains two types of information : a geometric information related to the geometry of the objects contained in the image, and a statistical information that represents the texture. Thus, it is reasonable to think that deterministic and non deterministic approaches should be combined for a good reconstruction of an occluded region. This paper is a contribution to the geometric aspect of the reconstruction and one can easily think to a way of combining it with a method like in [7], [19]. In a first step, one could connect only the most significant level lines for the recovery of image geometry (see [6] for the notion of significant level lines). Then the texture could be restored applying the methods of [7], [19] in the regions enclosed by the interpolated lines.

III. Mathematical analysis of the method

The occlusion Ω is supposed to be an open, bounded and simply connected set such that $\partial\Omega$ is a rectifiable Jordan curve. In this section, we shall prove that there exists an optimal set of curves connecting the T-junctions with respect to the criterion (1) with $\alpha = \beta = 1$. The result can be equally proven for any $\alpha > 0$ and $\beta \geq 0$ with exactly the same arguments.

Since our image is modeled as a BV-function, T-junctions must be carefully defined. First, let us fix some notations (readers may refer to [1] for more details): $d_{\mathcal{H}}$ denotes the usual Hausdorff distance, $\mathcal{H}^N$ is the N-dimensional Hausdorff measure. For some countably $\mathcal{H}^m$-rectifiable subset K of $\mathrm{I\!R}^{\mathrm{N}}$, $m \leq N$, the lower m-dimensional density of K at $x \in \mathrm{I\!R}^{\mathrm{N}}$ is

$$\underline{\mathrm{D}}^m(x,K) = \liminf_{r \downarrow 0} \frac{\mathcal{H}^m(B_r(x) \cap K)}{w_m r^m},$$

where w_m denotes the volume of the unit ball in $\mathrm{I\!R}^m$ and $B_r(x)$ is the ball in $\mathrm{I\!R}^{\mathrm{N}}$ with radius r and centered at x. The upper m-dimensional density $\overline{\mathrm{D}}^m$ is defined analogously replacing lim inf by lim sup. In case both limits coincide we define

$$\mathrm{D}^m(x,K) := \underline{\mathrm{D}}^m(x,K) = \overline{\mathrm{D}}^m(x,K)$$

An important property of countably $\mathcal{H}^m$-rectifiable sets is that

$$\begin{array}{ll} \mathrm{D}^m(x,K) = 0 & \text{for } \mathcal{H}^m\text{-a.e. } x \in \mathrm{I\!R}^{\mathrm{N}} \setminus K \\ \mathrm{D}^m(x,K) = 1 & \text{for } \mathcal{H}^m\text{-a.e. } x \in K \end{array}$$

The measure-theoretic boundary of K is the set

$$\partial_M K := \{x \in \mathrm{I\!R}^{\mathrm{N}} : \overline{\mathrm{D}}^m(x,K) > 0 \text{ and } \underline{\mathrm{D}}^m(x,K) > 0\}$$

For the special case of sets of finite perimeter in $\mathrm{I\!R}^{\mathrm{N}}$ - the sets whose characteristic function is in $\mathrm{BV}(\mathrm{I\!R}^{\mathrm{N}})$ (see [1], [8]) - the reduced boundary $\partial^* K$ is made of all those points where the generalized inner normal to K exists. In addition, $\partial^* K \subset \partial_M K$ and $\mathcal{H}^{N-1}(\partial_M K \setminus \partial^* K) = 0$.

Turning back to our problem, the following lemma ensures that there exists a simple rectifiable curve Γ arbitrarily close to Ω such that the one-dimensional restriction of u to Γ has bounded variation. In addition, claims (ii), (iii), (iv), (v) and (vi) roughly mean that we can find a dense set of values (λ) assumed by u such that :

1. the lines of level λ "intersecting" the occlusion boundary have finite length (claim (ii)).
2. they can be associated with a finite number of points on the occlusion boundary (claims (iii), (v)) where they have unit density (claim (vi)).
3. these points can be approximated by sequences of points associated with lower and greater levels (claim (iv), where convergence is meant with respect to $\mathcal{H}^1$).

All these properties allow a reliable definition of T-junctions and shall be used for proving the existence of an optimal disocclusion.

