Disordered systems and the replica method in AdS/CFT

Yasuaki Hikida (KEK)

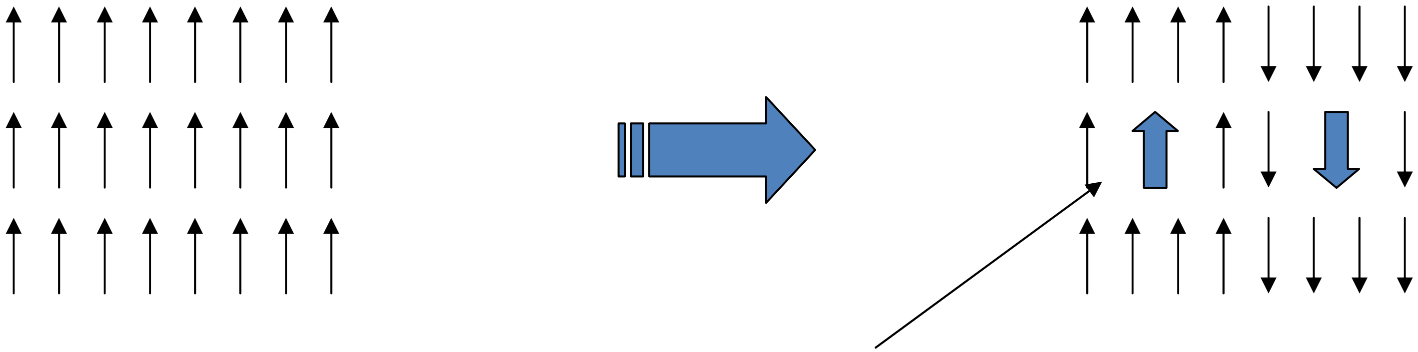
Ref. Fujita, YH, Ryu, Takayanagi,
JHEP12(2008)065

March 16@KEK workshop 2009

1. Introduction

Disordered systems

- Impurities



Impurities may induce large effects

- Disordered systems
  - Real materials
  - Spin glass systems
  - Quantum Hall effects

➡ Strongly coupled physics, AdS/CFT correspondence

[Hartnoll-Herzog, Fujita-YH-Ryu-Takayanag

AdS/CFT correspondence

[Maldacena]

- The duality

d-dim. CFT at the boundary *z*=0 ⟺ Gravity on (*d*+*1*)-dim. AdS

$$ds^2 = \frac{1}{z^2}(dz^2 + \sum_{i=1}^{d}(dx^i)^2)$$

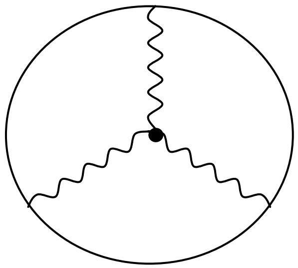
$\mathcal{O}, \ \mathcal{J}_\mu, \ \mathcal{T}_{\mu\nu}$ $\qquad$ $\phi, \ A_\mu, g_{\mu\nu}$

– Partition function [Gubser-Klebanov-Polyakov, Witten]

$$\left\langle \exp \int \phi_0 \mathcal{O} \right\rangle_{\mathrm{CFT}} = e^{-I_{\mathrm{SUGRA}}(\phi)}\Big|_{\phi(\partial \mathrm{AdS})=\phi_0}$$

