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Dissipative reaction diffusion systems with quadratic growth

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Abstract

We introduce a class of reaction diffusion systems of which weak solution exists global-in-time with relatively compact orbit in L^1. Reaction term in this class is quasi-positive, dissipative, and up to with quadratic growth rate. If the space dimension is less than or equal to two, the solution is classical and uniformly bounded. Provided with the entropy structure, on the other hand, this weak solution is asymptotically spatially homogeneous.

Keywords. reaction diffusion equation, weak solution, duality argument, entropy, asymptotic behavior.
MSC(2010) 35K57, 35B40

1 Introduction

The purpose of the present paper is to study global-in-time behavior of the solution to the reaction diffusion system. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and $\tau_j > 0$ and $d_j > 0$, $1 \leq j \leq N$, be constants. We consider the system

$$\tau_j \frac{\partial u_j}{\partial t} - d_j \Delta u_j = f_j(u) \quad \text{in } Q_T = \Omega \times (0,T),\ 1 \leq j \leq N$$
$$\left.\frac{\partial u_j}{\partial \nu}\right|_{\partial\Omega} = 0, \quad u_j|_{t=0} = u_{j0}(x) \geq 0, \tag{1}$$

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where $u=(u_j)$ and $T>0$.

We assume that

$$f_j:\mathbb{R}^N\to\mathbb{R} \text{ is locally Lipschitz continuous, } 1\leq j\leq N, \tag{2}$$

and therefore, system (1) admits a unique classical solution local-in-time if the initial value $u_0=(u_{j0}(x))$ is sufficiently smooth. Also, the nonlinearity is assumed to be quasi-positive, which means

$$f_j(u_1,\cdots,u_{j-1},0,u_{j+1},\cdots,u_n)\geq 0,\ 1\leq j\leq N,\ 0\leq u=(u_j)\in\mathbb{R}^N. \tag{3}$$

Here and henceforth, we say $u=(u_j)\geq 0$ if and only if $u_j\geq 0$ for any $1\leq j\leq N$. From this condition, the solution satisfies $u=(u_j(\cdot,t))\geq 0$ as long as it exists.

The solution which we handle with, however, is mostly weak solution defined as follows.

Definition 1 *We say that*

$$0\leq u=(u_j(\cdot,t))\in L^\infty_{loc}([0,T),L^1(\Omega)^N)\cap L^1_{loc}(0,T;W^{1,1}(\Omega)^N)$$

is a weak solution to (1) if $f(u)\in L^1_{loc}(\overline{\Omega}\times(0,T))$,

$$\tau_j\frac{d}{dt}\int_\Omega u_j\varphi\ dx+d_j\int_\Omega\nabla u_j\cdot\nabla\varphi\ dx=\int_\Omega f_j(u)\varphi\ dx,\quad 1\leq j\leq N$$

for any $\varphi\in W^{1,\infty}(\Omega)$ *in the sense of distributions with respect to* t, *and*

$$u_j|_{t=0}=u_{j0}(x),\quad 1\leq j\leq N$$

in the sense of measures on $\overline{\Omega}$.

Remark 1 *Similarly to the case of Dirichlet boundary condition in (1) (see, e.g., [2] and also Lemma 5.1 of [17]), the above weak solution* $u=(u_j(\cdot,t))$ *is in* $C((0,T),L^1(\Omega)^N)$ *and it holds that*

$$u_j(\cdot,t)=e^{t\tau_j^{-1}d_j\Delta}u_j(\cdot,\tau)+\int_\tau^t e^{(t-s)\tau_j^{-1}d_j\Delta}f_j(u(\cdot,s))\ ds,\quad 1\leq j\leq N \tag{4}$$

for any $0<\tau\leq t<T$. *Furthermore, we have*

$$\left[\int_\Omega u_j\varphi(\cdot,t)\right]_{t=t_1}^{t=t_2}=\iint_{\Omega\times(t_1,t_2)}\tau_ju_j\varphi_t-d_j\nabla u_j\cdot\nabla\varphi+f_j(u)\varphi\ dxdt$$

for any $1\leq j\leq N$, $0<t_1\leq t_2<T$, *and* $\varphi=\varphi(x,t)\in C^1(\overline{\Omega}\times[t_1,t_2])$.

Henceforth, C_i, $i = 1, 2, \cdots, 47$, denote positive constants. Besides (2)-(3) we assume at most quadratic growth of the nonlinearity $f(u) = (f_j(u))$,

$$|f(u)| \leq C_1(1 + |u|^2), \quad u = (u_j) \geq 0, \tag{5}$$

and also its dissipativity indicated by

$$\sum_{j=1}^{N} f_j(u) \leq 0, \quad u = (u_j) \geq 0. \tag{6}$$

We also assume

$$\frac{\partial f_j}{\partial u_j}(u) \geq -C_2(1 + |u|), \ 1 \leq j \leq N, \quad 0 \leq u = (u_j) \in \mathbb{R}^N. \tag{7}$$

For such a system, global-in-time existence of the weak solution is known as in Theorem 1 below, where $\| \ \|_p$, $1 \leq p \leq \infty$, stands for the standard L^p norm.

Theorem 1 (Pierre-Rolland [19]) *Assume (2), (3), (5), (6), and (7), and let*

$$0 \leq u_0 = (u_{j0}(x)) \in L^1(\Omega)^N$$

be given. Then there is a weak solution to (1) global-in-time, denoted by $0 \leq u = (u_j(\cdot, t)) \in C([0, +\infty), L^1(\Omega)^N)$, *which satisfies*

$$\begin{aligned} & u \in L^2_{loc}(\overline{\Omega} \times (0, +\infty))^N, \\ & \nabla u_j \in L^p_{loc}(\overline{\Omega} \times (0, +\infty))^N, \ 1 \leq p < \frac{4}{3}, \ 1 \leq j \leq N, \\ & \|u(\cdot, t)\|_1 \leq C_3 \|u_0\|_1 \quad \text{for } t \geq 0. \end{aligned} \tag{8}$$

Remark 2 *Provided with (2), (3), (5), and (6), global-in-time existence of the weak solution to (1)is proven for* $u_0 = (u_{j0}) \in L^2(\Omega)^N$ *in [17]. Theorem 1 is an extension of this result, in the sense that it admits general* $0 \leq u_0 \in L^1(\Omega)^N$.

Remark 3 *Inequality (6) is used to guarantee for the limit of approximate solutions to be a sub-solution to (1) (see also Theorem 5.14 of [18]). This inequality may be relaxed as*

$$\sum_{j=1}^{N} f_j(u) \leq C_4(b \cdot u + 1), \quad 0 \leq u = (u_j) \in \mathbb{R}^N$$

for Theorem 1 to hold, where $0 \leq b = (b_j) \in \mathbb{R}^N$.

Remark 4 *Inequality (7) may be so relaxed as (H6) in [19]. This inequality, however, is used also in the proof of Theorem 3 below.*

Generally, weak solution can include blowup time and may not be unique. The first result proven in this paper is concerned with the orbit constructed in Theorem 1.

Theorem 2 *The orbit* $\mathcal{O} = \{u(\cdot,t) \mid t \geq 0\}$ *made by the solution* $u = (u_j(\cdot,t))$ *in Theorem 1 is relatively compact in* $L^1(\Omega)^N$.

The second result is the regularity of this solution.

Theorem 3 *Assume (7) in addition to (2), (3), (5), and (6), and let* $n \leq 2$ *and* $0 \leq u_0 = (u_{j0}(x))$ *be sufficiently smooth. Then the weak solution* $u = (u_j(\cdot,t))$ *to (1) obtained in Theorem 1 is classical, and takes relatively compact orbit* $\mathcal{O} = \{u(\cdot,t) \mid t \geq 0\}$ *in* $C(\overline{\Omega})^N$.

Remark 5 *Since the classical solution is unique, Theorem 3 assures the existence of a unique classical solution to (1), which is global-in-time and uniformly bounded.*

The first example covered by Theorems 1-3 is the four-component system describing chemical reaction $A_1 + A_3 \leftrightarrow A_2 + A_4$:

$$N = 4, \quad f_j(u) = (-1)^j(u_1u_3 - u_2u_4), \ 1 \leq j \leq 4. \tag{9}$$

There is a weak solution global-in-time (9) which converges exponentially to a unique spatially homogeneous stationary state in L^1 norm [4, 5, 6, 8, 7]. Similar results hold for the renormalized solution [11] involving higher growth rate [20]. Also, this solution is classical even in higher space dimensions when the diffusion coeffcients are quasi-uniform [10].

The second example is the Lotka-Volterra system, where

$$f_j(u) = (-e_j + \sum_k a_{jk}u_k)u_j, \quad 1 \leq j \leq N, \tag{10}$$

in (1). For (10) the assumptions of Theorem 1 are fulfilled if

$$0 \leq (e_j) \in \mathbb{R}^N \tag{11}$$

and

$$(Au, u) \leq 0, \quad 0 \leq u = (u_j) \in \mathbb{R}^N \tag{12}$$

where $A = (a_{jk})$.

This system, (1) with (10), is studied in [25], and an analogous result to Theorem 3 is obtained under a stronger condition than (11)-(12), that is,

$$0 \le (e_j) \in \mathbb{R}^N, \quad {}^tA + A = 0, \ A = (a_{jk}). \tag{13}$$

Here, equality ${}^tA + A = 0$ in (11) was applied to prevent blowup in infinite time. Theorem 2, therefore, provides a natural extension of our previous work [25] even to (10), in the sense that the condition (13) is relaxed as (11)-(12).

Remark 6 *The nonlinearities (9) and (10) with (13) for $(e_j) = 0$ satisfy the equality in (6):*

$$\sum_{j=1}^{N} f_j(u) = 0, \quad 0 \le u = (u_j) \in \mathbb{R}^N. \tag{14}$$

Under this condition, blowup in finite time is excluded if $n \le 2$ (see [12] and also Proposition 3.2 of [4]). Blowup in infinite time is also excluded by the proof of Proposition 5.1 of [25], replacing (5.4) by (3.12) with (3.19) there. Hence Theorem 3 is still valid without (7) if (14) is assumed for (6). This result holds even if $-e_j u_j$ is added to $f_j(u)$ satisfying (14) for each $1 \le j \le N$, where $e_j \ge 0$ is a constant.

We recall that a fundamental property derived from (6) is the total mass control, indicated by

$$\frac{d}{dt}\int_\Omega \tau \cdot u \ dx \le 0, \quad \tau = (\tau_j) > 0. \tag{15}$$

Besides (15), blowup analysis is used in [25] for the study of (10)-(11), based on the scaling

$$u_\mu(x,t) = \mu^2 u(\mu x, \mu^2 t), \quad \mu > 0. \tag{16}$$

At this process, the inequality

$$\sum_{j=1}^{N} f_j(u) \log u_j \le C_5(1 + |u|^2), \quad u = (u_j) \ge 0 \tag{17}$$

is confirmed, and plays a key role in establishing a priori estimates of the solution in [25]. Actually, (17) is valid for general $f = (f_j(u))$ satisfying (7).

