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Dispersion function computations for unlimited frequency values

Anas Abo-Zena *Gulf Science and Technology Company, Pittsburgh, Pennsylvania 15230, USA*

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Summary. Progress in the matrix method for the calculation of the seismic surface wave dispersion function for a layered elastic media started with the beginning of the electronic digital computer age.

The use of Thomson–Haskell formulation, Knopoff's method or any other published method has had the persistent problem of loss of precision at high frequencies. The severity of the high-frequency limitation problem varies among the approaches but exists in all of them.

In this paper we present a novel method to determine the dispersion function for Rayleigh waves in layered elastic media. In this method there is no limitation on the value of the frequency.

Introduction

The determination of the dispersion function of seismic surface waves in layered elastic media is an essential part of solving the wave propagation problem in that media.

Since the electronic digital computer has become available for geophysical research, great efforts have been directed towards further developing the numerical determination of the dispersion function of layered elastic media. The theoretical formulation for this function was developed by Thomson (1950) and Haskell (1953). The numerical efforts started by Press, Harkrider & Seafeldt (1961) were followed by Knopoff and his co-workers in the years 1964 to 1972, as well as by other investigators including Dunkin (1965), Thrower (1965) and Watson (1970).

The use of the original form of the Thomson–Haskell formulation, Knopoff's method or any other published method has had the persistent problem of loss of precision at high frequency. The severity of the high-frequency limitation problem varies among the published approaches but exists in all of them.

The calculation of the dispersion function requires the subtraction of two quantities. At very high frequencies where the thickness of a layer is greater than several wavelengths, these two quantities become very close to each other and differ only in the less significant digits. The determination of each quantity separately would make the difference lose some of its significant digits. This loss of significant digits would occur as a chain reaction in the computation. Algebraic subtraction should be made so that the computer calculates only the difference and not the original quantities.

In our approach, we have succeeded in avoiding this subtraction error by transforming the problem to one where the different frequency factors are separated. Using matrix manipulation and taking advantage of all the symmetry, we believe that this approach offers the fastest algorithm at the high-frequency range.

Notation

ρ = density,
d = thickness,
λ, μ = Lamé elastic constants,
α = $[(\lambda + 2\mu)/\rho]^{1/2}$,
β = $[\mu/\rho]^{1/2}$,
f = frequency,
c = phase velocity of the free wave along the x axis,
ω = $2\pi f$,
k = ω/c,
i = $\sqrt{-1}$,

$$r_\alpha = \begin{cases} i[(c/\alpha)^2 - 1]^{1/2} & c > \alpha, \\ [1 - (c/\alpha)^2]^{1/2} & c < \alpha, \end{cases}$$

$$r_\beta = \begin{cases} i[(c/\beta)^2 - 1]^{1/2} & c > \beta, \\ [1 - (c/\beta)^2]^{1/2} & c < \beta, \end{cases}$$

γ = $2(\beta/c)^2$,
u = x component of displacement,
w = z component of displacement,
σ_z = normal stress,
τ_{xz} = tangential stress,
P = $\exp(r_\alpha kd)$,
Q = $\exp(r_\beta kd)$.

Formulation of the problem

Consider a horizontally layered elastic solid half-space with plane wave travelling in the positive x direction as shown in Fig. 1. Each layer is assumed to be isotropic, homogeneous and perfectly elastic. The mth layer is bounded by the two boundaries $(m - 1)$ and (m).

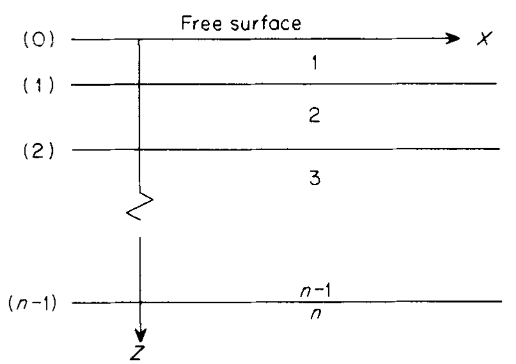

Figure 1. Diagram of layered structure and the numbering of the layers and the boundaries.

Within each medium, the displacement of u and ω and the stresses σ_z and τ_{xz} can all be derived from a scalar potential ϕ and a vector potential ψ (Ewing, Jardeszky & Press 1957). These potentials are obtained for each layer as solutions of the equations:

$$\nabla^2\phi = \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2}, \tag{1}$$

$$\nabla^2\psi = \frac{1}{\beta^2}\frac{\partial^2\psi}{\partial t^2}. \tag{2}$$

Subject to boundary conditions at each surface of the layer. Assuming time periodicity of both ϕ and ψ and recognizing that the waves are plane then the potentials can be taken as

$$\phi = \phi_0(z)\exp[i(\omega t - kx)] \tag{3}$$

$$\psi = \psi_0(z)\exp[i(\omega t - kx)]. \tag{4}$$

Equations (1) and (2) then becomes

$$\frac{d^2\phi_0}{dz^2} = k^2(1 - c^2/\alpha^2)\phi_0 \tag{5}$$

$$\frac{d^2\psi_0}{dz^2} = k^2(1 - c^2/\beta^2)\psi_0. \tag{6}$$

Solutions to equations (5) and (6) are

$$\phi_0 = b_1\cosh(r_\alpha kz) + b_2\sinh(r_\alpha kz) \tag{7}$$

$$\psi_0 = b_3\cosh(r_\beta kz) + b_4\sinh(r_\beta kz). \tag{8}$$

The four constants b_1, appearing in equations (7) and (8) are to be determined for each layer from the boundary conditions at its surfaces. The stresses σ_z and τ_{xz} and the displacements u and ω are related to ϕ and ψ as follows:

