Thesis proposal

Distance-based Formulations for the Position Analysis of Kinematic Chains

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September 22, 2010

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1 Introduction

The International Federation for the Promotion of Mechanism and Machine Science (IFToMM) defines a mechanism as a constrained system of bodies designed to convert motions of, and forces on, one or several bodies into specific motions of, and forces on, the remaining ones [55]. Then, depending on the use of these movements or forces, a mechanism is either itself a machine or a part of one. In the theory of mechanisms the study of motion, or kinematics, can be broadly divided into two big branches: *kinematic analysis* and *kinematic synthesis*. Kinematic analysis is the examination and determination of the motion (position, velocity, and acceleration) of a mechanism, kinematic synthesis is the development of a mechanism whose motion holds a desired set of characteristics.

The proposed thesis addresses the kinematic analysis problem, in particular, the research is interested in the position analysis of mechanisms with rigid bodies (links) interconnected by kinematic pairs (joints) —*i.e.*, kinematic chains. This problem, of completely geometrical nature, consists in finding the feasible configurations that a kinematic chain can adopt within the specified ranges for its degrees of freedom [68], a configuration is an assignment of positions and orientations to all links that satisfy the kinematic constraints imposed by all joints. In mechanical engineering the solution of this problem is a fundamental issue, however its relevance is even greater because important problems of other research areas can be reduced to it.

For example, the position analysis problem of kinematic chains arises in robot kinematics when solving the inverse/forward displacement analysis of serial/parallel manipulators [22, 36, 47, 73], in grasping and path planning when planning the coordinated manipulation of an object or the locomotion of a reconfigurable robot [105], or in simultaneous localization and map-building (SLAM) [63]. Other domains where the problem appears include at least the simulation and control of complex deployable structures [41], the theoretical study of rigidity [6, 7], the location of points on a plane in Computer Aided Design (CAD) programs [56], and the conformational analysis of biomolecules [44, 69, 94]. The proposed research is specially devoted to, but not limited to, the position analysis of the kinematic chains that emerge in mechanical engineering and robot kinematics.

Broadly speaking, the methods developed in mechanical engineering and robot kinematics for the position analysis of kinematic chains can be classified as graphical, analytical, and numerical [34]. The graphical approaches are completely geometrical and specially designed to solve particular problems. The analytical and numerical methods deal, in general, with kinematic chains of any topology and translate the original geometric problem into a system of equations, typically nonlinear, that defines the location of each link —*i.e.*, loop-closure equations or constraint equations. The difference between analytical and numerical approaches lies in the procedure used to solve the resulting system of nonlinear equations.

In the analytical approaches the system of nonlinear equations that characterize the valid configurations of the analyzed kinematic chain is reduced to a univariate polynomial using variable elimination. This procedure is implemented using either Gröbner bases [18, 21] or elimination and resultant methods —Sylvester resultant, Dixon resultant, or Macaulay resultant— [54, 90]. In any case, the analytical approaches are complete —*i.e.*, they are able to find all valid configurations of the analyzed kinematic chain. Instead, the numerical methods can be or not be complete. The complete numerical methods are, for example, the approaches that solve the system of nonlinear equations using polynomial continuation [72, 88] or interval-based techniques based on branch-and-prune methods [11, 68, 74]. The incomplete numerical methods are those that find only one solution of the system using local searching algorithms [34].

Classically, the set of nonlinear equations associated with a kinematic chain is directly stated in terms of Cartesian poses —*i.e.*, location and orientation— of the links using, for instance, methods with quaternions, rotation matrices, or Euler angles. This widely accepted approach has two major disadvantages: (a) arbitrary reference frames has to be introduced, and (b) all formulas involve translations and rotations simultaneously. The proposed thesis departs from this usual formulation by expressing the original position problem as a system of distance constraints between points, namely, as a graph with vertices subject to edge lengths constraints for then determining its feasible distinct embeddings in the corresponding problem's dimension.

The embedding problem of a graph of distance constraints, a topic of interest in the study of isostatic frameworks, is equivalent to the problem of determining the point conformation (configuration, relative position or location) by inference from interpoint Euclidean distance information. This problem is studied by the so-called distance geometry, a term coined by the American mathematician Leonard Blumenthal during the first half of the twentieth century [4, 5]. This relatively new branch of mathematics concerns about the classification and study of geometric spaces by means of the metrics —*i.e.*, distances— which can be defined on them [29]. Distance geometry has successfully been used to intrinsically characterize Euclidean spaces, providing proves of theorems of Euclidean geometry without imposing external reference frames [28].

The Graph Distance Matrix (GDM), also called the all-pairs shortest path matrix, is a square matrix with zero diagonal entries consisting of all distances from vertex i to vertex j. For a graph of distance constraints, the GDM corresponds to a symmetric partial matrix —*i.e.*, a symmetric matrix in which only some of its entries are specified. Then, the embedding problem in a specific dimension of this graph reduces, in Euclidean Distance Geometry (EDG), to determine whether its GDM can be completed to a proper Euclidean Distance Matrix (EDM), a semidefinite matrix of Euclidean squared distances between points whose rank gives the embedding dimension [77]. This problem is called the Euclidean Distance Matrix Completion Problem (EDMCP).

The reported approaches in the literature for the solution of the EDMCP can be divided into global methods (complete) and local methods (incomplete), depending whether they can or can not find all solutions. The global methods, that are completely numerical, include the algorithms that use criteria based on the theory of Cayley-Menger determinants [12, 50] to transform the problem into a system of multilinear equalities and inequalities [64, 67], and the algorithms based on the reduction and expansion of the problem's dimension [82, 85]. The local methods are the algorithms based mainly on semidefinite programming, a subfield of convex optimization [2, 46].

The total number of pairwise distances between n points is $\frac{n(n-1)}{2}$, therefore finding all possible solutions of the EDMCP is extremely complex in general, in fact, Saxe in [76] showed that this problem is NP-complete for dimension 1 and NP-hard for higher dimensions. However, as indicated in [106], determining all the set of unknown squared distances is normally unnecessary because, for instance, to solve a spatial constraint between n points, only $3\,n-6$ distances are usually enough. An example of this characteristic is the approach presented in [65] where the set of the serial and parallel manipulators whose EDM can be derived following a constructive geometric process through trilaterations is identified.