Proposition 1 *If the nonlinearity $f = (f_j(u))$, $u = (u_j)$, satisfies (2), (3), (6), and (7), then inequality (17) holds true.*

Without the scaling property (16), we use the point-wise inequality derived from (6),

$$\frac{\partial}{\partial t}(\tau \cdot u) - \Delta(d \cdot u) \leq 0 \text{ in } Q_T, \quad \left.\frac{\partial}{\partial \nu}(d \cdot u)\right|_{\partial\Omega} \leq 0, \quad d = (d_j) > 0. \tag{18}$$

(We actually have the equality for the boundary condition on $d \cdot u$ in (18).) Obviously, (15) is a direct consequence of (18), which, however, deduces several other important properties. The estimate below is obtained by the duality argument recently developed (see [18]).

Proposition 2 (Pierre [18]) *If $0 \leq u = (u_j(x,t))$ is smooth on $\overline{\Omega} \times [0,T]$ and satisfies (18), then it follows that*

$$\|u\|_{L^2(Q_T)} \leq C_6 T^{1/2} \|u_0\|_2, \quad u|_{t=0} = u_0. \tag{19}$$

By the argument developed in our previous work [25], inequality (19) guarantees global-in-time existence of the classical solution, indicated by $T = +\infty$, under the assumptions of Theorem 3. The next proposition, on the other hand, is a refinement of the above Proposition 2, and may be used alternatively to derive a key inequality for the uniform boundedness of this global-in-time solution, that is, inequality (85) in section 3. See Remark 11.

Proposition 3 *Under the assumptions of Proposition 2, it holds that*

$$\|u\|_{L^2(Q(\eta,T))} \leq C_7(\eta,T) \|u_0\|_1^{1/2} \|u\|_{L^1(Q_T)}^{1/2} \tag{20}$$

for any $0 < \eta < T$ where $Q(\eta,T) = \Omega \times (\eta,T)$.

Spatially asymptotic homogenization is observed for (1) with (10)-(11) under the presence of entropy [16, 25]. The final result in this paper shows that this phenomenon is extended to the weak solution.

Theorem 4 *Assume (2), (3), (5), and (6), and let*

$$0 \leq u = (u_j(\cdot,t)) \in C([0,+\infty), L^1(\Omega)^N) \tag{21}$$

be the global-in-time weak solution to (1) in Theorem 1. Define its ω-limit set by

$$\omega(u_0) = \{u_* \in L^1(\Omega)^N \mid \exists t_k \uparrow +\infty, \ \lim_{k\to\infty} \|u(\cdot, t_k) - u_*\|_1 = 0\}.$$

Then we have the following properties:

1. *Assume $f_j(u) = u_j g_j(u)$, $1 \leq j \leq N_1$, with*

$$|g_j(u)| \leq C_8(1+|u|), \ \sum_{j=1}^{N_1} b_j \tau_j^{-1} g_j(u) \geq 0, \quad 0 \leq u = (u_j) \in \mathbb{R}^N, \tag{22}$$

   where $0 < b = (b_j) \in \mathbb{R}^{N_1}$ and $1 \leq N_1 \leq N$. Assume, furthermore,

$$\log u_{j0} \in L^1(\Omega), \quad 1 \leq j \leq N_1. \tag{23}$$

   Then it holds that

$$P_1\omega(u_0) \subset \mathbb{R}_+^{N_1} = \{u = (u_1, \cdots, u_{N_1}) \in \mathbb{R}^N \mid u_1, \cdots, u_{N_1} > 0\}$$

   where $P_1 : (u_1, \cdots, u_N) \mapsto (u_1, \cdots, u_{N_1})$.

2. *Assume that inequality (6) is improved as*

$$\sum_{j=1}^{N} f_j(u) \leq -e \cdot u, \quad 0 \leq u = (u_j) \in \mathbb{R}^N \tag{24}$$

   with $0 \leq e = (e_j) \in \mathbb{R}^N$ satisfying $e_{N_2+1}, \cdots, e_N > 0$ for $N_2 \geq N_1$. Then it holds that $P_2\omega(u_0) = \{0\}$, where $P_2 : (u_1, \cdots, u_N) \mapsto (u_{N_2+1}, \cdots, u_N)$.

Remark 7 *The second inequality of (22) provides with a Lyapunov function to (1). Instead of (23), on the other hand, we may assume $u_{j0} \in L^\infty(\Omega)$ with $u_{j0} \not\equiv 0$, $1 \leq j \leq N_1$, by the strong maximum principle and the parabolic regularity.*

Remark 8 *Theorem 4 is applicable to the Lotka-Volterra system. Thus we have a wide class of (6) with (13) provided with $(N-2)$ entropies, where any non-stationary spatially homogeneous solutions are periodic-in-time [13]. For such a system, the ω-limit set $\omega(u_0)$ forms a spatially homogeneous periodic solution or a unique spatially homogeneous stationary state. In particular, the ω-limit set $\omega(u_0)$ in Theorem 4 is not always contained in the set of stationary solutions.*

This paper is composed of four sections and five appendices. Theorems 2, 3, and 4 are proven in Sections 2, 3, and 4, respectively. Then Propositions 1, 2, and 3 are proven in Sections A, B, and C, respectively.

We shall use the duality argument, relying on the study of the parabolic problem

$$\frac{\partial v}{\partial t} - \Delta(av) = f \text{ in } Q_T, \quad \left.\frac{\partial}{\partial \nu}(av)\right|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0(x) \tag{25}$$

where

$$0 < C_9^{-1} \leq a = a(x,t) \leq C_9, \quad f \in L^2(Q_T), \quad v_0 \in L^2(\Omega) \tag{26}$$

to which Section D is devoted. This study takes a significant role in this paper, because (18) implies

$$\frac{\partial v}{\partial t} - \Delta(av) \leq 0 \text{ in } Q_T, \quad \left.\frac{\partial}{\partial \nu}(av)\right|_{\partial\Omega} \leq 0$$

for $v = \tau \cdot u + 1$ and $a = \frac{d \cdot u + 1}{\tau \cdot u + 1}$.

Section E is concerned with the regularity of the weak solution to the heat equation

$$\frac{\partial w}{\partial t} = \Delta w + H \text{ in } Q_T, \quad \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega} = 0, \quad w|_{t=0} = w_0(x) \tag{27}$$

for

$$w_0 \in L^1(\Omega), \quad H \in L^1(Q_T). \tag{28}$$

Here, compactness of the mapping (Proposition 10)

$$(w_0, H) \in L^1(\Omega) \times L^1(Q_T) \mapsto w \in L^1(Q_T)$$

is particularly important for the proof of Theorem 2.

2 Proof of Theorem 2

Outline of this section: Global-in-time existence of the weak solution is known under the assumptions of Theorem 2. Here we shall show that this orbit is relatively compact in $L^1(\Omega)$. Given $t_k \uparrow +\infty$, we construct a compact family of functions in $L^1(Q_0)^N$ which dominates $u_k = u_k(x,t) = u(x, t + t_k) \geq 0$ above, where $Q_0 = \Omega \times (-\eta_0, 1)$ for $\eta_0 > 0$. We prove that this dominating sequence is bounded in $L^2(Q_{\eta_0})$ which implies that $\{f_j(u_k)\}$ is bounded *in* $L^1(Q_{\eta_0})$*. This bound implies the compactness of* $\{u_k\}$ *in* $L^1(Q_{\eta_0})$ *due to the compactness of the mapping* $(w_0, H) \in L^1(\Omega) \times L^1(Q_T) \mapsto w \in L^1(Q_T)$ *in (27). Then, we even prove that the dominating sequence is relatively compact in* $L^2(Q_\eta)$*,* $\eta \in (0, \eta_0)$*. From dominating convergence, it follows that* $\{u_k\}$ *is itself relatively compact in* $L^2(Q_\eta)$*. Then a sub-sequence of* $f_j(u_k)$ *converges in* $L^1(Q_\eta)$ *so that* u_k *converges in* $C([-\eta, 1]; L^1(\Omega))$*. In particular,* $u(\cdot, t_k)$ *converges in* $L^1(\Omega)$ *which is our main objective.*

First, we confirm the scheme [19] to construct the global-in-time weak solution to (1) (see Remark 2 in §1 for a historical note). In fact, the initial value $0 \leq u_0 = (u_{0j}) \in L^1(\Omega)^N$ is approximated by smooth $\tilde{u}_0^\ell = (\tilde{u}_{j0}^\ell)$, $\ell = 1, 2, \cdots$, satisfying

$$\begin{aligned} &\tilde{u}_{j0}^\ell = \tilde{u}_{j0}^\ell(x) \geq \max\{\frac{1}{\ell}, u_{j0}(x)\} \quad \text{a.e. in } \Omega \\ &\tilde{u}_{j0}^\ell \to u_{j0} \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega, \quad 1 \leq j \leq N. \end{aligned} \tag{29}$$

Second, the nonlinearity is modified by a smooth, non-decreasing truncation $T_\ell : [0, +\infty) \to [0, \ell + 1]$, such that $T_\ell(s) = s$ for $0 \leq s \leq \ell$. Then the nonlinearity $f^\ell = (f_j \circ T_\ell)$ satisfies (2), (3), and (6) for $f = (f_j^\ell)$. Then we take the unique global-in-time classical solution $\tilde{u}^\ell = (\tilde{u}_j^\ell(\cdot, t))$ to

$$\begin{aligned} &\tau_j \frac{\partial \tilde{u}_j^\ell}{\partial t} - d_j \Delta \tilde{u}_j^\ell = f_j^\ell(\tilde{u}^\ell) \quad \text{in } \Omega \times (0, +\infty) \\ &\left.\frac{\partial \tilde{u}_j^\ell}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \left.\tilde{u}_j^\ell\right|_{t=0} = \tilde{u}_{j0}^\ell(x) \end{aligned} \tag{30}$$

to obtain

$$\|\tau \cdot \tilde{u}^\ell(\cdot, t)\|_1 \leq \|\tau \cdot \tilde{u}^\ell(\cdot, s)\|_1, \quad 0 \leq s \leq t < +\infty \tag{31}$$

and in particular,

$$\sup_{t \geq 0} \|\tilde{u}^\ell(\cdot, t)\|_1 \leq C_{10}. \tag{32}$$

Third, we have

$$\|\tilde{u}_j^\ell\|_{L^2(Q(\eta,T))} + \|\nabla \tilde{u}_j^\ell\|_{L^p(Q(\eta,T))^N} \leq C_{11}(\eta, T, p, \|u_0\|_1), \quad 1 \leq j \leq N \tag{33}$$

for $0 < \eta < T$ and $1 \leq p < \frac{4}{3}$, recalling $Q(\eta, T) = \Omega \times (\eta, T)$. Finally, up to a subsequence we have

$$\tilde{u}^\ell \to u \quad \text{in } L^1_{loc}(\overline{\Omega} \times [0, +\infty))^N \text{ and a.e. in } \Omega \times (0, +\infty). \tag{34}$$

See the proof of Theorem 1 of [19] for (33)-(34).