– Correlation function

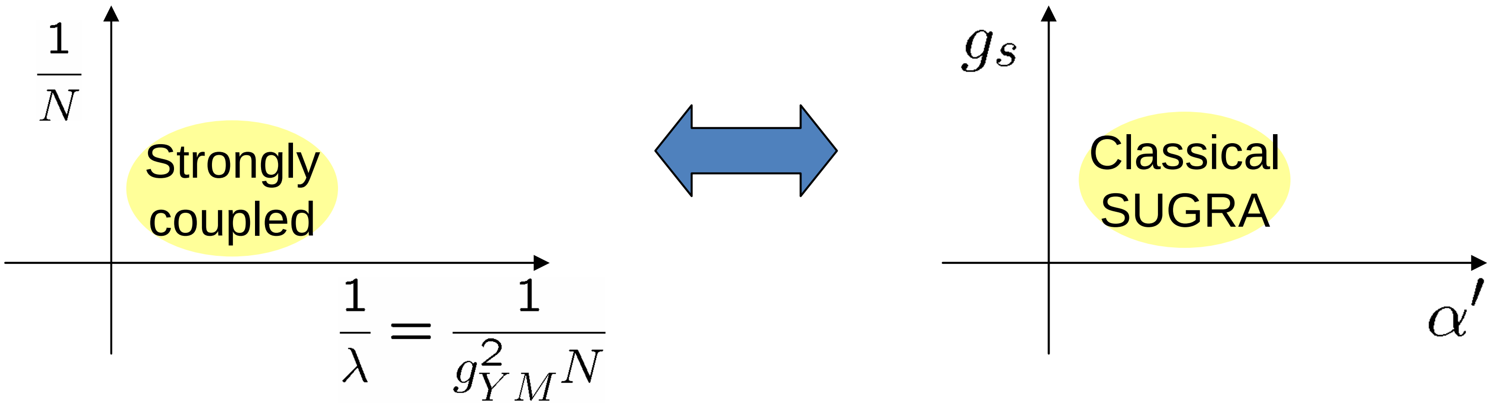
$$\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle_{\mathrm{CFT}} = \frac{\delta^3}{\delta \phi_0^3} e^{-I(\phi_0)}\Big|_{\phi_0=0} =$$



Coupling regions

- Relation between coupling regions

U(N) *N*=4 4d SYM IIB string on AdS_5



- Strong coupling physics from AdS/CFT
  - Quark gluon plasma

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

  - Strongly correlated physics in condensed matter

Examples of AdS/CMP (I)

- AdS superconductor

  [Gubser, Hartnoll-Herzog-Horowitz, Maeda-Okamura, Herzog-Kovtun-son, ...]

  – Scalar fields can condense near the black hole horizon in AdS space ($\Leftrightarrow$ no hair theorem).

  – In the dual CFT, it can be interpreted as a condensation of cooper pair

    - High T_c superconductur
    - A second order phase transition, infinite DC conductivity, energy gap, ...

- Qauntum Hall effects

  [Keski-Cakkuri-Kraus, Davis-Kraus-Shah, Fujita-Li-Ryu-Takayanagi, Hikida-Li-Takayanagi]

  – Chern-Simons theory as an effective theory

Examples of AdS/CMP (II)

- Non-relativistic CFT

$$t \to \lambda^z t, \;\; x \to \lambda x$$

– Schrödinger group

[Son, Balasubramanian-McGreevy, Sakaguchi-Yoshida, Herzog-Rangamani-Ross, Maldacena-Martelli-Tachikawa, Adams-Balasubramanian-McGreevy, Nakayama-Ryu-Sakaguchi-Yoshida, ...]

- Galilean + Dilatation + Special conformal (z=2)
- Cold atom at criticality (BCS-BEC crossover)

– Lifshitz-like model

[Kachru-Liu-Mulligan, Horava, Taylor]