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z} \tag{9}$$

$$w = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x} \tag{10}$$

$$\sigma_z = \lambda\nabla^2\phi + 2\mu\frac{\partial^2\phi}{\partial z^2} + 2\mu\frac{\partial^2\psi}{\partial x\partial z} \tag{11}$$

$$\tau_{xz} = \mu\left(2\frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right). \tag{12}$$

Inserting equations (7) and (8) into (9)–(12) produces a set of equations relating the stress displacement vector to b_1, b_2, b_3 and b_4 as follows:

$$u = -ikb_1\cosh(r_\alpha kz) - ikb_2\sinh(r_\alpha kz)$$
$$- r_\beta kb_3\sinh(r_\beta kz) - r_\beta kb_4\cosh(r_\beta kz) \tag{13}$$

$$w = r_\alpha kb_1\sinh(r_\alpha kz) + r_\alpha kb_2\cosh(r_\alpha kz)$$
$$- ikb_3\cosh(r_\beta kz) - ikb_4\sinh(r_\beta kz) \tag{14}$$

$$\sigma_z = \rho(\gamma-1)k^2c^2b_1\cosh(r_\alpha kz) + \rho(\gamma-1)k^2c^2b_2\sinh(r_\alpha kz) - ik^2r_\beta 2\mu b_3\sinh(r_\beta kz) - ir_\beta 2\mu k^2b_4\cosh(r_\beta kz) \tag{15}$$

$$\tau_{xz} = -2i\rho\beta^2 r_\alpha k^2b_1\sinh(r_\alpha kz) - 2ir_\alpha\rho\beta^2k^2b_2\cosh(r_\alpha kz) - (\gamma-1)\rho k^2c^2b_3\cosh(r_\beta kz) - (\gamma-1)k^2c^2\rho b_4\sinh(r_\beta kz) \tag{16}$$

where the factor $\exp[i(\omega t - kx)]$ has been suppressed.

Equations (13)–(16) could be arranged into matrix form

$$[W] = [U_m(z)]\,[B] \tag{17}$$

where

$$[W]^{\mathrm{T}} = [iu/k,\ w/k,\ \sigma_z/k^2c^2,\ i\tau_{xz}/k^2c^2] \tag{18}$$

$$[B]^{\mathrm{T}} = [b_1, b_2, -ib_3, -ib_4] \tag{19}$$

and

$$U_m(z) = \begin{bmatrix} 1 & 1 & r_\beta & r_\beta \\ r_\alpha & r_\alpha & 1 & 1 \\ \rho(\gamma-1) & \rho(\gamma-1) & \rho\gamma r_\beta & \rho\gamma r_\beta \\ \rho\gamma r_\alpha & \rho\gamma r_\alpha & \rho(\gamma-1) & \rho(\gamma-1) \end{bmatrix} \cdot \begin{bmatrix} \exp(r_\alpha kz) & 0 & 0 & 0 \\ 0 & \exp(r_\alpha kz) & 0 & 0 \\ 0 & 0 & \exp(r_\beta kz) & 0 \\ 0 & 0 & 0 & \exp(r_\beta kz) \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & -1 & r_\beta & -r_\beta \\ -r_\alpha & r_\alpha & -1 & 1 \\ \rho(\gamma-1) & -\rho(\gamma-1) & \rho\gamma r_\beta & \rho\gamma r_\beta \\ -\rho\gamma r_\alpha & \rho\gamma r_\alpha & -\rho(\gamma-1) & \rho(\gamma-1) \end{bmatrix} \cdot \begin{bmatrix} \exp(-r_\alpha kz) & 0 & 0 & 0 \\ 0 & \exp(-r_\alpha kz) & 0 & 0 \\ 0 & 0 & \exp(-r_\beta kz) & 0 \\ 0 & 0 & 0 & \exp(-r_\beta kz) \end{bmatrix}. \tag{20}$$

Boundary conditions

Due to the continuity of the quantities of the matrix W at the boundary of any layer, a relation between the displacement at the boundary (m) and the boundary $(m-1)$ would be

$$[W]_{(m)} = A_m\,[W]_{(m-1)} \tag{21}$$

where

$$A_m = [U_m(d_m)]\,[U_m(0)]^{-1} \tag{22}$$

$$[U_m(0)]^{-1} = \tfrac{1}{2}\begin{bmatrix} \gamma & 0 & -1/\rho & 0 \\ 0 & -(\gamma-1)/r_\alpha & 0 & 1/\rho r_\alpha \\ -(\gamma-1)/r_\beta & 0 & 1/\rho r_\beta & 0 \\ 0 & \gamma & 0 & -1/\rho \end{bmatrix}. \tag{23}$$