Summig up, we obtain

$$\|\tau \cdot u(\cdot,t)\|_1 \leq \|\tau \cdot u(\cdot,s)\|_1, \quad 0 \leq s \leq t < +\infty$$
$$\sup_{t\geq 0} \|u(\cdot,t)\|_1 \leq C_{10} \tag{35}$$

by (31)-(32). It holds also that

$$\|u_j\|_{L^2(Q(\eta,T))} + \|\nabla u_j\|_{L^p(Q(\eta,T))^N} \leq C_{11}(\eta,T,p,\|u_0\|_1), \quad 1 \leq j \leq N \tag{36}$$

by (33), and this $u = (u_j(\cdot,t))$ is a weak solution to (1) satisfying (8). In particular, we obtain $u = (u_j(\cdot,t)) \in C([0,+\infty), L^1(\Omega)^N)$ by Remark 1.

Given $t_k \uparrow +\infty$, let

$$u_{jk}(\cdot,t) = u_j(\cdot,t+t_k), \quad u_k = (u_{jk}(\cdot,t)), \quad Q = \Omega \times (-2,1). \tag{37}$$

It holds that

$$\|u_k\|_{L^2(Q)^N} \leq C_{12} \tag{38}$$

by (36) and hence

$$\|f(u_k)\|_{L^1(Q)^N} \leq C_{13}.$$

Since

$$\|u_k(\cdot,-2)\|_1 \leq C_{10}$$

holds by (35), passing to a subsequence, we have

$$u_k \to u_\infty \quad \text{in } L^1(Q)^N \text{ and a.e. in } Q \tag{39}$$

by Proposition 10 in §E. From (36), furthermore, this u_∞ is a weak solution to (1) (for a different initial value) satisfying (8). In particular, it holds that

$$u_k \rightharpoonup u_\infty \text{ weakly in } L^2(Q)^N, \quad \|u_\infty\|_{L^2(Q)^N} \leq C_{12} \tag{40}$$

by (38).

The coefficients

$$\underline{a} \leq a_k(x,t) \equiv \frac{d \cdot u_k + 1}{\tau \cdot u_k + 1} \leq \overline{a}, \quad \underline{a} \leq a_\infty(x,t) \equiv \frac{d \cdot u_\infty + 1}{\tau \cdot u_\infty + 1} \leq \overline{a} \tag{41}$$

are well-defined, provided with the property

$$a_k \to a_\infty \quad \text{a.e. in } Q \tag{42}$$

where

$$\underline{a} = \inf_{s>0} \frac{\underline{d}s+1}{\overline{\tau}s+1} > 0, \quad \overline{a} = \sup_{s>0} \frac{\overline{d}s+1}{\underline{\tau}s+1} < +\infty$$

for $\underline{d} = \min_j d_j$, $\overline{d} = \max_j d_j$, $\underline{\tau} = \min_j \tau_j$, and $\overline{\tau} = \max_j \tau_j$.

Since the first convergence in (39) means

$$\lim_{k\to\infty} \int_{-2}^{1} \|u(\cdot, t+t_k) - u_\infty(\cdot, t)\|_1 \, dt = 0, \tag{43}$$

we have

$$\lim_{k\to\infty} \|u_k(\cdot, t) - u_\infty(\cdot, t)\|_1 = 0 \quad \text{for a.e. } t \in (-2, 1),$$

passing to a subsequence. In particular, there is $\eta_0 \in (1, 2)$ such that

$$u_k(\cdot, -\eta_0) \to u_\infty(\cdot, -\eta_0) \quad \text{in } L^1(\Omega) \tag{44}$$

as $k \to \infty$.

Remark 9 *The convergence (44), combined with (40), is not sufficient to apply Proposition 5 in Section D for the proof of the strong convergence*

$$u_k \to u_\infty \quad \text{in } L^2(Q_0), \quad Q_0 = \Omega \times (-\eta_0, 1).$$

By Lemma 2 of [19], in fact, the family $\{u_k\}$ is relatively compact in $L^p(Q_0)$ for $1 \leq p < 2$. Therefore, we could replace the convergence in (44) by a convergence in $L^p(\Omega)$ for all $p < 2$, but it is not clear how to obtain the conclusion of Proposition 5 directly with this better convergence. We instead bound u_k from above by the solution w_k of an appropriate majorizing system, and prove that w_k is compact in $L^2(Q_0)$. For justification purposes, furthermore, we do it on regularized approximate systems, see the introduction of w_k^ℓ below.

First, similarly to (44), we may assume

$$\tilde{u}_k^\ell(\cdot, -\eta_0) \to u_k(\cdot, -\eta_0) \quad \text{in } L^1(\Omega), \ k = 1, 2, \cdots \tag{45}$$

as $\ell \to \infty$ by (34), where

$$\tilde{u}_k^\ell(\cdot, t) = \tilde{u}^\ell(\cdot, t+t_k).$$

Now we take smooth $w_k^\ell = w_k^\ell(x,t)$, satisfying

$$\frac{\partial w_k^\ell}{\partial t} - \Delta(a_k^\ell w_k^\ell) = 0 \quad \text{in } Q_0 = \Omega \times (-\eta_0, 1)$$
$$\frac{\partial}{\partial \nu}(a_k^\ell w_k^\ell)\Big|_{\partial\Omega} = 0, \quad w_k^\ell\Big|_{t=-\eta_0} = \tau \cdot \tilde{u}_k^\ell(\cdot, -\eta_0), \tag{46}$$

where

$$a_k^\ell(x,t) = \frac{d \cdot \tilde{u}_k^\ell + 1}{\tau \cdot \tilde{u}_k^\ell + 1}.$$

Since $w_k^\ell(\cdot,t) \geq 0$ it follows that

$$\|w_k^\ell(\cdot,t)\|_1 \leq \|\tau \cdot \tilde{u}_k^\ell(\cdot,-\eta_0)\|_1 \leq C_{10}, \quad -\eta_0 \leq t \leq 1 \tag{47}$$

from (46). Therefore, by Proposition 7 in §D, each $\eta_1 \in (1, \eta_0)$ admits the estimate

$$\left\| \int_{-\eta_1}^1 a_k^\ell w_k^\ell \; dt \right\|_\infty + \|w_k^\ell\|_{L^2(Q_1)^N} \leq C_{14}(\eta_1), \quad Q_1 = \Omega \times (-\eta_1, 1). \tag{48}$$

Furthermore, inequality

$$\sum_{j=1}^N f_j^\ell(u) \leq 0, \quad 0 \leq u = (u_j) \in \mathbb{R}^N$$

implies

$$\frac{\partial}{\partial t}(\tau \cdot \tilde{u}_k^\ell + 1) - \Delta(a_k^\ell(\tau \cdot \tilde{u}_k^\ell + 1)) \leq 0, \quad \frac{\partial}{\partial \nu}(\tau \cdot \tilde{u}_k^\ell + 1)\Big|_{\partial\Omega} = 0,$$

and hence

$$\tau \cdot \tilde{u}_k^\ell + 1 \leq w_k^\ell \quad \text{in } Q_0 \tag{49}$$

by the classical maximum principle.

In the following, first, we shall show that $\{w_k^\ell\}_\ell$ is relatively compact in $L^2_{loc}(\overline{\Omega} \times (-\eta_0, 1])$ for each $k = 1, 2, \cdots$ (Lemma 5). Second, assuming $w_k^\ell \to w_k^\infty$ in $L^2_{loc}(\overline{\Omega} \times (-\eta_0, 1])$ up to a subsequence, we shall show that $\{w_k^\infty\}$ is relatively compact in $L^2_{loc}(\overline{\Omega} \times (-\eta_0, 1])$ (Lemma 6). Since

$$0 \leq \tau \cdot u_k + 1 \leq w_k^\infty \quad \text{a.e. in } Q_0 \tag{50}$$

this property implies the relatively compactness of $\{\tau \cdot u_k\}$ (and hence that of $\{u_k\}$) in $L^2_{loc}(\overline{\Omega} \times (\eta_0, 1])$, by $u_k = (u_{jk}) \geq 0$ and $\tau = (\tau_j) > 0$.

Lemma 5 *For each $k = 1, 2, \cdots$, the family $\{w_k^\ell\}_\ell \subset L^2(Q_1)^N$ is relatively compact.*

Proof: In the following proof, we fix k and let $\ell \to \infty$. By (34), we have

$$\underline{a} \leq a_k^\ell(x,t) \leq \overline{a}, \ a_k^\ell(x,t) \to a_k(x,t) \equiv a(x, t+t_k) \text{ for a.e. } (x,t) \in Q_1. \quad (51)$$

Since (48) holds, there is a subsequence satisfying

$$w_k^\ell \rightharpoonup w_k^\infty \quad \text{weakly in } L^2(Q_1).$$

From (51) and standard duality argument, it follows also that

$$\left\| \int_{-\eta_1}^1 a_k w_k^\infty \ dt \right\|_\infty + \|w_k^\infty\|_{L^2(Q_1)} \leq C_{14}(\eta_1). \quad (52)$$

First, we shall show

$$w_k^\ell(\cdot,t) \to w_k^\infty(\cdot,t) \quad \text{in } L^1(\Omega) \text{ and for a.e. } t \in (-\eta_0, 1). \quad (53)$$

For this purpose, we take smooth $r_0 = r_0(x)$ and define $z_k^\ell = z_k^\ell(x,t)$ by

$$\begin{aligned} &\frac{\partial z_k^\ell}{\partial t} - \Delta(a_k^\ell z_k^\ell) = 0 \quad \text{in } Q_0 \\ &\left.\frac{\partial z_k^\ell}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \left. z_k^\ell \right|_{t=-\eta_0} = r_0. \end{aligned} \quad (54)$$

By (46) and (54) we obtain

$$\sup_{-\eta_0 \leq t \leq 1} \|w_k^\ell(\cdot,t) - z_k^\ell(\cdot,t)\|_1 \leq \|\tau \cdot \tilde{u}_k^\ell(\cdot, -\eta_0) - r_0\|_1, \quad (55)$$

using Proposition 6 in §D.

