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Dissipativity of general Duhem hysteresis models

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***Abstract*— In this paper, we discuss the dissipativity property of Duhem hysteresis models. Under some sufficient conditions on the functions which defines the Duhem model, an explicit construction of the storage functions is given which takes into account the data on the anhysteresis function. We present an example on the semi-linear Duhem model.**

I. INTRODUCTION

Hysteresis is a nonlinear operator with memory that is commonly found in a wide range of physical systems, such as, magnetic material, piezo-electric material and mechanical friction. For studying the influence of hysteresis in a system, numerous hysteresis models have been proposed, such as the backlash, elastic-plastic, Preisach and Duhem models [2], [10].

In the literature related to magnetic materials, the hysteresis behavior is caused by the friction/pinning of the magnetic domain-walls [7], [15]. When the influence of the friction/pinning of magnetic domain-walls is neglected, the relation between the magnetization and the external magnetic field is defined as anhysteresis function. In this case, Jiles and Atherton [7] propose a hysteresis model for describing the magnetization that is composed of an anhysteresis part and another component which is due to the pinning of magnetic domain-walls. Similarly, Coleman and Hodgdon [3] propose another hysteresis model to describe the same phenomenon. The general model that contains both models is the Duhem model [10], i.e., the Coleman-Hodgdon model [3] and Jiles-Atherton model [7] are a class of Duhem model. Since the anhysteresis function has played a role in the description of hysteresis in magnetic material, we take it into account in our study.

Hysteresis phenomenon that is due to the friction, either between the magnetic domain-walls or between mechanical surfaces, dissipates energy by heat. This is related to the concept of dissipativity in the systems theory literature [14], [16], [18]. For linear electrical and mechanical systems, the energy loss can be described by constructing a storage function [14], [16], [18] whose time-derivative is less than or equal to the quantity of the supplied power to the systems. However, constructing storage function for hysteresis operator is not straightforward.

The existence of the storage function for hysteresis operator can be useful to analyze the stability of systems which contain a hysteretic element. In Gorbet *et al* [4], a storage function is constructed for Preisach operator with non-negative weighting function, and it is employed to show the stability of electro-mechanical systems with a hysteretic piezo-actuator. For relay and backlash operators, the corresponding storage function has been proposed in Brokate and Sprekels [2]. However, the construction of storage function for the Duhem model remains limited and the paper [6] has presented a preliminary result only for a small class of Duhem model. In this paper, we extend the result in [6] by admitting a larger class of Duhem model.

This paper is organized as follows. In Section II, we introduce the Duhem hysteresis operator and formulate the dissipativity problem. In Section III, we give sufficient conditions for a Duhem model to admit a storage function. An example on the construction of such storage function is presented in Section IV.

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II. DUHEM HYSTERESIS OPERATOR

We denote by $C^1(\mathbb{R}_+)$ the space of continuously differentiable functions $f : \mathbb{R}_+ \to \mathbb{R}$.

Using the same description as in [10], [12], [17], the Duhem model $\Phi : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+), u \mapsto \Phi(u) =: y$ is described by

$$\dot{y}(t) = f_1(y(t), u(t))\dot{u}_+(t) + f_2(y(t), u(t))\dot{u}_-(t),$$
$$y(0) = y_0, \quad (1)$$

where $\dot{u}_+(t) := \max\{0, \dot{u}(t)\}$, $\dot{u}_-(t) := \min\{0, \dot{u}(t)\}$. We refer to [10], [12], [17] for the detail discussion on the solution of (1). Note that the differential equation (1) can also be put into state-space form where the state consists of both variables u, y. In this paper, we analyze the dissipativity using the behavioral framework where the analysis is done directy on the manifest variables u, y, in the same vein as that for linear systems in [16]. Let $U, Y \subset \mathbb{R}$ be the domain of $u(t)$ and $y(t)$, respectively.

The functions f_1 and f_2 in (1) are defined appropriately according to the hysteresis curve obtained from experimental data.

We can now state our problem as follows.

Definition 2.1: The Duhem model as in (1)-(3) is said to be *dissipative with respect to the supply rate* $\langle \dot{y}, u\rangle$ if there exists a positive definite storage function $H(y, u)$ such that for every $u \in C^1(\mathbb{R}_+)$ and $y := \Phi(u)$,

$$\frac{\mathrm{d}H(y(t), u(t))}{\mathrm{d}t} \le \langle \dot{y}(t), u(t)\rangle. \quad (2)$$

We remark that the supply-rate function as given in the Definition 2.1 has been used in the study of dissipativity for

Preisach model [4] and in the study of counter-clockwise systems [1], [13]. This supply-rate function also belongs to the family of general supply-rate functions as described in [16] which studies general dissipativity theory for linear systems using the behavioural framework.

In Section III, we show that under some assumptions on f_1, f_2, we have a family of storage functions which satisfies (2). It is constructed using only the data on f_1, f_2.

The motivation of the dissipativity property in (2) stems from the physical law governing an electrical inductor. The magnetic flux ϕ and the electric current I in an inductor can be related by an operator Φ, i.e., $\phi = \Phi(I)$ (for instance, with a linear inductor model, $\Phi(I) = LI$ where L is the inductance). Basic electrical law yields that $\dot{\phi} = V$ where V is the voltage across the inductor. Hence the electrical power (defined by $\langle V(t), I(t)\rangle$) transferred to the inductor is equal to $\langle (\dot{\overbrace{\Phi(I)}})(t), I(t)\rangle$. Since inductor is a passive electrical element and there is energy loss due to hysteresis, the power being stored in the inductor has to be less than or equal to the amount of power being transferred into the inductor. In this case, (2) holds with $u = I$.