By multiplying $[U_m(d_m)]$ by $[U_m(0)]^{-1}$ and defining $P = \exp(r_\alpha kd)$ and $Q = \exp(r_\beta kd)$ one can obtain

$$A_m = \tfrac{1}{2}[PV_1H_1 + (1/P)V_2H_2 + QV_3H_3 + (1/Q)V_4H_4] \tag{24}$$

where

$$[V_1, V_2, V_3, V_4] = \begin{bmatrix} 1 & 1 & r_\beta & r_\beta \\ r_\alpha & -r_\alpha & 1 & -1 \\ \rho(\gamma-1) & \rho(\gamma-1) & \rho\gamma r_\beta & \rho\gamma r_\beta \\ \rho\gamma r_\alpha & -\rho\gamma r_\alpha & \rho(\gamma-1) & -\rho(\gamma-1) \end{bmatrix} \tag{25}$$

and

$$\begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix} = \begin{bmatrix} \gamma & -(\gamma-1)/r_\alpha & -1/\rho & 1/(\rho r_\alpha) \\ \gamma & (\gamma-1)/r_\alpha & -1/\rho & -1/(\rho r_\alpha) \\ -(\gamma-1)/r_\beta & \gamma & 1/(\rho r_\beta) & -1/\rho \\ -(\gamma-1)/r_\beta & -\gamma & 1/(\rho r_\beta) & 1/\rho \end{bmatrix}. \tag{26}$$

It may be useful to note that $\tfrac{1}{2}[V_1V_2V_3V_4]\,[H_1H_2H_3H_4]^{\mathrm{T}}$ is a unit matrix.

By repeated application of equation (21) one obtains

$$[W]_{(n-1)} = A_{n-1}\ldots A_2A_1\,[W]_{(0)}. \tag{27}$$

Then by applying the inverse of equation (17) with $z = 0$, equation (27) becomes

$$[B]_n = [U_n(0)]^{-1}A_{n-1}\ldots A_2A_1[W]_{(0)}. \tag{28}$$

In the case where there is no source at infinity $b_{1_n} = -b_{2_n}$ and $b_{3_n} = -b_{4_n}$. In addition, if $\sigma_0 = 0$ and $\tau_0 = 0$ then equation (28) becomes

$$\begin{bmatrix} b_1 \\ -b_1 \\ -ib_3 \\ ib_3 \end{bmatrix}_n = \begin{bmatrix} J_{11} & J_{21} \\ J_{12} & J_{22} \\ J_{13} & J_{23} \\ J_{14} & J_{24} \end{bmatrix} \cdot \begin{bmatrix} iu/k \\ w/k \end{bmatrix}_{(0)} \tag{29}$$

where

$$J = [U_n(0)]^{-1}A_{n-1}A_{n-2}\ldots A_2A_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{30}$$

From equation (29) follows

$$\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}(J_{11}+J_{12}) & (J_{21}+J_{22})\\(J_{13}+J_{14}) & (J_{23}+J_{24})\end{bmatrix}\begin{bmatrix}iu/k\\w/k\end{bmatrix}_{(0)}. \tag{31}$$

It follows that

$$\frac{iu_0}{w_0}=-\frac{J_{21}+J_{22}}{J_{11}+J_{12}}=-\frac{J_{23}+J_{24}}{J_{13}+J_{14}} \tag{32}$$

and that the matrix

$$[\mathrm{LS}]=\begin{bmatrix}(J_{11}+J_{12}) & (J_{21}+J_{22})\\(J_{13}+J_{14}) & (J_{23}+J_{24})\end{bmatrix} \tag{33}$$

should be singular.

From the definition of J in equation (30) one can express the matrix LS as

$$[\mathrm{LS}]=\begin{bmatrix}\mathrm{EA}\\\mathrm{EB}\end{bmatrix}[\mathrm{K}]\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix} \tag{34}$$

where

$$\begin{bmatrix}\mathrm{EA}\\\mathrm{EB}\end{bmatrix}=½\begin{bmatrix}\gamma & -(\gamma-1)/r_\alpha & -1/\rho & 1/\rho r_\alpha\\-(\gamma-1)/r_\beta & \gamma & 1/\rho r_\beta & -1/\rho\end{bmatrix}_n \tag{35}$$

and

$$\mathrm{K}=A_{n-1}A_{n-2}\ldots A_2A_1. \tag{36}$$

From equation (34) it follows that

$$[\mathrm{LS}]=\begin{bmatrix}[\mathrm{EA}]\,[\mathrm{K}]\begin{bmatrix}1\\0\\0\\0\end{bmatrix} & [\mathrm{EA}]\,[\mathrm{K}]\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\\[\mathrm{EB}]\,[\mathrm{K}]\begin{bmatrix}1\\0\\0\\0\end{bmatrix} & [\mathrm{EB}]\,[\mathrm{K}]\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\end{bmatrix} \tag{37}$$

from which the dispersion function D could be expressed as

$$D=[\mathrm{EA}]\,[\mathrm{K}]\begin{bmatrix}1\\0\\0\\0\end{bmatrix}[\mathrm{EB}]\,[\mathrm{K}]\begin{bmatrix}0\\1\\0\\0\end{bmatrix}-[\mathrm{EA}]\,[\mathrm{K}]\begin{bmatrix}0\\1\\0\\0\end{bmatrix}[\mathrm{EB}]\,[\mathrm{K}]\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \tag{38}$$

