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Dispersion Relations in Nucleon-Nucleon Scattering

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Two-pion contribution to the absorptive part of nucleon-nucleon scattering amplitudes in the unphysical region is calculated using the dispersion relations for pion-nucleon scattering. The dispersion relations with this absorptive part are used for analyzing nucleon-nucleon scattering data at low energy and at moderate energy, and we find good agreement if we choose the coupling constant as

$$\frac{f^2}{4\pi}=0.08\pm0.01.$$

§ 1. Introduction

Although the dispersion relations have proved so successful in the problem of pion-nucleon phenomena, the corresponding relationship for nucleon-nucleon case has not been exploited in detail.[1] This is due to the appearance of large unphysical regions which do not correspond to directly observable quantities. However, it is possible to calculate this unphysical contribution in terms of the meson theory with sufficient accuracy to compare with experimental information. The application of this relation to experimental data will be an important test for our understanding of local field theories and this is the purpose of this paper.

The validity of dispersion relations for nucleon-nucleon scattering has not been proved with the same rigor as the pion-nucleon case, but for forward scattering the relation has been proved in every order in perturbation expansion.[2] We expand the S-matrix in the number of exchanged pions and calculate one- and two-pion exchange terms of its absorptive part. In doing so, use is made of the dispersion relations for pion-nucleon scattering. Three- and more-pion exchange terms, after one subtraction, turned out to be unimportant for moderate nucleon energies.

The dispersion relation with this absorptive part in the unphysical region is used for analyzing proton-proton scattering below 40 Mev where phase shift analysis was carried out.[3] It is also applied to low energy neutron-proton scattering. We find good agreement if we choose the coupling constant as

$$\frac{f^2}{4\pi}=0.08\pm0.01.$$

In § 2 we calculate the two-pion contribution to the absorptive part in the unphysical region. This dispersion relation is applied to proton-proton scattering in § 3, and to low energy nucleon-nucleon scattering in § 4. Conclusion and discussions are given in § 5.

§ 2. One- and two-pion contribution to the absorptive part

In this section we calculate the absorptive part of the forward scattering amplitudes in the unphysical region. Here the momenta of the incoming particles are in general complex and the scattering matrices are not defined in the usual sense. However, there are various ways of extending into complex variables. For example, in Feynman's perturbation expansion the scattering matrices are written as rational functions in the momentum variables and they are well defined even for complex momenta.

The S-matrix elements are divided into two parts, i.e. dispersive part D and absorptive part A:

$$\begin{aligned} S(E) &= (2\pi)^4\delta^4(\Sigma p)\, i(D(E)+iA(E)) \quad E\geq m, \\ &= (2\pi)^4\delta^4(\Sigma p)\, i(D(E)-iA(E)) \quad E<m, \end{aligned} \tag{2·1}$$

where E is the laboratory energy of the incoming nucleon and m is the nucleon mass. D and A are written in the form

$$D(E)=\bar{u}(p)\bar{u}(q)\,d(E)\,u(p)\,u(q)$$

$$A(E)=\bar{u}(p)\bar{u}(q)\,a(E)\,u(p)\,u(q)$$

and $d(E)$ and $a(E)$ satisfy dispersion relations.[1] In this paper we are interested only in low energy phenomena in which relativistic effects are unimportant. We often make non-relativistic approximations whenever it is convenient and we do not distinguish between d and D or a and A since energy dependence of the Dirac spinor u, $\bar{u}$, is small. For the case of forward scattering in definite spin state, D and A are both real and

$$A(E)=\frac{1}{(2\pi)^4\delta^4(0)}\mathrm{Re}\,S(E), \quad E<m. \tag{2·2}$$

The problem is thus the calculation of the real part of the S-matrix for $E<m$.

The corresponding Feynman diagrams are drawn in Fig. 1 and Fig. 2. α, α', β or β' stand for spin and isotopic spin state of each particle As is easily seen,

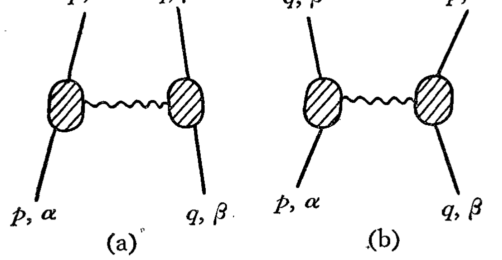

Fig. 1.

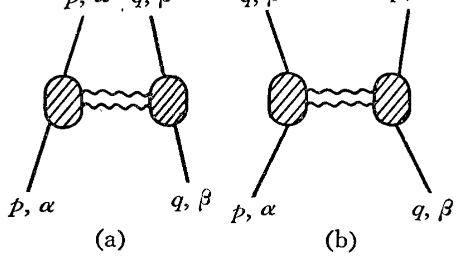

Fig. 2

Fig. 1(a) and Fig. 2(a) give no contribution to the absorptive part in the unphysical region[1] in which we are interested (Fig 1(a) is actually zero), so we concern ourselves with exchange type diagrams, i.e. Fig. 1(b) and Fig. 2(b). The momentum p is held fixed and q is varied in such a way that

$$p_\mu=(m\,;0),$$
$$q_\mu=(E\,;q\boldsymbol{e}),\quad q^2=E^2-m^2.$$

The absorptive part for one-pion exchange, $A_{1\pi}$, is rather trivial and*

$$A_{1\pi}(E)=\frac{f^2}{4\pi}\frac{2\pi^2}{m}(\alpha'|\boldsymbol{\sigma e}\tau_i|\beta)\,(\beta'|\boldsymbol{\sigma e}\tau_i|\alpha)\,\delta\Big(E-m+\frac{1}{2m}\Big) \qquad (2\cdot3)$$

where f is the renormalized coupling constant.**

For the calculation of the two-pion part, $A_{2\pi}$, it is convenient to choose the center-of-mass system. In this case

$$p_\mu'=\Big(E'\,;-\frac{q}{2}\boldsymbol{e}\Big),\quad q_\mu'=\Big(E'\,;\frac{q}{2}\boldsymbol{e}\Big).$$
$$E'^2=\frac{q^2}{4}+m^2.$$

