Distance Between Beros of Certain Differential Equations Having Delayed Arguments (*) (**).

STANLEY B. ELIASON (Norman, Oklahoma, U.S.A.)

Summary. – *For the continuous real valued functions, p, m and g, with $p(x)\geqslant 0$, and $m(x)\geqslant 0$, and $\mu>0$, $\nu>0$ being reals, the differential equations $y''(x)+p(x)|y(x)|^{\mu}\operatorname{sgn} y(x)=m(x)|y(g(x))|^{\nu}\operatorname{sgn} y(g(x))$ is considered. Lyapunov type integral inequalities are established which yield implicit lower bounds on the distance between consecutive zeros of a nontrivial solution of the above equation, and several others. The same is done for a problem involving the distance from a zero of a solution y to the next greater zero of its derivative y'. Special conditions are placed on the corresponding initial functions. They allow for application of results to oscillatory solutions of the given equation, and also to non-trivial solutions having a zero initial function. When $p(x)\equiv 0$. the results take on a special form; and when in addition $m(x)>0$, $g(x)<x$ and $\nu=1$, one result establishes a necessary condition for the existence of an oscillatory solution having infinitely many small semicycles. This condition is the weak form of a strict integral inequality, due to G. Ladas et al., which establishes a sufficient condition for the oscillation of all bounded solutions.*

1. – Introduction.

In this paper we wish to consider the differential equation

$$y''(x)+p(x)|y(x)|^{\mu}\operatorname{sgn} y(x)=\int_a^b |y(u)|^{\nu}\operatorname{sgn} y(u)\, d_u\alpha(x,u)\,, \tag{1.1}$$

on an interval I of reals having end points $a<b$, ($a=-\infty$ or $b=+\infty$ are possible). In the (proper or improper) Riemann-Stieltjes integral, $d_u\alpha(x,u)$ signifies that for each x, $\alpha(x,\cdot)$ is the integrator function. For the remainder of the paper we drop the subscript u on d. Throughout we assume hypotheses

(H_1): $g, h, p\colon I\to R$ are all continuous with $p(x)\geqslant 0$ and $h(x)\leqslant g(x)$ on I, and $\mu>0$ and $\nu>0$ are reals;

and

(H_2): $\alpha\colon I\times R\to R$ is such that for each x in I, $\alpha(x,s)$ is monotone increasing in s with $\alpha(x,s)\equiv\alpha(x,h(x)-)$ for $s<h(x)$ and $\alpha(x,s)\equiv\alpha(x,g(x)+)$ for $s>g(x)$; and for each continuous $k\colon I\to R$, $\alpha(x,k(x))$, $\alpha(x,k(x)+)$ and $\alpha(x,k(x)-)$ are bounded and Lebesgue integrable on compact subintervals of I.

(*) Entrata in Redazione il 28 marzo 1974.
(**) Based on research supported in part by the U.S. Army Research Office-Durham through Grant Number DA-ARO-31-124-72-G154 with The University of Oklahoma Research Institute.

In (H_2) and elsewhere we denote right and left hand limits by $+$ and $-$.

Problems studied here involve placing implicit lower bounds on the distance between zeros of certain solutions y, (and/or derivatives y'), of (1.1), and other equations, by means of establishing Lyapunov type inequalities. This also leads to establishing a necessary condition for the existence of certain oscillatory solutions of a special case of (1.1), i.e., where $p(x) \equiv 0$ and the integral on the right is replaced by $m(x)|y(g(x))|^{\nu}$. $\operatorname{sgn} y(g(x))$, with $m(x) > 0$, g strictly increasing and $g(x) < x$ on $I = [a, \infty)$.

A special case of (1.1) is

$$(1.2) \qquad y''(x) + p(x)|y(x)|^{\mu} \operatorname{sgn} y(x) = m(x)|y(g(x))|^{\nu} \operatorname{sgn} y(g(x)) \, ,$$

where we assume

(H_3): $m: I \to R$ is nonnegative and continuous, and p, g, μ and ν satisfy (H_1).

The basic theorems which are established for equation (1.1) can also be modified to handle the equation

$$(1.3) \qquad y''(x) + \sum_{i=1}^{K} \int_a^b |y(u)|^{\nu_i} \operatorname{sgn} y(u)\, d\alpha_i(x, u) = 0 \, ,$$

under

(H_4): for each $1 \leqslant i \leqslant K$, $\nu_i > 0$ and either α_i or $-\alpha_i$ satisfies (H_2) for appropriate functions h_i and g_i satisfying (H_1).

Thus, as special cases of (1.3) under (H_4), we have equation (1.1) under (H_1) and (H_2), but where « $p(x) \geqslant 0$ » is deleted and the phrase « is monotone increasing » may be changed to « is of bounded variation »; and equation (1.2) where g, p and m are any continuous functions, and $\nu > 0$ hold.

The work of this paper continues that of the author [2]. There in section 3, equation (1.1) above is studied under (H_1), but with $-\alpha$ satisfying (H_2) and $\mu = 1$. Also there, equation (1.2) above is studied under (H_3), but with m being nonpositive.