Given the location of n points in a space, n-lateration is a method to determine the location of another point whose distance to these n points is known. In three-dimensional space, for $n < 3$ the problem is indeterminate, for $n = 3$ the problem, called trilateration, has two feasible solutions, and for $n > 3$ the problem is overconstrained and has only one feasible solution for generic cases. The aim of the proposed thesis is to extend the ideas developed in [65] for, instead of solving the EDMCP, obtaining the system of equations that characterize the valid configurations of a system of distance constraints between points using only n-laterations and applications of the theory of Cayley-Menger determinants. This system of equations will be solved using analytical and numerical procedures adapted to its particularities. In this way, the proposed research can be seen as a new approach for solving the position analysis of kinematic chains.

The rest of this thesis proposal is organized as follows. Section 2 defines the main objectives and scope of the proposed research. Section 3 details the contributions that are expected to achieve with the thesis. In Section 4 the state of the art related with the thesis topic is reviewed. Section 5 presents the scheduling of the thesis development in three years of work, indicating the tasks already completed. In Section 6 the required resources to carry out the research are presented. Finally, Section 7 lists the achieved and submitted publications resulting from the current state of research.

2 Objectives and scope

2.1 Objectives

The main goal of the proposed thesis is to develop complete methods for the position analysis of kinematic chains that exclusively use n-laterations and applications of the theory of Cayley-Menger determinants for the formulation of the system of equations that constraints the location of each link. Analytical and numerical procedures, adapted to the particularities of the resulting system of equations, will be implemented for solving them. In any case, the proposed research is intended to obtain methods that, based on the foundations of Distance Geometry, satisfy the following key characteristics:

a. Intrinsic. In geometry, an intrinsic or natural equation is an equation which specifies a curve independent of any choice of coordinates or parameterization [96, 97]. An objective of the proposed research work is to obtain methods that are not linked to any artificial coordinate frame.

b. Distance-based. An aim of this thesis proposal is to develop methods that do not involve simultaneously translations and rotations in their formulas —*i.e.*, free of trigonometric functions and identities— and that exclusively dependent of the known distance constraints of the kinematic chain under study.

c. Algebraic. A function is algebraic if it can be constructed using only a finite number of elementary operations —*i.e.*, addition, subtraction, multiplication, division, and integer (or rational) root extraction— together with the inverses of functions capable of being so constructed [98]. Nonalgebraic functions are called transcendental functions. It is a priority of this work to derive methods of a completely algebraic nature.

d. Simple. A plan of the proposed thesis is to get position analysis methods following the *Keep it Simple and Straightforward* principle. It states that simplicity should be a key goal in design, and that unnecessary complexity should be avoided.

2.2 Scope

The thesis scope includes both planar and spatial kinematic chains with one or more loops, focusing mainly on the most important and challenging kinematic chains with no mobility —*i.e.*, structures— that emerge in mechanical engineering and robot kinematics, principally, the following ones:

1. Baranov trusses. A Baranov truss is a closed kinematic chain with no mobility from which it is not possible to obtain another kinematic chain of the same mobility if one or more links are suppressed [20]. The simplest Baranov truss is the well-known triad, a one-loop structure with three links and two feasible configurations, there is one Baranov truss with two loops, the pentad, and twenty eight Baranov trusses of four loops or nine links. The objective of the proposed thesis is to derive a general algorithm for the generic Baranov truss of n links.

2. Serial manipulators. The inverse kinematics problem in robot serial manipulators is to find the values of the joint positions given the position and orientation of the end-effector relative to the base and the values of all of the geometric link parameters [78]. This problem can be seen as to determine the feasible configurations of a closed kinematic chain of one loop. The objective of the proposed thesis is to obtain formulations for the position analysis of the general 6R serial manipulator and all its specializations.

3. Parallel manipulators. The forward kinematics problem in parallel manipulators is to find all poses of platform —relative to the base— that are compatible with the specified leg lengths. A parallel robot with its legs fixed is a kinematic chain with no mobility, therefore, the forward kinematics problem in parallel manipulators is equivalent to the position analysis problem of structures. The aim of the proposed thesis is to obtain formulations for the general 6-6 Stewart-Gough platform and all its specializations.

In addition, the proposed research work is also aimed at extending its results to the analysis of kinematic chains with mobility, namely:

4. Kinematic chains with mobility. Any kinematic chain with mobility is equivalent to a structure when its input joints variables are fixed. Then, the curve traced by a coupler point —*i.e.*, a point of a link that is not connected directly to the ground— can be parametrized over these variables and the solution of the position analysis problem of the corresponding structure. The plan of the proposed thesis is to obtain formulations for the coupler curve of planar kinematic chains as the double butterfly linkage, the Watt-II six bar linkage, or the Stephenson-III six bar linkage.

Due to the relevance of the position analysis problem of kinematic chains to other research areas, this proposal is also aimed at applying the resulting procedures to CAD/CAM and molecular conformation.

3 Expected contributions

The achievement of the objectives presented in Section 2 would allow to:

1. Extend the methods available in the literature for the position analysis of Baranov trusses with novel and simple formulations. The resulting procedures would be the first effort in the kinematics community for solving the position analysis problem of planar kinematic chains in terms of distances. All current procedures combine simultaneously translations and rotations in their formulas using trigonometric functions and identities.

2. Contribute to the development of free-coordinate formulations not only for the kinematics analysis of mechanisms and robots but for the other problems arising in these areas. In particular, contribute to the implementation of new algorithms for the kinematics analysis of complex robot mechanisms without using parameters linked to artificial reference frames (such as the Denavit-Hartenberg parameters) as well as establishing the foundations for the development of a theory of spatial rigid-body kinematics based completely on distances.

3. Determine the equivalences between the inverse kinematics problem of robot serial manipulators and the forward kinematics problem of Stewart-Gough platforms. It is widely known that the duality between parallel and serial robots is complete for the velocity and static aspects, but it remains difficult to establish it for the position analysis [52].
4. Obtain fast, novel and simple methods for the two historical most important problems in robot kinematics: the inverse kinematics of the general 6R serial manipulator and the forward kinematics of the 6-6 Stewart-Gough platform.
5. Facilitate and simplify the teaching of kinematics of mechanisms. Nowadays, since the current complete methods for the position analysis of kinematic chains are based on elimination procedures and/or numerical solutions of systems of transcendental equations, they are cumbersome and/or complicated [36, 47, 89, 90]. Therefore, the teaching of this topic in both undergraduate and graduate courses is, in general, limited to the position analysis of simple kinematic chains as the four-bar linkage, the pentad or the typical industrial serial robots.
6. Enlarge the number of kinematic chains whose position analysis can be solved in practice. Nowadays, all analytical and numerical methods for the position analysis of kinematic chains have a practical limit because the number of equations to be solved is proportional to the number of loops of the kinematic chain under study. In the proposed thesis, since the system of equations for a kinematic chain is not based in loop-closure equations, the position analysis of unreported complex kinematic chains, such a five-loop Baranov truss, could be successfully implemented.