The S-matrix is expressed as[4]

$$S_{2\pi}=-\frac{1}{2}\frac{1}{(2\pi)^8}\iint\frac{\langle j,l|S^1|i,k\rangle\langle j,-l|S^2|i,-k\rangle}{(k_0^2-\omega_k^2)\,(l_0^2-\omega_l^2)}d^4k\,d^4l, \qquad (2\cdot4)$$

where $\langle j,l|S^1|i,k\rangle$ is the scattering amplitude of an i-th pion with momentum k into a j-th pion with momentum l by the first nucleon. Similarly, $\langle j,-l|S^2|i,-k\rangle$ describes the scattering by the second nucleon. The momentum transfer is given by

$$l-k=p'-q'=t,$$
$$t_\mu=(0\,;-q\boldsymbol{e}).$$

$A_{2\pi}$ has non-vanishing value for $q^2<-4$.[1] In this unphysical region the momentum transfer is purely imaginary and the scattering matrices $\langle|S^i|\rangle$ must be defined in some way. Here we proceed as follows. The scattering matrix with constant momentum transfer can be expressed in a dispersion formula in the pion energy Since we are interested only in the region of fairly small t_μ^2, it can be sufficiently well approximated by the following expression[4]:

$$\langle j,l|S|i,k\rangle=(2\pi)^4 i\delta^4(l-k-t)[\boldsymbol{A}(k_0)\tau_i\tau_j\boldsymbol{\sigma k\sigma l}+\boldsymbol{B}(k_0)$$
$$\times(\tau_i\tau_j\boldsymbol{\sigma l\sigma k}+\tau_j\tau_i\boldsymbol{\sigma k\sigma l})+\boldsymbol{C}(k_0)\tau_j\tau_i\boldsymbol{\sigma l\sigma k}] \qquad (2\cdot5)$$

* We take $\hbar=c=pion\ mass=1$.

** $2mf=g$ is the pseudoscalar coupling constant.

$$A(k_0)=f^2\frac{1}{k_0-\varepsilon}+\frac{1}{4\pi}\int_0^\infty\frac{dp}{\omega_p}\frac{\sigma_{33}(p)}{\omega_p-k_0}+\frac{1}{36\pi}\int_0^\infty\frac{dp}{\omega_p}\frac{4\sigma_{11}+4\sigma_{13}+\sigma_{33}}{\omega_p+k_0},$$

$$B(k_0)=\frac{1}{12\pi}\int_0^\infty\frac{dp}{\omega_p}\frac{\sigma_{33}+2\sigma_{13}}{\omega_p-k_0}+\frac{1}{12\pi}\int_0^\infty\frac{dp}{\omega_p}\frac{\sigma_{33}+2\sigma_{13}}{\omega_p+k_0},$$

$$C(k_0)=-f^2\frac{1}{k_0+\varepsilon}+\frac{1}{36\pi}\int_0^\infty\frac{dp}{\omega_p}\frac{4\sigma_{11}+4\sigma_{13}+\sigma_{33}}{\omega_p-k_0}+\frac{1}{4\pi}\int_0^\infty\frac{dp}{\omega_p}\frac{\sigma_{33}}{\omega_p+k_0},\quad (2\cdot6)^*$$

$$\varepsilon=\frac{\Delta^2-2}{4m},\quad \Delta^2=-q^2>0$$

Except for the first terms of A and C, (2·6) is equal to the static formula. However, the ε in the denominator cannot be neglected since $A_{2\pi}$ (Eq. (2·8)) is divergent without this term. In this sense $A_{2\pi}$ does not have a static limit as $1/m\to0$. This scattering matrix is defined also for complex momenta.

The momentum transfer t is time-like in the sense that $t_\mu^2=t_0^2-t^2=\Delta^2>0$, and it is possible to choose a Lorentz frame in which $t_\mu'=(\Delta, 0)$. Notice that the S-matrix as defined by the Feynman method is invariant under (complex) Lorentz transformation. In this reference frame the calculation of the absorptive part is straightforward since it is identical with the calculation of total cross section for the production of two (physical) pions.** The momenta of the two pions are

$$l_\mu'=\left(\frac{\Delta}{2};k_1,k_2,k_3\right),\quad -k_\mu'=\left(\frac{\Delta}{2};-k_1,-k_2,-k_3\right),$$

$$k_1^2+k_2^2+k_3^2=\frac{\Delta^2}{4}-1,$$

or in the original reference frame,

$$k_\mu=\left(ik_3;k_1,k_2,i\frac{\Delta}{2}\right),\quad l_\mu=\left(ik_3;k_1,k_2,-i\frac{\Delta}{2}\right),$$

if $\boldsymbol{e}$ is chosen as the third axis In this way, if

$$S_{2\pi}=\delta^4(0)\int\frac{d^4k\,d^4l}{(k_0^2-\omega_k^2)(l_0^2-\omega_l^2)}\delta^4(l-k-t)f(k,l),$$

its absorptive part in the unphysical region is given by

$$A_{2\pi}=-\frac{1}{16\pi^2\Delta}\int d^3k\,\delta\left(k^2-\left(\frac{\Delta^2}{4}-1\right)\right)f\left(ik_3;k_1,k_2,i\frac{\Delta}{2},ik_3;k_1,k_2,-i\frac{\Delta}{2}\right),$$

$$\Delta^2\geq4.\quad (2\cdot7)$$

(2·4), (2·5), (2·6) and (2·7) yield

* In the following, we neglect σ_{11} and σ_{13}.