- Time reversal symmetry

$$\mathcal{L} = \int d^2x dt ((\partial_t \phi)^2 - \kappa (\nabla^2 \phi)^2)$$

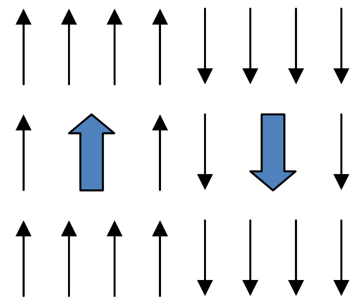
Plan of talk

1. Introduction
2. The replica method
3. Field theory analysis
4. Holographic replica method
5. Conclusion
6. Appendix

2. The replica method

Disordered systems

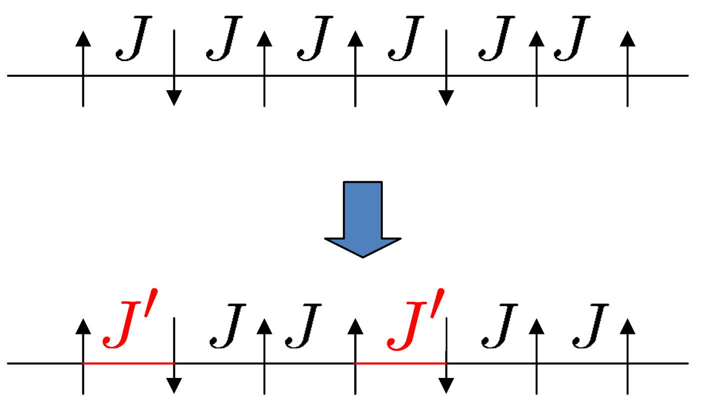
- Types of disorder
  - Annealed disorder
    - Impurities are in thermal equilibrium.
  - Quenched disorder
    - Impurities are fixed.



- An example: Random bond Ising model

$$H = -\frac{1}{2}\sum_i J S_i S_{i+1}$$

$$H = -\frac{1}{2}\sum_i J_i S_i S_{i+1}$$



Set up

- Prepare a *d*-dim. quantum field theory
  - Ex. U(N) *N*=4 4d SYM

$$S_0[\varphi] = \int d^d x \, \mathcal{L}_0(\varphi)$$

- Perturb the theory by a operato $\mathcal{O}(x)$
  - Ex. a single trace operato $\mathrm{Tr}[\varphi\varphi\cdots\varphi]$

$$S = S_0[\varphi] + \int d^d x \, g(x)\mathcal{O}(x)$$

The disorder configuration depends on *x*

- Take an average over the disorder

$$P[g(x)] \propto \exp\left[-\frac{1}{2f}\int d^d x g(x)^2\right]$$

The replica method

- Free energy

$$\ln Z[J] = \lim_{n \to 0} \frac{\overline{(Z[J])^n} - 1}{n} = \frac{d}{dn}(Z[J])^n|_{n=0}$$

- The replica method
  - Prepare *n* copies, take an average, then set

$$\overline{(Z_g)^n} = \int [\mathcal{D}g(x)] P[g(x)] \prod_{i=1}^{n} [\mathcal{D}\varphi_i] \times$$
$$\times \exp\left[- \sum_{i=1}^{n} S_0[\varphi_i] - \int d^d x\, g(x) \sum_{i=1}^{n} \mathcal{O}_i(x) \right]$$
$$= \int \prod_{i=1}^{n} [\mathcal{D}\varphi_i] \exp\left[- \sum_{i=1}^{n} S_0[\varphi_i] + \frac{f}{2} \int d^d x (\sum_{i=1}^{n} \mathcal{O}_i(x))^2 \right]$$

Correlation functions

- The effective action

$$S_{\text{eff}} = \sum_{i=1}^{n} S_0[\varphi_i] - \frac{f}{2}\int d^d x (\sum_{i=1}^{n} \mathcal{O}_i(x))^2$$

Relevant ⇔ $\text{dim}\mathcal{O}(x) < d/2$ ⇔ Harris criteria

- Correlation functions

$$\overline{\langle \mathcal{O}(z)\mathcal{O}(w)\rangle_g} = \overline{\left\langle \frac{1}{Z_g}\int [\mathcal{D}\varphi] e^{-S_0 - \int d^d x g(x)\mathcal{O}(x)} \mathcal{O}(z)\mathcal{O}(w) \right\rangle}$$

$$= \lim_{n\to 0}\int \prod_{i=1}^{n} [\mathcal{D}\varphi_i] e^{-S_{\text{eff}}} \mathcal{O}_1(z)\mathcal{O}_1(w)$$

(cf. the supersymmetric method)

3. Field theory analysis

Set up

- Original theory without disorder
  - *d*-dim. conformal field theory in the large *N* limit
- Our disordered system
  - Deform the theory by a singlet operator $\mathcal{O}(x)$

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{c}{|x|^{2\Delta}}, \quad \frac{d-2}{2} < \Delta < \frac{d}{2}$$

← Harris criteria

Conformal dimension

Unitarity

$$\langle \prod_{p=1}^{n} \mathcal{O}(x_p)\rangle = O(N^{2-n})$$

← Higher point functions can be neglected.