Lemma 1: Let $\Omega \subset \mathrm{I\!R}^2$ be an occlusion satisfying the assumptions above and $u \in \mathrm{BV}(\mathrm{I\!R}^2 \setminus \overline{\Omega})$. There exists an open set $\Omega' \supset\supset \Omega$ arbitrarily close to Ω such that $\Gamma := \partial\Omega'$ is a C^∞ Jordan curve and, denoting by $\tilde{u}$ the restriction of u to Γ,

$$\begin{array}{lll} & \tilde{u} \in \mathrm{BV}(\Gamma) & (i) \\ \exists \mathcal{R} \subset \mathrm{I\!R}, & \mathcal{H}^1(\mathrm{I\!R} \setminus \mathcal{R}) = 0 \quad \text{and} & \\ \forall \lambda \in \mathcal{R}, & \bullet\ \mathcal{H}^1(\partial_{\mathrm{M}} X_\lambda u) < +\infty & (ii) \\ & \bullet\ \mathcal{H}^0(\partial_{\mathrm{M}} X_\lambda \tilde{u}) < +\infty & (iii) \\ & \bullet\ X_\lambda \tilde{u} = \lim_{\mu \to \lambda} X_\mu \tilde{u} & (iv) \\ & \bullet\ \mathcal{H}^0(\partial_{\mathrm{M}} X_\lambda u \cap \Gamma) < +\infty & (v) \\ & \bullet\ \forall x \in \partial_{\mathrm{M}}(X_\lambda \tilde{u}),\ D^1(x, \partial_{\mathrm{M}}(X_\lambda u)) = 1 & (vi) \end{array}$$

If $x \in A_\lambda := \partial_{\mathrm{M}}(X_\lambda \tilde{u})$ for some $\lambda \in \mathcal{R}$, we say that x is an *admissible T-junction.*

The reader may refer to [13] for a proof of this lemma, based on the properties of BV functions and sets of finite perimeter. In the sequel, we call *admissible occlusion* any set Ω' like above. According to the lemma, we can deduce an admissible occlusion from any occlusion.

Let x be an admissible T-junction on Γ. In view of the previous lemma, we can define for almost every λ such that $x \in A_\lambda$ an average direction

$$\nu_\lambda(x) := \frac{1}{\mathcal{H}^1(B \cap \partial^* X_\lambda u)} \int_{B \cap \partial^* X_\lambda u} \nu_{X_\lambda u}\, d\mathcal{H}^1$$

where $B := B_{r_0}(x)$, r_0 is such that $d(\Gamma, \Omega) > r_0$ and $\nu_{X_\lambda u}$ denotes the generalized inner normal at every point of the reduced boundary $\partial^* X_\lambda u$. Without loss of generality, we can assume that the integral goes over the measure-theoretic connected component of x within $B \cap \partial^* X_\lambda u$, that is the set $C \subset B \cap \partial^* X_\lambda u$ such that for $\mathcal{H}^1$-almost every $y \in C$, $D^1(y, C) = 1$ and C contains a curve joining x to y. Consequently, each admissible T-junction x is associated for almost every λ such that $x \in A_\lambda$ with an average direction $\nu_\lambda(x)$ and the orientation $o_\lambda(x) = \pm 1$, which refers to the orientation of the normal along C. We take as a convention that when C is watched from the occlusion boundary, $o_\lambda = 1$ if ν_λ points toward the left of the curve and

$o_\lambda = -1$ otherwise. Now, recall from Lemma 1 that for every $\lambda \in \mathcal{R}$, A_λ is a finite set. Moreover, we deduce from the properties of level lines that card A_λ is even and

$$\operatorname{card}\{x \in A_\lambda : o_\lambda(x) = 1\} = \operatorname{card}\{x \in A_\lambda : o_\lambda(x) = -1\}$$

In the sequel we shall denote

$$\begin{array}{lcl} A_\lambda^1 & := & \{x \in A_\lambda : o_\lambda(x) = 1\} \\ A_\lambda^{(-1)} & := & \{x \in A_\lambda : o_\lambda(x) = -1\}. \end{array}$$

We define $\mathcal{F}$ as the set of all measurable functions $\gamma : [0,1] \to \overline{\Omega'}$ such that $\gamma \in \mathrm{W}^{1,1}(0,1)$, $|\gamma'(t)|$ is constant and strictly positive almost everywhere on $[0,1]$ and the curvature of γ as a function of arc-length is a vector-valued Radon measure with finite total variation, or, in other words, γ' as a function of arc-length is in $\mathrm{BV}(0, \mathcal{L}(\gamma))$. In addition, we assume that the trace of γ in $\overline{\Omega'}$ is a simple curve.
Denoting by $\int_0^{\mathcal{L}(\gamma)} |\gamma''(s)|ds$ the total variation of γ' in $[0, \mathcal{L}(\gamma)]$, we get that

$$\int_0^{\mathcal{L}(\gamma)} (1 + |\gamma''(s)|)ds = \int_0^1 \left(|\frac{d\gamma}{dt}| + \frac{1}{\mathcal{L}(\gamma)} |\frac{d^2\gamma}{dt^2}| \right) dt < \infty,$$

from which we deduce that $\gamma' \in \mathrm{BV}(0,1)$. Then, we add to $\mathcal{F}$ all those curves of zero length whose endpoints coincide on Γ and finally we define

$$\mathcal{M} = \{\gamma \in \mathcal{F} : \exists \lambda \in \mathcal{R} \text{ such that } \gamma(0) \in A_\lambda^1,\ \gamma(1) \in A_\lambda^{(-1)}\}.$$