Since (51), we have

$$z_k^\ell \to z_k^\infty \quad \text{in } L^2(Q_0) \quad (56)$$

by Proposition 5 in §D. In particular, it follows that

$$z_k^\ell(\cdot,t) \to z_k^\infty(\cdot,t) \quad \text{in } L^2(\Omega)^N \text{ and for a.e. } t \in (-\eta_0, 1). \quad (57)$$

Here, $z_k^\infty = z_k^\infty(x,t)$ is the L^2 solution to

$$\frac{\partial z_k^\infty}{\partial t} - \Delta(a_k z_k^\infty) = 0 \text{ in } Q_0, \quad \left.\frac{\partial z_k^\infty}{\partial \nu}\right|_{\partial\Omega} = 0, \quad z_k^\infty|_{t=-\eta_0} = r_0.$$

Using

$$\begin{aligned}
&\|w_k^\ell(\cdot,t) - w_k^{\ell'}(\cdot,t)\|_1 \\
&\quad \leq \|w_k^\ell(\cdot,t) - z_k^\ell(\cdot,t)\|_1 + \|z_k^\ell(\cdot,t) - z_k^{\ell'}(\cdot,t)\|_1 + \|z_k^{\ell'}(\cdot,t) - w_k^{\ell'}(\cdot,t)\|_1 \\
&\quad \leq \|z_k^\ell(\cdot,t) - z_k^{\ell'}(\cdot,t)\|_1 + 2\|\tau \cdot \tilde{u}_k^\ell(\cdot,-\eta_0) - r_0\|_1, \ -\eta_0 \leq t \leq 1, \qquad (58)
\end{aligned}$$

we obtain

$$\limsup_{\ell,\ell'\to\infty} \|w_k^\ell(\cdot,t) - w_k^{\ell'}(\cdot,t)\|_1 \leq 2\|\tau \cdot u_k(\cdot,-\eta_0) - r_0\|_1 \ \text{ for a.e. } t \in (-\eta_0, 1)$$

by (45) and (57). Since r_0 is an arbitrary smooth function, there holds that

$$\limsup_{\ell,\ell'\to\infty} \|w_k^\ell(\cdot,t) - w_k^{\ell'}(\cdot,t)\|_1 \leq 0 \ \text{ for a.e. } t \in (-\eta_0, 1)$$

and hence (53). In particular, we may assume

$$\lim_{\ell\to\infty} \|w_k^\ell(\cdot,-\eta_1) - w_k^\infty(\cdot,-\eta_1)\|_1 = 0. \qquad (59)$$

Reducing (46) to

$$\begin{aligned}
&\left[w_k^\ell(\cdot,t)\right]_{t=t_1}^{t=t_2} = \Delta \int_{t_1}^{t_2} a_k^\ell w_k^\ell(\cdot,t)\ dt \\
&\frac{\partial}{\partial\nu} \int_{t_1}^{t_2} a_k^\ell w_k^\ell(\cdot,t)\ dt \Big|_{\partial\Omega} = 0, \quad -\eta_1 < t_1, t_2 < 1,
\end{aligned}$$

we obtain

$$\begin{aligned}
&[w_k^\infty(\cdot,t)]_{t=t_1}^{t=t_2} = \Delta \int_{t_1}^{t_2} a_k w_k^\infty(\cdot,t)\ dt \\
&\frac{\partial}{\partial\nu} \int_{t_1}^{t_2} a_k w_k^\infty(\cdot,t)\ dt \Big|_{\partial\Omega} = 0 \quad \text{for a.e. } t_1, t_2 \in (-\eta_1, 1),
\end{aligned}$$

in the sense of distributions on $\overline{\Omega}$, recalling (52). It thus follows that

$$\begin{aligned}
&\left[w_k^\ell(\cdot,t) - w_k^\infty(\cdot,t)\right] - \Delta \int_{-\eta_1}^{t} \left[a_k^\ell w_k^\ell - a_k w_k^\infty\right](\cdot,t')\ dt' \\
&\quad = \left[w_k^\ell(\cdot,-\eta_1) - w_k^\infty(\cdot,-\eta_1)\right] \\
&\frac{\partial}{\partial\nu} \int_{-\eta_1}^{t} \left[a_k^\ell w_k^\ell - a_k w_k^\infty\right](\cdot,t')\ dt' \Big|_{\partial\Omega} = 0 \quad \text{for a.e. } t \in (-\eta_1, 1) \qquad (60)
\end{aligned}$$

in the same sense. From the elliptic regularity, (48), and (52), we get

$$\int_{-\eta_1}^{t} \left[a_k^\ell w_k^\ell - a_k w_k^\infty\right](\cdot,t')\ dt' \in H^2(\Omega) \quad \text{for a.e. } t \in (-\eta_1, 1).$$

Then, taking $L^2(Q)$ inner product of the first equation of (60) with $a_k^\ell w_k^\ell - a_k w_k^\infty$ leads to

$$\begin{aligned}
&\iint_{Q_1} (w_k^\ell - w_k^\infty)(a_k^\ell w_k^\ell - a_k w_k^\infty)\ dxdt \\
&\quad \leq \int_\Omega (w_k^\ell(\cdot,-\eta_1) - w_k^\infty(\cdot,-\eta_1))\ dx \cdot \int_{-\eta_1}^{1} [a_k^\ell w_k^\ell - a_k w_k^\infty](\cdot,t)\ dt.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\iint_{Q_1} (w_k^\ell - w_k^\infty)(a_k^\ell w_k^\ell - a_k w_k^\infty)\ dxdt \\
&\quad \leq 2C_{14}(\eta_1)\|w_k^\ell(\cdot,-\eta_1) - w_k^\infty(\cdot,-\eta_1)\|_1
\end{aligned}$$

from (48) and (52). We thus end up with

$$\limsup_{\ell\to\infty} \iint_{Q_1} (w_k^\ell - w_k^\infty)(a_k^\ell w_k^\ell - a_k w_k^\infty)\ dxdt \leq 0 \tag{61}$$

by (59).

Here, we use

$$\begin{aligned}
&\underline{d}\|w_k^\ell - w_k^\infty\|_{L^2(Q_1)^N}^2 \leq \iint_{Q_1} a_k^\ell (w_k^\ell - w_k^\infty)^2\ dxdt \\
&= \iint_{Q_1} (w_k^\ell - w_k^\infty)(a_k^\ell w_k^\ell - a_k w_k^\infty) + (w_k^\ell - w_k^\infty) w_k^\infty (a_k - a_k^\ell)\ dxdt \\
&\leq \iint_{Q_1} (w_k^\ell - w_k^\infty)(a_k^\ell w_k^\ell - a_k w_k^\infty) + \frac{\underline{d}}{2}(w_k^\ell - w_k^\infty)^2 \\
&\quad + \frac{1}{2\underline{d}}(w_k^\infty)^2 (a_k - a_k^\ell)^2\ dxdt
\end{aligned}$$

to deduce

$$\begin{aligned}
\underline{d}\|w_k^\ell - w_k^\infty\|_{L^2(Q_1)^N}^2 &\leq \iint 2(w_k^\ell - w_k^\infty)(a_k^\ell w_k^\ell - a_k w_k^\infty) \\
&\quad + \frac{1}{\underline{d}}(w_k^\infty)^2 (a_k - a_k^\ell)^2\ dxdt.
\end{aligned}$$

Then it follows that

$$w_k^\ell \to w_k^\infty \quad \text{in } L^2(Q_1)^N$$

from (51), (61), and the dominated convergence theorem. □

By Lemma 5, passing to a subsequence, we have

$$w_k^\ell \to w_k^\infty \quad \text{in } L^2_{loc}(\overline{\Omega} \times (-\eta_0, 1]) \text{ and a.e. in } \Omega \times (-\eta_0, 1) \tag{62}$$

as $\ell \to \infty$, where $k = 1, 2, \cdots$.

Lemma 6 *The family $\{w_k^\infty\}$ is relatively compact in $L^2_{loc}(\overline{\Omega} \times (-\eta_0, 1])^N$.*

Proof: We have only to repeat the proof of the previous lemma, replacing w_k^ℓ by w_k^∞. First, we have (52) for any $\eta_1 \in (1, \eta_0)$. Second, it follows that

$$\begin{aligned} &\frac{\partial w_k^\infty}{\partial t} - \Delta(a_k w_k^\infty) = 0 \quad \text{in } Q_0 = \Omega \times (-\eta_0, 1) \\ &\left.\frac{\partial}{\partial \nu}(a_k w_k^\infty)\right|_{\partial\Omega} = 0, \quad w_k^\infty|_{t=-\eta_0} = \tau \cdot u_k(\cdot, -\eta_0) \end{aligned} \tag{63}$$

from (46). We define $z_k^\ell = z_k^\ell(x,t)$ by (54) for smooth $r_0 = r_0(x)$. Passing to a subsequence, we obtain (56), where $z_k^\infty = z_k^\infty(x,t)$ is the L^2 solution to

$$\frac{\partial z_k^\infty}{\partial t} - \Delta(a_k z_k^\infty) = 0 \quad \text{in } Q_0, \quad \left.\frac{\partial z_k^\infty}{\partial \nu}\right|_{\partial\Omega} = 0, \quad z_k^\infty|_{t=-\eta_0} = r_0$$

defined by Proposition 4 in §D. Then, Proposition 5 guarantees

$$z_k^\infty \to z_\infty \quad \text{in } L^2(Q_0) \tag{64}$$

by (41)-(42). Here, $z_\infty = z_\infty(x,t)$ is the L^2 solution to

$$\frac{\partial z_\infty}{\partial t} - \Delta(a_\infty z_\infty) = 0 \quad \text{in } Q_0, \quad \left.\frac{\partial z_\infty}{\partial \nu}\right|_{\partial\Omega} = 0, \quad z_\infty|_{t=-\eta_0} = r_0.$$

We modify (58) as

$$\begin{aligned} &\|w_k^\ell(\cdot,t) - w_{k'}^\ell(\cdot,t)\|_1 \\ &\leq \|w_k^\ell(\cdot,t) - z_k^\ell(\cdot,t)\|_1 + \|z_k^\ell(\cdot,t) - z_{k'}^\ell(\cdot,t)\|_1 + \|z_{k'}^\ell(\cdot,t) - w_{k'}^\ell(\cdot,t)\|_1 \\ &\leq \|z_k^\ell(\cdot,t) - z_{k'}^\ell(\cdot,t)\|_1 + \|\tau \cdot \tilde{u}_k^\ell(\cdot,-\eta_0) - r_0\|_1 + \|\tau \cdot \tilde{u}_{k'}^\ell(\cdot,-\eta_0) - r_0\|_1, \end{aligned}$$

so that letting $\ell \to \infty$ leads to

$$\begin{aligned} \|w_k^\infty(\cdot,t) - w_{k'}^\infty(\cdot,t)\|_1 &\leq \|z_k^\infty(\cdot,t) - z_{k'}^\infty(\cdot,t)\|_1 + \|\tau \cdot u_k(\cdot,-\eta_0) - r_0\|_1 \\ &\quad + \|\tau \cdot u_{k'}(\cdot,-\eta_0) - r_0\|_1 \quad \text{for a.e. } t \in (-\eta_0, 1). \end{aligned} \tag{65}$$

From (44), and (64), (65), it follows that

$$\lim_{k,k'\to\infty} \|w_k^\infty - w_{k'}^\infty\|_1 = 0 \quad \text{for a.e. } t \in (-\eta, 1) \tag{66}$$

because r_0 is arbitrary. Inequality (52), and equations of (63) and (66) imply the result as in the proof of Lemma 5. □

Proof of Theorem 2: Since (50) follows from (34), (49), and (62), we obtain

$$0 \leq u_{jk} + 1 \leq \underline{\tau}^{-1} w_k^\infty \ \text{ a.e. in } Q_0, \quad 1 \leq j \leq N \tag{67}$$

where $\underline{\tau} = \min_j \tau_j > 0$. It also holds that

$$w_k^\infty \to w_\infty \quad \text{in } L^2_{loc}(\overline{\Omega} \times (-\eta_0, 1])^N \text{ and a.e. in } \Omega \times (-\eta_0, 1), \tag{68}$$

passing to a subsequence. From (39), (67)-(68), and the dominated convergence theorem it follows that

$$\iint_{\Omega\times(-\eta_1,1)} (u_{jk})^2 \ dxdt \to \iint_{\Omega\times(-\eta_1,1)} (u_{j\infty})^2 \ dxdt, \quad u_\infty = (u_{j\infty}),$$

for any $\eta_1 \in (\eta_0, 2)$. See Theorem 4 in p.21 of [9] and its proof.