  - *n* de

$$S_{\text{int}} = -f \int d^d x \Phi_{\text{pert}}, \quad \Phi_{\text{pert}} = \frac{1}{2}(\sum_{i=1}^{n} \mathcal{O}_i(x))^2$$

Double trace deformation

- Perturbation by a double trace operator
  - A simpler case for an exercise

$$S_{\text{int}} = \lambda \int d^d x \Phi_{\text{pert}}$$

$$\Phi_{\text{pert}} = \frac{1}{2}\mathcal{O}^2(x), \quad \Phi_{\text{pert}}(x)\Phi_{\text{pert}}(0) \sim \frac{2v}{|x|^{2\Delta}}\Phi_{\text{pert}}(0)$$

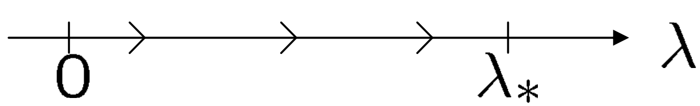
  - Beta function

$$\frac{d}{d \ln |k|}\lambda(k) = \beta_\lambda = -(d-2\Delta)\lambda(k) + (\lambda(k))^2$$

One-loop exact in large N limit

$$\Rightarrow \quad \lambda(k) = \frac{(d-2\Delta)\lambda_0}{|k|^{d-2\Delta} + \lambda_0}$$

$$\begin{array}{ll} \text{UV } (|k| \to \infty) & \lambda \to 0 \\ \text{IR } (|k| \to 0) & \lambda \to \lambda_* = (d-2\Delta) \end{array}$$



Non-trivial fixed point

Two point function

- Anomalous dimension

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_\lambda = \left\langle \mathcal{O}(x)\mathcal{O}(0)\exp\left[-\lambda\int d^dx\Phi_{\mathrm{pert}}(x)\right]\right\rangle_0$$

$$\Rightarrow \quad \gamma_{\mathcal{O}} = \Delta + \tfrac{1}{2}\lambda(k)$$

- RG flow equation

$$\left[\frac{d}{d\ln|k|} - \beta_\lambda\frac{d}{d\lambda} + d - 2\gamma_{\mathcal{O}}\right]\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle = 0$$

$$\Rightarrow \quad \langle\mathcal{O}(k)\mathcal{O}(-k)\rangle = C\exp\left[-\int^{\ln|k|} d\ln|k|'(d-2\gamma_{\mathcal{O}})\right]$$

$$= \frac{|k|^{2\Delta-d}}{1+\lambda_0|k|^{2\Delta-d}}$$

$$\mathrm{UV}: \quad \langle\mathcal{O}^2\rangle \sim |k|^{2\Delta-d}$$

$$\mathrm{IR}: \quad \langle\mathcal{O}^2\rangle \sim |k|^{d-2\Delta} \quad \leftarrow \quad \Delta \leftrightarrow d-\Delta$$

Large N disordered system

- Replica theory
  - *n* CFTs; $CFT_1 \otimes CFT_2 \otimes ... \otimes CFT_n$
  - Single trace operators $\mathcal{O}_i(x)$

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \delta_{i,j}\frac{v}{|x|^{2\Delta}}$$

- Double trace deformation

$$S_{\text{int}} = -\frac{f}{2}\int d^dx(\sum_{i=1}^{n}\mathcal{O}_i(x))^2$$

⇓ Regularization with λ

$$S_{\text{int}} = -\frac{f}{2}\int d^dx(\sum_{i=1}^{n}\mathcal{O}_i(x))^2 + \frac{\lambda}{2}\int d^dx\sum_{i=1}^{n}(\mathcal{O}_i(x))^2$$