Even though several equations studied in [2] are special cases of (1.3) under (H_4), we cannot claim that the theorems of [2] are special cases of the results here. This is due to the fact that, in the main theorems of each paper, certain restrictions are placed on the initial and terminal portions of solutions. For equation (1.1) above, the restrictions vary accordingly as α or $-\alpha$ is monotone increasing in its second variable. It is for a common area where these conditions overlap that equation (1.3) under (H_4) may be studied.

Here, as in [2], the Lyapunov type inequalities are motivated by work of ÈL'BERT [1]

Other recent work concerning (1.2) with $p(x) \equiv 0$, $g(x) \to +\infty$ and $g(x) \leqslant x$ can be found in [3], [4], [5] and [6]. They also consider certain generalizations.

Examples 1.1 through 1.6 of GUSTAFSON [3] show several such equations and point out how certain equations may have oscillatory and nonoscillatory solutions both existing in quite interesting combinations even when $p(x) \equiv 0$ and $\nu = 1$.

Some results of MYŠKIS, appearing in [6] and [7] deal with lower bounds on the distance between zeros of solutions of (1.1) where $p(x) \equiv 0$ and $\nu = 1$. The Riemann-Stieltjes integral here in (1.1) is expressed in somewhat different form.

Finally, for later reference, if y is a solution of (1.1) on an interval J, having *initial segment* J^- and *terminal segment* J^+, as in (3.3) of [2], we have

(1.4)
$$\int_a^b |y(u)|^\nu \operatorname{sgn} y(u)\, d\alpha(x, u) = \int_{\underline{h(x)}}^{\overline{g(x)}} |y(u)|^\nu \operatorname{sgn} y(u)\, d\alpha(x, u)\,,$$

where the upper and lower bars on $g(x)$ and $h(x)$ are given by

$$\overline{g(x)} = \begin{cases} g(x)\,, & \text{if } g(x) = \sup J \cup J^+ \\ g(x)+\,, & \text{if } g(x) < \sup J \cup J^+\,, \end{cases}$$

and

$$\underline{h(x)} = \begin{cases} h(x)\,, & \text{if } h(x) = \inf J \cup J^- \\ h(x)-\,, & \text{if } h(x) < \inf J \cup J^- \end{cases}$$

Certainly (1.4) may be applied to equation (1.3) under (H_4) as well. That is, each g_i and h_i satisfy (1.4) provided J^+ and J^- are defined as unions of J_i^+ and J_i^- respectively.

2. – Lyapunov type integral inequalities I.

In this section we wish to establish Lyapunov type integral inequalities which yield implicit lower bounds on the lengths of *left quarter-cycles*, $[c, d]$, of solutions y of (1.1), i.e., $y(c) = 0 = y'(d)$, $y(x) \neq 0$ on (c, d). They may be used as well to place lower bounds on the first proper value λ, if such exists and is nonnegative, of a Sturm-Liouville type problem

(2.1)
$$\begin{aligned} &y''(x) + \lambda y(x) = m(x)\, y\big(g(x)\big) \\ &y(x) \equiv 0 \qquad \text{on } [c, d]^-,\ y'(d) = 0 \end{aligned}$$

where $[c, d] \subseteq I$, m is continuous and $g(x) \leqslant x$ on I. Such problems are considered in Chapter III of [8].

Notationally, as in [2] when q and r are reals, we let $q \vee r \equiv \max\{q, r\}$ and $q \wedge r \equiv \min\{q, r\}$. Also, a quarter-cycle, (later, a semicycle), $[c, d]$ is called *positive* or *negative* depending on $y(x) > 0$ or $y(x) < 0$ on (c, d).

In order to apply our first theorem to (1.1), fairly specific conditions are assumed on the initial and terminal segments of the left quarter-cycle. When the quarter-cycle is positive these are given by (2.3) below. These conditions correspond to (2.8) and (2.9) of [2], which of course, deal with a different equation. Condition (2.3)′ below denotes (2.3) with the inequalities changed in direction; and (2.4)′ denotes (2.4) with «max» changed to «min». Finally $y'(\delta)$ denotes $y'(\delta+)$ when used in (2.4) or (2.4)′.

THEOREM 2.1. – *Let $[c, d] \subseteq I$ be a positive, (negative), left quartercycle of a solution y of* (1.1) *under* (H_1) *and* (H_2). *Assume γ and δ satisfy*

$$\delta \in I, \quad \delta \leqslant c \quad \textit{and} \quad \gamma \in [\delta, d]^-, \quad (\gamma = \delta \textit{ if } [\delta, d]^- \textit{ is null}), \tag{2.2}$$

and that y satisfies

(2.3) *y is a solution of* (1.1) *on $[\delta, d]$, $(x-\gamma)y(x) \leqslant 0$ on $[\delta, d]^- \cup [\delta, c]$, $y(x) \geqslant 0$ on $[\delta, d]^+$, the restriction of y to $[\gamma, \delta]$ is absolutely continuous and $y'(x) \leqslant 0$ a.e. on $[\gamma, \delta)$,*

((2.3)′). *Let $\psi \in [\delta, d]$ satisfy*

$$y'(\psi) = \max\{y'(x) : \delta \leqslant x \leqslant d\} \tag{2.4}$$

((2.4)′).