** This procedure is not always justified when unphysical threshold appears.

$$A_{2\pi}(E) = -\frac{1}{16\pi\Delta}\int_{-\sqrt{\Delta^2/4-1}}^{\sqrt{\Delta^2/4-1}} dk_3 \Big\{ (\boldsymbol{\tau}^1\boldsymbol{\tau}^2)^2 \Big[\Big(\frac{\Delta^2}{2}-1-k_3^2\Big)^2 + \frac{1}{2}\Delta^2\Big(\frac{\Delta^2}{4}-1-k_3^2\Big)$$

$$\times ((\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2)-(\boldsymbol{\sigma}^1\boldsymbol{e})(\boldsymbol{\sigma}^2\boldsymbol{e}))\Big]\Big[\boldsymbol{A}(ik_3)\boldsymbol{C}(ik_3)+\boldsymbol{B}(ik_3)\boldsymbol{B}(ik_3)\Big]$$

$$+2\tau_i^1\tau_j^1\tau_j^2\tau_i^2\Big[\Big(\frac{\Delta^2}{2}-1-k_3^2\Big)^2+\frac{1}{2}\Delta^2\Big(\frac{\Delta^2}{4}-1-k_3^2\Big)((\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2)-(\boldsymbol{\sigma}^1\boldsymbol{e})(\boldsymbol{\sigma}^2\boldsymbol{e}))\Big]$$

$$\times \boldsymbol{A}(ik_3)\boldsymbol{B}(ik_3)$$

$$+2(\boldsymbol{\tau}^1\boldsymbol{\tau}^2)^2\Big[\Big(\frac{\Delta^2}{2}-1-k_3^2\Big)^2-\frac{1}{2}\Delta^2\Big(\frac{\Delta^2}{4}-1-k_3^2\Big)((\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2)-(\boldsymbol{\sigma}^1\boldsymbol{e})(\boldsymbol{\sigma}^2\boldsymbol{e}))\Big]$$

$$\times \boldsymbol{A}(ik_3)\boldsymbol{B}(ik_3)$$

$$+\tau_i^1\tau_j^1\tau_j^2\tau_i^2\Big[\Big(\frac{\Delta^2}{2}-1-k_3^2\Big)^2-\frac{1}{2}\Delta^2\Big(\frac{\Delta^2}{4}-1-k_3^2\Big)((\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2)-(\boldsymbol{\sigma}^1\boldsymbol{e})(\boldsymbol{\sigma}^2\boldsymbol{e}))\Big]$$

$$\times\Big[\boldsymbol{A}(ik_3)\boldsymbol{A}(ik_3)+\boldsymbol{B}(ik_3)\boldsymbol{B}(ik_3)\Big]\Big\}, \qquad (2\cdot8)$$

where $E=m-\Delta^2/2m$ and spin and isospin operator X^1Y^2 means $(\alpha'|X^1|\beta)(\beta'|Y^2|\alpha)$ This matrix element can be written as a linear combination of $(\alpha'|X^1|\alpha)(\beta'|Y^2|\beta)$, which we simply write as XY Then

$$A_{1\pi}(E) = \Big(-\frac{3}{4}+\frac{1}{4}(\boldsymbol{\tau\tau})+\frac{3}{4}(\boldsymbol{\sigma\sigma})-\frac{1}{4}(\boldsymbol{\tau\tau})(\boldsymbol{\sigma\sigma})-\frac{3}{2}(\boldsymbol{\sigma e})(\boldsymbol{\sigma e})$$

$$+\frac{1}{2}(\boldsymbol{\tau\tau})(\boldsymbol{\sigma e})(\boldsymbol{\sigma e})\Big)\frac{f^2}{4\pi}\frac{2\pi^2}{m}\delta\Big(E-m+\frac{1}{2m}\Big) \qquad (2\cdot9)$$

and

$$A_{2\pi}(E) = A_{2\pi}^{(1)}+(\boldsymbol{\tau\tau})A_{2\pi}^{(2)}+(\boldsymbol{\sigma\sigma})A_{2\pi}^{(3)}+(\boldsymbol{\tau\tau})(\boldsymbol{\sigma\sigma})A_{2\pi}^{(4)}$$

$$+(\boldsymbol{\sigma e})(\boldsymbol{\sigma e})A_{2\pi}^{(5)}+(\boldsymbol{\tau\tau})(\boldsymbol{\sigma e})(\boldsymbol{\sigma e})A_{2\pi}^{(6)}. \qquad (2\cdot10)$$

The final expression for $A_{2\pi}$ is given in the Appendix and numerical results are given in Table I. In general, the forward scattering amplitudes D and A can be expanded in the following way,

$$D(E) = D^{(1)}+(\boldsymbol{\tau\tau})D^{(2)}+(\boldsymbol{\sigma\sigma})D^{(3)}+(\boldsymbol{\tau\tau})(\boldsymbol{\sigma\sigma})D^{(4)}+(\boldsymbol{\sigma e})(\boldsymbol{\sigma e})D^{(5)}$$

$$+(\boldsymbol{\tau\tau})(\boldsymbol{\sigma e})(\boldsymbol{\sigma e})D^{(6)}, \qquad (2\cdot11)$$

and

$$A(E) = A^{(1)}+(\boldsymbol{\tau\tau})A^{(2)}+(\boldsymbol{\sigma\sigma})A^{(3)}+(\boldsymbol{\tau\tau})(\boldsymbol{\sigma\sigma})A^{(4)}+(\boldsymbol{\sigma e})(\boldsymbol{\sigma e})A^{(5)}$$

$$+(\boldsymbol{\tau\tau})(\boldsymbol{\sigma e})(\boldsymbol{\sigma e})A^{(6)}.$$

Each coefficient $D^{(i)}(E)$ and $A^{(i)}(E)$ separately satisfies dispersion relations[1)]

$$D^{(i)}(E) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{A^{(i)}(E')}{E'-E}dE' \qquad (2\cdot12)$$

$$=\frac{P}{\pi}\int_m^\infty \frac{A^{(i)}(E')}{E'-E}dE'+\frac{f^2}{4\pi}\frac{2\pi}{m}\frac{\lambda_i}{m-\frac{1}{2m}-E}$$

$$-\frac{32\pi}{m^2}\frac{R^{-1}}{1-R^{-1}\rho}\frac{\mu_i}{E-E_D}+\frac{1}{\pi}\int_{-\infty}^{m-2/m}\frac{A^{(i)}(E')}{E'-E}dE',$$

where

$$\lambda_1=-\frac{3}{4},\quad \lambda_2=\frac{1}{4},\quad \lambda_3=\frac{3}{4},\quad \lambda_4=-\frac{1}{4},\quad \lambda_5=-\frac{3}{2},\quad \lambda_6=\frac{1}{2},$$

$$\mu_1=\frac{3}{16},\quad \mu_2=-\frac{3}{16},\quad \mu_3=\frac{1}{16},\quad \mu_4=-\frac{1}{16},\quad \mu_5=\mu_6=0,$$

and R is the radius of the deuteron, $R^{-1}=\sqrt{mB}$, and $\rho(=\rho(-B,-B))$ is the effective range for spin triplet state, and $E_D=m-2B+B^2/2m$, B being the binding energy of the deuteron.