Start with a CFT with deformation λ, then introduce the disorder

RG flow

- Beta functions

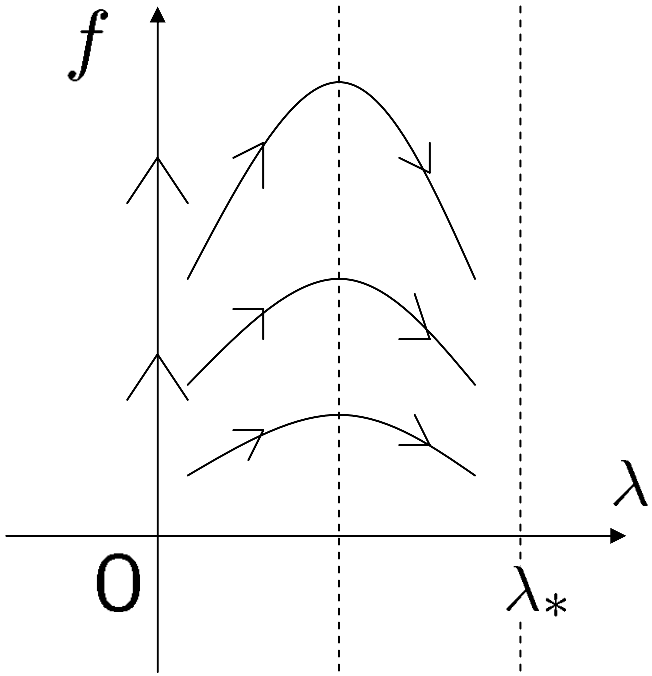
$$\beta_\lambda = -(d-2\Delta)\lambda(k) + (\lambda(k))^2$$
$$\beta_f = -(d-2\Delta)f(k) - n(f(k))^2 + 2f(k)\lambda(k)$$

- Flow of couplings

$$\lambda(k) = \frac{(d-2\Delta)\lambda_0}{|k|^{d-2\Delta}+\lambda_0}$$
$$f(k) - \frac{\lambda(k)}{n} = \frac{(d-2\Delta)(f_0-\lambda_0/n)}{|k|^{d-2\Delta}-nf_0+\lambda_0}$$

$$n \to 0$$

$$\text{UV}: \quad \lambda \to 0, f \to 0$$
$$\text{IR}: \quad \lambda \to \lambda_* = (d-2\Delta), f \to 0$$



Two pint function

- Redefinition of operators

$$\hat{\mathcal{O}}_0(x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathcal{O}_i(x),\ \hat{\mathcal{O}}_j(x) = \mathcal{O}_j(x) - \frac{1}{n}\sum_{i=1}^{n}\mathcal{O}_i(x)$$

$$\Rightarrow \gamma_{\hat{\mathcal{O}}_0} = \Delta + \frac{1}{2}\lambda - \frac{n}{2}f,\ \gamma_{\hat{\mathcal{O}}_j} = \Delta + \frac{1}{2}\lambda$$

- Two point functions

1. In hated basis of replicated theor $\langle \hat{\mathcal{O}}_I(k)\hat{\mathcal{O}}_J(-k)\rangle$
2. In the original basis $\langle \mathcal{O}_1(k)\mathcal{O}_1(-k)\rangle$
3. In the limit of $n \to 0$

$$\overline{\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_g} = \frac{(1+(f_0+\lambda_0)|k|^{2\Delta-d})|k|^{2\Delta-d}}{(1+\lambda_0|k|^{2\Delta-d})^2}$$

$$\text{UV}: \quad \langle \mathcal{O}^2\rangle \sim |k|^{2\Delta-d}$$

$$\text{IR}: \quad \langle \mathcal{O}^2\rangle \sim |k|^{d-2\Delta}\ (\lambda_0 > 0)$$