It follows that

$$1 < |y'(\psi)|^{\mu-1} \int\limits_{c\vee\psi}^{d} (t-c)^{\mu} p(t)\, dt + |y'(\psi)|^{\nu-1} \int\limits_{\Psi}^{D} \Bigg(\int\limits_{\underline{H(t)}}^{\overline{G(t)}} [c - (\delta \vee u)]^{\nu} d\alpha(t, u) \Bigg) dt \tag{2.5}$$
$$\leqslant (d-c)^{\mu} |y'(\psi)|^{\mu-1} \int\limits_{c\vee\psi}^{d} p(t)\, dt + (c-\delta)^{\nu} |y'(\psi)|^{\nu-1} \int\limits_{\Psi}^{D} \big[\alpha\big(t, \overline{G(t)}\big) - \alpha\big(t, \underline{H(t)}\big)\big]\, dt\,,$$

where

$$G(t) \equiv (\gamma \vee g(t)) \wedge c\,, \qquad H(t) \equiv (\gamma \vee h(t)) \wedge c \tag{2 6}$$

with upper and lower bars following the convention of (1.4) *for $J = [\delta, d]$. Also, in general Ψ and D satisfy*

$$\Psi = \psi \in [\delta, d] \quad \textit{and} \quad D = d\,, \tag{2.7}$$

but when g and h are monotone increasing they may be chosen to satisfy

$$\Psi = d \wedge \sup\{\psi\} \cup \{t : g(t) < \gamma\} \quad \text{and} \quad D = \psi \vee \inf\{d\} \cup \{t : h(t) > c\}\,. \tag{2.7$_{\text{mon}}$}$$

PROOF. – From (1.1) and (1.4), for any s, x in $[\delta, d]$ it follows that

$$y'(s) = y'(x) + \int\limits_{s}^{x} p(t) |y(t)|^{\mu} \operatorname{sgn} y(t)\, dt - \int\limits_{s}^{x} \Bigg(\int\limits_{\underline{h(t)}}^{\overline{g(t)}} |y(u)|^{\nu} \operatorname{sgn} y(u)\, d\alpha(t, u) \Bigg) dt\,. \tag{2.8}$$

Assuming now that the left quarter cycle is positive, by (2.3), (2.4), (2.6), (2.7), $(2.7)_{\mathrm{mon}}$, we have from equation (2 8) with $s=\psi$, $x=d$

$$\begin{aligned}(2.9)\qquad y'(\psi) &\leqslant \int_{\psi}^{d} p(t)\,y^{\mu}(c\vee t)\,dt + \int_{\Psi}^{D}\Big(\int_{\underline{H(t)}}^{\overline{G(t)}} [-y(u)]^{\nu}\,d\alpha(t,u)\Big)\,dt \\ &\leqslant \int_{c\vee\psi}^{d} p(t)\Big(\int_{c}^{t} y'(\tau)\,d\tau\Big)^{\mu} dt + \int_{\Psi}^{D}\Big(\int_{\underline{H(t)}}^{\overline{G(t)}}\Big(\int_{\delta\vee u}^{c} y'(\tau)\,d\tau\Big)^{\nu} d\alpha(t,u)\Big)\,dt \\ &< [y'(\psi)]^{\mu}\int_{c\vee\psi}^{d}(t-c)^{\mu}p(t)\,dt + [y'(\psi)]^{\nu}\int_{\Psi}^{D}\Big(\int_{\underline{H(t)}}^{\overline{G(t)}}[c-(d\vee u)]^{\nu}\,d\alpha(t,u)\Big)\,dt \\ &\leqslant (d-c)^{\mu}[y'(\psi)]^{\mu}\int_{c\vee\psi}^{d} p(t)\,dt + (c-\delta)^{\nu}[y'(\psi)]^{\nu}\int_{\Psi}^{D}\big[\alpha\big(t,\overline{G(t)}\big)-\alpha\big(t,\underline{H(t)}\big)\big]\,dt\,.\end{aligned}$$

It is since y is nonnegative outside of $[\gamma, c]$ on $[c, d]\cup[c, d]^{+}\cup[c, d]^{-}$ that G and H may be chosen to satisfy (2.6); and also that allows Ψ and D to be as in (2.7) or $(2.7)_{\mathrm{mon}}$. Similar computations are found in (2.11), (2.12), (3.10), (3.11), (3.12), and $(3.12)_{\mathrm{mon}}$ of [2].

The theorem now follows.

REMARK 1. – For equation (1.2) under (H_3), take $h(x)\equiv g(x)$. Then the computations (3.15), (3.16), and preceding of [2] allow the existence of an α satisfying (H_2), and also

$$(2.10)\qquad \int_{\underline{h(x)}}^{\overline{g(x)}} |y(u)|^{\nu}\operatorname{sgn} y(u)\,d\alpha(x,u) \equiv m(x)\big|y\big(g(x)\big)\big|^{\nu}\operatorname{sgn} y\big(g(x)\big)\,,$$

and

$$(2.11)\qquad 0\leqslant\alpha\big(x,\overline{G(x)}\big)-\alpha\big(x,\underline{H(x)}\big)\leqslant\alpha\big(x,\overline{g(x)}\big)-\alpha\big(x,\underline{h(x)}\big)\equiv m(x)$$

on $[\delta, d]$.