4 State of the art

Next, the most relevant topics related to the position analysis of kinematic chains, in the context of mechanical engineering and robot kinematics, are reviewed, paying attention to the applications in which a contribution is expected to be done. Likewise, the Euclidean distance matrix completion problem and the reported approaches for its solution are discussed.

4.1 The position analysis of kinematic chains

The position analysis problem in kinematic chains consists in finding the feasible configurations that the mechanism can adopt within the specified ranges for its degrees of freedom [68]. A configuration is an assignment of positions and orientations to all links that holds the kinematic constraints imposed by all joints. In kinematic chains with no mobility, a configuration is also called an assembly mode. In mechanical engineering and robot kinematics, the methods developed for the position analysis problem of kinematic chains can be classified as graphical, analytical, and numerical. Next, each of them is succinctly explained.

4.1.1 Graphical methods

Before the computers and the doctoral thesis of Ferdinand Freudenstein, the "Father of Modern Kinematics", who developed the kinematic theory that would be utilized in the computer age, the position analysis of kinematic chains was mainly based on grapho-analytical methods and was essentially limited to planar mechanisms. These methods, developed with a geometrical approach and designed to solve specific problems, principally use Dyadic Decomposition (DD) for solving the position analysis. In a kinematic chain, a dyad is any connection of two links with a revolute or a prismatic pair. The DD approach consists in the identification of a four bar loop in the kinematic chain under study for then calculating the position of the other dyads using arc intersections, the procedure can be combined with other techniques —*e.g.*, interpolation methods— for improving its results and scope.

Nowadays, the graphical methods can be considered as out of date, they have been overcomed by the analytical and numerical methods. A complete review of graphical approaches can be found in [34].

4.1.2 Analytical and numerical methods

The analytical and numerical methods work, in general, with kinematic chains of any topology. In both methods the position analysis problem of kinematic chains, fundamentally a geometric problem, is translated to a system of equations —*i.e.*, loop-closure equations or constraint equations. These equations, typically highly nonlinear, defines the location of each link in the kinematic chain. The difference between the analytical and numerical methods lies in the procedure used to solve the resulting set of nonlinear equations.

The system of equations

The equations that characterize the valid configurations of a kinematic chain are classically stated in terms of Cartesian poses. They are formulated using algebra, linear algebra, and/or trigonometric properties and identities. In these equations the position and orientation of the links is represented either separately or jointly. In the first case, the position is normally represented with vectors and the orientation is represented using for instance:

1. **Quaternions.** Quaternions were discovered by Sir William Rowan Hamilton in 1843 and after improved by the representations of Josiah Willard Gibbs and Hermann Grassmann [78]. A quaternion γ is defined to have the form $\gamma = \gamma_0 + \gamma_1\, i + \gamma_2\, j + \gamma_3\, k$, where the components γ_0, γ_1, γ_2, and γ_3 are scalars, and i, j, and k are operators that satisfy the following rules:

$$\begin{aligned} &ii = jj = kk = -1,\\ &ij = k,\ jk = i,\ ki = j,\\ &ji = -k,\ kj = -i,\ ik = -j. \end{aligned}$$

   The conjugate of quaternion γ is $\hat{\gamma} = \gamma_0 - \gamma_1\, i - \gamma_2\, j - \gamma_3\, k$, a unit quaternion holds that the quaternion product $\gamma\,\hat{\gamma}$ is equal to 1. Unit quaternions can be used to describe orientations. For example, a spatial vector $\mathbf{p} = (p_x, p_y, p_z)^T$ can be defined in quaternion notation as $\mathbf{p} = p_x\, i + p_y\, j + p_z\, k$ then, if ϕ is a unit quaternion, the quaternion product $\phi\,\mathbf{p}\,\hat{\phi}$ performs a rotation of the vector $\mathbf{p}$ about the direction $(\phi_1, \phi_2, \phi_3)^T$.

2. **Rotation matrices.** A rotation matrix $\mathbf{R}$ is a squared matrix with real entries that performs a rotation in Euclidean space, it can be defined for any dimension and holds that $\mathbf{R}^T = \mathbf{R}^{-1}$ and $\det(\mathbf{R}) = 1$. For instance, a spatial rotation of a coordinate frame i about the axis z of a different coordinate frame j through an angle θ can be represented as

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. **Euler angles.** The orientation of a coordinate frame i relative to a coordinate frame j can be denoted as a vector of three angles. When each of these angles represents a rotation about an axis of a moving coordinate frame, they are called Euler angles. Thus, the location of the axis of each successive rotation depends upon the preceding rotations, so the order of the rotations must be appended to the three angles to define the orientation [78].

Altough the separate representation of position and orientation is widely used in the kinematics of planar mechanisms, its application to spatial kinematic chains is not, in general, fairly straightforward. Therefore, important efforts have been carried out in the literature to develop representations that combine together position and orientation in a compact notation in order to implement systematic methods for the formulation of the system of equations. Examples of these representations include:

1. **Homogeneous transformations.** In homogeneous transformations, a spatial vector ${}^{A}\mathbf{p} = (p_x, p_y, p_z)^T$, defined in a local coordinate system A, is expressed with respect to a reference coordinate system U as

$$\left(\begin{array}{c} {}^{U}\mathbf{p} \\ \hline 1 \end{array}\right) = \left(\begin{array}{ccc|c} & {}^{U}\mathbf{R}_A & & {}^{U}\mathbf{p}_{AO} \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c} {}^{A}\mathbf{p} \\ \hline 1 \end{array}\right), \tag{1}$$

where ${}^{U}\mathbf{p}_{AO}$ and ${}^{U}\mathbf{R}_A$ are the position vector and orientation matrix of the local system A with respect to frame U, respectively [3].