$$\sim |k|^{4\Delta-2d}\ (\lambda_0 = 0)$$

$$\Delta \leftrightarrow 2\Delta - \frac{d}{2}$$

$$\frac{d}{2} - 2 < 2\Delta - \frac{d}{2} < \frac{d}{2}$$

Unitrity bound is violated

4. Holographic replica method

AdS/CFT dictionary

- The map

d-dim. CFT at the boundary *z*=0

$\mathcal{O}$: a spin-less operator

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{m^2 + \frac{d^2}{4}}$$

$$\frac{d-2}{2} < \Delta = \Delta_- < \frac{d}{2}$$

Unitarity bound ↗ ↖ Harris criteria

Gravity on (*d*+*1*)-dim. AdS

$$ds^2 = \frac{1}{z^2}(dz^2 + \sum_{i=1}^{d}(dx^i)^2)$$

ϕ : a scalar field

m : mass of the scalar

$$-\frac{d^2}{4} < m^2 < \frac{1-d^2}{4}$$

BF bound ↗ ↖ Normalisability

- Boundary behavior at *z*$\sim$0
  - A scalar field satisfying KG eq. and the regularity at z=∞

$$\phi(z,x) \sim z^{d-\Delta}(\alpha(x) + O(z^2)) + z^{\Delta}(\beta(x)/(2\Delta - d) + O(z^2))$$

Source to $\mathcal{O}(x)$ → $\alpha(x)$

Legendre transform

$\langle \mathcal{O}(x) \rangle = \beta(x)$

Legendre transform

[Klebanov-Witten]

- Evaluation of action
  - Start from the (*d*+*1*) dim. action for the scalar

$$S[\phi] = \frac{1}{2}\int d^{d+1}x\sqrt{g}[\partial^\mu\phi\partial_\mu\phi + m^2\phi^2]$$

  - Insert the solution and partially integrate over z

$$S[\alpha] = -\frac{1}{2}\int d^dk\,\alpha(k)G(k)\alpha(-k)$$

$$G(k) = \frac{(2\Delta-d)\Gamma(d/2-\Delta)}{\Gamma(\Delta-d/2)}\left[\frac{|k|}{2}\right]^{2\Delta-d}$$

  - L

$$S[\beta] = S[\alpha] - \int d^dk\frac{\delta S[\alpha]}{\delta\alpha}\alpha = \frac{1}{2}\int d^dk\,\frac{\beta(k)\beta(-k)}{G(k)}$$

$$\beta = -\frac{\delta S[\alpha]}{\delta\alpha} \quad \left(\langle\mathcal{O}\rangle_\alpha = \frac{\delta}{\delta\alpha}\langle e^{\int\alpha\mathcal{O}}\rangle_0\right)$$

Double trace deformation

[Witten]

- A simpler case with one CFT
  - Deformation by a double trace operator

$$S_{\rm int} = \frac{\lambda}{2}\int d^d x \mathcal{O}(x)\mathcal{O}(x)$$

  - The deformed action in the gravity side

$$S[\beta, J] = \int d^d k \left[\frac{\beta(k)\beta(-k)}{2G(k)} + \frac{\lambda}{2}\beta(k)\beta(-k) + \beta(-k)J(k)\right]$$

EOM for β $\quad \frac{\beta(k)}{G(k)} + \lambda\beta(k) + J(k) = 0$

$$S[J] = -\frac{1}{2}\int d^d k \left[J(k)\left(\frac{G(k)}{1+\lambda G(k)}\right)J(-k)\right]$$

  - Two point function

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle = \frac{\delta^2}{\delta J^2} e^{-S[J]}\Big|_{J=0} = \frac{G(k)}{1+\lambda G(k)}$$

⇨ reproduces the field theory result

Holographic replica method

[Aharony-Clark-Karch, Kiritsis, Kiritsis-Niarchos

- Set up
  - – *n* CFTs wit $S_{\rm int} = \int d^d x \left[-\frac{f}{2} (\sum_{i=1}^{n} \mathcal{O}_i(x))^2 + \frac{\lambda}{2} \sum_{i=1}^{n} (\mathcal{O}_i(x))^2 \right]$