Numerical analysis

Using equation (38) to determine D shows that at the high-frequency range the two terms on the right side become very close to each other as the frequency increases. This causes a loss of the significant digits. The closeness actually occurs because both numbers become dominated by the same exponential frequency factor. The basic element in the computation is a linear expression of the terms P, Q, $1/P$ and $1/Q$. The dispersion function is a sum of the products of two of these elements. This makes the dispersion function's final composition a linear expression of the terms P^2, Q^2, $1/P^2$, $1/Q^2$, PQ, $1/PQ$, P/Q, Q/P and a constant. The largest factor would be P^2 if the real part of r_α is positive.

In general without any limitation on α, β or k the largest factor would be P^2, Q^2, $1/P^2$ or $1/Q^2$. Such a factor would overshadow the smaller factors. The resultant value D would lose some or maybe all of the significant digits at high frequencies. To avoid this difficulty we use the factored matrix form, equation (24), to determine the constant and the coefficients of the factors which represent the dispersion function D.

With some matrix manipulation the function D could be expressed as

$$D = [\mathrm{EA}]\,[\mathrm{K}] \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} [\mathrm{K}]^{\mathrm{T}}\,[\mathrm{EB}]^{\mathrm{T}}$$

or

$$D = [1 \quad 0 \quad 0 \quad 0]\,[\mathrm{K}]^{\mathrm{T}}\,\{[\mathrm{EA}]^{\mathrm{T}}[\mathrm{EB}] - [\mathrm{EB}]^{\mathrm{T}}[\mathrm{EA}]\}\,[\mathrm{K}] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

In the analysis we will use the second expression as it is more useful in extending this work to study reflectivity (Fuchs 1968; Kind 1976). If we define

$$Y_m = A_m^{\mathrm{T}} \ldots A_{n-1}^{\mathrm{T}}\,\{[\mathrm{EA}]^{\mathrm{T}}[\mathrm{EB}] - [\mathrm{EB}]^{\mathrm{T}}[\mathrm{EA}]\}\,A_{n-1} \ldots A_m \tag{40}$$

then

$$Y_m = A_m^{\mathrm{T}}\, Y_{m+1} A_m. \tag{41}$$

For the computation start with

$$Y_{m+1} = Y_n = \{[\mathrm{EA}]^{\mathrm{T}}[\mathrm{EB}] - [\mathrm{EB}]^{\mathrm{T}}[\mathrm{EA}]\} \tag{42}$$

and A_{n-1} to obtain Y_{n-1} and repeat applying equation (41), $n-1$ times, to obtain Y_1. The intersection of the first row with the second column in Y_1 is the value of the dispersion function.

In the computation, one should note that

$[\mathrm{EA}]^{\mathrm{T}}[\mathrm{EB}] - [\mathrm{EB}]^{\mathrm{T}}[\mathrm{EA}]$

is an antisymmetric matrix and if M is antisymmetric then $C^{\mathrm{T}}MC$ is also antisymmetric. By substituting from equation (24) in (41) one obtains

$$Y_m = \tfrac{1}{4}\left[PH_1^{\mathrm{T}}V_1^{\mathrm{T}} + \frac{1}{P}H_2^{\mathrm{T}}V_2^{\mathrm{T}} + QH_3^{\mathrm{T}}V_3^{\mathrm{T}} + \frac{1}{Q}H_4^{\mathrm{T}}V_4^{\mathrm{T}}\right][Y_{m+1}]\left[PV_1H_1 + \frac{1}{P}V_2H_2 + QV_3H_3 + \frac{1}{Q}V_4H_4\right]. \tag{43}$$

Since Y_{m+1} is antisymmetric and V_i is a column vector, then $V_i^{\mathrm{T}}[Y_{m+1}]V_i$ is a scalar quantity equal to zero. It follows that one can put Y_m in the following form

$$
\begin{aligned}
4Y_m = {} & [H_1^{\mathrm{T}}V_1^{\mathrm{T}} + H_2^{\mathrm{T}}V_2^{\mathrm{T}}][Y_{m+1}][V_2H_2 + V_1H_1] \\
& + [H_3^{\mathrm{T}}V_3^{\mathrm{T}} + H_4^{\mathrm{T}}V_4^{\mathrm{T}}][Y_{m+1}][V_4H_4 + V_3H_3] \\
& + PQ[H_1^{\mathrm{T}}V_1^{\mathrm{T}} + H_3^{\mathrm{T}}V_3^{\mathrm{T}}][Y_{m+1}][V_3H_3 + V_1H_1] \\
& + \frac{P}{Q}[H_1^{\mathrm{T}}V_1^{\mathrm{T}} + H_4^{\mathrm{T}}V_4^{\mathrm{T}}][Y_{m+1}][V_4H_4 + V_1H_1] \\
& + \frac{Q}{P}[H_2^{\mathrm{T}}V_2^{\mathrm{T}} + H_3^{\mathrm{T}}V_3^{\mathrm{T}}][Y_{m+1}][V_3H_3 + V_2H_2] \\
& + \frac{1}{PQ}[H_2^{\mathrm{T}}V_2^{\mathrm{T}} + H_4^{\mathrm{T}}V_4^{\mathrm{T}}][Y_{m+1}][V_4H_4 + V_2H_2].
\end{aligned}
\tag{44}
$$

The above expression shows that the coefficients of P^2, Q^2, $1/P^2$ and $1/Q^2$ are all zero. The absence of these terms is also a feature of the work by Knopoff (1964), Thrower (1965) and Dunkin (1965).