2. **Screw transformations.** Chasles' theorem states that any displacement of a body in space can be accomplished by means of a rotation of the body about a unique line in space accompanied by a translation of the body parallel to that line [78]. This line is called a screw axis, in this, the ratio of a linear displacement d to the rotation θ is referred to as the pitch h of the screw axis ($d = h\,\theta$). In any reference frame, a screw axis is represented by means of a unit vector $\hat{w}$ parallel to it and the position vector ρ of any point lying on it.

   A spatial vector ${}^{A}\mathbf{p} = (p_x, p_y, p_z)^T$, defined in a local coordinate system A, is expressed with respect to a reference coordinate system U, using screw transformations, as

$$ {}^{U}\mathbf{p} = {}^{U}\mathbf{R}_A \, ({}^{A}\mathbf{p} - \rho) + d\,\hat{w} + \rho, $$

   where ${}^{U}\mathbf{R}_A$ is the orientation matrix of the local system A with respect to frame U.

3. **Dual quaternions.** A dual number is defined as $z = a + \epsilon\, b$ with $\epsilon^2 = 0$ but $\epsilon \neq 0$, where a is called the real part and b the dual part. A dual quaternion $\breve{q}$ is an usual quaternion with dual number components, it can be reformulated as a dual number as $\breve{q} = q + \epsilon\, q^o$, where q and q^o are both quaternions, and ϵ is the dual factor [102]. Dual quaternions follow the rules of quaternion algebra with the condition $\epsilon^2 = 0$.

   A spatial displacement consisting of a rotation by θ and slide by d around and along a screw axis S can be written as the dual quaternion

$$ \breve{S}(\breve{\theta}) = \sin\frac{\breve{\theta}}{2}\,\mathsf{S} + \cos\frac{\breve{\theta}}{2}, $$

   where $\breve{\theta} = \theta + \epsilon\, d$ and S is the dual vector formed from the Plücker coordinates of the screw axis [60]. A dual vector is a special class of dual number whose real and dual parts are both vectors. A straight line in three-dimensional space is represented in Plücker coordinates by six numbers, the first three correspond to the coordinates of the direction vector for the straight line and the second three are the moments of this vector about the origin [33].

Analytical techniques

In the analytical techniques, the system of nonlinear equations of the kinematic chain under study is transformed to a system of polynomial equations that after is reduced to a univariate polynomial using variable elimination. The polynomial transformation is performed using substitution of variables and the variable elimination is implemented using either Gröbner bases or elimination and resultant methods. In any case, the analytical techniques are able to find all the solutions of the system of nonlinear equations, that is, they are complete methods. Next, the variable elimination procedures are explained.

Gröbner bases technique

In 1966 the concept of Gröbner Bases was introduced by Bruno Buchberger in his doctoral thesis [9]. The basic idea of the method is to eliminate the highest-ordered terms in a given set of polynomial equations by adding multiples of the other equations in the set, this process is known as reduction [54]. In the method, the system of polynomial equations $f_1 = 0, f_2 = 0, \ldots, f_n = 0$ in the variables $x_1, x_2, \ldots, x_n$ is written as a triangular form $g_n(x_n) = 0, g_{n-1}(x_n, x_{n-1}) = 0, \ldots, g_1(x_1, x_2, \ldots, x_n)$, called a Gröbner basis [52]. The process of Gröbner basis, that can be applied to every set of polynomial equations, generalizes three familiar techniques: Gaussian elimination for solving linear systems of equations, the Euclidean algorithm for computing the greatest common divisor of two univariate polynomials, and the Simplex Algorithm for linear programming [81].

The Gröbner bases technique has two main drawbacks. First, the method may generate a large number of complex intermediate polynomials before converging to the Gröbner basis, this imply that the computation time is heavily dependent upon the size of the system and may be prohibitively long [52, 54]. Second, the calculations with real numbers are numerically unstable [52]. However, the technique has been successfully used in the position analysis of different kinematic chains [17, 18, 21]. In addition, most modern computer algebra systems, as *Maple* and *Mathematica*, include implementations of this technique.

Elimination and resultant methods

One of the objectives of elimination theory, a key branch of classical algebra, is to develop algorithms for computing the solutions of a system of polynomial equations in several variables based on resultants. Resultants are polynomial expressions in the coefficients of a system of polynomial equations that are derived after eliminating variables. The importance of resultants lies in that their vanishing is a necessary and sufficient condition for the system to have a solution [95]. Resultants expressions for two polynomials $f(x) = a_m x^m + \ldots + a_0$ and $g(x) = b_n x^n + \ldots + b_0$ of degrees m and n, respectively, are:

1. **The Sylvester resultant.** The Sylvester matrix is an $(m+n) \times (m+n)$ matrix formed by filling the matrix beginning with the upper left corner with the coefficients of $f(x)$, then shifting down one row and one column to the right and filling in the coefficients starting there until they hit the right side. The process is then repeated for the coefficients of $g(x)$ [99]. The determinant of this matrix is the Sylvester resultant.

2. **The Bézout determinant.** Assuming that $m > n$, the Bézout determinant is formed with a system of m equations derived from $f(x)$ and $g(x)$. The first $m - n$ equations are formed from $g(x)$ by multiplication with $x^{m-n-1}, x^{m-n-1}, \ldots, x^0$ in sequence. The remaining n equations are derived from $f(x)$ and $x^{m-n}\, g(x)$, both of which are of degree m. These latter polynomials are set equal to zero, and each of the resulting equations, $f(x) = 0$

and $x^{m-n}\,g(x) = 0$, is solved explicitly for its highest degree term in x, its two highest degree terms in x, and so on. After taking ratios and cross multiplying these equations, n polynomials of degree $m-1$ are obtained. The Bézout determinant is the determinant of the coefficient matrix of these m equations [45].