For $E\geq m$, $A(E)$ can be related by the so-called optical theorem to the total scattering cross section $\sigma_{tot}(E)$ as

$$A(E)=\frac{q}{2E}\sigma_{tot}(E). \qquad (2\cdot13)$$

Table I.

Δ*	$A^{(1)}_{2\pi}$	$A^{(2)}_{2\pi}$	$A^{(3)}_{2\pi}$	$A^{(4)}_{2\pi}$	$A^{(5)}_{2\pi}$	$A^{(6)}_{2\pi}$
2.0	0.00	0.00	0.00	0.00	0.00	0.00
2.05	−0.18	0.45	−0.10	0.39	0.08	−0.06
2.1	−0.31	0.55	−0.16	0.43	0.15	−0.12
2.2	−0.48	0.72	−0.21	0 49	0.27	−0.24
2.4	−0.70	0.97	−0.27	0.56	0.43	−0.41
2.6	−0.89	1.13	−0.32	0.60	0.57	−0.54
2.8	−1.05	1.25	−0.37	0.61	0.69	−0.64
3.0	−1.21	1.31	−0.42	0.60	0.80	−0.71

* $E=m-\Delta^2/2m$

§ 3. Application to proton-proton scattering

Probably no subtraction will be necessary for the dispersion relation for two-nucleon scattering. In practical applications, however, it is more convenient to make one subtraction to suppress the high energy effect which is less known. In this paper we use the dispersion relation of the type

$$D(E)-D(m)=\frac{E-m}{\pi}P\left[\int_m^\infty+\int_{-\infty}^m\right]\frac{A(E')dE'}{(E'-m)(E'-E)} \qquad (3\cdot1)$$

to bring in the experimental data. The second integral of (3·1) is the unphysical contribution, arising from the exchange of pions. In the last section we have calculated one- and two-pion exchange term, the latter starting from $E=E_{2\pi}=m-(2/m)=m-40$ Mev to the left. For $E<-m$, the absorptive part A is related to the total nucleon-antinucleon cross section. Between $-m$ and $E_{3\pi}=m-(9/2m)=m-90$ Mev, from which three-pion term $A_{3\pi}$ starts, the absorptive part is very complicated and hard to evaluate. However, since $A(E_{3\pi})$ and $A(-m)$ are nearly equal, an interpolation by a straight line would give a rough estimate for three- and more-pion contribution. This term turned out to be small for low and moderate energies (see Table II).

In this section, the relation is used for analyzing proton-proton scattering data up to 40 Mev. In Eq. (3·1) the low energy S-wave scattering gives large contribution to both sides. In order to eliminate this, we subtract

$$\frac{E-m}{\pi}P\int_m^\infty \frac{A_{eff}(E')}{(E'-m)(E'-E)}dE'$$

from both sides, where A_{eff} is the absorptive part calculated from the effective range formula. Scattering length and effective range used here are

$$\begin{aligned} {}^1a(pp) &= -15.6\times 10^{-13}\,\text{cm} \\ {}^1r(pp) &= 2\cdot 65\times 10^{-13}\,\text{cm}. \end{aligned} \qquad (3\cdot 2)$$

Taking unpolarized p–p beam, $D=D^{(1)}+D^{(2)}$ and $A=A^{(1)}+A^{(2)}$. The final formula is

$$\begin{aligned} D(E) = &-\frac{2\pi}{m}\left(a+\frac{a^2}{\sqrt{a^2-2ar}}\right)+\frac{\sigma_{eff}(E)}{m}\frac{1+\frac{ar}{2}q_c^2}{\sqrt{a^2-2ar}} \\ &+\frac{(E-m)}{\pi}P\int_m^\infty \frac{\frac{q'}{2E'}(\sigma(E')-\sigma_{eff}(E'))}{(E'-E)(E'-m)}dE' \qquad (3\cdot 3) \\ &+\frac{E-m}{\pi}\int_{-\infty}^{m}\frac{A(E')}{(E'-E)(E'-m)}dE'. \end{aligned}$$

The right-hand side is compared with $D(E)$, the real part of forward scattering amplitude,

$$\begin{aligned} D(E) = \frac{2\pi}{mq_c}(&\sin\delta_S\cos\delta_S+\sin\delta_P^{J=0}\cos\delta_P^{J=0}+3\sin\delta_P^{J=1}\cos\delta_P^{J=1} \\ &+5\sin\delta_P^{J=2}\cos\delta_P^{J=2}+5\sin\delta_D\cos\delta_D+\cdots) \\ A(E) = \frac{2\pi}{mq_c}(&\sin^2\delta_S+\sin^2\delta_P^{J=0}+3\sin^2\delta_P^{J=1}+5\sin^2\delta_P^{J=2} \\ &+5\sin^2\delta_D+\cdots).\quad E\geq m \end{aligned} \qquad (3\cdot 4)^*$$

* We neglect the difference between the nuclear bar phase shifts used here and true nuclear phase shifts which we do not know.

The total cross section used are those cited in the review article by Hess[5] and the values cited in (3·2). Results are given in Table II and plotted in Fig. 3. In this energy region, the experimental data show no disagreement with the theory. It is hoped that the more data on phase shifts, both for *p–p* and *n–p* be obtained shortly.

Table II.

E_{lab} (Mev)	D	1st line in (2)	2nd line	3rd line			Sum
				1-π	2π	3- or more-π	
9.68	0.88	1.09	0.02	−.23	.01	.01	.90
9.73	0.74	1.08	0.02	−.23	.01	.01	.89
14.16	0.80	1.04	0.02	−.28	.02	.01	.81
18.2	1.04	1.02	0.03	−.32	.02	.01	.76
19.8	0.71	1.02	0.03	−.33	.02	.01	.75
31.8	0.38 (0.52)*	1.04	0.05	−.38	.02	.02	.75
39.4	0.83	1.06	0.12	−.40	.03	.02	.83

* Result from the phase shift analysis from which the 14° point is dropped.