For the special case $\beta < c$, the magnitude of Q and $1/Q$ would be one. For this case one could define

$$SQ = QV_3H_3 + \frac{1}{Q}V_4H_4. \tag{45}$$

Then equation (42) becomes

$$4Y_m = \left[PH_1^{\mathrm{T}}V_1^{\mathrm{T}} + \frac{1}{P}H_2^{\mathrm{T}}V_2^{\mathrm{T}} + SQ^{\mathrm{T}}\right][Y_{m+1}]\left[PV_1H_1 + \frac{1}{P}V_2H_2 + SQ\right]. \tag{46}$$

By using equation (43) one obtains

$$
\begin{aligned}
4Y_m = {} & SQ^{\mathrm{T}}[Y_{m+1}]SQ \\
& + [H_1^{\mathrm{T}}V_1^{\mathrm{T}} + H_2^{\mathrm{T}}V_2^{\mathrm{T}}][Y_{m+1}][V_1H_1 + V_2H_2] \\
& + P\{[H_1^{\mathrm{T}}V_1^{\mathrm{T}} + SQ^{\mathrm{T}}][Y_{m+1}][V_1H_1 + SQ] - SQ^{\mathrm{T}}[Y_{m+1}]SQ\} \\
& + \frac{1}{P}\{[H_2^{\mathrm{T}}V_2^{\mathrm{T}} + SQ^{\mathrm{T}}][Y_{m+1}][V_2H_2 + SQ] - SQ^{\mathrm{T}}[Y_{m+1}]SQ\}.
\end{aligned}
\tag{47}
$$

For the special case $c > \alpha > \beta$ the magnitude of P and Q would be one. In this case the direct use of the Thomson–Haskell matrix would be justified. But if one would prefer to follow the present scheme, a new matrix should be defined

$$SP = PV_1H_1 + \frac{1}{P}V_2H_2. \tag{48}$$

Then equation (42) becomes

$$4Y_m = [SP^{\mathrm{T}} + SQ^{\mathrm{T}}][Y_{m+1}][SP + SQ]. \tag{49}$$

To facilitate writing a computer program to carry the computation, we have listed in the Appendix all the matrices referred to in this text.

Numerical results

In an attempt to avoid the difficulty associated with the loss of precision in calculating the dispersion function at high frequencies, there are at least three independent investigators (Knopoff 1964, Dunkin 1965 and Thrower 1965) who have succeeded in omitting mathematically the exponential terms P^2, $1/P^2$, Q^2 and $1/Q^2$ from the computation process. The approach of Thrower as well as Dunkin is usually known as the delta matrix technique, which is given in detail by Pestel & Leckie (1963).

From reading Knopoff (1964) and Dunkin (1965) as well as Thrower (1965), I received the impression that the problem of the loss of precision had already been solved by each of the scientists. However, Schwab & Knopoff (1970) and Schwab (1970) reported limitations on the frequency values due to the loss of precision when an actual computer program was developed utilizing the best of the work published by Knopoff (1964), Dunkin (1965), Thrower (1965), Randall (1967) and Watson (1970). Schwab & Knopoff (1970) measured the limitation by the ratio of the total thickness of the layers above the half space to the wavelength. They found that a loss of precision occurs if such a ratio is greater than 5¼ with the use of a single precision eight decimal digit computer (IBM 7094).

In the delta matrix technique and Knopoff's method, the layer computation is started by linear expressions of the terms $PQ, P/Q, Q/P$ and $1/PQ$ for each matrix element. Only the largest term in each of these expressions would take advantage of the total number of the decimal digits allowed in the computer. The smaller terms may lose some of their significant digits. As these expressions interact with each other during matrix multiplication, a loss of precision may occur.

The advantage of our method over all other published methods is that the matrix multiplications for a layer computation are done independently of its thickness (see equation (44) and Appendix).

We have written a computer program using our method to determine the dispersion function. The program is in single-precision mode. We chose one of the layered structures used by Haskell (1953) to demonstrate the use of this program. The model consists of two layers on the top of a half-space. The parameter for the first layer, the second layer and the half-space are ($\alpha = 6.14$, 5.50 and 8.26 km/s), ($\beta = 3.39$, 3.18 and 4.65 km/s), ($\rho = 2.7$, 2.7 and 3.0) and ($d = 13.60$, 11.85 and ∞). We have calculated the dispersion function for a grid of frequency and phase velocity values. The frequency range is 0_+ to 10 Hz and the phase velocity range is from 3.1 to 3.5 km/s. The data are presented in Fig. 2 where the positive and negative values are represented by the white and dark bands respectively. The boundary between these two bands represents the zero values of the dispersion function. Fig. 3 is an expansion of the dispersion function data near the shear velocity of the first layer. The frequency range in Fig. 3 is from 0_+ to 20 Hz. In the program the actual values of the dispersion function are kept on the computer storage unit for further use. Information drawn from the storage unit can be used to find the roots of the dispersion function and/or to determine the group velocity.