Given a system of $n+1$ polynomial equations $P_i(X) = f_i(x_1, x_2, \ldots x_n) = 0$, $i = 1, ..., n+1$, the methods that simultaneously and efficiently eliminate several variables from it at a time include:

1. The Dixon resultant. Let $\hat{X} = \{\hat{x}_1, \hat{x}_2, \ldots \hat{x}_n\}$ be a new set of variables and

$$\delta(\hat{\mathbf{X}}) = \begin{vmatrix} Q_{1,1} & \cdots & Q_{1,n+1} \\ Q_{2,1} & \cdots & Q_{2,n+1} \\ \vdots & \vdots & \vdots \\ Q_{n,1} & \cdots & Q_{n,n+1} \\ P_1(\hat{X}) & \cdots & P_{n+1}(\hat{X}) \end{vmatrix},$$

where $Q_{j,1} = (f_i(\hat{x}_1, \ldots, \hat{x}_{j-1}, x_j, \ldots, x_n) - f_i(\hat{x}_1, \ldots, \hat{x}_j, x_{j+1}, \ldots, x_n))\,/\,(x_j - \hat{x}_j)$. $\delta(\hat{X})$ is known as the Dixon polynomial. Now, if F is a set of polynomials formed by the set of all coefficients (which are polynomials in X) of terms in $\delta(\hat{X})$, then the coefficient matrix of F is the Dixon matrix and its determinant is known as the Dixon resultant [43].

2. The Macaulay resultant. For $1 \leq i \leq n+1$, let d_i the total degree of polynomial $P_i(X)$ and $d_m = 1 + \sum_1^{n+1}(d_i - 1)$. Let T denotes the set of all terms of degree d_m in the variables $x_1, x_2, \ldots x_n$, that is, $T = \{{x_1}^{\alpha_1}\,{x_2}^{\alpha_2}\,\ldots\,{x_n}^{\alpha_n} | \alpha_1 + \alpha_2 + \ldots + \alpha_n = d_m\}$. Now, let $T^{(i)}$ the terms of degree $d_m - d_i$ that are not divisible by $\{{x_1}^{d_1}, {x_2}^{d_2}, \ldots, {x_{i-1}}^{d_{i-1}}\}$. Then, the multiplication of terms in $T^{(i)}$ by $P_i(X)$ gives a set of polynomials whose coefficients formed the Macaulay matrix. The determinant of this matrix is the Macaulay resultant [107].

Numerical techniques

The numerical techniques developed in the literature for solving the system of nonlinear equations that define the feasible configurations of a kinematic chain can be divided in incomplete and complete methods. The incomplete methods, that only provide some solutions (typically one) of the system of equations, commonly are gradient-based iterative methods that require an initial guess of a solution [10]. The complete methods —*i.e.*, procedures that find all solutions of a system— are, for instance, the approaches that solve the problem using polynomial continuation or interval-based techniques based on branch-and-prune methods. Next, these two last strategies are explained.

1. Polynomial continuation. The basic premise of this procedure, originally known as "bootstrap method" and developed by B. Roth and F. Freudenstein in 1963 [75], is that small perturbations in the coefficients of a system of nonlinear equations lead to small changes in the solutions [54]. The method begins with an initial system whose solutions are all known, then the system is modified, in a step-by-step process, to the system whose solutions are sought, while tracking all solutions paths along the way [68].

2. Interval-based techniques. The branch-and-prune approach, a technique developed to solve optimization problems, consists in using approximate bounds of the solution set for discarding the parts of the search space that contain no solution [27]. It employs a successive decomposition of the initial problem into smaller disjoint subproblems that are solved iteratively until a criterion is achieved and the optimal solution is found [80]. The convergence of this approach is guaranteed because the bounds get tighter as the intermediate domains get smaller [70].

The interval-based techniques develop iterative algorithms that combine interval methods with the branch-and-prune principle for determining all solutions of a system of equations within a given search space. The interval methods integrate interval arithmetic with analytic estimation techniques to solve a system of equations, two main classes of interval methods have been explored in the robotics literature: those based on the interval version of the Newton method [11] and those based on polytope approximations of the solution set [68].

4.1.3 Applications

Baranov trusses At the beginning of the twentieth century, the Russian mathematician L. Assur proposed a structural classification of planar kinematic chains based on the smallest kinematic chains which, when added to, or subtracted from the original one, results in a mechanism that has the same mobility. Thereafter, these elementary kinematic chains have been called Assur groups. The relevance of these mechanisms become evident when analyzing a complex planar kinematic chain, because it is always possible to decompose it into Assur groups which can be analyzed one by one. A kinematic chain, with no mobility, from which an Assur kinematic group is obtained by removing any one of its links is defined as an Assur Kinematic Chain (AKC) or Baranov truss when no slider joints are considered. Hence any Baranov truss corresponds to multiple Assur groups.

Considering only revolute pairs —a prismatic pair can be modeled as a limit case of a revolute pair—, the simplest Baranov truss is the well-known triad, a one-loop structure with three links and two assembly modes. There is one Baranov truss with two loops, the pentad, a five-link structure whose position analysis leads to up to 6 assembly modes. E. Peysah is credited to be the first researcher in obtaining an analytic form solution for this problem in 1985 [57], the same result was obtained independently at least by G. Pennock and D. Kassner [59], K. Wohlhart [100], and C. Gosselin et al. [22]. Regarding three loops, or seven links, there are three types of Baranov trusses (see Fig. 1), namely, I) a kinematic chain with three binary links and four ternary links with one ternary link connected to the other three, II) a kinematic chain with three binary links and four serially-connected ternary links, and III) a kinematic chain with four binary links, two ternary links, and one quaternary link. The position analysis of these kinematic chains leads to up to 14, 16, and 18 assembly modes, respectively. C. Innocenti, in [40], [38] and [39], obtained these results using resultant elimination techniques. Alternatively, for the type I three-loop Baranov truss, a solution based on homotopy continuation was presented by A. Liu and T. Yang in [48].

Concerning four loops, or nine links, there are twenty eight types of Baranov trusses. The position analysis of these kinematic chains can lead to up 56 assembly modes depending on the type of Baranov truss. Hang et. al. in [26] presented solutions for the position analysis of all these kinematic chains based on polynomial continuation. Solutions based on elimination and resultant methods of some of these Baranov trusses have been reported at least in [8, 24, 25, 91, 92, 93, 101]. To the knowledge of the author of this thesis proposal, there is neither a reported classification of the Baranov trusses with more than four loops nor the position analysis of any of them has been solved.