As a result, for equation (1.2) under (H_3), the two right hand integrands in the two lines of (2.5) may be replaced by the corresponding integrands below in (2.12). The limits of integration Ψ and D still, in general satisfy (2.7) or $(2.7)_{\mathrm{mon}}$. In the next corollary we see that when $p(x)\equiv 0$ and g is increasing on I, then ψ can be determined in terms of g, γ and δ.

REMARK 2. – The purpose of stating (2.2) and (2.3) in the form given is to allow initial conditions such as in (2.1), where $\gamma=\delta=c$, and also initial conditions which are likely fulfilled when considering an oscillatory solution of (1.1) where the zeros of the solution are isolated.

REMARK 3. – As in Remark 1 of Section 2 of [2], the change of variable $T(x) \equiv \equiv d + c - x$ allows the above theorem to be applied to right quartercycles of solutions. All conditions of the theorem take on a « mirror » form. The reader can see from [2] what needs to be done here. This applies as well to be corollary below, but we note that in this case it is necessary that g be at times an advanced argument. At the end of Section 3 we have further comments relating to this.

REMARK 4. - The larger the values of γ and δ, satisfying (2.2) and (2.3), (or (2.3)$'$), the sharper the inequalities (2.5) become. Thus if $\gamma < \delta < c$ and if $y'(\delta)$ exists as a two sided derivative it is natural to assume $y'(\delta) = 0$.

COROLLARY 2.2. – *Let $[c, d] \subseteq I$ be a positive, (negative), left quartercycle of a solution y of (1.2) under (H_3), where in addition, g is increasing on I. Assume γ and δ satisfy (2.2), y satisfies (2.3), ((2.3)$'$), and $p(x) \equiv 0$ on $[\delta, d]$.*

Then it follows that

$$(2.12)\qquad 1 < |y'(\delta \vee g^{-1}(\gamma))|^{\nu-1} \int_{\delta \vee g^{-1}(\gamma)}^{d \wedge g^{-1}(c)} [c - (\delta \vee g(t))]^{\nu} m(t)\, dt$$
$$\leqslant (c-\delta)^{\nu} |y'(\delta \vee g^{-1}(\gamma))|^{\nu-1} \int_{\delta \vee g^{-1}(\gamma)}^{d \wedge g^{-1}(c)} m(t)\, dt\,.$$

PROOF. – The computations (2.10) and (2.11) yield the integrands displayed in (2.12). Consequently we need only verify the choice of limits of integration and the value of ψ.

When the left quarter-cycle is positive, $p(x) \equiv 0$ on $[\delta, d]$, g is increasing on I, and (2.2) and (2.3) hold, then y is convex on $[\delta, \delta \vee g^{-1}(\gamma)] \cap [\delta, d]$ and on $[\delta, d] \cap \cap g^{-1}([c, d] \cup [\delta, d]^{+})$. It is concave on $[g^{-1}(\gamma), g^{-1}(c)] \cap [\delta, d]$. Furthermore it is necessary that $g(\gamma) < c$, $\gamma < g(d)$, $g(c) < c$ and $g(\delta) < c$ hold.

The first two of the above inequalities are easily seen to be true. For the third, we note that a necessary condition that $[c, d]$ be a positive left quarter-cycle is that $y''(x_0) < 0$ hold for some x_0 in (c, d). Thus for $x_1 = g(x_0)$, we must have $y(x_1) < 0$ and $\gamma < x_1 < c$, so that $g(c) < g(x_0) = x_1 < c$ holds. Finally we have $g(\delta) < g(c) < c$, yielding the last inequality.

By studying the results of the two previous paragraphes we see that ψ may be selected as $\delta \vee g^{-1}(\gamma)$, and also that $\gamma \vee g^{-1}(\gamma) < d \wedge g^{-1}(c)$.

Of course, if $\delta \vee g^{-1}(\gamma) = \delta$, the derivatives in (2.21) denotes a right hand one. This establishes the Corollary.

REMARK 5. – The inequalities (2.5) and (2.12) do provide an implicit lower bound on the length of a left quarter-cycle, but at times they can be interpreted in a larger sense as showing an interdependence among two or more various types of cycles. For instance, when $\gamma < \delta < c$ and y is a solution on $[\gamma, d]$, since we may assume $y'(\delta) = 0$, the inequalities show a relationship between the right and left quarter-cycles $[\delta, c]$ and $[c, d]$. If $y(\gamma) = 0$ we then have an additional left quarter-cycle $[\gamma, \delta]$ involved.

REMARK 6. – We are able, in a certain sense, to establish the sharpness of the inequality provided by the extremes of (2.12). For this example, however, we are basically required to have g be a delay, i.e. $g(x) < x$ on I. We establish this sharpness when $[c, d]$ is a minimal left quarter-cycle, i.e., it properly contains none other such. By considering the proof of Corollary 2.2 we see that it is natural to assume $g(d) \leqslant c$ in this case.