Therefore, it holds that

$$u_k \to u_\infty \quad \text{in } L^2_{loc}(\overline{\Omega} \times (-\eta_0, 1])^N \text{ and a.e. in } \Omega \times (-\eta_0, 1) \tag{69}$$

by (40), and hence

$$f(u_k) \to f(u_\infty) \quad \text{in } L^1_{loc}(\overline{\Omega} \times (-\eta_0, 1])^N \tag{70}$$

by (5) and the dominated convergence theorem.

From (39), on the other hand, there is $\eta \in (1, \eta_0)$ such that

$$u_k(\cdot, -\eta) \to u_\infty(\cdot, -\eta) \quad \text{in } L^1(\Omega)^N. \tag{71}$$

Proposition 9, combined with (70) and (71), now implies

$$u_k \to u_\infty \quad \text{in } C([-\eta, 1], L^1(\Omega)^N),$$

and hence

$$u_k(\cdot, 0) = u(\cdot, t_k) \to u_\infty(\cdot, 0) \quad \text{in } L^1(\Omega)^N.$$

Thus, any $t_k \uparrow +\infty$ admits a subsequence such that $\{u(\cdot, t_k)\}$ converges in $L^1(\Omega)^N$, and the proof is complete. □

3 Proof of Theorem 3

Outline of this section: Since the case $n = 1$ is easier, we assume $n = 2$. As is noted in our previous work [25], $n = 2$ is the critical dimension for the uniform boundedness of the classical solution $u = (u_j(\cdot, t))$ to (1) with (5)-(6). We have, therefore, $T = +\infty$ and $\sup_{t\geq 0} \|u(\cdot, t)\|_\infty < +\infty$, provided that $\|u_0\|_1$ is sufficiently small. By this property, called ε-regularity, and the monotonicity formula noticed in [23, 24], we have the formation of finitely many delta-functions to $u = (u_j(\cdot, t))$ as the blowup time approaches. To show Theorem 3, first, we derive a bound on $\sup_{0\leq t<T} \|u(\cdot, t)\|_{L \log L}$, using (17) and (19). This bound is improved to the one on $\sup_{0\leq t<T} \|u(\cdot, t)\|_2$ by the Gagliardo-Nirenberg inequality. Once this estimate is achieved, we get a bound of $\sup_{0\leq t<T} \|u(\cdot, t)\|_\infty$ by the semi-group estimate and bootstrap argument, which implies $T = +\infty$. Since these bounds are not uniform in T, we exclude the possibility of blowup in infinite time in the second step. For this purpose we assume the contrary, and derive the above described blowup mechanism for the solution sequence, obtained by the translation in time of the original global-in-time and classical solution. Then this property, formation of finitely many delta functions, contradicts Theorem 2, the relative compactness of the orbit in $L^1(\Omega)$ made by this classical solution.

Assuming the smooth initial value $0 \leq u_0 = (u_{j0}(x))$, we have the unique local-in-time classical solution denoted by $u = (u_j(\cdot, t))$, $0 \leq t < T$. We may assume $u_{j0} = u_{j0}(x) > 0$, $1 \leq j \leq N$, on $\overline{\Omega}$ by the strong maximum principle, which implies $u_j(\cdot, t) > 0$ on $\overline{\Omega}$ for any $1 \leq j \leq N$. Below we shall take the case $n = 2$.

The fundamental estimate is (35), particularly,

$$\sup_{0\leq t<T} \|u(\cdot, t)\|_1 \leq C_{10}. \tag{72}$$

First, we show the a priori estimate

$$\sup_{0\leq t<T} \|u(\cdot, t)\|_\infty \leq C_{15}(T), \tag{73}$$

which guarantees for this $u = u(\cdot, t)$ to be global-in-time. To this end, we multiply (1) by $\log u_j$. Then (17) implies

$$\frac{d}{dt}\sum_{j=1}^N \tau_j \int_\Omega \Phi(u_j)\ dx + \underline{d}\sum_{j=1}^N \int_\Omega u_j^{-1}|\nabla u_j|^2\ dx$$
$$\leq C_{16}\left(\int_\Omega |u|^2\ dx + 1\right) \quad \text{with } \underline{d} = \min_j d_j > 0, \tag{74}$$

where

$$\Phi(s) = s(\log s - 1) + 1, \quad s > 0.$$

Inequality (74) coincides with (3.18) in [25] for $\varphi \equiv 1$.

This inequality, combined with Proposition 1, implies

$$\sup_{0 \le t < T} \|\Phi(u_j(\cdot, t))\|_1 \le C_{17}(T), \quad 1 \le j \le N. \tag{75}$$

Here we use ineuality (22) of [3], of which local version is presented as in Lemma 11.1 of [23], that is,

$$\|w\|_3^3 \le \varepsilon \|w\|_{H^1}^2 \|w \log w\|_1 + C_{18}(\varepsilon), \quad 0 \le w \in L^3(\Omega) \tag{76}$$

for any $\varepsilon > 0$. *In fact, inequality (5) implies*

$$\frac{\tau_j}{2} \frac{d}{dt} \|u_j\|_2^2 + d_j \|\nabla u_j\|_2^2 \le C_{19}(\|u\|_3^3 + 1).$$

Then we obtain

$$\tau_j \frac{d}{dt} \|u_j\|_2^2 + d_j \|\nabla u_j\|_2^2 \le C_{20}(T), \quad 1 \le j \le N$$

by (72), (75)-(76), and Poincaré-Wirtinger's inequality, and hence

$$\sup_{0 \le t < T} \|u(\cdot, t)\|_2 \le C_{21}(T). \tag{77}$$

Once (77) is proven, the semigroup estimate (see [21])

$$\|e^{t\Delta} w\|_r \le C_{22} \max\{1, t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{r})}\} \|w\|_q, \quad 1 \le q \le r \le \infty$$

applied to (4) implies (73) by the quadratic growth (5). More precisely, we put

$$g_j = \mu u_j + C_1(1 + |u|^2)$$

for $\mu \gg 1$, *and define* $\tilde{u}_j = \tilde{u}_j(\cdot, t)$ *by*

$$\tau_j \frac{\partial \tilde{u}_j}{\partial t} - d_j \Delta \tilde{u}_j + \mu \tilde{u}_j = g_j(\cdot, t), \quad \left. \frac{\partial \tilde{u}_j}{\partial \nu} \right|_{\partial\Omega} = 0, \quad \tilde{u}_j|_{t=0} = u_{j0}(x).$$

Then the comparison principle guarantees $0 \le u_j \le \tilde{u}_j$, *and it holds also that*

$$\tilde{u}_j(\cdot, t) = e^{tL_j} u_{j0} + \int_0^t e^{(t-s)L_j} \tau_j^{-1} g_j(\cdot, s) \, ds,$$

where $L_j = \tau_j^{-1}[-d_j\Delta + \mu]$ *provided with the Neumann boundary condition. Then inequality (73) follows from the iteration scheme used in pp. 10-11 of [25]. More precisely, assuming* $\sup_{t\in[0,T)} \|u(\cdot,t)\|_q \leq C_{23}(T)$ *for* $q \geq 2$*, we obtain* $\sup_{t\in[0,T)} \|\tilde{u}_j(\cdot,t)\|_r \leq C_{24}(T)$ *for* $q \leq r \leq \infty$ *satisfying* $\frac{2}{q} - \frac{1}{r} < 1$*, by* $n = 2$*. Repeating this argument twice, we reach (73).*

Second, we show that (73) is improved as

$$\sup_{t\geq 0} \|u(\cdot,t)\|_\infty \leq C_{25}. \tag{78}$$

If this is not the cas, we have the non-empty blowup set

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid 1 \leq \exists j \leq N,\ \exists x_k \to x_0,\ \exists t_k \uparrow +\infty,\ \lim_{k\to\infty} u_j(x_k,t_k) = +\infty\}.$$

Given $x_0 \in \mathcal{S}$*, we have* $t_k \uparrow +\infty$ *and* $x_k \to x_0$ *such that*

$$\lim_{k\to\infty} |u|(x_k,t_k) = +\infty, \tag{79}$$

where $|u| = \sqrt{\sum_{j=1}^N u_j^2}$*. By Theorem 2 and its proof, we have a subsequence denoted by the same symbol, satisfying (69) and*

$$u_k \to u_\infty \quad \text{in } C([-1,1], L^1(\Omega)^N) \tag{80}$$

for $u_k = u_k(\cdot,t)$ *defined by (37).*

Given $x_0 \in \overline{\Omega}$ *and* $0 < R \ll 1$*, let* $0 \leq \varphi = \varphi_{x_0,R}(x) \in C^\infty(\overline{\Omega})$ *be the cut-off function introduced by [22], that is,*

$$\varphi_{x_0,R}(x) = \begin{cases} 1, & x \in \Omega \cap B(x_0,R/2) \\ 0, & x \in \Omega \setminus B(x_0,R), \end{cases} \qquad \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0, \tag{81}$$

and

$$|\nabla\varphi| \leq C_{26}R^{-1}\varphi^{5/6}, \quad |\Delta\varphi| \leq C_{26}R^{-2}\varphi^{2/3}, \tag{82}$$

which is also used in our previous work [25]. To define this function, first, we take $0 \leq \psi = \psi_{x_0,R} \in C^\infty(\overline{\Omega})$ *satisfying*

$$\psi_{x_0,R}(x) = \begin{cases} 1, & x \in \Omega \cap B(x_0,R/2) \\ 0, & x \in \Omega \setminus B(x_0,R), \end{cases} \qquad \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0. \tag{83}$$

Then, setting $\varphi = \psi_{x_0,R}^6$*, we obtain (81)-(82). Second, to define* $\psi = \psi_{x_0,R}$ *satisfying (83) we distinguish two cases,* $x_0 \in \Omega$ *and* $x_0 \in \partial\Omega$*. If* $x_0 \in \Omega$*, we take* $\psi_{x_0,R}$ *as the standard radially symmetric cut-off function, assuming* $0 < R \ll 1$*. If* $x_0 \in \partial\Omega$*, on the other hand, this* $\psi = \psi_{x_0,R}$ *is constructed by* *a composition of the standard radially symmetric cut-off function and the conformal diffeomorphism* $X : \overline{\Omega \cap B(x_0, 2R)} \to \overline{\mathbf{R}^2_+}$. *See p.91 of [23].*

Given $\varepsilon > 0$, *we take sufficiently small* $R > 0$ *such that*

$$\|u_\infty(\cdot, 0)\|_{L^1(\Omega\cap B(x_0,4R))} < \frac{\varepsilon}{4}.$$

Then we obtain

$$\int_\Omega u^j_\infty(\cdot, 0)\varphi_{x_0,4R}\ dx < \frac{\varepsilon}{4} \quad \text{for } 1 \le j \le N.$$

Since the mapping

$$t \mapsto \int_\Omega u^j_\infty(\cdot, t)\varphi_{x_0,4R}\ dx$$

is continuous by $u_\infty \in C([-1, 1], L^1(\Omega)^N)$, *there exists* $\delta \in (0, 1)$ *such that*

$$\int_\Omega u^j_\infty(\cdot, t)\varphi_{x_0,4R}\ dx < \frac{\varepsilon}{2}, \quad |t| < \delta$$

which implies

$$\sup_{|t|\le\delta} \|u_\infty(\cdot, t)\|_{L^1(\Omega\cap B(x_0,2R))} < \frac{\varepsilon}{2}. \tag{84}$$

By (80), inequality (84) implies

$$\sup_{|t|\le\delta} \|u_k(\cdot, t)\|_{L^1(\Omega\cap B(x_0,R))} < \varepsilon \tag{85}$$

for $k \gg 1$, *similarly. Henceforth, we assume (85) for* $k = 1, 2, \cdots$.