Each curve of $\mathcal{M}$ is denoted as $\gamma(x, \lambda)$ where $x = \gamma(0) \in A_\lambda^1$. Let us now emphasize that we cannot ensure γ to be continuously differentiable on $[0,1]$. Thus, the behavior of the angular terms at endpoints for a subsequence of curves is hardly controllable. The simplest way to avoid this problem is to artificially modify the boundary conditions. For every $\lambda \in \mathcal{R}$ and $x \in A_\lambda$ we define the line of level λ arriving at x as the segment $S(x, \lambda)$ of length $\eta \ll 1$ making the angle $(\nu_\lambda(x))^\perp$ with Γ at x. With each curve $\gamma \in \mathcal{M}$ related to the level λ we associate the curve $\tilde{\gamma} : [0, \mathcal{L}(\gamma) + 2\eta] \to \mathbb{R}^2$ with respect to arc-length such that

$$\tilde{\gamma} = \begin{cases} S(\gamma(0), \lambda) & \text{on } [0, \eta] \\ \gamma & \text{on } [\eta, \mathcal{L}(\gamma) + \eta] \\ S(\gamma(1), \lambda) & \text{on } [\mathcal{L}(\gamma) + \eta, \mathcal{L}(\gamma) + 2\eta] \end{cases}$$

and we define the energy of γ as

$$E(\gamma) = \int_0^{\mathcal{L}(\gamma)+2\eta} (1 + |\tilde{\gamma}''(s)|)ds.$$

Remark that

$$E(\gamma) \leq \int_0^{\mathcal{L}(\gamma)} (1 + |\gamma''(s)|)ds + 2\eta + 2\pi$$

so that $\gamma \in \mathrm{W}^{1,1}(0,1)$ and $\gamma' \in \mathrm{BV}(0,1)$ imply $E(\gamma) < +\infty$. In the sequel, unless specified, we shall implicitly deal with the extension $\tilde{\gamma}$ of any curve $\gamma \in \mathcal{M}$.

Let us consider a set of curves in $\mathcal{M}$ connecting the admissible T-junctions on Γ two by two – or possibly with themselves – and such that two different curves do not cross within Ω'. Remark that the non-crossing property holds as well at curves endpoints in the sense that the sign of $\tilde{u}'$ along Γ determines the relative position of two curves starting at x. Now, observe that by Lemma 1, each level $\lambda \in \mathcal{R}$ is associated with a finite number of curves $(\gamma_{i,\lambda})$ where $i \in I$ and card $I < +\infty$. For any $i \in I$, $\gamma_{i,\lambda}$ induces a partition of Ω' in two sets. Walking the curve $\gamma_{i,\lambda}$ from $\gamma_{i,\lambda}(0)$, we define $X_{i,\lambda}$ as the left set if $o_\lambda(\gamma_{i,\lambda}(0)) = 1$ and the right set otherwise. Then we define $X_\lambda := \cup_{i \in I} X_{i,\lambda}$ and the reconstructed function u_d can be obtained by setting for every $x \in \Omega$,

$$u_d(x) := \sup\{\lambda \in \tilde{\mathcal{R}} : x \in X_\lambda\},$$

where $\tilde{\mathcal{R}}$ is a countable and dense subset of $\mathcal{R}$. It is straightforward that u_d is a measurable function on Ω whose upper level sets coincide with the X_λ's up to Lebesgue negligible sets. Finally we define the reconstructed function associated with u as

$$u_r(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^2 \setminus \overline{\Omega'} \\ u_d(x) & \text{if } x \in \Omega' \end{cases}$$

We call disocclusion of u with respect to Ω any reconstructed function u_d obtained like above and such that the corresponding set of curves, denoted by $\mathcal{D}$, has finite total energy $E(\mathcal{D})$ with

$$E(\mathcal{D}) = \int_{\mathcal{R}} \sum_{x \in A_\lambda^1} E(\gamma(x, \lambda)) d\lambda \tag{3}$$

where we restrict to A_λ^1 in order to avoid redundancy. For simplicity we shall also call $\mathcal{D}$ a disocclusion. Of course, this definition makes sense only when there are T-junctions. Otherwise, $\tilde{u}$ is constant on $\partial\Omega'$, say $\tilde{u} \equiv \lambda_0$, and the disocclusion is naturally defined as

$$u_r(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^2 \setminus \overline{\Omega'} \\ \lambda_0 & \text{if } x \in \Omega' \end{cases}$$

Theorem 1: Under the assumptions above, let $\Omega \subset \mathbb{R}^N$ be an occlusion and $u \in \mathrm{BV}(\mathbb{R}^2 \setminus \overline{\Omega})$ such that $|u| < M$. Then there exists a disocclusion of u with minimal energy and the extended function u_r is in $\mathrm{BV}(\mathbb{R}^2)$.