EXAMPLE 2.3. – Let there be given an increasing function g satisfying $g(x) < x$ on $I = [a, \infty)$ and (H_3). Suppose $g(d) < d$ are in I. Define $\delta = g(d)$ and $c = (\delta + d)/2$.

Now for any $\varepsilon > 0$ we can construct an equation (1.2) with $p(x) \equiv 0$ on I and (H_3) is satisfied, and for which $[c, d]$ is a minimal positive left quarter-cycle of some solution y satisfying (2.2) and (2.3) for $y = g(c) > g(\delta)$, and for which the last functional on the right in (2.12) is dominated by $1 + \varepsilon$.

To construct the equation (1.2) we first construct the solution. For $\varepsilon > 0$ as given, and for each $\eta > 0$, sufficiently small in that $c < d - \eta$ and $g(d + \eta) < c$ hold, let y be defined in a piecewise fashion as follows: on $[g(d + \eta), d - \eta]$ let $y(x) \equiv \equiv x - c$; on $(d - \eta, d + \eta]$ let $y(x) \equiv d - c - \eta + 2\eta\pi^{-1}\sin[\pi(2\eta)^{-1}(x - (d - \eta))]$; on $(d + \eta, \infty)$ let $y(x) \equiv (d - c - \eta) + (d + \eta - x)$; on $[g(\delta), g(d - \eta)]$ let $y(x) \equiv A(x - \gamma)$ where $A = [g(d + \eta) - c][g(d - \eta) - \gamma]^{-1}$ and, as we recall $\gamma = g(c)$; and finally on $\big(g(d - \eta), g(d + \eta)\big)$ let y be any convex function such that $y'(\delta) = 0$ and such that the resulting function y defined on $[g(\delta), \infty)$ has a continuous second derivative. Since $y\big(g(d - \eta)\big) = y\big(g(d + \eta)\big)$, $y'\big(g(d - \eta)\big) < 0$ and $y'\big(g(d + \eta)\big) = 1 > 0$ we are allowed to make this construction of y on $\big(g(d - \eta), g(d + \eta)\big)$.

With y so defined, on $[\delta, g(d + \eta)]$ and on $[d - \eta, d + \eta]$ let

$$m(x) \equiv y''(x)/|y(g(x))|^{\nu}\operatorname{sgn} y(g(x)) .$$

Elsewhere on $[\delta, \infty)$, since $y''(x) \equiv 0$, we define $m(x) \equiv 0$.

We now note that m satisfies (H_3) and y is a solution of (1.2) on $[\delta, \infty)$, with $p(x) \equiv 0$, and having $[c, d]$ as a positive left quarter-cycle.

Also in the last functional of (2.12), since $c = g^{-1}(\gamma) > \delta$, we have

$$\begin{aligned}(2.13)\qquad (c - \delta)^{\nu}\big|y'\big(g^{-1}(\gamma)\big)\big|^{\nu-1}\int\limits_{g^{-1}(\gamma)}^{d} m(t)\,dt \\ = [c - g(d)]^{\nu}\,1^{\nu-1}\int\limits_{d-\eta}^{d}[-y(g(t))]^{-\nu}\pi(2\eta)^{-1}\sin\big[\pi(2n)^{-1}\big(t - (d - \eta)\big)\big]\,dt \\ \leqslant [c - g(d)]^{\nu}\big[-y\big(g(d + \eta)\big)\big]^{-\nu}\int\limits_{d-\eta}^{d}\pi(2\eta)^{-1}\sin\big[\pi(2\eta)^{-1}\big(t - (d - \eta)\big)\big]\,dt \\ = [c - g(d)]^{\nu}[c - g(d + \eta)]^{-\nu}.\end{aligned}$$

By the continuity of g, for the $\varepsilon > 0$ given, it is now clear that the last term in (2.13) can be dominated by $1 + \varepsilon$ by taking η sufficiently small.

3. – Lyapunov type inequalities II.

Inequalities similar to (2.5) and (2.12) are provided below for *semicycles*, i.e., $y(c) = y(d) = 0$, $y(x) \neq 0$ on (c, d). The arguments used are again modifications of those used in [1] and [2]. The choice of ψ satisfying (2.4) is somewhat more restrictive in the following theorem. Also there is a restriction that $[\delta, d]^+ \subseteq \{d\}$.

THEOREM 3.1. – *Let* $[c, d]$ *be a positive, (negative), semicycle of a solution* y *of* (1.1) *under* (H_1) *and* (H_2). *Assume* γ *and* δ *satisfy* (2.2), y *satisfies* (2.3), $((2.3)')$, *and* $[\delta, d]^+ \subseteq \{d\}$. *Let* $\psi \in [c, d]$ *satisfy* (2.4), $((2.4)')$.