By this inequality we can deduce

$$\|u(\cdot, t_k)\|_{L^\infty(\Omega\cap B(x_0,R/8))} \le C_{27}, \quad k = 1, 2, \cdots, \tag{86}$$

using Lemma 5.2 of [25] applied to $u_k(\cdot, t) = u(\cdot, t + t_k)$, *which contradics (79). Thus the uniform boundedness (78) has been shown. We complete the proof of Theorem 3 with this inequality, becuase it implies relative compactness of the orbit* $\mathcal{O} = \{u(\cdot, t) \mid t \ge 0\}$ *in* $C(\overline{\Omega})^N$.

For the sake of completeness, we describe how to derive (86). In fact, in our setting, we can take $s_k \in (0, \delta)$ *satisfying*

$$\|u_k(\cdot, -s_k)\|_2 \le C_{28} \tag{87}$$

by (69). This property makes the proof simpler; it suffices to apply the argument in p.14-15 of [25].

More precisely, by inequality (3.19) in [25], or Lemma 11.1 of [23], it holds that

$$\int_\Omega u_j^3 \varphi_{x_0,R}\ dx \leq C_{29}\|u_j\|_{L^1(\Omega\cap B(x_0,R))}\cdot \int_\Omega |\nabla u_j|^2\varphi_{x_0,R}\ dx + C_{29}\|u_j\|_1 \quad (88)$$

for any smooth $u = (u_j(\cdot,t)) \geq 0$*. Furthermore, the inequality*

$$\begin{aligned} &\frac{\tau_j}{2}\frac{d}{dt}\int_\Omega u_j^2\varphi_{x_0,R}\ dx + d_j\int_\Omega |\nabla u_j|^2\ \varphi_{x_0,R}\ dx \\ &\quad \leq C_{30}(R)\left(\int_\Omega |u|^3\varphi_{x_0,R}\ dx + 1\right), \end{aligned} \quad (89)$$

follows from (5), as in (3.8) of [25]. We thus end up with

$$\begin{aligned} &\sup_{t\in[-s_k,\delta]} \|u_k(\cdot,t)\|^2_{L^2(\Omega\cap B(x_0,R/2))} \\ &\quad + \int_{-s_k}^{\delta} \|\nabla u_k(\cdot,t)\|^2_{L^2(\Omega\cap B(x_0,R/2))}\ dt \leq C_{31} \end{aligned} \quad (90)$$

by (87)-(89), recalling $u_k = (u_{jk}(\cdot,t)) = (u_j(\cdot,t+t_k))$*. Then we take* $0 < s_k' < s_k$ *such that*

$$\|\nabla u_k(\cdot,s_k')\|_{L^2(\Omega\cap B(x_0,R/2))} \leq C_{32},$$

using (90), which implies

$$\|u_k(\cdot,s_k')\|_p \leq C_{33}(p), \quad 1\leq p<\infty \quad (91)$$

by (72) and Sobolev's embedding theorem. Using an analogous inequality to (89), with u_j *replaced by* $u_j^{3/2}$*, that is, (3.12) of [25], we obtain*

$$\sup_{t\in[-s_k',\delta]} \|u_k(\cdot,t)\|_{L^3(\Omega\cap B(x_0,R/4))} \leq C_{34}.$$

This inequality is improved as

$$\sup_{t\in[-s_k',\delta]} \|u_k(\cdot,t)\|_{L^4(\Omega\cap B(x_0,R/4))} \leq C_{35} \quad (92)$$

by repeating the same argument.

Here we use

$$\tau_j\frac{\partial \tilde u_{jk}}{\partial t} - d_j\Delta\tilde u_{jk} = \tilde g_{jk}, \quad \left.\frac{\partial \tilde u_j^k}{\partial \nu}\right|_{\partial\Omega} = 0$$

with $\tilde{u}_{jk} = u_{jk}\varphi$ *and* $\varphi = \varphi_{x_0, R/4}$*, where*

$$\tilde{g}_{jk} = -d_j(u_{jk}\Delta\varphi + 2\nabla u_{jk}\cdot\nabla\varphi) + f_j(u_k)\varphi.$$

We have

$$\int_{-s_k'}^{\delta} \|\tilde{g}_{jk}(\cdot,t)\|_2^2 \ dt \leq C_{36}$$

by (90) and (92). Then, using

$$\tilde{u}_{jk}(\cdot,t) = e^{(t+s_k)\tau_j^{-1}d_j\Delta}\tilde{u}_{jk}(\cdot,-s_k') + \int_{-s_k'}^{t} e^{(t-s)\tau_j^{-1}d_j\Delta}\tau_j^{-1}\tilde{g}_{jk}(\cdot,s) \ ds$$

for $t \in (-s_k, \delta)$*, and the following semi-group estimate [21], that is,*

$$\|\nabla e^{t\Delta}w\|_r \leq C_{37}(q,r)\max\{1, t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\}\|w\|_q, \quad 1 \leq q \leq r \leq \infty$$

with $n = 2$*, we obtain*

$$\sup_{t\in[-s_k'',\delta]} \|\nabla u_{jk}(\cdot,t)\|_r \leq C_{38}$$

for $0 < s_k'' < s_k$ *and* $1 \leq r < \infty$*, and hence (86) by (72).* □

Remark 10 *In the above proof, inequality (7) is used to exclude blowup in finite time. This condition can be replaced by (14) as is described in Remark 6.*

Remark 11 *Inequality (85) can be shown alternatively by the relative compactness of* $\{u(\cdot,t_k)\} \subset L^1(\Omega)$ *and an inequality derived from (5), (20), and (72), that is,*

$$\int_{-1}^{1} \left| \frac{d}{dt}\int_\Omega u_j(\cdot,t+t_k)\varphi \ dx \right| \ dt \leq C_{39}\|\varphi\|_{W^{2,\infty}}, \quad k \gg 1 \qquad (93)$$

valid to $\varphi \in C^2(\overline{\Omega})$ *with* $\left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0$*. We note that inequality (93) is callled the monotonicity formula by [23, 24].*

4 Proof of Theorem 4

Outline of this section: Theorem 4 says that the solution becomes spatially homogeneous under the presense of an entropy functional. This assertion follows from the LaSalle principle and the relatively compactness of the orbit. In our previous work [25] we developed this argument in the framework of the classical solution. Here, since we are concerned with the weak solution, we use the approximate solution to complete the proof of this theorem.

Lemma 7 *Under the assumptions of the first case of Theorem 4, it holds that*

$$\log u_j \in L^1_{loc}(\overline{\Omega} \times [0, +\infty)), \ \nabla \log u_j \in L^2(\Omega \times (0, +\infty))^N, \quad 1 \leq j \leq N_1$$

and

$$\frac{d}{dt} H(u) \geq \sum_{j=1}^{N_1} b_j \tau_j^{-1} d_j \int_\Omega |\nabla \log u_j|^2 \ dx \geq 0 \tag{94}$$

in the sense of distributions with respect to t, where

$$H(u) = \sum_{j=1}^{N_1} \int_\Omega b_j \log u_j \ dx.$$

Proof: *Let $\tilde{u}^\ell = (\tilde{u}^\ell_j(\cdot, t))$ be the approximate solution of $u = (u_j(\cdot, t))$ defined by (30). It satisfies (34), and also $\tilde{u}^\ell_j(\cdot, t) > 0$ on $\overline{\Omega}$ for $1 \leq j \leq N$. Letting $g^\ell_j = g_j \circ T_\ell$, we have*

$$\begin{aligned} \frac{d}{dt} H(\tilde{u}^\ell) &= \sum_{j=1}^{N_1} b_j \tau_j^{-1} \int_\Omega |\nabla \log \tilde{u}^\ell_j|^2 + g^\ell_j(\tilde{u}^\ell) \ dx \\ &\geq \sum_{j=1}^{N_1} \int_\Omega b_j \tau_j^{-1} |\nabla \log \tilde{u}^\ell_j|^2 \ dx \geq 0 \end{aligned}$$

and hence

$$H(\tilde{u}^\ell(\cdot, t)) \geq H(\tilde{u}^\ell_0) \geq H(u_0) > -\infty \tag{95}$$

by (23) and (29). Therefore, using

$$\log_+ \tilde{u}^\ell_j \leq \tilde{u}^\ell_j \to u_j \ \textit{in} \ L^1_{loc}(\overline{\Omega} \times [0, +\infty)) \ \textit{and a.e. in} \ \Omega \times (0, +\infty) \tag{96}$$

valid to $1 \leq j \leq N$ *and Fatou's lemma, we have*

$$\begin{aligned} &\log u_j \in L^1_{loc}(\overline{\Omega} \times [0,+\infty)), \quad 1 \leq j \leq N_1 \\ &H(u(\cdot,t)) \geq H(u_0) \quad \textit{for a.e. } t, \end{aligned} \tag{97}$$

where $\log_+ s = \max\{\log s, 0\}$*. Furthermore, (32) implies*

$$H(\tilde{u}^\ell(\cdot,t)) \leq C_{40},$$

and, therefore,

$$\iint_{\Omega\times(0,+\infty)} |\nabla \log \tilde{u}_j^\ell|^2 \; dxdt \leq C_{41}, \quad 1 \leq j \leq N_1. \tag{98}$$

Thus $\{\nabla \log \tilde{u}_j^\ell\}$*,* $1 \leq j \leq N_1$*, is weakly relatively compact in* $L^2(\Omega \times (0,+\infty))^N$*. Consequently, it holds that*

$$\nabla \log u_j \in L^2(\Omega \times (0,+\infty))^N, \quad 1 \leq j \leq N_1 \tag{99}$$

and (94) in the sense of distributions with respect to t*.* □

We have already shown (69) for $u_k = (u_{jk}(\cdot,t))$*,* $u_{jk}(\cdot,t) = u_j(\cdot,t+t_k)$*, and* $\eta_0 \in (1,2)$*. Let* $u_\infty = (u_{j\infty}(\cdot,t))$*. We take* $\eta_1 \in (1,\eta_0)$ *and put* $Q_1 = \Omega \times (-\eta_1, 1)$*.*