Proof: Assuming that there are T-junctions (otherwise the result is trivial), the existence of an optimal disocclusion can be proven in five steps, starting from an admissible occlusion Ω' containing Ω and given by Lemma 1. The full details of the proof can be found in [13].
Step#1 : there exists a trivial (non optimal) disocclusion $\mathcal{D}_0$.

This can be checked by simply setting $u_d \equiv -M$ in Ω' and using the fact that the admissible occlusion boundary $\Gamma := \partial\Omega'$ is smooth. Accordingly, one can define a minimizing sequence $(\mathcal{D}_n)_{n \in \mathbb{N}}$ of disocclusions, each disocclusion $\mathcal{D}_n$ being made of curves $\gamma^n(x, \lambda)$ where $\lambda \in \mathcal{R}$ and $x \in A_\lambda^1$.
Step#2 : there exists a subsequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$, and a countable and dense set $\Lambda \subset \mathcal{R}$ such that

$$\forall \lambda \in \Lambda,\ \forall x \in A_\lambda^1,\ \sup_m E(\gamma^m(x, \lambda)) < +\infty$$

This follows by a diagonal extraction argument.
Step#3 : there exists a subsequence $(\mathcal{D}_k)_{k \in \mathbb{N}}$ of $(\mathcal{D}_m)$ such that $\forall \lambda \in \Lambda$ and $x \in A_\lambda^1$, $\gamma^k(x, \lambda)$ is converging to some curve $\gamma(x, \lambda)$. This result comes again by a diagonal extraction and the use of the relative compactness of BV in L^1. It says that the limit disocclusion can be defined for a countable set of levels. The extension to the whole set $\mathcal{R}$ is done in the following step, using the same kind of argument.
Step#4 : for almost every $\lambda \in \mathcal{R}$ and every $x \in A_\lambda^1$, the limit curve $\gamma(x, \lambda)$ can be defined as limit of curves $\gamma(x, \lambda)$ where $\lambda \in \Lambda$.
Step#5 : the limit disocclusion $\mathcal{D}$ is the disocclusion with minimal energy.

This last step is proven using the lower semicontinuity of the total variation. Finally, the fact that the reconstructed function has bounded variation comes from a trace theorem for BV functions. ■

Remark: Simple examples show that the optimal solution needs not be unique in general.

IV. A practical algorithm for disocclusion

A. Occlusion boundary computation

Recall first that we assume the occlusion to be without hole. We shall use for image representation a grid with integer and half-integer coordinates. Every image pixel is associated with a point with integer coordinates surrounded by eight points whose at least one coordinate is a half-integer. These points allow to define lines passing between image pixels and thus it is possible to represent the occlusion boundary with a polygonal line whose vertices are points with half-integer coordinates. In order to ensure that this polygonal line is a Jordan curve (i.e. without self-crossings), we shall slightly modify the coordinates of the vertices corresponding to a concavity of the occlusion. More precisely, it is easily seen that the occlusion boundary can be represented by a polygonal line whose vertices are points where the direction changes; but at each point of concavity, we shall replace its coordinates by those of a very close point, say at a distance less than 10^{-3}, outside the occlusion. Let us illustrate this with an example. In Figure 2 the occlusion to remove corresponds to the three white pixels. The large black disks are the centers of image pixels whereas the small black disks correspond to the points whose at least one coordinate is half-integer. The occlusion boundary is the thick solid line. Starting from the point $(1.5, 3.5)$, we can build the list of all those points where the boundary direction changes :

$$\{(1.5,3.5),\ (1.5,2.5),\ (2.5,2.5),\ (2.5,1.5),\ (4.5,1.5),\\ (4.5,2.5),(2.5,2.5),\ (2.5,3.5)\}\ .$$

After processing of the concavities, the list we shall keep for describing the occlusion boundary is

$$\{(1.5,3.5),\ (1.5,2.5),\ (2.4999,2.4999),\ (2.5,1.5),\ (4.5,1.5),\\ (4.5,2.5),\ (2.5001,2.5001),\ (2.5,3.5)\}\ ,$$

which corresponds to a simple and closed polygonal line. Notice that we use 8-connectedness to define the occlusion.