  - – *n* AdS spaces sharing the same boundary
    - coupled to each other by boundary conditions for ϕ_i
- The deformed action in the gravity side

$$S[\beta, J] = \int d^d k \left[\frac{\sum_{i=1}^{n} \beta_i(k)\beta_i(-k)}{2G(k)} - \frac{f}{2} (\sum_{i=1}^{n} \beta_i(k))(\sum_{i=1}^{n} \beta_i(-k)) + \frac{\lambda}{2} \sum_{i=1}^{n} \beta_i(k)\beta_i(-k) + \beta_1(-k) J_1(k) \right]$$

$$\xrightarrow{\text{EOM for } \beta_i} S[J] \xrightarrow{\left. \frac{\delta^2}{\delta J_1^2} e^{-S[J]} \right|_{J=0}} \langle \mathcal{O}_1^2 \rangle_n \xrightarrow{n \to 0} \overline{\langle \mathcal{O}^2 \rangle}$$

5. Conclusion

Summary and discussions

- Summary
  - Disordered systems and the replica method
    - Prepare *n* QFTs, introduce disorder, then take *n*->0 limit
  - RG flow and the two point function
    - Conformal perturbation theory
  - Holographic replica method
    - Multiple AdS spaces coupled though the boundary
- Open problems
  - Quantum disordered system
    - Dual geometry is AdS black hole
  - Other quantities
    - E.g. two point function of currents
  - Holographic supersymmetric method
    - OSP(N|N) or U(N|N) supergroup structure

6. Appendix

Beta function

- Perturbation from CFT

$$S_{\text{int}} = f l^{2\Delta-d} \int d^d x \Phi(x), \;\; \Phi(x)\Phi(0) \sim \frac{2v}{|x|^{2\Delta}} \Phi(0)$$

- Shift of cut off length
  - UV cut off length *l* is shifted to *l*(1 + ε)

$$\begin{aligned} \delta W &\sim \epsilon(2\Delta - d) f l^{2\Delta-d} \int d^d x \Phi(x) \\ &+ \frac{f^2 l^{4\Delta-d}}{2!} 2 \int d^d x \int_{x+l}^{x+l(1+\epsilon)} d^d y \Phi(x)\Phi(y) \\ &\sim \epsilon l^{2\Delta-d}((2\Delta - d) f + 2\Omega^d v f^2) \int d^d x \Phi(x) \end{aligned}$$

- Beta function

$$-\frac{df}{d\ln(L/l)} = \beta_f = (2\Delta - d) f + 2\Omega^d v f^2$$

Anomalous dimension

- Perturbation from CFT

$$S_{\text{int}} = f l^{2\Delta - d} \int d^d x \Phi(x), \;\; \Phi(x)\mathcal{O}(0) \sim \frac{v}{|x|^{2\Delta}} \mathcal{O}(0)$$

- Wave function renormalization
  - Two point function

$$G_2(z) = l^{2\Delta} \langle \mathcal{O}(z)\mathcal{O}(0) \rangle$$

  - Shift of UV cut off

$$\begin{aligned} \delta G_2(z) &\sim \epsilon 2\Delta l^{2\Delta} \langle \mathcal{O}(z)\mathcal{O}(0) \rangle \\ &\quad + 2 f l^{4\Delta - d} \int_l^{l(1+\epsilon)} d^d x \langle \mathcal{O}(z)\Phi(x)\mathcal{O}(0) \rangle \\ &\sim \epsilon l^{2\Delta} (2\Delta + 2\Omega^d v f) \langle \mathcal{O}(z)\mathcal{O}(0) \rangle \end{aligned}$$

  - Anomalous dimension

$$-\frac{1}{2}\frac{dG_2(z)}{d\ln(L/l)} = \gamma_{\mathcal{O}} = \Delta + \Omega^d v f$$