Lemma 8 *Under the assumptions of fhe first case of Theorem 4, it holds that*

$$\log u_{j\infty} \in L^1(Q_1), \quad \log u_{jk} \to \log u_{j\infty} \;\textit{in}\; L^1(Q_1)$$

as $k \to \infty$ *for* $1 \leq j \leq N_1$*.*

Proof: We take $\eta_2 \in (\eta_1, \eta_0)$ *and put* $Q_2 = \Omega \times (-\eta_2, 1)$*. By (94) we have*

$$\iint_{Q_2} \sum_{j=1}^{N_1} b_j \log u_{jk} \; dxdt \geq (1+\eta_2)\cdot H(u_0) > -\infty, \tag{100}$$

recalling (23). Then $\log u_{j\infty} \in L^1(Q_2)$*,* $1 \leq j \leq N_1$*, follow from (69), (96), and Fatou's lemma. In particular, we obtain*

$$\log u_{jk} \to \log u_{j\infty} \quad \textit{a.e. in}\; Q_2,\; 1 \leq j \leq N_1. \tag{101}$$

By (30) we obtain

$$\tau_j \frac{\partial}{\partial t} \log \tilde{u}_j^\ell - d_j \Delta \log \tilde{u}_j^\ell \geq g_j^\ell(\tilde{u}^\ell), \quad \frac{\partial}{\partial \nu} \log \tilde{u}_j \Big|_{\partial\Omega} = 0,$$

which implies

$$\tau_j \frac{\partial}{\partial t} \log u_{jk} - d_j \Delta \log u_{jk} \geq g_j(u_k), \quad \frac{\partial}{\partial \nu} \log u_{jk}\Big|_{\partial\Omega} = 0, \quad 1 \leq j \leq N_1$$

in the sense of distributions in Q_1*, recalling (22), (34), and (98)-(99).*

By (100) there is $\eta \in (\eta_1, \eta_2)$ *such that* $\{\log u_{jk}(\cdot, -\eta)\}$, $1 \leq j \leq N_1$*, is bounded in* $L^1(\Omega)$*. Then we take the solution (see Proposition 8 in §E)*

$$w_j^k = w_j^k(\cdot, t) \in L^\infty(-\eta, 1; L^1(\Omega)) \cap L^1_{loc}(-\eta, 1; W^{1,1}(\Omega))$$

to

$$\tau_j \frac{\partial w_j^k}{\partial t} - d_j \Delta w_j^k = g_j(u_k) \quad \text{in } \Omega \times (-\eta, 1) \equiv Q_\eta$$

$$\frac{\partial w_j^k}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad w_j^k\Big|_{t=-\eta} = \log u_{jk}(\cdot, -\eta).$$

Then we obtain

$$w_j^k \leq \log u_{jk} (\leq u_{jk}) \text{ in } Q_\eta, \quad 1 \leq j \leq N_1 \tag{102}$$

from the comparison principle (Lemma 3.4 of [2]). By (22) and (69) we have

$$g_j(u_k) \to g_j(u_\infty) \quad \text{in } L^1(Q_\eta)$$

by the dominated convergence theorem which implies

$$w_j^k \to w_j \quad \text{in } L^1(Q_\eta) \tag{103}$$

with some w_j *by Proposition 10. The result follows from (101)-(103) and the dominated convergence theorem.* □

Proof of Theorem 4: Since $\{u(\cdot, t) \mid t \geq 0\}$ *is relatively compact in* $L^1(\Omega)^N$*, the* ω*-limit set* $\omega(u_0)$ *is non-empty. Let* $t_k \uparrow +\infty$ *and* $u(\cdot, t_k) \to u_*$ *in* $L^1(\Omega)^N$*. Passing to a subsequence, we obtain (80) for* $u_k(\cdot, t) = (u_j(\cdot, t + t_k))$*.*

Under the assumptions of the first case, we have the existence of

$$\lim_{t \uparrow +\infty} H(u(\cdot, t))$$

by (35) and (94), which implies the LaSalle principle,

$$\lim_{k \to \infty} \int_{t_k - 1}^{t_k + 1} dt \cdot \sum_{j=1}^{N_1} b_j \tau_j^{-1} d_j \int_\Omega |\nabla \log u_j|^2 \, dx = 0$$

again by (94). Then we obtain

$$\nabla \log u_{j\infty} = 0 \quad in\ \Omega \times (-1,1), \quad 1 \leq j \leq N_1$$

in the sense of distributions, recalling Lemma 8. Then it follows that $0 < u_{j\infty} \in \mathbb{R}$ *for* $1 \leq j \leq N_1$.

In the second case we use (1) in the form of

$$\tau_j \frac{\partial u_j}{\partial t} + e_j u_j = d_j \Delta u_j + f_j(u) + e_j u_j, \quad \left.\frac{\partial u_j}{\partial \nu}\right|_{\partial\Omega} = 0.$$

It holds that

$$\frac{d}{dt}\int_\Omega \tau \cdot u\ dx + \int_\Omega e \cdot u\ dx \leq 0$$

in the sense of distributions with respec to t*, and hence there exsits*

$$\lim_{t \uparrow +\infty} \int_\Omega \tau \cdot u\ dx.$$

Then we obtain

$$\iint_{\Omega \times (-1,1)} e \cdot u_\infty(x,t)\ dxdt = 0$$

from the LaSalle principle, and hence

$$u_{j*} = 0, \quad N_2 + 1 \leq j \leq N$$

for $u_* = (u_{j*})$*. The proof is complete.* □

A Proof of Proposition 1

Assuming (2), (3), (5), (6), and (7), we shall show (17). Put

$$\tilde{f}_j(u) = f_j(u_1, \cdots, u_{j-1}, 0, u_{j+1}, \cdots, u_N) \geq 0, \quad 0 \leq u = (u_j) \in \mathbb{R}^N.$$

If $|u| \leq 1$ *is the case, we have* $0 \leq u_j \leq 1$ *for* $1 \leq j \leq N$*. Then, for* $u_j > 0$ *it holds that*

$$\begin{aligned} f_j(u) \log u_j &= (f_j(u) - \tilde{f}_j(u)) \log u_j + \tilde{f}_j(u) \log u_j \\ &\leq (f_j(u) - \tilde{f}_j(u)) \log u_j \leq C_{42} u_j |\log u_j| \leq C_{43}, \end{aligned}$$

and hence

$$\sum_{j=1}^N f_j(u) \log u_j \leq N C_{36}, \quad |u| \leq 1. \tag{104}$$

Assume $|u| > 1$*, and put* $s_j = u_j/|u| \in (0,1]$*. It holds that*

$$\sum_{j=1}^{N} s_j^2 = 1 \tag{105}$$

and

$$\begin{aligned} \sum_{j=1}^{N} f_j(u)\log u_j &= \log|u|\cdot\sum_{j=1}^{N} f_j(u) + \sum_{j=1}^{N} f_j(u)\log s_j \\ &\leq \sum_{j=1}^{N} f_j(u)\log s_j \end{aligned} \tag{106}$$

by (6). Here we have

$$\begin{aligned} f_j(u)\log s_j &= (f_j(u) - \tilde{f}_j(u))\log s_j + \tilde{f}_j(u)\log s_j \\ &\leq (f_j(u) - \tilde{f}_j(u))\log s_j \end{aligned} \tag{107}$$

and

$$\begin{aligned} & f_j(u) - \tilde{f}_j(u) \\ & \quad = \int_0^1 \frac{d}{ds} f_j(s_1|u|, \cdots, s_{j-1}|u|, s\cdot s_j|u|, s_{j+1}|u|, \cdots, s_N|u|)\ ds \\ & \quad = \int_0^1 \frac{\partial f_j}{\partial u_j}(u(s))\ ds \cdot s_j|u|, \end{aligned}$$

where

$$u(s) = (s_1|u|, \cdots, s_{j-1}|u|, s\cdot s_j|u|, s_{j+1}|u|, \cdots, s_N|u|).$$

Since

$$|u(s)| \leq |u|, \quad 0 \leq s \leq 1$$

it follows from (7) that

$$\begin{aligned} (f_j(u) - \tilde{f}_j(u))\log s_j &\leq C_2(1+|u|)|u|\cdot s_j|\log s_j| \\ &\leq C_{44}|u|^2, \quad |u| \geq 1. \end{aligned} \tag{108}$$

Inequalities (106)-(108) imply

$$\sum_{j=1}^{N} f_j(u)\log u_j \leq NC_{44}|u|^2, \quad |u| \geq 1 \tag{109}$$

and then we obtain (17) by (104) and (109).

B Proof of Proposition 2

Let $u_0 = u|_{t=0}$. By (18) we have

$$\tau \cdot u(\cdot,t) - \tau \cdot u_0 \leq \int_0^t \Delta(d \cdot u(\cdot,s))\ ds,$$

and hence

$$\begin{aligned}
&(\tau \cdot u(\cdot,t), d \cdot u(\cdot,t)) - (\tau \cdot u_0, d \cdot u(\cdot,t)) \\
&\quad \leq -(\nabla d \cdot u(\cdot,t), \nabla \int_0^t d \cdot u(\cdot,s)\ ds) \\
&\quad = -\frac{1}{2}\frac{d}{dt}\|\nabla \int_0^t d \cdot u(\cdot,s)\ ds\|_2^2,
\end{aligned} \tag{110}$$

where (,) denotes the L^2-inner product. Integration of (110) over $(0,T)$ implies

$$\begin{aligned}
&\int_0^T (\tau \cdot u(\cdot,t), d \cdot u(\cdot,t))\ dt \\
&\quad \leq \|\tau \cdot u_0\|_2 \cdot \int_0^T \|d \cdot u(\cdot,t)\|_2\ dt \\
&\quad \leq T^{1/2}\|\tau \cdot u_0\|_2 \cdot \left(\int_0^T \|d \cdot u(\cdot,t)\|_2^2\ dt\right)^{1/2},
\end{aligned}$$

and hence (19) holds by $u = (u_j(\cdot,t)) \geq 0$.