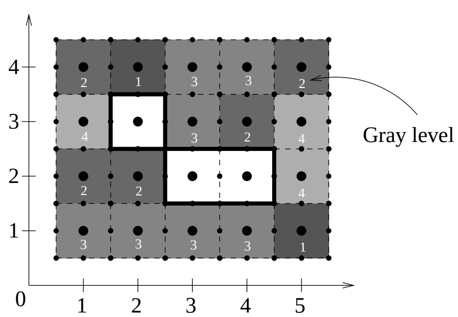

Fig. 2. Using a grid with integer and half-integer coordinates makes it easy to describe the boundary of the occlusion made of three white pixels.

B. T-junctions computation

The other structure that will be useful to describe the occlusion is the set of T-junctions. T-junctions are those points of the boundary which are a common vertex to two pixels not belonging to the occlusion and with different gray levels. The T-junctions list is built in two steps. In the first step, we simply enumerate all the points with this property; in the second step we insert in the previous list all the intermediate gray levels. In other words, a line separating the levels 1 and 4 actually corresponds to three level lines : the line separating the levels 1 and 2, the line separating 2 and 3 and, finally, the line that separates 3 and 4. Each T-junction is described by its coordinates, the two corresponding levels and the angle between the associated entering level line and the horizontal direction. However, it must be taken into account that the direction of the corresponding level line is not reliable due to discretization and noise. It is therefore better to compute an average direction of the level line on a neighborhood around the T-junction, say a ball with radius four.

C. Interpolation rules and interpolation energy

The disocclusion principle we shall use consists in pairwise connecting the T-junctions with respect to some rules related to the mathematical properties of level lines :

1. Two T-junctions can be connected by a line if they are associated with the same gray levels and have the same orientation so that the gradient orientation does not change along the line. In other words, a line cannot twist. Two T-junctions with these properties are said *compatible*.
2. Two level lines connecting two pairs of T-junctions cannot cross.

Both rules limit the number of T-junctions that can be connected with a given one. As an example, we have listed in the table below all the T-junctions on the occlusion boundary of Figure 2. From the first constructing rule we deduce two distinct columns and a T-junction from one column can be connected only with a T-junction of the other column whose gray levels are the same.

TABLE I

Every T-junction in the first column can be associated with a junction with the same gray levels in the second column.

Coord.	Levels		Coord.	Levels	
(1.5 ; 3.5)	1	2	(4.5 ; 1.5)	2	1
(4.5 ; 1.5)	1	2	(2.5 ; 3.5)	2	1
(1.5 ; 3.5)	2	3	(1.5 ; 2.5)	3	2
(2.5 ; 1.5)	2	3	(4.5 ; 1.5)	3	2
(4.5 ; 1.5)	2	3	(4.5 ; 2.5)	3	2
(3.5 ; 2.5)	2	3	(2.5 ; 3.5)	3	2
(1.5 ; 3.5)	3	4	(1.5 ; 2.5)	4	3
(4.5 ; 1.5)	3	4	(4.5 ; 2.5)	4	3

Now, an admissible solution to our disocclusion problem is a set of level lines corresponding to a one-to-one map between the two columns, satisfying the two construction rules and such that every level line stays in the occlusion. Given a one-to-one map between the two columns, there is a very simple way to check that the second rule is satisfied, in the sense that there exists a corresponding set of lines that do not cross. Every T-junction belongs to the occlusion boundary represented as a Jordan curve. Two T-junctions that are connected induce a split of the Jordan curve in two arcs that intersect only at their endpoints. Two other T-junctions can be connected only if they belong to the same arc. Indeed, it is easily seen that otherwise any corresponding pair of level lines will cross. Therefore, every admissible solution satisfies a causality principle, in the sense that every given association between two T-junctions coerces any new association. We shall say that two pairs of compatible T-junctions are compatible between them if they can be associated with two non-crossing lines.

Denoting by $L_{i,j}$ an admissible line connecting the junctions i and j, it can be associated with the following energy,

$$C_{i,j} = \int_{L_{i,j}} (\alpha + \beta|\kappa|)ds, \quad \alpha > 0, \beta \geq 0$$

The term $\int_{L_{i,j}} |\kappa| ds$ denotes the angular total variation along the line, to which we implicitly add the angles at i and j between $L_{i,j}$ and the level lines to be continued. Among all the lines connecting i and j and living in the occlusion domain, it is easy to check that the geodesic path is the line with minimal energy $C_{i,j}$. By geodesic path, we mean here the shortest line connecting i and j and contained in the closure of the occlusion domain. Since this closure is supposed to be simply connected, the geodesic path between i and j is unique. In addition, it is easily seen that two pairs of T-junctions compatible between them are associated with two geodesics that do not cross.