The computations for Figs 2 and 3 were done on an eight decimal digit computer (Univac 1108). For the highest frequency in Fig. 3, the thickness of one of the layers corresponds to more than 80 wavelengths. This result is a partial confirmation of the title of this paper, that there is no frequency limitation with our technique as there has been with all previously published methods.

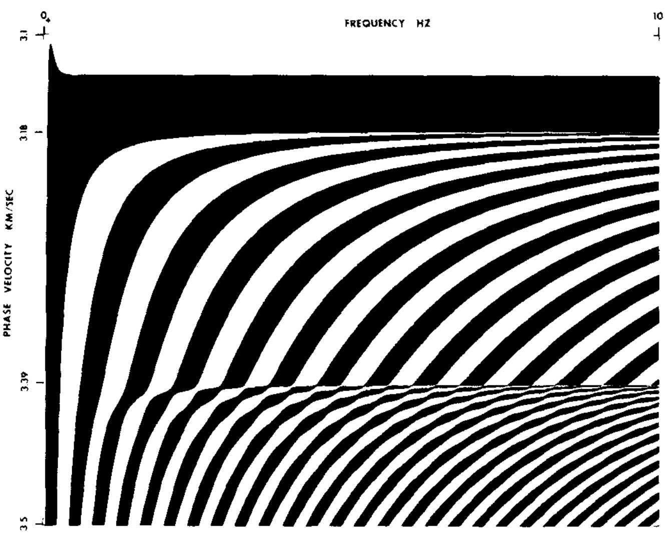

Figure 2. Dispersion function of the model described in the text. The frequency range is (0_+–10 Hz). The phase velocity range is (3.1–3.5 km/s).

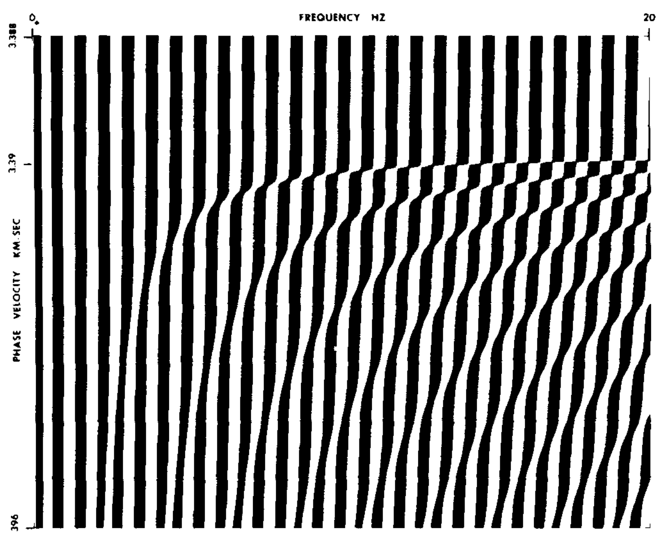

Figure 3. Expansion of the dispersion function data near the shear velocity of the first layer. The frequency range is (0_+–20 Hz). The phase velocity range is (3.388–3.396 km/s).

One of the reviewers of this paper claimed that Knopoff's method has no loss of precision but only an overflow. The reviewer provides us with modification to take care of the overflow problem. The modification consists of adding the following block of code in front of GO TO 230 and in front of 230 CONTINUE in the computer program (Schwab & Knopoff 1972, p. 126).

```
AMXINV = 1.0D + 00/DMAX1(DABS(UKNP),
            DABS(VKNP), DABS(SKNP),
            DABS(WKNP), DABS(RKNP))
UKNP = AMXINV*UKNP
VKNP = AMXINV*VKNP
SKNP = AMXINV*SKNP
WKNP = AMXINV*WKNP
RKNP = AMXINV*RKNP
```

To support our claim that all the published methods for calculating the dispersion function suffer from the loss of precision problem, we have carried numerical tests to examine the delta matrix method and Knopoff's method in its published form as well as in its form after the reviewer modification.

To examine the delta matrix method we used a program developed for using this method at Gulf Research Center. To test Knopoff's method we used the published computer program (Schwab & Knopoff 1972, p. 126), with and without the modification suggested by the reviewer.

The numerical test was to reproduce Fig. 3 of this paper, using the same computer, Univac 1108, in its single precision mode. The result of the test is that all the three programs lost precision beyond certain frequency value. Specifically, the delta matrix holds up to 2.5 c/s which corresponds to about 10 wavelengths in the thickest layer. Knopoff's method produced correct results up to 2.1 c/s. By incorporating the modification for the overflow suggested by the reviewer in Schwab & Knopoff's program, the correct computations extended to 3.9 c/s which corresponds to about 15 wavelengths in the thickest layer.