III. MAIN RESULTS

Before we can state our main results in Theorem 3.3 and 3.5, we need to introduce three functions: an anhysteresis function f_{an}, the function ω_Φ (or ν_Φ) and the intersection function Ω (or Υ). These functions play important roles in the construction of storage function and in the characterization of dissipativity. They are defined based on the data on f_1 and f_2.

A. Curves definition

1) The anhysteresis function: In order to define the anhysteresis function, we rewrite f_1 and f_2 as follows

$$\left.\begin{aligned} f_1(y(t), u(t)) &= F(y(t), u(t)) + G(y(t), u(t)), \\ f_2(y(t), u(t)) &= -F(y(t), u(t)) + G(y(t), u(t)), \end{aligned}\right\} \tag{3}$$

where $F, G : \mathbb{R}^2 \to \mathbb{R}$. We assume that the implicit function $F(\sigma, \xi) = 0$ can be represented by an explicit function $\sigma = f_{an}(\xi)$ or $\xi = g_{an}(\sigma)$. Such function f_{an} (or g_{an}) is called an anhysteresis function and the corresponding graph $\{(\xi, f_{an}(\xi)) | \xi \in \mathbb{R}\}$ is called an anhysteresis curve. Using f_{an}, it can be checked that $f_1(f_{an}(\xi), \xi) = f_2(f_{an}(\xi), \xi)$ holds.

By employing the implicit function F for representing anhysteresis, we can include Duhem models that admits $f_{an} = 0$ or $g_{an} = 0$. For instance, the dissipativity property for Duhem model with $g_{an} = 0$ was presented in our preliminary work [6]. Note also that the functions F and G in (3) are defined by

$$F = \frac{f_1 - f_2}{2} \qquad G = \frac{f_1 + f_2}{2}.$$

2) The functions ω_Φ and ν_Φ: For every pair $(y_0, u_0) \in Y \times U$, let $\omega_{\Phi,1}(\cdot, y_0, u_0) : [u_0, \infty) \to \mathbb{R}$ be the solution of

$$x(\tau) - x(u_0) = \int_{u_0}^{\tau} f_1(x(\sigma), \sigma)\, \mathrm{d}\sigma, \qquad x(u_0) = y_0 \quad \forall \tau \in [u_0, \infty),$$

and let $\omega_{\Phi,2}(\cdot, y_0, u_0) : (-\infty, u_0] \to \mathbb{R}$ be the solution of

$$x(\tau) - x(u_0) = \int_{u_0}^{\tau} f_2(x(\sigma), \sigma)\, \mathrm{d}\sigma, \qquad x(u_0) = y_0 \quad \forall \tau \in (-\infty, u_0].$$

Using the above definitions, for every pair $(y_0, u_0) \in Y \times U$, the function $\omega_\Phi(\cdot, y_0, u_0) : \mathbb{R} \to \mathbb{R}$ is defined by the concatenation of $\omega_{\Phi,2}(\cdot, y_0, u_0)$ and $\omega_{\Phi,1}(\cdot, y_0, u_0)$:

$$\omega_\Phi(\tau, y_0, u_0) = \begin{cases} \omega_{\Phi,2}(\tau, y_0, u_0) & \forall \tau \in (-\infty, u_0) \\ \omega_{\Phi,1}(\tau, y_0, u_0) & \forall \tau \in [u_0, \infty). \end{cases} \tag{4}$$

Similarly, we can introduce the function ν_Φ which is dual to construction of ω_Φ.

For every pair $(y_0, u_0) \in Y \times U$, let $\nu_{\Phi,1}(\cdot, y_0, u_0) : [u_0, \infty) \to \mathbb{R}$ be the solution of

$$x(\tau) - x(u_0) = \int_{u_0}^{\tau} f_2(x(\sigma), \sigma)\, \mathrm{d}\sigma, \qquad x(u_0) = y_0 \quad \forall \tau \in [u_0, \infty),$$

and let $\nu_{\Phi,2}(\cdot, y_0, u_0) : (-\infty, u_0] \to \mathbb{R}$ be the solution of

$$x(\tau) - x(u_0) = \int_{u_0}^{\tau} f_1(x(\sigma), \sigma)\, \mathrm{d}\sigma, \qquad x(u_0) = y_0 \quad \forall \tau \in (-\infty, u_0].$$

Using the above definitions, for every pair $(y_0, u_0) \in X$, the function $\nu_\Phi(\cdot, y_0, u_0) : \mathbb{R} \to \mathbb{R}$ is defined by the concatenation of $\nu_{\Phi,2}(\cdot, y_0, u_0)$ and $\nu_{\Phi,1}(\cdot, y_0, u_0)$:

$$\nu_\Phi(\tau, y_0, u_0) = \begin{cases} \nu_{\Phi,2}(\tau, y_0, u_0) & \forall \tau \in (-\infty, u_0) \\ \nu_{\Phi,1}(\tau, y_0, u_0) & \forall \tau \in [u_0, \infty). \end{cases} \tag{5}$$

3) Intersection function: The function, which describes the intersection between f_{an} and ω_Φ, is characterized in the following lemma.