Let us now consider all the sets of level lines pairwise connecting the T-junctions with respect to the above construction rules. In the continuous plane there are infinitely many such sets, since it is always possible to derive a set from another by a simple modification of the level lines conformation within the occlusion domain (see Figure 3).

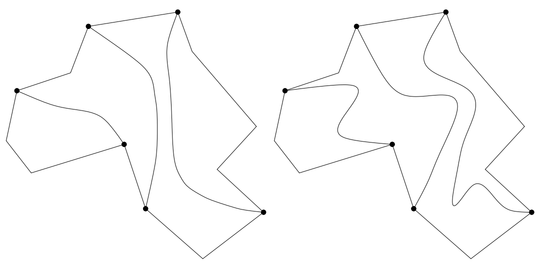

Fig. 3. Two different admissible solutions associated with the same pairwise connections of T-junctions.

Denote by E the energy of an admissible solution, defined as the sum of the $C_{i,j}$ over all connected couples (i, j). Obviously, among all admissible sets of level lines, the set with minimal energy contains only geodesic paths. Thus, in order to compute the optimal solution to our disocclusion problem, it is enough to enumerate the sets of all one-to-one maps between T-junctions that satisfy the construction rules, to compute for every map the corresponding admissible solution made of geodesic paths only and to keep among all those admissible solutions the one with smallest energy.

D. Computing a geodesic path

We shall now describe a method to compute the geodesic path connecting two compatible T-junctions. This method needs a prior triangulation of the occlusion domain, which we recall to be without hole and whose boundary can be represented by a Jordan curve. Such a triangulation can be theoretically performed in $\mathcal{O}(N)$, where N is the number of vertices of the boundary, using the Chazelle algorithm [5]. This algorithm is however so complex that, to our knowledge, it is still unimplemented. Some methods can be found in the literature that perform a triangulation in $\mathcal{O}(N \log N)$ [18]. We used for our algorithm a less efficient but simpler method, the so-called *ear-clipping* method, whose complexity is $\mathcal{O}(N^2)$ [18].

Once the polygon representing the occlusion has been triangulated, the Hershberger and Snoeyink [10] method computes the geodesic path connecting two vertices of the polygon boundary and requires $\mathcal{O}(N)$ operations. This method is based on the *funnel* algorithm due to Lee and Preparata [12]. Practically, we used an implementation due to J. Mitchell and his students [16].

E. Using dynamic programming for energy minimization

We are now in position to compute the cost for connecting two compatible T-junctions i and j. Let us address the problem of finding an optimal set of connections, that is a set with minimal energy. The dynamic programming approach provides solutions to this otherwise exponential problem and is well suited with the causality property satisfied by successive connections of T-junctions. Denote by $(t_1, t_2, \ldots, t_{2m})$ the set of T-junctions to be processed and recall that there is always an even number of T-junctions per level. The dynamic programming is based on the iterative computation of optimal energies $E_{i,i+k}$ corresponding to energies of optimal pairwise connections of all T-junctions in the interval $[i, i+k]$. These energies can be defined only whenever k is even since it is not possible to pairwise connect oddly many T-junctions. $C_{i,j}$ denotes the energy of the geodesic path connecting i and j and we take as a convention that $C_{i,j}$ is infinite whenever i and j are not compatible. In our case, the dynamic programming approach reduces to the following iterations.

Step#1 : Processing of intervals of length 2; all the following energies are computed

$$E_{1,2} = C_{1,2}, \ldots, E_{i,i+1} = C_{i,i+1}, \ldots, E_{2m,1} = C_{2m,1}$$

Step#2 : Processing of intervals of length 4; using energies $E_{i,i+1}$ computed at previous step, one can compute :

$$\begin{aligned}
E_{1,4} &= \min\{C_{1,4} + E_{2,3} \,;\, E_{1,2} + E_{3,4}\} \\
&\vdots \\
E_{i,i+3} &= \min\{C_{i,i+3} + E_{i+1,i+2} \,;\, E_{i,i+1} + E_{i+2,i+3}\} \\
&\vdots \\
E_{2m,3} &= \min\{C_{2m,3} + E_{1,2} \,;\, E_{2m,1} + E_{2,3}\}
\end{aligned}$$