Then with G and H given by (2.6), and Ψ and D given by (2.7) or $(2.7)_{\text{mon}}$, it follows that

$$(3.1) \qquad 1 < (d-\psi)^{-1}\Big\{|y'(\psi)|^{\mu-1}\int_{\psi}^{d}(d-t)(t-c)^{\mu}p(t)\,dt + |y'(\psi)|^{\nu-1}\int_{\Psi}^{D}(d-t)\Big(\int_{\underline{H(t)}}^{\overline{G(t)}}[c-(\delta\vee u)]^{\nu}d\alpha(t,u)\Big)dt\Big\}$$
$$\leqslant |y'(\psi)|^{\mu-1}(d-\psi)^{-1}\mu^{\mu}(\mu+1)^{-(\mu+1)}(d-c)^{\mu+1}\int_{\psi}^{d}p(t)\,dt + |y'(\psi)|^{\nu-1}(c-\delta)^{\nu}\int_{\Psi}^{D}\big[\alpha\big(t,\overline{G(t)}\big)-\alpha\big(t,\underline{H(t)}\big)\big]dt\,.$$

PROOF. – For $s = \psi$ we integrate both sides of (2.8) for x between ψ and d. This yields

$$(3.2) \qquad 0 \geqslant -y(\psi) = \int_{\psi}^{d}y'(x)\,dx$$
$$= (d-\psi)y'(\psi) - \int_{\psi}^{d}\int_{\psi}^{x}p(t)|y(t)|^{\mu}\operatorname{sgn}y(t)\,dt\,dx + \int_{\psi}^{d}\int_{\psi}^{x}\Big(\int_{\underline{H(t)}}^{\overline{G(t)}}|y(u)|^{\nu}\operatorname{sgn}y(u)\,d\alpha(t,u)\Big)dt\,dx.$$

By (3.2) and one integration by parts, as in the computations (2.9), we have

$$(3.3) \qquad (d-\psi)y'(\psi) \leqslant \int_{\psi}^{d}(d-x)p(x)y^{\mu}(x)\,dx + \int_{\psi}^{d}(d-x)\Big(\int_{\underline{h(t)}}^{\overline{g(t)}} - |y(u)|^{\nu}\operatorname{sgn}y(u)\,d\alpha(x,u)\Big)dx$$
$$< [y'(\psi)]^{\mu}\int_{\psi}^{d}(d-x)(x-c)^{\mu}p(x)\,dx + [y'(\psi)]^{\nu}\int_{\Psi}^{D}(d-x)\Big(\int_{\underline{H(x)}}^{\overline{G(x)}}[c-(\delta\vee u)]^{\nu}d\alpha(x,u)\Big)dx\,.$$

Finally, $\mu^{\mu}(\mu+1)^{-(\mu+1)}(d-c)^{\mu+1}$ maximizes $(d-t)(t-c)^{\mu}$ for $c\leqslant t\leqslant d$. This provides the last inequality in (3.1), and establishes the theorem.

The computations (2.10) and (2.11) may again be used to provide a corresponding result for equation (1.2), from (3.1).

For equation (1.2) under $(\mathbf{H}_3)$ with $p(x)\equiv 0$ on $[\delta, d]$ and g is increasing, we have a situation here similar to Corollary 2.2. Recall that from the proof of that corollary $g(c)<c$ holds, so that our assumption of $\psi\in[c,d]$ is assured if $\gamma\leqslant g(c)$. In order to provide an integral inequality similar to (2.12), we incorporate this assumption in the following corollary. The case of $\psi\in[c,d]$, but $\gamma<g(c)$ is still covered in the main theorem.

COROLLARY 3.2. – *Let $[c,d]\subseteq I$ be a positive, (negative), semicycle of a solution y of (1.2) under $(\mathbf{H}_3)$, where in addition, g is increasing on I. Assume γ and δ satisfy (2.2) with $\gamma\geqslant g(c)$, y satisfies (2.3), $\big((2.3)'\big)$ with $[\delta, d]^{+}\subseteq\{d\}$, and $p(x)\equiv 0$ on $[\delta, d]$.*

Then it follows that

$$1<[d-g^{-1}(\gamma)]^{-1}\left|y'\big(g^{-1}(\gamma)\big)\right|^{\nu-1}\int\limits_{g^{-1}(\gamma)}^{d\vee g^{-1}(c)}(d-x)[c-(\delta\vee g(x))]^{\nu}m(x)\,dx \tag{3.4}$$
$$\leqslant(c-\delta)^{\nu}\left|y'\big(g^{-1}(\gamma)\big)\right|^{\nu-1}\int\limits_{g^{-1}(\gamma)}^{d\vee g^{-1}(c)}m(x)\,dx\,.$$

PROOF. – Under the assumptions here, (3.4) follows from (3.1) and a modification of the proof of Corollary 2.2.

REMARK 1. – It is interesting to note that as inequalities involving γ, δ, c and d, if $g^{-1}(\gamma)\geqslant\delta$ and $g(d)\leqslant d$, then we may compare (3.4) and (2.12). We see that the first inequality of (3.4) is stronger than the first one of (2.12), but the extreme inequalities of (2.12) and (3.4) are identical. This naturally leads us to the question of sharpness of the extreme inequality in (3.4). The following modification of Example 2.3 does provide this sharpness, when as previously, we assume g is an increasing delayed deviation.