Lemma 3.1: Let Y, U be an open set. Assume that f_1 and f_2 in (3) be such that f_1, f_2 and f_{an} are continuously differentiable[1]. Moreover, assume that f_{an} is strictly increasing and there exists a positive real number ϵ such that for all $(\sigma, \xi) \in Y \times U$ the following inequality holds

$$f_1(\sigma, \xi) < \frac{\mathrm{d} f_{an}(\xi)}{\mathrm{d}\xi} - \epsilon \quad \text{whenever} \quad \sigma > f_{an}(\xi)\ , \tag{6}$$

$$f_2(\sigma, \xi) < \frac{\mathrm{d} f_{an}(\xi)}{\mathrm{d}\xi} - \epsilon \quad \text{whenever} \quad \sigma < f_{an}(\xi)\ , \tag{7}$$

[1] In the case in which the function f_{an} is uniquely defined we may obtain directly its smoothness from the property,

$$\frac{\mathrm{d} f_{an}}{\mathrm{d} b}(b) = \frac{\frac{\partial f_2}{\partial b}(f_{an}(b), b) - \frac{\partial f_1}{\partial b}(f_{an}(b), b)}{\frac{\partial f_1}{\partial a}(f_{an}(b), b) - \frac{\partial f_1}{\partial a}(f_{an}(b), b)}$$

Then there exists $\Omega \in C^1(Y \times U, \mathbb{R})$ such that

(1) $\Omega(\sigma, \xi) \geq \xi$ whenever $\sigma \geq f_{an}(\xi)$ and $\Omega(\sigma, \xi) < \xi$ otherwise.

(2) $\omega_\Phi(\Omega(\sigma, \xi), \sigma, \xi) = f_{an}(\Omega(\sigma, \xi))$. (8)

(3) Moreover, for all $u \in C^1$, $y := \Phi(u)$, $\frac{\mathrm{d}}{\mathrm{d}t}\Omega(y(t), u(t))$ exists.

Proof: Consider the continuous function $\varphi : \mathbb{R} \times Y \times U \to \mathbb{R}$ defined as $\varphi(\xi, y_0, u_0) = \omega_\Phi(\xi, y_0, u_0) - f_{an}(\xi)$. Consider also A_0 and A_1 the two subsets of $\mathbb{R}^3$ defined as,

$$A_0 = \{(\xi, y_0, u_0) \in \mathbb{R}^3, (y_0, u_0) \in X \ , \ y_0 > f_{an}(u_0) \ , \ \xi > u_0\} \ ,$$
$$A_1 = \{(\xi, y_0, u_0) \in \mathbb{R}^3, (y_0, u_0) \in X \ , \ y_0 < f_{an}(u_0) \ , \ \xi < u_0\} \ ,$$

Note that the function f_{an} being strictly increasing by assumption, implies that these sets are open sets. Also, the function ω_Φ satisfies

$$\frac{\partial \omega_\Phi}{\partial \xi}(\xi, y_0, u_0) = f_1(\omega_\Phi(\xi, y_0, u_0), \xi) \quad \forall(\xi, y_0, u_0) \in A_0 \ ,$$
$$\frac{\partial \omega_\Phi}{\partial \xi}(\xi, y_0, u_0) = f_2(\omega_\Phi(\xi, y_0, u_0), \xi) \quad \forall(\xi, y_0, u_0) \in A_1 \ .$$

Consequently, $\omega_\Phi(\xi, y_0, u_0)$ is solution of ordinary differential equations computed from C^1 vector field. With [5, Theorem V.3.1], it implies that ω_Φ is a C^1 function in $A_0 \cup A_1$. Moreover, the function f_{an} being C^1 implies that the function φ is C^1 in $A_0 \cup A_1$. With (6) and (7), the function φ satisfies,

$$\frac{\partial \varphi}{\partial \xi}(\xi, y_0, u_0) \quad < \quad -\epsilon \neq 0 \ , \ \forall(\xi, y_0, u_0) \ \in \ A_0 \cup A_1 \ .$$

Consequently, φ is a strictly decreasing function in its first argument in the set $A_0 \cup A_1$. This also implies that,

$$\varphi(\xi, y_0, u_0) < \varphi(u_0, y_0, u_0) - \epsilon(\xi - u_0) \quad \forall(\xi, y_0, u_0) \in A_0 \ ,$$
$$\varphi(\xi, y_0, u_0) > \varphi(u_0, y_0, u_0) - \epsilon(\xi - u_0) \quad \forall(\xi, y_0, u_0) \in A_1 \ .$$

Note that if $y_0 > f_{an}(u_0)$, then $\varphi(u_0, y_0, u_0) > 0$ and consequently there exists a unique real number u^* such that $\varphi(u^*, y_0, u_0) = 0$ and $(u^*, y_0, u_0) \in A_0$. On the other hand, if $y_0 < f_{an}(u_0)$, then $\varphi(u_0, y_0, u_0) < 0$ and consequently there exists a unique real number u^* such that $\varphi(u^*, y_0, u_0) = 0$ and $(u^*, y_0, u_0) \in A_3$. Therefore, by denoting $\Omega(y_0, u_0) = u^*$, by employing the implicit function theorem and using the fact that φ is C^1, it can be shown that Ω is C^1. □

Roughly speaking the function Ω satisfying **(1)–(3)** means that for all (σ, ξ), the two functions $\omega_\Phi(\cdot, \sigma, \xi)$ and $f_{an}(\cdot)$ intersect at a unique point larger or smaller than u_0 depending on the sign of $\sigma - f_{an}(\xi)$. Moreover, along the solutions of (1), the time derivative of the intersecting point exists.

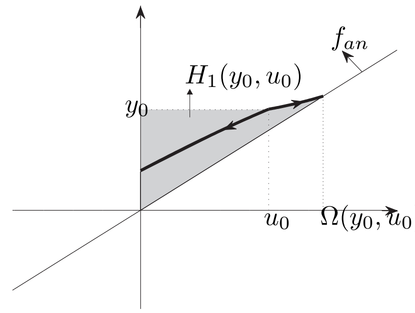

Fig. 1. Graphical interpretation of the storage function H_1 in (12). For a given (y_0, u_0), H_1 is equal to the area in light grey

Similarly, the following lemma characterizes the function that describes the intersection between f_{an} and ν_Φ.