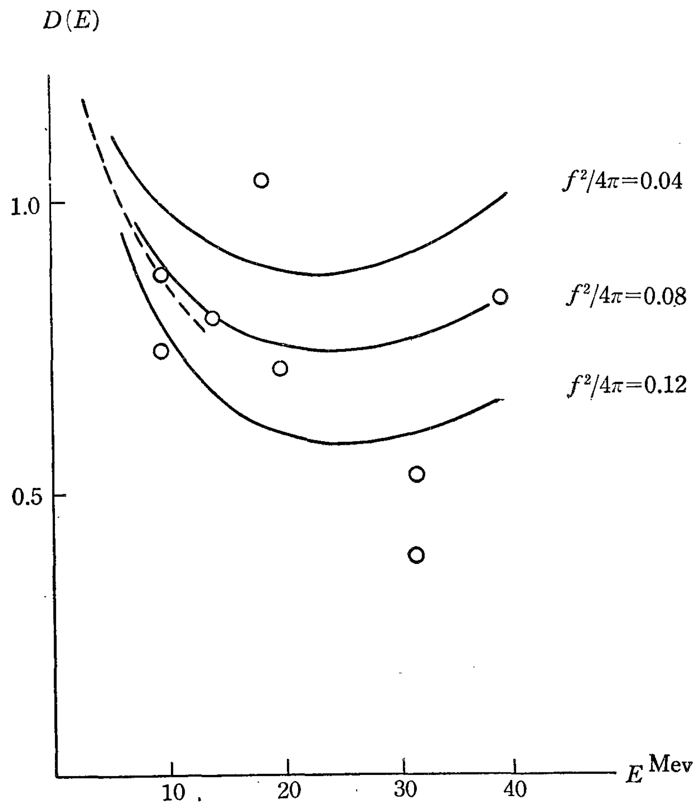

Fig. 3. Real part of the forward *p-p* scattering amplitude
Solid lines are theoretical curves (Eq. (3·3)) for various choices of $f^2/4\pi$. Circles are experimental points.[3] Dashed line is obtained with the effective range approximation.

§ 4. Application to nucleon-nucleon scattering at low energy*

In this section, we apply the dispersion relation to low energy nucleon-nucleon scattering. As proton-proton scattering has ambiguity due to Coulomb interaction, we concern ourselves with neutron-proton scattering in spin triplet, isospin singlet state which is experimentally well known. In this case,

$$D=D^{(1)}-3D^{(2)}+D^{(3)}-3D^{(4)}+\frac{1}{3}D^{(5)}-D^{(6)},$$

and (4·1)

$$A=A^{(1)}-3A^{(2)}+A^{(3)}-3A^{(4)}+\frac{1}{3}A^{(5)}-A^{(6)}.$$

The relation to be used is

$$\lim_{E\to m}\frac{D(E)-D(m)}{E-m}=\lim_{E\to m}\frac{P}{\pi}\int_m^\infty\frac{A(E')\,dE'}{(E'-E)\,(E'-m)}+\frac{1}{\pi}\int_{-\infty}^m\frac{A(E')\,dE'}{(E'-m)^2}+\frac{32\pi R^{-1}}{(1-R^{-1}\rho)\,m^2\,(m-E_D)^2}$$

$$=\lim_{E\to m}\frac{P}{\pi}\int_m^\infty\frac{\frac{q'}{2E'}\sigma(E')\,dE'}{(E'-E)\,(E'-m)}+\frac{1}{\pi}\int_{-\infty}^{m-2/m}\frac{A(E')\,dE'}{(E'-m)^2}-\frac{f^2}{4\pi}\frac{2\pi}{m}\frac{1}{\left(E-m+\frac{1}{2m}\right)^2}+\frac{32\pi R^{-1}}{(1-R^{-1}\rho)\,m^2\,(m-E_D)^2}. \quad (4·2)$$

The last term in the right-hand side comes from deuteron, and the third term comes from one-pion exchange process. The second term comes from two-pion exchange process which we have calculated in § 2, and from three- and more-pion exchange processes which we neglect here as its contribution may be small.

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* The problem discussed in this section is the same as the one discussed by S. Matsuyama.[6] He retained only one-pion exchange neglecting multi-pion processes, while we included also two-pion contribution here. He discussed n-p scattering in spin triplet state, so

$$D=D^{(1)}-D^{(2)}+D^{(3)}-D^{(4)}+\frac{1}{3}D^{(5)}-\frac{1}{3}D^{(6)},$$
$$A=A^{(1)}-A^{(2)}+A^{(3)}-A^{(4)}+\frac{1}{3}A^{(5)}-\frac{1}{3}A^{(6)}, \quad (4·1)'$$

and the relation used was

$$D(E)-D(m)=(E-m)\frac{P}{\pi}\int_m^\infty\frac{\frac{q'}{2E'}\sigma(E')\,dE'}{(E'-E)\,(E'-m)}-\frac{f^2}{4\pi}\frac{4\pi}{3m}\frac{(E-m)}{\left(E'-m+\frac{1}{2m}\right)^2}+\frac{16\pi R^{-1}(E-m)}{(1-R^{-1}\rho)m^2(m-E_D)\,(E-E_D)}. \quad (4·2)'$$

The second and third terms are functions of the pion-nucleon coupling constant, f^2. The left-hand side and the first term in the right-hand side can be expressed in terms of observable quantities,

$$D(E_c)=D_{S\text{-wave}}(E_c)+D_{D\text{-wave}}(E_c)+\cdots,$$

$$D_{S\text{-wave}}(E_c)=\frac{8\pi}{mq_c}\sin\delta_S\cos\delta_S,$$

$$\cdots\cdots\cdots\cdots,$$

and

$$A(E_c)=A_{S\text{-wave}}(E_c)+A_{D\text{-wave}}(E_c)+\cdots,$$

$$A_{S\text{-wave}}(E_c)=\frac{8\pi}{mq_c}\sin^2\delta_S,$$

$$\cdots\cdots\cdots\cdots,$$

and

$$q_c\cot\delta_S=-\frac{1}{a_S}+\frac{1}{2}r_Sq_c^2-Pr_S^3q_c^4+\cdots,$$

$$\cdots\cdots\cdots\cdots,$$

where the subscript c means "in the center-of-mass system".