EXAMPLE 3.3. – Again let $I=[a,\infty)$, y, d and g be as in Example 2.3. Define $y^*\colon I\to R$ in a piecewise fashion by letting $y^*(x)\equiv y(x)$ on $[a,d]$. On $[d,\infty)$ we let y^* be the solution of the initial value problem

$$\begin{aligned} &y^{*\prime\prime}(x)=m^*(x)\left|y^*\big(g(x)\big)\right|^{\nu}\operatorname{sgn}y^*\big(g(x)\big)\\ &y^*(x)\equiv y(x)\qquad\text{on }[g(d),d]\,,\qquad y^{*\prime}(d)=0\,,\end{aligned} \tag{3.5}$$

where, on $[a,d]$, $m^*(x)\equiv m(x)$ as in Example 2.3; on $[d+\eta^2,\infty)$, $m^*(x)\equiv 0$; and on $(d, d+\eta^2)$, m^* is a line satisfying $\lim\limits_{x\to d+}m^*(x)=m(d)$ and $\lim\limits_{x\to d+\eta^2-}m^*(x)=0$.

For this example, by using the computations (2.13) on $[a, d]$ and noting that y^* represents a line having slope on $[d+\eta^2, \infty)$, it follows that $y^*(d^*)=0$ for some $d^*>d$. Consequently $[c, d^*]$ is a positive semicycle of y^* and

$$(3.6)\qquad \begin{aligned} &(c-\delta)^\nu \big|[y^{*\prime}(g^{-1}(\gamma))]\big|^{\nu-1} \int\limits_{g^{-1}(\gamma)}^{d^*} m^*(x)\,dx \\ &\leqslant (c-\delta)^\nu 1^\nu \Big(\int\limits_{d-\eta}^{d} m(x)\,dx + \int\limits_{d}^{d+\eta^2} m^*(x)\,dx \Big) \\ &\leqslant [c-g(d)]^\nu [c-g(d+\eta)]^{-\nu} + [c-g(d)]^\nu (\pi/4)\,[-y(g(d))]^{-\nu}\eta\,. \end{aligned}$$

Now as $\eta\to 0$, the last term in (3.6) tends to zero and consequently the last line can be dominated by $1+\varepsilon$ for any preassigned $\varepsilon>0$.

PROPOSITION. – With m^* and y^* as in the previous examples, $[d, d^*]$ is a positive right quarter-cycle of y^* and

$$(3.7)\qquad \int\limits_{d}^{d^*} m^*(x)\,dx = \int\limits_{d}^{d+\eta^2} m^*(x)\,dx \to 0 \qquad \text{as } \eta^2\to 0\,.$$

Consequently it follows that for equation (1.2) under (H_3), with $p(x)\equiv 0$ and g increasing and $g(x)<x$ on I, no Lyapunov type integral inequality, having the same basic form as the extremes of (2.12) or (3.4), appears to be possible. Indeed, it appears that corresponding integral inequalities for such a right quarter-cycle, $[d, d^*]$, likely should involve a positive power of (d^*-d) in the functional on the right.

4. – Results for a general equation.

Here we examine modifications in Theorems 2.1 and 3.1 necessary for us to apply the results to equation (1.3) under (H_4). As indicated in the introduction we here place further restrictions on the initial and terminal portions of the solution. This is spelled out in (4.2) and (4.3) below, where we have used conditions (2.8) and (2.9) of [2] to modify (2.2) and (2.3). Here, (4.2)′ denotes (4.2) with inequalities changed in direction.

THEOREM 4.1. – *Let $[c, d]$ be a positive, (negative), left quarter-cycle of a solution of* (1.3) *under* (H_4). *Assume*

$$(4.1)\qquad \delta\in I \quad \textit{and} \quad \delta\leqslant c$$

and that

(4.2) *y is a solution of* (1.3) *on* $[\delta, d]$, $y(x) \leqslant 0$ *on* $[\delta, d]^- \cup [d, c]$, $y(x) \geqslant 0$ *on* $[\delta, d]^+$ *the restrictions of y to* $[\delta, d]^-$ *and* $[\delta, d]^+$ *are absolutely continuous with* $y'(x) \leqslant 0$, *a.e., on* $[\delta, d]^- \cup [\delta, d]^+$,

((4.2)′). *Let* $\psi \in [\delta, d]$ *satisfy* (2.4), ((2.4′).
Then it follows that

$$1 < \sum_{i=1}^{K} |y'(\psi)|^{\nu_i - 1} \int_{\Psi_i}^{D_i} \left(\int_{\underline{H_i(t)}}^{\overline{G_i(t)}} U_i(u)\, d\alpha_i(t, u) \right) dt\,, \tag{4.3}$$

where, for each $1 \leqslant i \leqslant K$; *if* α_i *satisfies* (H_2) *then*

$$(4.4)^+ \qquad G_i(t) \equiv c \vee g_i(t)\,, \quad H_i(t) \equiv c \vee h_i(t)\,, \quad U_i(u) \equiv [(u \wedge d) - c]^{\nu_i}\,,$$