$$\ldots\ldots$$

Step#k : Processing of intervals of length $2k$

$$\begin{aligned}
E_{1,2k} &= \min\{C_{1,2k} + E_{2,2k-1} \,;\, E_{1,2} + E_{3,2k} \,; \\
&\qquad E_{1,4} + E_{5,2k} \,; \ldots \,;\, E_{1,2k-2} + E_{2k-1,2k}\} \\
&\vdots \\
E_{i,i+2k-1} &= \min\{C_{i,i+2k-1} + E_{i+1,i+2k-2} \,;\, E_{i,i+1} + \\
&\qquad + E_{i+2,i+2k-1} \,; \ldots \,;\, E_{i,i+2k-3} + E_{i+2k-2,i+2k-1}\} \\
&\vdots \\
E_{2m,2k-1} &= \min\{C_{2m,2k-1} + E_{1,2k-2} \,;\, E_{2m,1} + E_{2,2k-1} \,; \\
&\qquad \ldots \,;\, E_{2m,2k-3} + E_{2k-2,2k-1}\}
\end{aligned}$$

$$\ldots\ldots$$

Step#m : $E_{1,2m}$ is the optimal energy.

$$\begin{aligned}
E_{1,2m} = \min\{&C_{1,2m} + E_{2,2m-1} \,;\, E_{1,2} + E_{3,2m} \,;\, E_{1,4} + \\
&+ E_{5,2m} \,; \ldots \,;\, E_{1,2m-2} + E_{2m-1,2m}\}
\end{aligned}$$

Denoting by $M = 2m$ the number of T-junctions and N the number of vertices of the polygonal line representing the occlusion boundary, the disocclusion algorithm we presented above can be achieved with a worst-case complexity of $\mathcal{O}(NM^2 + M^3)$ since the computation of each energy $C_{i,i+2k}$ costs $\mathcal{O}(M)$. Using a faster triangulation method in $\mathcal{O}(M \log M)$ would reduce the cost to $\mathcal{O}(NM \log M + M^3)$. Clearly, when the occlusion is convex, our disocclusion method can be achieved in $\mathcal{O}(M^3)$ since geodesic paths are straight lines. This cost remains high but it must be compared with a method involving a simple enumeration of all possible connections whose complexity is $\mathcal{O}(M!)$.

We used for implementing the disocclusion algorithm a function $Energy(i, i+k)$ which computes recursively the energy of the interval $[i, i+k]$. More precisely, the optimal energy is $Energy(1, 2m)$, which requires the computation of $Energy(2, 2m-1)$, $Energy(3, 2m-1)$, ..., which themselves require the computation of energy on smaller intervals. This method is equivalent to that described above, except that some computations can be avoided. Indeed, if one needs to compute $E_{i,i+3} + E_{i+4,i+2k-1}$ and it is already known that $E_{i,i+3} = \infty$, there is of course no need to compute $E_{i+4,i+2k-1}$. In the same way, it is unnecessary to go on the computation if $E_{i,i+3}$ is already larger than some minimal value obtained before. The complexity of such an implementation is theoretically the same as the method presented above but practically lower since many T-junctions are not compatible and thus many energies are infinite.

F. Geodesic propagation of restored levels to achieve the disocclusion process

Once the optimal set of connections has been computed, it remains only to "color" the occlusion. This can be done very simply in two steps. In the first step, one draws each geodesic path of positive length connecting two T-junctions forming an optimal pair. Actually one draws a two-pixels wide line where the two gray levels corresponding to the restored line are represented. In order to deal with covering problems, a simple method consists in drawing a first line between two consecutive junctions forming an optimal pair (there is at least one pair having this property). Then one goes on the occlusion boundary counterclockwise and draws the optimal geodesic paths by systematically coloring the right side of the line, whereas the left side is filled only at those points that have not been colored yet (see Figure 4). Once all the geodesic paths have been drawn, the disocclusion is achieved by a simple geodesic propagation of the values, that is a dilation with respect to occlusion domain.

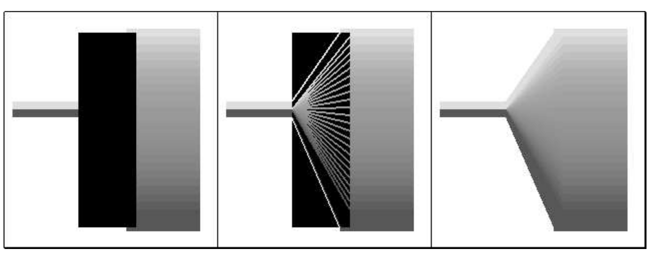

Fig. 4. Left, original image occluded by a black rectangle; Middle, the geodesic paths associated with the optimal connections; Right, the image obtained after disocclusion.

We summarize below the main steps of our disocclusion algorithm

Step 1 Computation of the polygonal line corresponding to the occlusion boundary.

Step 2 Computation of each T-junction on the occlusion boundary. A T-junction is determined by its coordinates, its position on the boundary, the related gray levels and the average direction of the corresponding level line.