Lemma 3.2: Let Y, U be an open set. Assume that f_1 and f_2 in (3) be such that f_1, f_2 and f_{an} are continuously differentiable. Moreover, assume that f_{an} is strictly increasing and assume that there exists a positive real number ϵ such that for all (y, u) in $Y \times U$ the following inequality holds

$$f_2(\sigma, \xi) < \frac{\mathrm{d}f_{an}(\xi)}{\mathrm{d}\xi} - \epsilon \quad \text{whenever} \quad \sigma > f_{an}(\xi), \quad (9)$$
$$f_1(\sigma, \xi) < \frac{\mathrm{d}f_{an}(\xi)}{\mathrm{d}\xi} - \epsilon \quad \text{whenever} \quad \sigma < f_{an}(\xi), \quad (10)$$

Then there exists $\Upsilon \in C^1(Y \times U, \mathbb{R})$ such that

(1) $\Upsilon(\sigma, \xi) \geq \xi$ whenever $\sigma \geq f_{an}(\xi)$ and $\Upsilon(\sigma, \xi) < \xi$ otherwise.

(2) $\nu_\Phi(\Upsilon(\sigma, \xi), \sigma, \xi) = f_{an}(\Upsilon(\sigma, \xi))$. (11)

(3) Moreover, for all $u \in C^1$, $y := \Phi(u)$, $\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon(y(t), u(t))$ exists.

B. Storage function using ω_Φ

We define a candidate storage function $H_1 : X \to \mathbb{R}$.

$$H_1(\sigma, \xi) = \sigma\xi - \int_0^\xi \omega_\Phi(\tau, \sigma, \xi) \ \mathrm{d}\tau + \int_0^{\Omega(\sigma, \xi)} \omega_\Phi(\tau, \sigma, \xi) - f_{an}(\tau) \ \mathrm{d}\tau, \quad (12)$$

where the function Ω is as given in Lemma 3.1. The graphic interpretation of H_1 is shown in Figure 1.

Theorem 3.3: Consider the Duhem hysteresis operator Φ defined in (1)-(3) with locally Lipschitz functions $F, G : \mathbb{R}^2 \to \mathbb{R}$ and with anhysteresis function f_{an}. Suppose that the hypotheses in Lemma 3.1 hold and the following condition holds for all (σ, ξ) in $Y \times U$

(A) $F(\sigma, \xi) \geq 0$ whenever $\sigma \leq f_{an}(\xi)$, and $F(\sigma, \xi) < 0$ otherwise.

Then for every $u \in C^1(\mathbb{R}_+)$ and for every $y_0 \in \mathbb{R}$, the function $t \mapsto H_1\big((\Phi(u))(t), u(t)\big)$ with H_1 as in (12) is differentiable and satisfies (2). In other words, Φ is dissipative with respect to the supply rate $\langle \dot{y}, u \rangle$ with the storage function H_1.

Proof: It can be checked that the hypothesis **(A)** on F implies that $f_1(\sigma,\xi) \geq f_2(\sigma,\xi)$ whenever $\sigma \leq f_{an}(\xi)$, and $f_1(\sigma,\xi) < f_2(\sigma,\xi)$ otherwise.

Let $u \in C^1(\mathbb{R}_+)$ and $y_0 \in \mathbb{R}$ and denote $u^* := \Omega(y,u)$. First, we would prove that for all $t \in \mathbb{R}_+$, $\dot{H}_1\big((\Phi(u))(t),u(t)\big)$ exists. Using (12) and with Leibniz derivative rule and denoting $y = (\Phi(u))$, we have

$$\begin{aligned}\frac{\mathrm{d}H_1\big(y(t),u(t)\big)}{\mathrm{d}t} &= \dot{y}(t)u(t)\\ &+ \Big[\omega_\Phi(u^*(t),y(t),u(t)) - f_{an}(u^*(t))\Big]\,\dot{u}^*(t)\\ &+ \int_{u(t)}^{u^*(t)} \frac{\mathrm{d}}{\mathrm{d}t}\omega_\Phi(\tau,y(t),u(t))\mathrm{d}\tau, \qquad (13)\end{aligned}$$

where we have invoked the relation $\omega_\Phi(u(t),y(t),u(t)) = y(t)$.

Let $t \geq 0$. The first term in the RHS of (13) exists for all $t \geq 0$ since $y(t)$ satisfies (1). Note that since $\omega_\Phi(u^*(t),y(t),u(t)) = f_{an}(u^*(t))$, the second term of (13) is zero since $\dot{u}^*(t)$ exists by Lemma 3.1. In order to get the dissipativity with the supply rate (2), it remains to check whether the last term of (13) exists, is finite and satisfies

$$\int_{u(t)}^{u^*(t)} \frac{\mathrm{d}}{\mathrm{d}t}\omega_\Phi(\tau,y(t),u(t))\mathrm{d}\tau \leq 0. \qquad (14)$$

It suffices to show that, for every $\tau \in [u(t),u^*(t)]$, the following Dini's derivative :

$$\lim_{\epsilon\searrow 0^+} \frac{1}{\epsilon}[\omega_\Phi(\tau,y(t+\epsilon),u(t+\epsilon)) - \omega_\Phi(\tau,y(t),u(t))], \quad (15)$$

exists and the limit is less or equal to zero when $u^*(t) > u(t)$ and the limit is greater or equal to zero elsewhere.