The left-hand side of Eq. (4·2) contains only S-wave scattering length a_S, and S-wave effective range r_S,

$$\lim_{E\to m}\frac{D(E)-D(m)}{E-m}=4\pi\left(a_S^3-\frac{1}{2}a_S^2r_S\right).$$

The first term in the right-hand side of Eq. (4·2) contains the contribution from all even partial waves, but we can neglect all contributions except that from low energy S-wave scattering in good approximation,

$$\lim_{E\to m}\frac{P}{\pi}\int_m^\infty\frac{A(E')\,dE'}{(E'-m)(E'-E)}\doteqdot-2\pi a_S^2\frac{2a_S^2+|a_Sr_S|-2a_Sr_S+4|a_Sr_S|\dfrac{r_SP_S}{a_S}}{\sqrt{a_S^2-a_Sr_S+|a_Sr_S|+4|a_Sr_S|\dfrac{r_SP_S}{a_S}}}.$$

The last term,

$$\frac{32\pi R^{-1}}{(1-R^{-1}\rho)m^2(m-E_D)^2}=\frac{8\pi}{(1-\gamma\rho)\gamma^3},$$

where

$$\frac{\gamma^2}{m}=B,$$

$$\gamma=1/a_S+\frac{1}{2}\rho\gamma^2-P\rho^3\gamma^4,$$

$$r_S=\rho-4P\rho^3/a_S^2.$$

Thus (4·2) is

$$4\pi\left(a_S^3-\frac{1}{2}a_S^2 r_S\right)+4\pi a_S^2\frac{a_S^2+\frac{1}{2}|a_Sr_S|-a_Sr_S+2|a_Sr_S|\frac{r_SP_S}{a_S}}{\sqrt{a_S^2-a_Sr_S+|a_Sr_S|+4|a_Sr_S|\frac{r_SP_S}{a_S}}}$$

$$-\frac{8\pi}{(1-\gamma\rho)\gamma^3}=-\frac{f^2}{4\pi}\frac{2\pi}{m}\frac{1}{\left(E-m+\frac{1}{2m}\right)^2}+\frac{1}{\pi}\int_{-\infty}^{m-2/m}\frac{A(E')}{(E'-m)^2}dE' \ . \qquad (4\cdot 3)$$

Inserting the experimental value for a_S and B known at present,

$$a_S=(5.377\pm 0.023)\times 10^{-13}\,\text{cm}$$

$$B=2.226\pm 0.004\,\text{Mev}$$

and an assumption,

$$P=0.00\pm 0.03,$$

we see the left-hand side of Eq. (4·3) is

$$-17.6\pm 2.2.^*$$

The right-hand side, the unphysical contribution, is

$$-11.7-2.6=-14.3 \quad \text{for} \quad f^2/4\pi=0.07$$

$$-13.4-3.8=-17.2 \quad \text{for} \quad f^2/4\pi=0.08$$

$$-15.1-4.8=-19.9 \quad \text{for} \quad f^2/4\pi=0.09.$$

The first numbers in the left represent the one-pion exchange contribution, and the second numbers represent the two-pion exchange contribution.

Thus we find as the pion-nucleon coupling constant

$$f^2/4\pi=0.082\pm 0.008.$$

§ 5. Conclusion and discussions

The results of the previous sections confirm the correctness of dispersion theoretic approach to the two-nucleon problem. The conventional meson theory of nuclear forces is as follows. One calculates nuclear potential perhaps by adiabatic approximation or including recoil corrections by expanding in p/m. Then he solves the Schrödinger equation with this potential to see if the theoretical phase shifts agree with experiment. This way of approach, however, meets with several difficulties. First, it is difficult to define " potential " without the adiabatic approximation. Second, the Schrödinger equation is non-relativistic and the rela-

* If P is negative, the coupling constant decreases.

tivistic two-body equation is not well established. Finally, we have to introduce cutoff since meson potential is badly singular at the origin.

In our approach all of these difficulties are removed. It is free from the adiabatic approximation and it is possible to formulate the theory in a relativistic way although we did it non-relativistically. It is our feeling that no subtraction is necessary in two-nucleon dispersion relations so that no more additional constant is needed. Taketani[7] has developed with success a theory in which he divided nuclear forces into inner and outer regions and proposed to deal only with the outer region meson theoretically. Taketani's idea is reproduced here by making one subtraction thereby eliminating less known high energy effects.

In this paper we have been concerned only with the forward scattering. By including the momentum transfer as another variable more information can be obtained from dispersion relations and it may even be possible to use them for solving the two-nucleon problem.[8]

This type of dispersion relations together with similar ones for n–p scattering could be used to determine the accurate value of the pion-nucleon coupling constant if the nucleon-nucleon scattering is measured with sufficient accuracy.

Appendix

Using the identities,

$$(\boldsymbol{\tau}^1\boldsymbol{\tau}^2)^2=3-2(\boldsymbol{\tau}^1\boldsymbol{\tau}^2),$$

and

$$\tau_i^1\tau_j^1\tau_j^2\tau_i^2=3+2(\boldsymbol{\tau}^1\boldsymbol{\tau}^2),$$