C Proof of Proposition 3

It follows from (18) that

$$\tau \cdot u(\cdot,T) - \tau \cdot u(\cdot,t) \leq \int_t^T \Delta(d \cdot u(\cdot,s))\ ds, \quad 0 \leq t \leq T. \tag{111}$$

It holds that

$$V_t = -d \cdot u$$

for

$$V(\cdot,t) = \int_t^T d \cdot u(\cdot,s)\ ds, \tag{112}$$

and hence (111) implies

$$\Delta V \geq -\tau \cdot u(\cdot,t) \geq \tilde{\tau} V_t \text{ in } Q_T, \ \left.\frac{\partial V}{\partial \nu}\right|_{\partial\Omega} \leq 0 \quad \text{for } \tilde{\tau} = \max_j \tau_j d_j^{-1} \tag{113}$$

by $u = (u_j(\cdot,t)) \geq 0$. It follows also that

$$\|V(\cdot,0)\|_1 \leq \int_0^T \|d \cdot u(\cdot,s)\|_1 \ ds \leq \tilde{d}\|u\|_{L^1(Q_T)} \quad \text{for } \tilde{d} = \max_j d_j$$

from (112). Therefore, the parabolic regularity to (113) implies

$$\begin{aligned} \sup_{\eta \leq t \leq T} \|V(\cdot,t)\|_\infty &\leq C_{45}(\eta,\tilde{\tau})\|V(\cdot,0)\|_1 \\ &\leq C_{45}(\eta,\tilde{\tau}) \cdot \tilde{d} \cdot \|u\|_{L^1(Q_T)} \end{aligned} \tag{114}$$

by $u = (u_j(\cdot,t)) \geq 0$.

Taking $0 \leq t_0 \leq t \leq T$, we apply (18) again, to obtain

$$\tau \cdot u(\cdot,t) \leq \tau \cdot u(\cdot,t_0) + \int_{t_0}^t \Delta(d \cdot u)(\cdot,s)) \ ds.$$

Then it follows that

$$(\tau \cdot u(\cdot,t), d \cdot u(\cdot,t)) \leq (\tau \cdot u(\cdot,t_0), d \cdot u(\cdot,t)) - \frac{1}{2}\frac{d}{dt}\|\nabla \int_{t_0}^t d \cdot u(\cdot,s) \ ds\|_2^2$$

where $(\ , \ )$ denotes the L^2-inner product. Integrating the above inequality with respect to $t \in [0,T]$ leads to

$$\begin{aligned} &\iint_{\Omega\times(t_0,T)} (\tau \cdot u)(d \cdot u) \ dxdt + \frac{1}{2}\|\nabla \int_{t_0}^T d \cdot u(\cdot,s) \ ds\|_2^2 \\ &\leq \int_{t_0}^T (\tau \cdot u(\cdot,t_0), d \cdot u(\cdot,t)) \ dt = (\tau \cdot u(\cdot,t_0), \int_{t_0}^T d \cdot u(\cdot,t) \ dt) \\ &\leq \|\tau \cdot u(\cdot,t_0)\|_1 \cdot \|V(\cdot,t_0)\|_\infty. \end{aligned} \tag{115}$$

Inequality (20) is a direct consequence of (114)-(115) and (15).

D Parabolic problem (25)

We confirm the following fact shown in the proof of Lemma 2.3 of [15].

Proposition 4 *For (26), there is a unique solution* $v = v(x,t) \in L^2(Q_T)$ *to (25) such that* $\int_0^t av \in L^2(0,T;H^2(\Omega))$ *in the sense that*

$$\begin{aligned} & v - \Delta\left(\int_0^t av(\cdot,s)\ ds\right) = v_0 + \int_0^t f(\cdot,s)\ ds \\ & \frac{\partial}{\partial\nu}\int_0^t av(\cdot,s)\ ds\Big|_{\partial\Omega} = 0. \end{aligned} \tag{116}$$

Similarly to (19), the estimate

$$\|v\|_{L^2(Q_T)} \leq C_{46}T^{1/2}(\|v_0\|_2 + \|f\|_{L^2(Q_T)}) \tag{117}$$

is proven for the above $v = v(x,t)$*, which ensures the following result by the dominated convergence theorem.*

Proposition 5 *Let* $0 < C_9^{-1} \leq a_k = a_k(x,t) \leq C_9$*,* $v_{k0} \in L^2(\Omega)$*, and* $f_k \in L^2(Q_T)$*,* $k = 1,2,\cdots$*, be sequences of coefficients, initial values, and inhomogeneous terms, respectively, satisfying*

$$\begin{aligned} & a_k \to a \quad \text{a.e. in } Q_T = \Omega\times(0,T) \\ & v_{k0} \to v_0 \ \text{in } L^2(\Omega), \quad f_k \to f \quad \text{in } L^2(Q_T). \end{aligned} \tag{118}$$

Let $v_k = v_k(x,t) \in L^2(Q_T)$ *be the solution to*

$$\frac{\partial v_k}{\partial t} - \Delta(a_k v_k) = f_k, \quad \frac{\partial}{\partial\nu}(a_k v_k)\Big|_{\partial\Omega} = 0, \quad v_k|_{t=0} = v_{k0}(x) \tag{119}$$

in the sense of Propsition 4. Then it holds that

$$v_k \to v \quad \text{in } L^2(Q_T),$$

where $v = v(x,t)$ *is the solution to (25).*

Proposition 5 implies the following proposition.

Proposition 6 *The solution* $v = v(x,t)$ *to (25) in Proposition 4 satisfies*

$$\|v(\cdot,t)\|_1 \leq \|v_0\|_1 + \int_0^t \|f(\cdot,s)\|_1\ ds \quad \text{for a.e. } t \in (0,T). \tag{120}$$

Proof: *Letting* $v_0^\pm = \max\{0, \pm v\}$, $f^\pm = \max\{0, \pm f\}$, *we take smooth* $C_9^{-1} \leq a_k = a_k(x,t) \leq C_9$, $f_{\pm k} = f_{\pm k}(x,t)$, *and* $v_{\pm 0k} = v_{\pm 0k}(x)$, $k = 1, 2, \cdots$, *such that*

$$a_k \to a, \ a.e., \quad v_{\pm k0} \to v_0^\pm \ in \ L^2(\Omega), \quad f_{\pm k} \to f^\pm \ in \ L^2(Q_T).$$

There is a unique classical solution $v_{\pm k} = v_{\pm k}(x,t) \geq 0$ *to*

$$\frac{\partial v_{\pm k}}{\partial t} - \Delta(a_k v_{\pm k}) = f_{\pm k} \ in \ Q_T, \quad \frac{\partial}{\partial \nu}(a_k v_{\pm k})\Big|_{\partial\Omega} = 0, \quad v_{\pm k}|_{t=0} = v_{\pm k0}(x) \tag{121}$$

which satisfies

$$\|v_{\pm k}(\cdot,t)\|_1 = \|v_{\pm k0}\|_1 + \int_0^t \|f_{\pm k}(\cdot,s)\|_1 \ ds, \quad 0 \leq t \leq T. \tag{122}$$

Here we have $v_{\pm k} \to v_\pm$ *in* $L^2(Q_T)$ *by Proposition 5, which solves*

$$\frac{\partial v_\pm}{\partial t} - \Delta(a v_\pm) = f^\pm \ in \ Q_T, \quad \frac{\partial}{\partial \nu}(a v_\pm)\Big|_{\partial\Omega} = 0, \quad v_\pm|_{t=0} = v_0^\pm,$$

in the sense of (116). Hence it follows that $v = v_+ - v_-$ *from the uniqueness of the solution and also*

$$\|v_\pm(\cdot,t)\|_1 = \|v_0^\pm\|_1 + \int_0^t \|f^\pm(\cdot,s)\|_1 \ ds, \quad 0 \leq t \leq T.$$

Then we obtain (120) by

$$\begin{aligned}
&\|v(\cdot,t)\|_1 = \|v_+(\cdot,t) - v_-(\cdot,t)\|_1 \leq \|v_+(\cdot,t)\|_1 + \|v_-(\cdot,t)\|_1 \\
&\|v_0\|_1 = \|v_0^+\|_1 + \|v_0^-\|_1 \\
&\|f(\cdot,s)\|_1 = \|f^+(\cdot,s)\|_1 + \|f^-(\cdot,s)\|_1.
\end{aligned}$$

□

Finally, the following proposition is derived similarly to (114) and (115).

Proposition 7 *Let* $0 < C_9^{-1} \leq a = a(x,t) \leq C_9$ *and let* $v = v(x,t) \geq 0$ *be a smoth function on* $\overline{\Omega} \times [0,T]$ *satisfying*

$$\frac{\partial v}{\partial t} - \Delta(av) \leq 0 \ in \ Q_T, \quad \frac{\partial}{\partial \nu}(av)\Big|_{\partial\Omega} \leq 0.$$

Then it holds that

$$\|v\|_{L^2(\Omega\times(\eta,T))} + \|\int_\eta^T av(\cdot,s) \ ds\|_\infty \leq C_{47}(\eta,T)\|v\|_{L^1(Q_T)}$$

for any $0 < \eta < T$.

E Linear heat equation (27)

The description of Remark 1 is a direct consequence of the following proposition. It is proven by the comparison principle (Lemma 3.4 of [2]).

Proposition 8 *Given $w_0 \in L^1(\Omega)$ and $H \in L^1(Q_T)$, let*

$$w = w(\cdot, t) \in L^\infty(0, T; L^1(\Omega)) \cap L^1_{loc}(0, T; W^{1,1}(\Omega))$$

be the solution to (27). More precisely, for any $\varphi \in W^{1,\infty}(\Omega)$ it holds that

$$\frac{d}{dt} \int_\Omega w\varphi \; dx + \int_\Omega \nabla w \cdot \nabla \varphi \; dx = \int_\Omega H\varphi \; dx$$

in the sense of distributions with respec to t and

$$\lim_{t \downarrow 0} w(\cdot, t) = w_0$$

in the sense of measures on $\overline{\Omega}$. Then it follows that

$$w(\cdot, t) = e^{t\Delta} w_0 + \int_0^t e^{(t-s)\Delta} H(\cdot, s) \; ds, \quad 0 \leq t \leq T. \tag{123}$$

In particular, $w \in C([0, T], L^1(\Omega))$ and this solution exists uniquely.

The existence of the solution in the above proposition may be proven by the duality argument (Lemma 3.3 of [2]). By (123), a result comparable to Proposition 5 is obtained.

Proposition 9 *The mapping $\mathcal{F} : (w_0, H) \in L^1(\Omega) \times L^1(Q_T) \mapsto w \in C([0, T], L^1(\Omega))$ is continuous, where $w = w(x, t)$ is the solution to (27) in Proposition 8.*

The following compactness result is known even to the nonlinear contraction semigroup [1] (see also Lemma 3.3 of [2]).

Proposition 10 *The mapping $\mathcal{F} : (w_0, H) \in L^1(\Omega) \times L^1(Q_T) \mapsto w \in L^1(Q_T)$ is compact, where $w = w(x, t)$ is the solution to (27) in Proposition 8. In other words, image of each bounded set in $L^1(\Omega) \times L^1(Q_T)$ by $\mathcal{F}$ is relatively compact in $L^1(Q_T)$.*

Proof: By (123), the dual operator

$$\mathcal{F}^* : L^\infty(Q_T) \to L^\infty(\Omega) \times L^\infty(Q_T)$$

is realized as $\mathcal{F}^*(h) = (\theta|_{t=0}, \theta)$, where $\theta = \theta(\cdot, t)$ is the solution to the backward heat equation

$$\frac{\partial \theta}{\partial t} + \Delta\theta = h \ in \ Q_T, \quad \left.\frac{\partial \theta}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \theta|_{t=T} = 0.$$

Then the assertion follows because $\mathcal{F}^*$ is compact by the parabolic regularity. □

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