For any $\epsilon \geq 0$, let us introduce the continuous function $\omega_\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$\omega_\epsilon(\tau) = \omega_\Phi(\tau, y(t+\epsilon), u(t+\epsilon)). \qquad (16)$$

More precisely, for every $\epsilon \geq 0$, ω_ϵ is the unique solution of

$$\omega_\epsilon(\tau) = \begin{cases} y(t+\epsilon) + \displaystyle\int_{u(t+\epsilon)}^{\tau} f_1(\omega_\epsilon(s),s)\mathrm{d}s \\ \qquad\qquad\qquad \forall \tau \geq u(t+\epsilon), \\ y(t+\epsilon) + \displaystyle\int_{u(t+\epsilon)}^{\tau} f_2(\omega_\epsilon(s),s)\mathrm{d}s \\ \qquad\qquad\qquad \forall \tau \leq u(t+\epsilon), \end{cases} \qquad (17)$$

Note that $\omega_0(\tau) = \omega_\Phi(\tau,y(t),u(t))$ for all $\tau \in \mathbb{R}$ and

$$\omega_\epsilon(u(t+\epsilon)) = y(t+\epsilon) \qquad \forall\, \epsilon \in \mathbb{R}_+ . \qquad (18)$$

In order to show the existence of (15) and the validity of (14), we consider several cases depending on the sign of $\dot{u}(t)$ and $y(t) - f_{an}(u(t))$.

First, we assume that $\dot{u}(t) > 0$. This implies that there exists a sufficiently small $\gamma > 0$ such that for every $\epsilon \in (0,\gamma]$, we have $u(t+\epsilon) > u(t)$ and

$$\omega_0(u(t+\epsilon)) = y(t) + \int_{u(t)}^{u(t+\epsilon)} f_1(\omega_0(s),s)\,\mathrm{d}s.$$

Moreover, with the change of integration variable $s = u(v)$ [2] we obtain

$$\omega_0(u(t+\epsilon)) = y(t) + \int_t^{t+\epsilon} f_1(\omega_0(u(v)),u(v))\,\dot{u}(v)\,\mathrm{d}v,$$

for all $\epsilon \in [0,\gamma]$.

The functions $\epsilon \mapsto w_0(u(t+\epsilon))$ and $\epsilon \mapsto y(t+\epsilon)$ with $\epsilon \in (0,\gamma]$ are two C^1 functions which are solutions of the same locally Lipschitz ODE and with the same initial value. By uniqueness of solution, we get $\omega_0(u(t+\epsilon)) = y(t+\epsilon)$. This fact together with (18) shows that

$$\omega_\epsilon(u(t+\epsilon)) = \omega_0(u(t+\epsilon)) \qquad \forall \epsilon \in [0,\gamma].$$

Let us evaluate (15) when $y(t) \geq f_{an}(u(t))$. In this case, Lemma 3.1(**1**) shows that $u(t) < u^*(t)$. Also, since for every $\epsilon \in (0,\gamma]$ the two functions $\omega_\epsilon(\tau)$ and $\omega_0(\tau)$ satisfy the same ODE for[3] $\tau \in [u(t+\epsilon),u^*(t)]$, we have

$$\omega_\epsilon(\tau) = \omega_0(\tau) \qquad \forall \tau \in [u(t+\epsilon),u^*(t)],$$

for all $\epsilon \in [0,\gamma]$. This implies that

$$\lim_{\epsilon\searrow 0^+} \frac{1}{\epsilon}[\omega_\epsilon(\tau) - \omega_0(\tau)] \;=\; 0, \qquad (19)$$

for all $\tau \in [u(t),u^*(t)]$. Therefore, for the case where $\dot{u}(t) > 0$ and $y(t) \geq f_{an}(u(t))$, the inequality (14) holds.

Now, we check (14) when $y(t) < f_{an}(u(t))$ and $\dot{u}(t) > 0$. Note that according to Lemma 3.1(**1**), $u^*(t) < u(t)$. Also, since $\dot{u}(t) > 0$, there exists $\gamma > 0$ such that we have $\tau \leq u(t) < u(s)$ and $\dot{u}(s) > 0$ for all s in $(t,t+\gamma)$. It follows from (17) and assumption (**A**) that for every $\epsilon \in (0,\gamma)$:

$$\begin{aligned}\frac{\mathrm{d}\omega_\epsilon(u(s))}{\mathrm{d}s} &= f_2(\omega_\epsilon(u(s)),u(s))\,\dot{u}(s)\\ &\leq f_1(\omega_\epsilon(u(s)),u(s))\,\dot{u}(s) \quad \forall s \in [t,t+\epsilon],\end{aligned}$$

and the function y satisfies

$$\frac{\mathrm{d}y(s)}{\mathrm{d}s} \;=\; f_1(y(s),u(s))\,\dot{u}(s) \qquad \forall s \;\in\; [t,t+\epsilon].$$

Since $\omega_\epsilon(u(t+\epsilon)) = y(t+\epsilon)$ and using the comparison principle (in reverse direction), we get that for every $\epsilon \in [0,\gamma)$:

$$\omega_\epsilon(u(s)) \;\geq\; y(s) \qquad \forall\, s \;\in\; [t,t+\epsilon].$$

Since the two functions $\omega_\epsilon(\tau)$ and $\omega_0(\tau)$ for $\tau \in [u^*(t),u(t)]$ are two solutions of the same ODE, it follows that [4] $\omega_\epsilon(\tau) \geq \omega_0(\tau)$ and we get that if it exists:

$$\lim_{\epsilon\searrow 0^+} \frac{1}{\epsilon}[\omega_\epsilon(\tau) - \omega_0(\tau)] \;\geq\; 0 \qquad \forall \tau \in [u^*(t),u(t)]. \quad (20)$$

[2]This change is allowed since for every $\epsilon \in [0,\gamma]$, u is a strictly increasing function from $[t,t+\epsilon]$ toward $[u(t),u(t+\epsilon)]$.