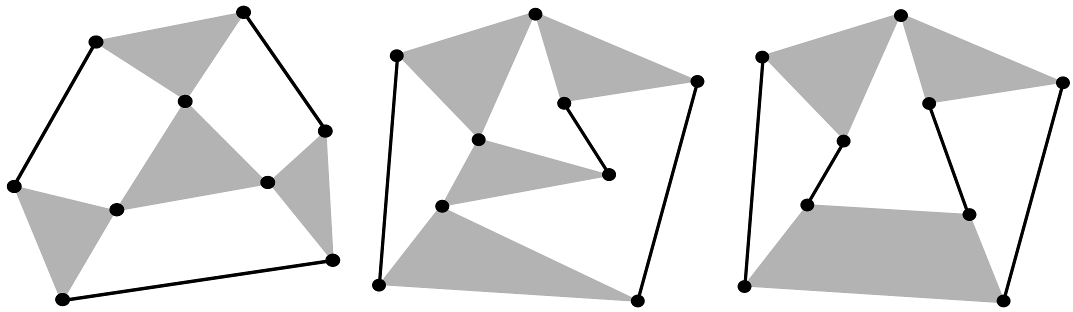

Figure 1: The three three-loop Baranov trusses.

The general 6R serial robot manipulator The inverse kinematics problem for the general 6R serial robot manipulator was described as "the Mount Everest of kinematics problems" by F. Freudenstein in 1973. The earliest systematic approach to this problem seems to have been made by Pieper and Roth [61, 62] who presented closed-formed solutions for manipulators with special geometries and showed that a naive elimination strategy for the general case would yield a univariate polynomial of degree $524,288$. In 1985, Tsai and Morgan [88] applied polynomial continuation to the problem and found only 16 solutions for various 6R manipulators of different geometries. They therefore conjectured that the inverse kinematics problem for the general 6R manipulator had at most 16 solutions.

In 1988, the work of Lee and Liang [47] confirmed the conjecture. They, using an extension of a method proposed by Joseph Duffy in 1980 [19], obtained by dialytic elimination a 16^{th} degree polynomial in the tangent of the half-angle of one of the joint variables. Thereafter, many authors have attempted to improved the inverse kinematics algorithm using elimination and resultant methods. Examples include the work of Raghavan and Roth [73], Manocha and Canny [49], and more recently, Husty et. al. [37], Xin et. al. [103], and Qiao et. al. [71].

The general 6-6 Stewart-Gough platform During the late 1980s and early 1990s, the forward kinematics problem was the focus of research of the Stewart-Gough platform. In 1993, M. Raghavan [72] presented, for the general case, the most successful numerical solution of this problem using polynomial continuation [14]. He concluded that the upper bound of the number of feasible configurations for the general 6-6 Stewart-Gough platform was 40. This fact was for the first time verified in 1996 by M. Husty [36] who using Study's parameters and resultant elimination techniques derived a univariate polynomial of degree 40. More recently, Gan et. al. [21] reported a univariate polynomial for the forward kinematics of the general 6-6 Stewart-Gough platform using rotation matrices and Gröbner bases.

4.2 The Euclidean distance matrix completion problem

An $n \times n$ matrix $\mathbf{D} = (d_{i,j})$ is called an Euclidean Distance Matrix (EDM) if and only if there are n points $p^1, p^2 \ldots p^n$ in some Euclidean space such that $d_{i,j} = \left\|p^i - p^j\right\|^2$, a proper EDM is a positive semidefinite matrix whose rank gives the dimension of the space in which the points are embedded [77]. A symmetric partial matrix is a symmetric matrix in which only some of its entries are specified, the unspecified entries are said to be free. Given a $n \times n$ symmetric partial matrix $\mathbf{A}$, a $n \times n$ matrix $\mathbf{D}$ is an EDM completion of $\mathbf{A}$ if and only if $\mathbf{D}$ is EDM.

The Euclidean Distance Matrix Completion Problem (EDMCP), in its general form, is the problem of determining whether or not a symmetric partial matrix $\mathbf{A}$ has an EDM completion [1]. It is convenient to indicate that in a graph of distance constraints, the Graph Distance Matrix (GDM), a square matrix with zero diagonal entries consisting of all distances from vertex i to vertex j, is a symmetric partial matrix. Following, the reported approaches in the literature for the solution of the EDMCP are presented.

4.2.1 Global methods

In the global methods for the EDMCP, all feasible EDM completions of a given symmetric partial matrix are determined in a specific dimension. For the position analysis of kinematic chains only the global methods developed for planar and spatial points are of interest. An example of solution for $\mathbb{R}^3$ is the branch-and-prune algorithm developed by Porta et. al. [64, 67]. In their algorithm the original distance constraint problem is transform into a system of multilinear equalities and inequalities using criteria based on the theory of Cayley-Menger determinants [12, 50], then all possible values for the unknown distances are established via a bound smoothing process [66].

In [82, 85], Thomas et. al. presented an elegant approach of a global method for the EDMCP in $\mathbb{R}^d$. The idea of their algorithm is to iteratively reducing and expanding the dimension of the problem using projection and backprojection operations based on the cosine theorem. In the approach, given a $n \times n$ symmetric partial matrix, all entries are converted to real compact intervals by putting lower and upper bounds for the unknown entries using triangle and/or tetrangle inequalities. The squared distance intervals —*i.e.*, the entries of the transformed symmetric partial matrix— are repetitively projected onto the hyperplane orthogonal to the axis defined by the entry $1, n$ and their bounds are reduced through backprojections or divided by bisection. The branches of the process that after d iterations of projection yield null matrices are finally backprojected, they correspond to all feasible EDM completions of the problem.

4.2.2 Local methods

The local methods only find a single EDM completion. The reported methods in the literature of this kind are the algorithms based on semidefinite programming, a subfield of convex optimization [2, 46, 104]. In this approach, given a partial symmetric matrix $\mathbf{A}$ with nonnegative elements and zero diagonal, one of its feasible EDM completions, the matrix $\mathbf{D}$, is computed by solving the convex optimization problem

$$\begin{aligned} &\text{minimize} \quad \|\mathbf{H} \circ (\mathbf{A} - \mathbf{D})\|_F^2 \\ &\text{subject to} \quad \mathbf{D} \in \xi, \end{aligned}$$

where $\mathbf{H}$ is an $n \times n$ symmetric matrix with nonnegative elements, $\circ$ denotes the Hadamard product, F indicates the Frobenius norm, and ξ represents the cone of EDMs. In this approach the dimension of the completion can not be specified. Other similar methods for a local solution of the EDMCP, but in which the desired rank of $\mathbf{D}$ can be constrained, include the dissimilarity parameterized formulation [23, 87] and the nonconvex position formulation [108].