The above numerical results confirm our claim that the use of the Thomson–Haskell formulation, Knopoff's method or the delta matrix approach, has had the persistent problem of loss of precision at high frequency. The severity of the high-frequency limitation problem varies among the approaches, but exists in all of them. The two tests on Knopoff's method show that even in the same approach with some manipulations in the computations as this supplied by the reviewer, the severity of the limitation could be improved by as much as a factor of 2, but there are still limitations.

Conclusions

The present approach offers the numerical solution for the dispersion of layered elastic media without restriction on the value of the frequency. The approach used here is not restricted to real or complex material constants, phase velocity and/or frequency. Such generality makes this approach useful for elastic as well as anelastic media (Schwab & Knopoff 1971), and also the analysis of normal mode as well as leaking modes (Phinney 1961, Gilbert 1964, Dainty 1971 and Watson 1972).

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Appendix: definition of the matrix elements

$$V_1H_1 = \begin{bmatrix} \gamma & -(\gamma-1)/r_\alpha & -1/\rho & 1/(\rho r_\alpha) \\ r_\alpha\gamma & -(\gamma-1) & -r_\alpha/\rho & 1/\rho \\ \rho\gamma(\gamma-1) & -\rho(\gamma-1)^2/r_\alpha & -(\gamma-1) & (\gamma-1)/r_\alpha \\ \rho\gamma^2 r_\alpha & -\rho\gamma(\gamma-1) & -\gamma r_\alpha & \gamma \end{bmatrix}$$

$$V_2H_2 = \begin{bmatrix} \gamma & (\gamma-1)/r_\alpha & -1/\rho & -1/(\rho r_\alpha) \\ -r_\alpha\gamma & -(\gamma-1) & r_\alpha/\rho & 1/\rho \\ \rho\gamma(\gamma-1) & \rho(\gamma-1)^2/r_\alpha & -(\gamma-1) & -(\gamma-1)/r_\alpha \\ -\rho r_\alpha\gamma^2 & -\rho\gamma(\gamma-1) & \gamma r_\alpha & \gamma \end{bmatrix}$$

$$\begin{bmatrix} -\rho\gamma(\gamma-1) & \rho\gamma^2 r_\beta & \gamma & -\gamma r_\beta \\ -\rho(\gamma-1)^2/r_\beta & \rho\gamma(\gamma-1) & (\gamma-1)/r_\beta & -(\gamma-1) \end{bmatrix}$$

$$V_4 H_4 = \begin{bmatrix} -(\gamma-1) & -\gamma r_\beta & 1/\rho & r_\beta/\rho \\ (\gamma-1)/r_\beta & \gamma & -1/(\rho r_\beta) & -1/\rho \\ -\rho\gamma(\gamma-1) & -\rho\gamma^2 r_\beta & \gamma & \gamma r_\beta \\ \rho(\gamma-1)^2/r_\beta & \rho\gamma(\gamma-1) & -(\gamma-1)/r_\beta & -(\gamma-1) \end{bmatrix}$$

$$\frac{V_1 H_1 + V_2 H_2}{2} = \begin{bmatrix} \gamma & 0 & -1/\rho & 0 \\ 0 & -(\gamma-1) & 0 & 1/\rho \\ \rho\gamma(\gamma-1) & 0 & -(\gamma-1) & 0 \\ 0 & -\rho\gamma(\gamma-1) & 0 & \gamma \end{bmatrix}$$

$$V_1 H_1 + V_3 H_3 =$$

$$\begin{bmatrix} 1 & -(\gamma-1)/r_\alpha + \gamma r_\beta & 0 & 1/(\rho r_\alpha) - r_\beta/\rho \\ \gamma r_\alpha - (\gamma-1)/r_\beta & 1 & -r_\alpha/\rho + 1/(\rho r_\beta) & 0 \\ 0 & -\rho(\gamma-1)^2/r_\alpha + \rho\gamma^2 r_\beta & 1 & (\gamma-1)/r_\alpha - \gamma r_\beta \\ \rho\gamma^2 r_\alpha - \rho(\gamma-1)^2/r_\beta & 0 & -\gamma r_\alpha + (\gamma-1)/r_\beta & 1 \end{bmatrix}$$

$$V_1 H_1 + V_4 H_4 =$$

$$\begin{bmatrix} 1 & -(\gamma-1)r_\alpha - \gamma r_\beta & 0 & 1/(\rho r_\alpha) + r_\beta/\rho \\ \gamma r_\alpha + (\gamma-1)/r_\beta & 1 & -r_\alpha/\rho - 1/(\rho r_\beta) & 0 \\ 0 & -\rho(\gamma-1)^2/r_\alpha - \rho\gamma^2 r_\beta & 1 & (\gamma-1)/r_\alpha + \gamma r_\beta \\ \rho\gamma^2 r_\alpha + \rho(\gamma-1)^2/r_\beta & 0 & -\gamma r_\alpha - (\gamma-1)/r_\beta & 1 \end{bmatrix}$$

$$V_2 H_2 + V_3 H_3 =$$

$$\begin{bmatrix} 1 & (\gamma-1)/r_\alpha + \gamma r_\beta & 0 & -1/(\rho r_\alpha) - r_\beta/\rho \end{bmatrix}$$