[3]we have for all $\tau \in [u(t+\epsilon),u^*(t)]$:

$$\frac{\mathrm{d}\omega_\epsilon(\tau)}{\mathrm{d}\tau} \;=\; f_1(\omega_\epsilon(\tau),\tau) \quad , \qquad \frac{\mathrm{d}\omega_0(\tau)}{\mathrm{d}\tau} \;=\; f_1(\omega_0(\tau),\tau)$$

[4]Otherwise there exist $\tau_1 < \tau_2$ such that $\omega_\epsilon(\tau_1) = \omega_0(u(\tau_1))$ and $\omega_\epsilon(\tau_2) > \omega_0(u(\tau_2))$ which contradict the uniqueness of the solution of the locally Lipschitz ODE.

In the following, to show the existence of the limit given in (20), we compute a bound on the function $\epsilon \mapsto \frac{1}{\epsilon}[\omega_\epsilon(\tau) - \omega_0(\tau)]$. Note that for every $\epsilon \in [0, \gamma]$,

$$
\begin{aligned}
|\omega_\epsilon(\tau) - \omega_0(\tau)| \le\ & |y(t+\epsilon) - y(t)| \\
& + \left| \int_{u(t+\epsilon)}^{u(t)} f_2(\omega_\epsilon(s), s)\, \mathrm{d}s \right| \\
& + \left| \int_{u(t)}^{\tau} f_2(\omega_\epsilon(s), s) - f_2(\omega_0(s), s)\, \mathrm{d}s \right| \\
\le\ & |y(t+\epsilon) - y(t)| + \int_{u(t)}^{u(t+\epsilon)} |f_2(\omega_\epsilon(s), s)|\, \mathrm{d}s \\
& + \int_{\tau}^{u(t)} |f_2(\omega_\epsilon(s), s) - f_2(\omega_0(s), s)|\, \mathrm{d}s,
\end{aligned}
$$

for all $\tau \in [u^*(t), u(t)]$. By the locally Lipschitz property of f_2, by the boundedness of f_2 and by the boundedness of ω_ϵ on $[\tau, u(t)]$ for all $\epsilon \in [0, \gamma]$, we obtain

$$
\begin{aligned}
|\omega_\epsilon(\tau) - \omega_0(\tau)| &\le |y(t+\epsilon) - y(t)| \\
&+ \int_{\tau}^{u(t)} L\, |\omega_\epsilon(s) - \omega_0(s)|\, \mathrm{d}s + \alpha |u(t+\epsilon) - u(t)|\ ,
\end{aligned}
$$

where α is a bound of f_2 on a compact set and L is the Lipschitz constant of f_2 on $[\omega_{\min}, \omega_{\max}] \times [\tau, u(t)]$ with

$$
\begin{aligned}
\omega_{\min} &= \min_{(c,s)\in[0,\gamma]\times[\tau,u(t)]} \omega_c(s), \\
\omega_{\max} &= \max_{(c,s)\in[0,\gamma]\times[\tau,u(t)]} \omega_c(s)\ .
\end{aligned}
$$

With Gronwall's lemma, this implies that for every $\epsilon \in [0, \gamma]$

$$
\begin{aligned}
&|\omega_\epsilon(\tau) - \omega_0(\tau)| \\
&\le \exp((u(t)-\tau)L)\Big[|y(t+\epsilon) - y(t)| + \alpha|u(t+\epsilon) - u(t)|\Big],
\end{aligned}
$$

for all $\tau \in [u^*(t), u(t)]$. Hence

$$
\begin{aligned}
&\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon} |\omega_\epsilon(\tau) - \omega_0(\tau)| \\
&\qquad \le \exp((u(t) - \tau)L)\Big[|f_1(y(t), u(t))| + \alpha\Big]\, \dot{u}(t),
\end{aligned}
$$

for all $\tau \in [u^*(t), u(t)]$. Consequently the limit given in (20) exists. It implies that the inequality (14) holds when $\dot{u}(t) > 0$ and $y(t) < f_{an}(u(t))$.

We can use similar arguments to prove that (14) is satisfied when $\dot{u}(t) < 0$.

Finally, when $\dot{u}(t) = 0$, we simply get

$$
\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon} |\omega_\epsilon(\tau) - \omega_0(\tau)| = 0,
$$

by continuity of the above bound. □

Remark 3.4: The storage function H_1 in the Theorem 3.3 is non-negative if f_1 is positive and f_{an} is strictly increasing or $g_{an} = 0$. To show this, let us consider the case when f_{an} is strictly increasing. If $u(t) \ge 0$ and $y(t) \ge f_{an}(u(t))$, we

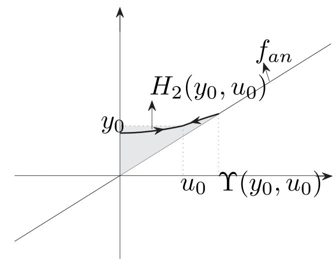

Fig. 2. Graphical interpretation of the storage function H_2 in (21). For a given (y_0, u_0), H_2 is equal to the difference between the area in light grey and the area in dark grey

have that $f_{an}(\tau) \le y(t)$ for all $\tau \in [0, u(t)]$ and $f_{an}(\tau) \le \omega_\Phi(\tau, y(t), u(t))$ for all $\tau \in [u(t), \Omega(y(t), u(t))]$. Hence

$$
\begin{aligned}
H_1(y(t), u(t)) &= \int_0^{u(t)} y(t) - f_{an}(\tau)\ \mathrm{d}\tau \\
&+ \int_{u(t)}^{\Omega(y(t),u(t))} \omega_\Phi(\tau, y(t), u(t)) - f_{an}(\tau)\ \mathrm{d}\tau \ge 0.
\end{aligned}
$$