5 Work planning

The following is a description of the foreseen tasks in the development of the proposed research:

5.1 Task 0: Background

The author of this thesis proposal studied Electronic Engineering in the Pontificia Universidad Javeriana, Cali, Colombia (a degree of 5 years) from 1999 to 2005 and a Research Master in Industrial Engineering in the Universidad de los Andes, Bogotá, Colombia from 2006 to 2008. Of interest for the development of the proposed research work, the theoretical background resulted from these studies include:

- Classical methods in robot kinematics.
- Robot dynamics with spatial vectors.
- Linear optimization and mixed integer programming.
- Metaheuristics —*e.g.*, Tabu search, genetic algorithms, and simulated annealing.

In February 2009, the author enrolled in the doctorate program Automàtica, Robòtica i Visió of the Universitat Politècnica de Catalunya, Barcelona, Spain, during that Spring semester, as part of the complementary ECTS credits required, he took six academic courses, three in control and automation, Robust Control, Multivariable Control, and Nonlinear Control, two in vision, Computer Vision and Pattern Recognition, and one in robotics, Computational Geometry Applied in Robotics. In September 2009, he joined to the Kinematics and Robot Design research group of the Institut de Robòtica i Informàtica Industrial (IRI) to develop his doctoral research under the supervision of Professor Federico Thomas.

5.2 Task 1: Euclidean distance geometry and mechanisms

The first step of the proposed thesis is accomplishing a deep understanding of Euclidean distance geometry, its foundations and the properties and tools applied to the position analysis of kinematic chains. To this end, the author will mainly concentrate his efforts in combining the reading of all the papers published by the Kinematics and Robot Design research group of IRI based on distance geometry and the theory of Cayley-Menger determinants [64, 65, 66, 67, 69, 82, 83, 84, 85, 86] together with a complete theoretical study about Euclidean distance geometry and its applications [15, 28, 29, 53, 79, 106].

Simultaneously to this study, the author will analyze the approaches reported in the literature for the position analysis of kinematic chains. This step include, for instance, the review of the general methods for planar kinematic chains, the complex-plane formulation [89, 90] and the Dixon determinant method [54], the particular solutions developed for the position analysis of Baranov trusses [25, 38, 39, 40, 91, 92], the polynomial solutions of the forward kinematics of the planar 3-R$\underline{\text{P}}$R parallel robot [13, 22, 30, 31, 32, 35, 42, 57, 59, 100], the inverse kinematics of the general 6R serial manipulator [37, 47, 49, 71, 73, 103] and the forward kinematics of the 6-6 Stewart-Gough platform [36, 51].

5.3 Task 2: The n-lateration problem

Given the location of n points in a space, the n-lateration problem consists in determining the location of another point whose distance to these n points is known. In the planar case, for $n < 2$ the problem is indeterminate, for $n = 2$ the problem, called bilateration, has two feasible solutions, and for $n > 2$ the problem is overconstrained and has at most one feasible solution for generic cases. Similarly, for the three-dimensional space when $n < 3$ the problem is indeterminate, when $n = 3$ the problem is called trilateration and has two feasible solutions, and when $n > 3$ the problem is overconstrained and has only one feasible solution for generic cases. Bilateration and trilateration are common problems solved by generations of students of trigonometry but, in this part of the proposed thesis, it is expected to:

Task 2.1 Obtain formulations for the bilateration and trilateration problems using Cayley-Menger determinants.

It is presumed that using Cayley-Menger determinants very compact expressions for bilateration and trilateration will be obtained and, as consequence, the further analyses and programming will be simplified.

The resulting formulations will be used to:

Task 2.2 Design an efficient algorithm for computing the distance ratios in trees of triangles.

Task 2.3 Design an efficient algorithm for computing the distance ratios in trees of tetrahedra.

5.4 Task 3: Position analysis of planar kinematic chains

This part of the proposed thesis is focused on the development of new formulations for the position analysis of planar kinematic chains based on the formulations and methods obtained in Task 2. The efforts will be concentrated mainly on the following problems:

Task 3.1 Solve the forward kinematics analysis of the general planar Stewart-Gough platform, the 3-R$\underline{\text{P}}$R parallel robot, including all its specializations and analytical cases.

Task 3.2 Study the position analysis of the Baranov trusses with five, seven, and nine links based on the computation of distance ratios in trees of triangles.

Task 3.3 Develop a general algorithm for the efficient position analysis of the generic Baranov truss of n links or the pin-jointed structural framework of $\frac{n-1}{2}$ loops based on the computation of distance ratios in trees of triangles.

5.5 Task 4: Position analysis of spatial kinematic chains

This part of the proposed thesis is focused on the development of new formulations for the position analysis of spatial kinematic chains by extending the results obtained in Task 2 and Task 3. The efforts will be concentrated mainly on the following problems:

Task 4.1 Solve in closed formed the inverse kinematics analysis of the general decoupled 6R serial robot manipulator (three intersecting axes), classifying its specializations and the most relevant industrial cases according to the maximum number of assembly modes.

Task 4.2 Develop an efficient algorithm for the inverse kinematics problem of the general 6R serial robot manipulator based on the computation of distance ratios in trees of tetrahedra.

Task 4.3 Derive a polynomial solution for the forward kinematics problem of the octahedral manipulator, the Stewart-Gough platform implemented in most applications.

Task 4.4 Develop an efficient algorithm for the forward kinematics analysis of the general 6-6 Stewart-Gough platform based on the computation of distance ratios in trees of tetrahedra.

5.6 Task 5: Resolution methods

Simultaneously to the development of the Tasks 3 and 4, the proposed thesis will aim at resolving the particular system of equations resulting from computing distance ratios by developing analytical and numerical procedures adapted to its characteristics. To this end, the following tasks will be carried out:

Task 5.1 Determine the canonical form of the system of equations that result from the application of the formulations derived in Task 2 to the position analysis of a kinematic chain. Since these formulations are based on distances and are free of trigonometric functions or identities, the resulting canonical system of equations is completely algebraic —*i.e.*, its equations are constructed only with the elementary operations $(+, -, *, \sqrt{})$.

Task 5.2 Study the applicability of the best known numerical and analytical methods for the solution of system of equations used in mechanical engineering and robot kinematics as well as other areas such as computer vision, CAD/CAM and molecular conformation to the particularities of the canonical system of equations obtained in Task 5.1.

Task 5.3 Develop of a specialized algorithm for the solution of the canonical system of equations derived in Task 5.1.