Step 3 Triangulation of the occlusion.

Step 4 Computation by dynamic programming of the optimal set of level lines pairwise connecting the T-junctions.

Step 5 Drawing of the corresponding geodesic paths.

Step 6 Use of geodesic propagation to build the restored image.

The complexity of this algorithm is

$$\mathcal{O}(N^2 + NM^2 + M^3 + P)$$

where N denotes the number of vertices of the polygonal line describing the occlusion boundary, M is the number of T-junctions and P is the number of pixels within occlusion. As mentioned before it can be reduced to $\mathcal{O}(N^2 + NM \log M + M^3 + P)$.

V. Experimental results

We present in Figures 5, 6, 7 below the performances of our algorithm for different types of problems involving occlusions. It appears that the results are acceptable despite the purely geometric nature of our approach and the use of $p = 1$ in the energy to minimize, which coerces the restored lines to be straight (or polygonal) lines. In addition and in contrast with the approaches in [2] and [3], it is not an issue to recover sharp edges.

Figure 5 shows how impulse noise can be removed : in a first step we identify occlusions due to impulse noise as all those image subsets that are modified by a contrast-invariant and idempotent denoising filter, the so-called *grain filter*, that was introduced in [13]. Then we apply our algorithm to these occlusions. Figure 6 shows that our method can be used for photographs restoration. Finally, Figure 7 illustrates the performances of our algorithm in some extreme case where 70% of the information has been removed : starting from the original image, we keep one line out of six and one column out of six. The remaining part of the image is turned white so that occluded regions are disjoint 5-by-5 squares. This last experiment may indicate that our disocclusion method could be used as part of a new multi-scale compression method. However, as it can be seen in Figure 7, our approach fails at recovering textures. As it was mentioned above, this is due to its purely geometric nature and it should be combined with a texture-oriented disocclusion method to give better results. This is the purpose of a future work.

Fig. 5. Left : noisy image (impulse noise, frequency = 10%). Right : occlusion detection is performed by means of the grain filter. Disocclusion is then achieved with our level-lines based algorithm.

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Fig. 6. Top : original image where occlusions are in white. Bottom : disocclusion performed by our level-lines based algorithm. The discontinuities are well recovered.

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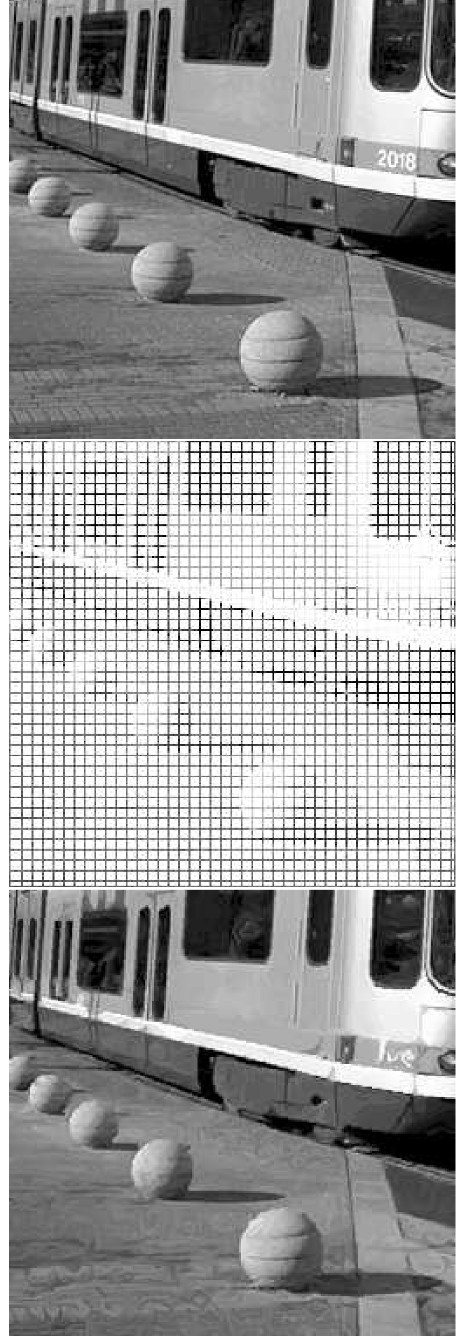

Fig. 7. Top : original image. Middle : image obtained by keeping only one line out of six and one column out of six and turning white the remaining part. Occlusions are the disjoint 5-by-5 white squares (this image may also illustrate the ability of our brain to perform amodal completion : remark indeed how we “see” despite the white squares). Bottom : restored image using our disocclusion method.

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