On the other hand, if $u(t) < 0$ and $y(t) \ge f_{an}(u(t))$, we have that $y(t) \le \omega_\Phi(\tau, y(t), u(t))$ for all $\tau \in [u(t), 0]$ (due to the positivity of f_1). Also, if $\Omega(y(t), u(t)) \ge 0$, $f_{an}(\tau) \le \omega_\Phi(\tau, y(t), u(t))$ for all $\tau \in [0, \Omega(y(t), u(t))]$ and if $\Omega(y(t), u(t)) < 0$, $f_{an}(\tau) > \omega_\Phi(\tau, y(t), u(t))$ for all $\tau \in [\Omega(y(t), u(t)), 0]$. Hence

$$
\begin{aligned}
H_1(y(t), u(t)) &= -\int_{u(t)}^{0} y(t) - \omega_\Phi(\tau, y(t), u(t))\ \mathrm{d}\tau \\
&+ \int_0^{\Omega(y(t),u(t))} \omega_\Phi(\tau, y(t), u(t)) - f_{an}(\tau)\ \mathrm{d}\tau \ge 0.
\end{aligned}
$$

For the case $y(t) < f_{an}(u(t))$, the non-negativity of H_1 can be checked using the same routine.

If we have $g_{an} = 0$, the non-negativity of H_1 can be checked directly (we refer also to [6]). △

C. Storage function using ν_Φ

Dual to the results from the previous subsection, we can also define storage functions based on ν_Φ. The candidate storage function $H_2 : Y \times U \to \mathbb{R}$ is given by

$$
\begin{aligned}
H_2(\sigma, \xi) = \sigma\xi &- \int_0^{\xi} \nu_\Phi(\tau, \sigma, \xi)\ \mathrm{d}\tau \\
&+ \int_0^{\Upsilon(\sigma,\xi)} \nu_\Phi(\tau, \sigma, \xi) - f_{an}(\tau)\ \mathrm{d}\tau, \quad (21)
\end{aligned}
$$

where the function Υ is as defined in Lemma 3.2.

Theorem 3.5: Consider the Duhem hysteresis operator Φ defined in (1)-(3) with locally Lipschitz functions $F, G : \mathbb{R}^2 \to \mathbb{R}$ and with anhysteresis function f_{an}. Suppose that the hypotheses in Lemma 3.2 and the assumption **(A)** in Theorem 3.3 hold. Then for every $u \in C^1(\mathbb{R}_+)$ and for every $y_0 \in \mathbb{R}$, the function $t \mapsto H_2\big((\Phi(u))(t), u(t)\big)$ with H_2 as in (21) is differentiable and satisfies (2). In other words, Φ

is dissipative with respect to the supply rate $\langle \dot{y}, u\rangle$ with the storage function H_2.

The proof of the theorem is similar to the proof of Theorem 3.3.

Remark 3.6: Similar to Remark 3.4, it can be checked that the storage function H_2 in the Theorem 3.5 is non-negative if f_2 is positive and f_{an} is strictly increasing. Since both Theorem 3.3 and Theorem 3.5 have the same assumptions, e.g., **(A)** and the hypotheses in Lemma 3.1, the convex combination of H_1 and H_2 is also a storage function which satisfies (2). Moreover, if additionally, it is assumed that f_1 and f_2 are positive and f_{an} is strictly increasing, the convex combination of H_1 and H_2 is also a non-negative storage function. △

IV. EXAMPLE

Let us consider an example of semilinear Duhem model [12] with $A_+ = -\alpha_1, B_+ = \alpha_2, E_+ = \alpha_3, A_- = \alpha_1, B_- = -\alpha_2, E_- = \alpha_3, C = 1$ and $D = 0$. In this case,

$$f_1(\sigma,\xi) = -\alpha_1\sigma+\alpha_2\xi+\alpha_3, \qquad f_2(\sigma,\xi) = \alpha_1\sigma-\alpha_2\xi+\alpha_3,$$

and it can be computed that $f_{an}(u(t)) = \frac{\alpha_2}{\alpha_1}u(t)$. Moreover, Lemma 3.1 holds if $\alpha_3 < \frac{\alpha_2}{\alpha_1}$.

A routine calculation of the curve ω_Φ gives us

$$\omega_\Phi(\tau, y(t), u(t)) = \tau\frac{\alpha_2}{\alpha_1} - \frac{\alpha_2}{\alpha_1^2} + \frac{\alpha_3}{\alpha_1} + \left[y(t) - u(t)\frac{\alpha_2}{\alpha_1} + \frac{\alpha_2}{\alpha_1^2} - \frac{\alpha_3}{\alpha_1}\right] e^{(-\alpha_1(\tau-u(t)))},$$

for $\tau \in [u(t),\infty)$ and

$$\omega_\Phi(\tau, y(t), u(t)) = \tau\frac{\alpha_2}{\alpha_1} + \frac{\alpha_2}{\alpha_1^2} - \frac{\alpha_3}{\alpha_1} + \left[y(t) - u(t)\frac{\alpha_2}{\alpha_1} - \frac{\alpha_2}{\alpha_1^2} + \frac{\alpha_3}{\alpha_1}\right] e^{(\alpha_1(\tau-u(t)))},$$

for $\tau \in (-\infty, u(t)]$.