Eq. (2·8) becomes

$$\begin{aligned}A=-\frac{1}{16\pi\Delta}\int_{-k}^{k}dk_3 &\Big\{\left(\frac{\Delta^2}{2}-1-k_3^2\right)^2[3\boldsymbol{A}(ik_3)\boldsymbol{C}(ik_3)+6\boldsymbol{B}(ik_3)\boldsymbol{B}(ik_3)\\ &\qquad+12\boldsymbol{A}(ik_3)\boldsymbol{B}(ik_3)+3\boldsymbol{A}(ik_3)\boldsymbol{A}(ik_3)]\\ &+(\boldsymbol{\tau}^1\boldsymbol{\tau}^2)\left(\frac{\Delta^2}{2}-1-k_3^2\right)^2[2\boldsymbol{A}(ik_3)\boldsymbol{A}(ik_3)-2\boldsymbol{A}(ik_3)\boldsymbol{C}(ik_3)]\\ &+\frac{1}{2}\Delta^2\left(\frac{\Delta^2}{4}-1-k_3^2\right)((\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2)-(\boldsymbol{\sigma}^1\boldsymbol{e})(\boldsymbol{\sigma}^2\boldsymbol{e}))[3\boldsymbol{A}(ik_3)\boldsymbol{C}(ik_3)\\ &\qquad-3\boldsymbol{A}(ik_3)\boldsymbol{A}(ik_3)]\\ &+\frac{1}{2}\Delta^2\left(\frac{\Delta^2}{4}-1-k_3^2\right)((\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2)-(\boldsymbol{\sigma}^1\boldsymbol{e})(\boldsymbol{\sigma}^2\boldsymbol{e}))(\boldsymbol{\tau}^1\boldsymbol{\tau}^2)\\ &\qquad\times[-2\boldsymbol{A}(ik_3)\boldsymbol{A}(ik_3)-4\boldsymbol{B}(ik_3)\boldsymbol{B}(ik_3)\\ &\qquad+8\boldsymbol{A}(ik_3)\boldsymbol{B}(ik_3)-2\boldsymbol{A}(ik_3)\boldsymbol{C}(ik_3)]\Big\}\end{aligned}$$

$$=-\frac{1}{16\pi\Delta}\int_{-k}^{k}dk_3$$

$$\left\{3\left(\frac{\Delta^2}{2}-1-k_3^2\right)^2\left[f^4\frac{1}{k_3^2+\varepsilon^2}+f^4\frac{1}{(\varepsilon-ik_3)^2}\right.\right.$$

$$\left.-\frac{16}{9\pi}f^2\frac{\varepsilon}{k_3^2+\varepsilon^2}\int_0^\infty\frac{\sigma_{33}}{\omega^2+k_3^2}dp+\frac{32}{81\pi^2}\left(\int_0^\infty\frac{\sigma_{33}dp}{\omega^2+k_3^2}\right)^2\right]$$

$$+2(\boldsymbol{\tau}^1\boldsymbol{\tau}^2)\left(\frac{\Delta^2}{2}-1-k_3^2\right)^2\left[-f^4\frac{1}{k_3^2+\varepsilon^2}+f^4\frac{1}{(\varepsilon-ik_3)^2}\right.$$

$$\left.+\frac{8}{9\pi}f^2\int_0^\infty\frac{\sigma_{33}dp}{\omega(\omega^2+k_3^2)}-\frac{8k_3^2}{81\pi^2}\left(\int_0^\infty\frac{\sigma_{33}dp}{\omega(\omega^2+k_3^2)}\right)^2\right]$$

$$+\frac{3}{2}\Delta^2\left(\frac{\Delta^2}{4}-1-k_3^2\right)((\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2)-(\boldsymbol{\sigma}^1\boldsymbol{e})(\boldsymbol{\sigma}^2\boldsymbol{e}))$$

$$\times\left[f^4\frac{1}{k_3^2+\varepsilon^2}-f^4\frac{1}{(\varepsilon-ik_3^2)}\right.$$

$$\left.-\frac{8}{9\pi}f^2\int_0^\infty\frac{\sigma_{33}dp}{\omega(\omega^2+k_3^2)}+\frac{8k_3^2}{81\pi^2}\left(\int_0^\infty\frac{\sigma_{33}dp}{\omega(\omega^2+k_3^2)}\right)^2\right]$$

$$+\Delta^2\left(\frac{\Delta^2}{4}-1-k_3^2\right)((\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2)-(\boldsymbol{\sigma}^1\boldsymbol{e})(\boldsymbol{\sigma}^2\boldsymbol{e}))(\boldsymbol{\tau}^1\boldsymbol{\tau}^2)$$

$$\times\left[-\frac{f^4}{(\varepsilon-ik_3)^2}-\frac{f^4}{k_3^2+\varepsilon^2}+\frac{4}{9\pi}f^2\frac{\varepsilon}{k_3^2+\varepsilon^2}\int_0^\infty\frac{\sigma_{33}}{\omega^2+k_3^2}dp\right.$$

$$\left.\left.-\frac{2}{81\pi^2}\left(\int_0^\infty\frac{\sigma_{33}dp}{\omega^2+k_3^2}\right)^2\right]\right\}$$

$$=I_1(E)+I_2(E)(\boldsymbol{\tau}^1\boldsymbol{\tau}^2)+I_3(E)((\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2)-(\boldsymbol{\sigma}^1\boldsymbol{e})(\boldsymbol{\sigma}^2\boldsymbol{e}))$$
$$+I_4(E)((\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2)-(\boldsymbol{\sigma}^1\boldsymbol{e})(\boldsymbol{\sigma}^2\boldsymbol{e}))(\boldsymbol{\tau}^1\boldsymbol{\tau}^2)$$

$$=I_1\left(-\frac{1}{4}-\frac{1}{4}(\boldsymbol{\tau}\boldsymbol{\tau})-\frac{1}{4}(\boldsymbol{\sigma}\boldsymbol{\sigma})-\frac{1}{4}(\boldsymbol{\tau}\boldsymbol{\tau})(\boldsymbol{\sigma}\boldsymbol{\sigma})\right)$$

$$+I_2\left(-\frac{3}{4}+\frac{1}{4}(\boldsymbol{\tau}\boldsymbol{\tau})-\frac{3}{4}(\boldsymbol{\sigma}\boldsymbol{\sigma})+\frac{1}{4}(\boldsymbol{\tau}\boldsymbol{\tau})(\boldsymbol{\sigma}\boldsymbol{\sigma})\right)$$