$V_2H_2+V_4H_4=$

$$\begin{bmatrix} 1 & (\gamma-1)/r_\alpha-\gamma r_\beta & 0 & -1/(\rho r_\alpha)+r_\beta/\rho \\ -\gamma r_\alpha+(\gamma-1)/r_\beta & 1 & r_\alpha/\rho-1/(\rho r_\beta) & 0 \\ 0 & \rho(\gamma-1)^2/r_\alpha-\rho\gamma^2 r_\beta & 1 & -(\gamma-1)/r_\alpha+\gamma r_\beta \\ -\rho\gamma^2 r_\alpha+\rho(\gamma-1)^2/r_\beta & 0 & \gamma r_\alpha-(\gamma-1)/r_\beta & 1 \end{bmatrix}$$

$$\frac{V_3H_3+V_4H_4}{2}=\begin{bmatrix} -(\gamma-1) & 0 & 1/\rho & 0 \\ 0 & \gamma & 0 & -1/\rho \\ -\rho\gamma(\gamma-1) & 0 & \gamma & 0 \\ 0 & \rho\gamma(\gamma-1) & 0 & -(\gamma-1) \end{bmatrix}$$

$$Y_n=\begin{bmatrix} 0 & \gamma^2-\dfrac{(\gamma-1)^2}{r_\alpha r_\beta} & \dfrac{1}{\rho r_\beta} & \dfrac{-\gamma}{\rho}+\dfrac{\gamma-1}{\rho r_\alpha r_\beta} \\ \dfrac{(\gamma-1)^2}{r_\alpha r_\beta}-\gamma^2 & 0 & \dfrac{-(\gamma-1)}{\rho r_\alpha r_\beta}+\dfrac{\gamma}{\rho} & \dfrac{-1}{\rho r_\alpha} \\ \dfrac{-1}{\rho r_\beta} & \dfrac{-\gamma}{\rho}+\dfrac{\gamma-1}{\rho r_\alpha r_\beta} & 0 & \dfrac{1}{\rho^2}-\dfrac{1}{\rho^2 r_\alpha r_\beta} \\ \dfrac{-(\gamma-1)}{\rho r_\alpha r_\beta}+\dfrac{\gamma}{\rho} & \dfrac{1}{\rho r_\alpha} & \dfrac{1}{\rho^2 r_\alpha r_\beta}-\dfrac{1}{\rho^2} & 0 \end{bmatrix}$$

$$SP=\begin{bmatrix} \gamma\left(P+\dfrac{1}{P}\right) & \dfrac{-(\gamma-1)}{r_\alpha}\left(P-\dfrac{1}{P}\right) & -\dfrac{1}{\rho}\left(P+\dfrac{1}{P}\right) & \dfrac{1}{\rho r_\alpha}\left(P-\dfrac{1}{P}\right) \\ \gamma r_\alpha\left(P-\dfrac{1}{P}\right) & -(\gamma-1)\left(P+\dfrac{1}{P}\right) & \dfrac{-r_\alpha}{\rho}\left(P-\dfrac{1}{P}\right) & \dfrac{1}{\rho}\left(P+\dfrac{1}{P}\right) \\ \rho\gamma(\gamma-1)\left(P+\dfrac{1}{P}\right) & \dfrac{-\rho(\gamma-1)^2}{r_\alpha}\left(P-\dfrac{1}{P}\right) & -(\gamma-1)\left(P+\dfrac{1}{P}\right) & \dfrac{(\gamma-1)}{r_\alpha}\left(P-\dfrac{1}{P}\right) \\ \rho\gamma^2 r_\alpha\left(P-\dfrac{1}{P}\right) & -\rho\gamma(\gamma-1)\left(P+\dfrac{1}{P}\right) & -\gamma r_\alpha\left(P-\dfrac{1}{P}\right) & \gamma\left(P+\dfrac{1}{P}\right) \end{bmatrix}$$

$$SQ = \begin{bmatrix} -(\gamma-1)\left(Q+\frac{1}{Q}\right) & \gamma r_\beta\left(Q-\frac{1}{Q}\right) & \frac{1}{\rho}\left(Q+\frac{1}{Q}\right) & -\frac{r_\beta}{\rho}\left(Q-\frac{1}{Q}\right) \\ -\frac{(\gamma-1)}{r_\beta}\left(Q-\frac{1}{Q}\right) & \gamma\left(Q+\frac{1}{Q}\right) & \frac{1}{\rho r_\beta}\left(Q-\frac{1}{Q}\right) & -\frac{1}{\rho}\left(Q+\frac{1}{Q}\right) \\ -\rho\gamma(\gamma-1)\left(Q+\frac{1}{Q}\right) & \rho\gamma^2 r_\beta\left(Q-\frac{1}{Q}\right) & \gamma\left(Q+\frac{1}{Q}\right) & -\gamma r_\beta\left(Q-\frac{1}{Q}\right) \\ -\rho(\gamma-1)^2\left(Q-\frac{1}{Q}\right) & \rho\gamma(\gamma-1)\left(Q+\frac{1}{Q}\right) & \frac{(\gamma-1)}{r_\beta}\left(Q-\frac{1}{Q}\right) & -(\gamma-1)\left(Q+\frac{1}{Q}\right) \end{bmatrix}.$$