5.7 Task 6: Extension of the resulting methods

This part of the proposed thesis is focused on applying the resulting methods to:

Task 6.1 Solve the position analysis of planar kinematic chains with mobility. Mechanisms as the double butterfly linkage [58], the Watt-II six bar linkage, the Stephenson-III six bar linkage, or the Dhingra-Almadi-Kohli mechanisms [16] will be studied.

Task 6.2 Solve problems of other research areas as:

- The problem of conformation of spatial distance-constrained molecular loops —*i.e.*, loops where some inter-atomic distances are held fixed, while others can vary.
- The problem in CAD of the location of points on a plane when a set of relative distances between them is given.
- The embedding problem of general planar bar and joint frameworks that arises in rigidity theory.

5.8 Task 7: Compilation of results

The last task of this thesis proposal is assigned to the elaboration of the dissertation and the preparation of its public defense.

Figure 2 presents the schedule for the tasks described through the Sections 5.1 to 5.8 in a Gantt chart that spans over three years, in this, T stands for a three months period, so that *T1* represents the first three months of a year. The tasks already attained are shown with crosshatch points.

6 Resources

All the proposed research work will be developed at the Institut de Robòtica i Informàtica Industrial-IRI (UPC-CSIC) from Barcelona, working in the framework of the Kinematics and

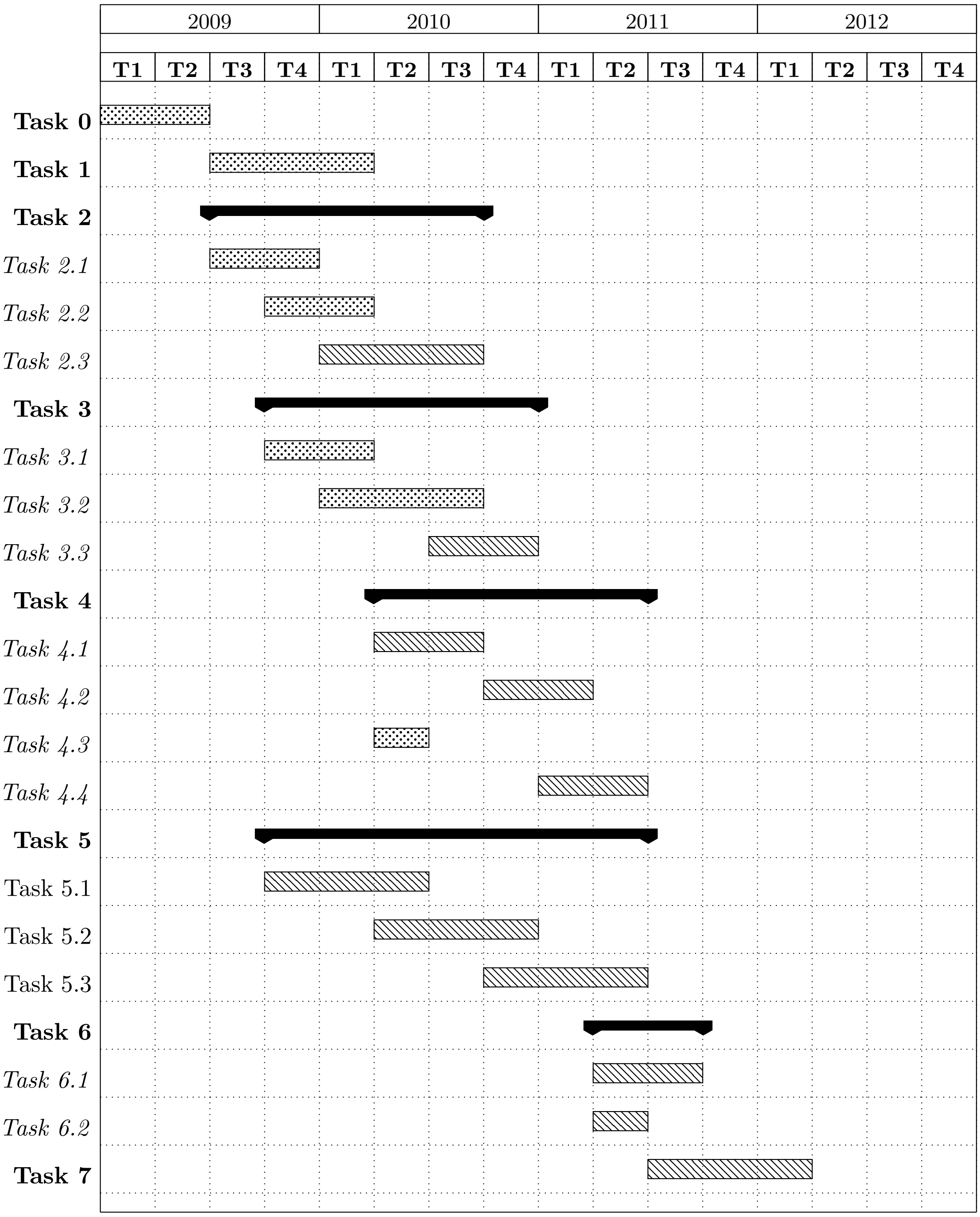

Figure 2: Work planning of the proposed thesis

Robot Design research group, financed by the Autonomous Government of Catalonia through the VALTEC programme and cofinanced with FEDER funds. The author will be financed by the Colombian Ministry of Communications and Colfuturo through the ICT National Plan of Colombia.

7 Publications

The following is the list of the achieved and submitted international publications resulting from the current state of research:

Journals

1. N. Rojas and F. Thomas, *The Forward Kinematics of 3-R$\underline{P}$R Planar Robots: A Review and a Distance-Based Formulation*, IEEE Transactions on Robotics (conditionally accepted).

2. N. Rojas and F. Thomas, *Distance-Based Position Analysis of the Three Seven-Link Assur Kinematic Chains*, submitted to Mechanism and Machine Theory.

Conferences

1. N. Rojas and F. Thomas, *A Robust Forward Kinematics Analysis of 3-R$\underline{P}$R Planar Platforms*, in Advances in Robot Kinematics, J. Lenarcic and M. Stanisic (editors), Springer Verlag, 2010.

2. N. Rojas, J, Borràs, and F. Thomas, *A Distance-Based Formulation of the Octahedral Manipulator Kinematics*, IFToMM Symposium on Mechanism Design for Robotics 2010, Mexico City, Mexico, September, 28-29, 2010.

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