Let us consider the case when $y(t) > f_{an}(u(t))$. In this case, the function $\Omega(y(t), u(t))$ is given by

$$\Omega(y(t), u(t)) = u(t) - \frac{1}{\alpha_1}\ln\left[\frac{\frac{\alpha_2}{\alpha_1^2} - \frac{\alpha_3}{\alpha_1}}{y(t) - u(t)\frac{\alpha_2}{\alpha_1} + \frac{\alpha_2}{\alpha_1^2} - \frac{\alpha_3}{\alpha_1}}\right]$$

Note that by the assumption on α_3 and since we consider $y(t) > \frac{\alpha_2}{\alpha_1}u(t)$, we have that $\Omega(y(t), u(t)) > u(t)$, i.e., the intersection point is located to the right of $u(t)$. On the other hand, when $y(t) \le \frac{\alpha_2}{\alpha_1}u(t)$, the function $\Omega(y(t), u(t))$ is given by

$$\Omega(y(t), u(t)) = u(t) + \frac{1}{\alpha_1}\ln\left[\frac{\frac{\alpha_3}{\alpha_1} - \frac{\alpha_2}{\alpha_1^2}}{y(t) - u(t)\frac{\alpha_2}{\alpha_1} - \frac{\alpha_2}{\alpha_1^2} + \frac{\alpha_3}{\alpha_1}}\right].$$

By the assumption on α_3 and since we consider $y(t) \le \frac{\alpha_2}{\alpha_1}u(t)$, we have that $\Omega(y(t), u(t)) \le u(t)$.

It follows from the above computation that $\frac{\mathrm{d}}{\mathrm{d}t}\Omega(y(t), u(t))$ exists and it is continuously differentiable.

As an example on the construction of storage function, let us denote $k_1 = \frac{\alpha_2}{\alpha_1}$, $k_2 = \frac{\alpha_2}{\alpha_1^2} - \frac{\alpha_3}{\alpha_1}$ and $u^*(t) = \Omega(y(t), u(t))$. Using the construction of storage function as in Theorem 3.3 and using ω_Φ and Ω as above, the storage function H_1 (when $y(t) > f_{an}(u(t))$ and $u(t) \le 0$) can be explicitly given by

$$H_1(y(t), u(t)) = y(t)u(t) - \frac{1}{2}k_1(u(t))^2 - k_2\big(u^*(t) - u(t)\big) + \frac{1}{\alpha_1}\left[y(t) - u(t)k_1 + k_2\right]\left[1 - e^{(-\alpha_1(u^*(t)-u(t)))}\right].$$

V. CONCLUSION

In this paper, we have presented a family of storage functions for the Duhem model by using the curves ω_Φ and ν_Φ we defined. Sufficient conditions for dissipativity on the Duhem model are also given which take into account the anhysteresis function.

REFERENCES

[1] D. Angeli, "Systems with counterclockwise input-output dynamics," IEEE Trans. Aut. Contr., Vol. 51, no. 7, pp. 1130-1143, 2006.

[2] M. Brokate & J. Sprekels, *Hysteresis and Phase Transitions*, Springer Verlag, New York, 1996.

[3] Bernard. D. Coleman, Marion. L. Hodgdon, *A constitutive relation for rate-independent hysteresis in ferromagnetically soft materials*, International Journal of Engineering Science, vol. 24, no. 6, pp. 897-919, 1986.

[4] R.B. Gorbet, K.A. Morris, D.W.L. Wang, "Passivity-based stability and control of hysteresis in smart actuators," *IEEE Trans. Contr. Syst. Techn.*, vol. 9, no. 1, pp. 5-16, 2001.

[5] P. Hartman, *Ordinary Differential Equations*. Second Edition, Birkhauser, 1982.

[6] B. Jayawardhana, V. Andrieu, "Sufficient conditions for dissipativity on Duhem hysteresis model," Proc. 48th IEEE Conf. Dec. Contr., Shanghai, 2009.

[7] D. C. Jiles, D. L. Atherton, *Theory of ferromagnetic hysteresis*, Journal of magnetism and magnetic material, vol. 61, no. 1-2, pp. 48-6, 1986.

[8] H.K. Khalil. *Nonlinear Systems*, 3rd edition, Prentice-Hall, Upper Saddle River, NJ, 2002.

[9] H. Logemann & E.P. Ryan, "Systems with hysteresis in the feedback loop: existence, regularity and asymptotic behaviour of solutions," *ESAIM Control, Optimiz. & Calculus of Variations*, vol. 9, pp. 169-196, 2003.

[10] J.W. Macki, P. Nistri, P. Zecca, "Mathematical models for hysteresis," *SIAM Review*, vol. 35, no. 1, pp. 94–123, 1993.

[11] I. Mayergoyz, *Mathematical Models of Hysteresis and Their Applications*, Academic Press, Boston, 2003.

[12] J. Oh, D.S. Bernstein, "Semilinear Duhem model for rate-independent and rate-dependent hysteresis," *IEEE Trans. Automat. Contr.*, vol. 50, pp. 631–645, 2005.

[13] A.K. Padthe, J-H. Oh, D.S. Bernstein, "Counterclockwise Dynamics of a Rate-independent Semilinear Duhem Model," Proc. 44th IEEE Conf. Dec. Contr. and Eur. Contr. Conf., Seville, 2005.

[14] A.J. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, Springer-Verlag, London, 2000.

[15] J. Takacs,"Mathematical proof of the definition of anhysteretic state", *Phicical B*, vol. 372, pp. 57–60, 2006.

[16] H.L. Trentelmann, J.C. Willems, "Every storage functions is a state function," Systems and Control Letters, vol. 32, pp. 249-259, 1997

[17] A. Visintin, *Differential Models of Hysteresis*, Springer-Verlag, New York, 1994.

[18] J.C. Willems, "Dissipative dynamical systems. Part I: general theory. Part II: linear systems with quadratic supply rates," Arch. Rat. Mech. Anal., vol. 45, pp. 321-393, 1972.