$$+I_3\left(-\frac{1}{2}-\frac{1}{2}(\boldsymbol{\tau}\boldsymbol{\tau})+\frac{1}{2}(\boldsymbol{\sigma}\boldsymbol{e})(\boldsymbol{\sigma}\boldsymbol{e})+\frac{1}{2}(\boldsymbol{\tau}\boldsymbol{\tau})(\boldsymbol{\sigma}\boldsymbol{e})(\boldsymbol{\sigma}\boldsymbol{e})\right)$$

$$+I_4\left(-\frac{3}{2}+\frac{1}{2}(\boldsymbol{\tau}\boldsymbol{\tau})+\frac{3}{2}(\boldsymbol{\sigma}\boldsymbol{e})(\boldsymbol{\sigma}\boldsymbol{e})-\frac{1}{2}(\boldsymbol{\tau}\boldsymbol{\tau})(\boldsymbol{\sigma}\boldsymbol{e})(\boldsymbol{\sigma}\boldsymbol{e})\right)$$

$$=A^{(1)}_{2\pi}(E)+(\boldsymbol{\tau\tau})A^{(2)}_{2\pi}(E)+(\boldsymbol{\sigma\sigma})A^{(3)}_{2\pi}(E)+(\boldsymbol{\tau\tau})(\boldsymbol{\sigma\sigma})A^{(4)}_{2\pi}(E)$$
$$+(\boldsymbol{\sigma e})(\boldsymbol{\sigma e})A^{(5)}_{2\pi}(E)+(\boldsymbol{\tau\tau})(\boldsymbol{\sigma e})(\boldsymbol{\sigma e})A^{(6)}_{2\pi}(E).$$

Thus

$$A^{(1)}_{2\pi}(E)=-\frac{1}{4}I_1(E)-\frac{3}{4}I_2(E)-\frac{1}{2}I_3(E)-\frac{3}{2}I_4(E),$$

$$A^{(2)}_{2\pi}(E)=-\frac{1}{4}I_1(E)+\frac{1}{4}I_2(E)-\frac{1}{2}I_3(E)+\frac{1}{2}I_4(E),$$

$$A^{(3)}_{2\pi}(E)=-\frac{1}{4}I_1(E)-\frac{3}{4}I_2(E),$$

$$A^{(4)}_{2\pi}(E)=-\frac{1}{4}I_1(E)+\frac{1}{4}I_2(E),$$

$$A^{(5)}_{2\pi}(E)=\frac{1}{2}I_3(E)+\frac{3}{2}I_4(E),$$

$$A^{(6)}_{2\pi}(E)=\frac{1}{2}I_3(E)-\frac{1}{2}I_4(E),$$

where

$$I_1=-\frac{3}{16\pi\Delta}\Bigg[f^4\Bigg\{4m(2k^2+1)\tan^{-1}\Big(\frac{2mk}{2k^2+1}\Big)-\frac{2}{3}k^3$$
$$+4k(2k^2+1)+\frac{2k(2k^2+1)^2}{k^2+(2k^2+1)^2/4m^2}\Bigg\}$$
$$+f^2\Bigg\{-\frac{32}{9\pi}(2k^2+1)^2\tan^{-1}\Big(\frac{2mk}{2k^2+1}\Big)\int_0^\infty\frac{\sigma_{33}dp}{\omega^2}\Bigg\}$$
$$+\frac{32}{81\pi^2}\int_{-k}^{k}(2k^2+1-x^2)^2\Big(\int_0^\infty\frac{\sigma_{33}}{\omega^2+x^2}dp\Big)^2dx\Bigg],$$

$$I_2=-\frac{1}{8\pi\Delta}\Bigg[f^4\Bigg\{-4m(2k^2+1)\tan^{-1}\Big(\frac{2mk}{2k^2+1}\Big)-\frac{2}{3}k^3$$
$$+4k(2k^2+1)+\frac{2k(2k^2+1)^2}{k^2+(2k^2+1)^2/4m^2}\Bigg\}$$
$$+f^2\Bigg\{\frac{8}{9\pi}\int_{-k}^{k}(2k^2+1-x^2)^2\Big(\int_0^\infty\frac{\sigma_{33}dp}{\omega(\omega^2+x^2)^2}\Big)dx\Bigg\}$$
$$-\frac{8}{81\pi^2}\int_{-k}^{k}(2k^2+1-x^2)^2\Big(\int_0^\infty\frac{x\sigma_{33}dp}{(\omega^2+x^2)\omega}\Big)^2dx\Bigg],$$

$$I_3=-\frac{3}{32\pi\Delta}\Big[f^4\Big\{-16k(k^2+1)+16m\frac{k^2(k^2+1)}{2k^2+1}\tan^{-1}\Big(\frac{2mk}{2k^2+1}\Big)\Big\}$$

$$+f^2\Big\{-\frac{32}{9\pi}(k^2+1)\int_{-k}^{k}(k^2-x^2)\Big(\int\frac{\sigma_{33}dp}{\omega(\omega^2+x^2)}\Big)dx\Big\}$$

$$+\frac{32}{81\pi^2}(k^2+1)\int_{-k}^{k}(k^2-x^2)\Big(\int_0^\infty\frac{x\sigma_{33}dp}{(\omega^2+x^2)\omega}\Big)dx\Big],$$

$$I_4=-\frac{1}{16\pi\Delta}\Big[f^4\Big\{-16k(k^2+1)-16m\frac{k^2(k^2+1)}{2k^2+1}\tan^{-1}\Big(\frac{2mk}{2k^2+1}\Big)\Big\}$$

$$+f^2\Big\{\frac{32}{9\pi}(k^2+1)k^2\tan^{-1}\Big(\frac{2mk}{2k^2+1}\Big)\int_0^\infty\frac{\sigma_{33}dp}{\omega^2}\Big\}$$

$$-\frac{8}{81\pi^2}(k^2+1)\int_{-k}^{k}(k^2-x^2)\Big(\int_0^\infty\frac{\sigma_{33}dp}{\omega^2+x^2}\Big)^2dx\Big],$$

and

$$k=\sqrt{\Delta^2/4-1}\,.$$

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Note added in proof: Two new sets of phase shift were reported besides those tabulated in Table II. They fit closely to the theoretical curve.

E_{lab} /Mev		D		Reference
9.69		0.91		9)
25.63	0.08	0.78	0.02	10)

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