and in general, Ψ_i *and* D_i *satisfy*

$$\Psi_i = \psi \in [\delta, d)\,, \quad D_i = d\,, \tag{4.5}$$

but when g_i *and* h_i *are monotone increasing, then*

$$(4.5)^+_{\mathrm{mon}} \qquad \psi_i = d \wedge \sup\{\psi\} \cup \{t\colon g_i(t) < c\}\,, \quad D_i = d\,;$$

but if $-\alpha_i$ *satisfies* (H_2) *then* $(4.4)^+$ *becomes*

$$(4.4)^- \qquad G_i(t) \equiv c \wedge g_i(t)\,, \quad H_i(t) \equiv c \wedge h_i(t)\,, \quad U_i(u) \equiv [c - (\delta \vee u)]^{\nu_i}\,,$$

(4.5) *remains unchanged, and* $(4.5)^+_{\mathrm{mon}}$ *becomes*

$$(4.5)^-_{\mathrm{mon}} \qquad \Psi_i = \psi\,, \quad D_i = \psi \vee \inf\{d\} \cup \{t\colon h_i(t) > c\}\,.$$

PROOF. – We integrate both sides of (1.3) and apply (1.4) to each term, much as in (2.8). Putting the values of ψ for s and d for x yields

$$y'(\psi) = \sum_{i=1}^{K} - \int_{\psi}^{d} \left(\int_{\underline{h_i(t)}}^{\overline{g_i(t)}} |y(u)|^{\nu_i} \operatorname{sgn} y(u)\, d\alpha_i(t, u) \right) dt\,. \tag{4.6}$$

Without loss of generality, assume the quarter-cycle $[c, d]$ is positive. Then depending on whether α_i is monotone increasing or monotone decreasing we delete the negative or positive portions of each integrand, as in the first step of (2.9). This gives the proper values found in (4.4), (4.5) and $(4.5)_{\mathrm{mon}}$. We omit the details since they are largely repetitive.

COROLLARY 4.2. – *If for each $1 \leqslant i \leqslant K$, either p_i is nonnegative or nonpositive and g_i is continuous on I, and if $\nu_i > 0$, then the theorem applies to*

$$y''(x) + \sum_{i=1}^{K} p_i(x) \left|y(g_i(x))\right|^{\nu_i} \operatorname{sgn} y(g_i(x)) = 0 \, . \tag{4.7}$$

In this case (4.3) *may be expressed as*

$$1 < \sum_{i=1}^{K} |y'(\psi)|^{\nu_i - 1} \int_{\psi}^{d} V_i(g_i(t))\, p_i(t)\, dt \, , \tag{4.8}$$

where, for each $1 \leqslant i \leqslant K$; if p_i is nonnegative then

$$V_i(t) \equiv \left[(c \vee (t \wedge d)) - c\right]^{\nu_i} \, ; \tag{4.9$^+$}$$

but if p_i is nonpositive then

$$V_i(t) \equiv \left[c - ((\delta \vee t) \wedge c)\right]^{\nu_i} . \tag{4.9$^-$}$$

PROOF. – The results here are obtained much as Corollary 2.2 follows from Theorem 2.1. That is, by applying (2.10) and (2.11). Even though (2.10) and (2.11) are given for m being nonnegative, similar results hold if m is nonpositive, but in this case the corresponding α is monotone decreasing in its second variable, so that the inequalities of (2.11) are reversed. Thus, if in this corollary, p_i is nonpositive then the corresponding α_i provides G_i, H_i and U_i satisfying (4.4)$^-$ in the theorem.

COROLLARY 4.3. – *If in equation* (4.7) *each p_i and g_i are merely assumed to be continuous and $\nu_1 > 0$ then* (4.8) *needs to be changed to*

$$1 < \sum_{i=1}^{K} |y'(\psi)|^{\nu_i - 1} \left\{ \int_{\psi}^{d} \left[W_{i+}(g_i(t))\, p_i^+(t) + W_{i-}(g_i(t))\, p_i^-(t)\right] dt \right\} \tag{4.10}$$

where W_{i+} is defined by (4.9)$^+$ *and W_{i-} is defined by* (4.9)$^-$.

PROOF. – This is obtained by writing equation (4.7) as

$$y''(x) + \sum_{i=1}^{K} \int_{\psi}^{d} \left[p_i^+(x) - p_i^-(x)\right] |y(g_i(x))|^{\nu_i} \operatorname{sgn} y(g_i(x)) = 0 \, , \tag{4.11}$$

where $p_i^+(x) \equiv 0 \vee p_i(x)$, $p_i^-(x) \equiv 0 \vee [-p_i(x)]$.

REMARK 1. – Each theorem and corollary above relates to Theorem 2.1 and left quarter-cycles. They may all be rewritten to relate to Theorem 3.1 and semicycles. Due to space demands, we simply remark that in Theorem 4.1, for semicycles, (4.2) needs modification to

$$(4.2)_s \qquad [\delta, d]^+ \subseteq \{d\} \quad \text{and the conditions of (4.2)}$$

and in (2.4), $\big((2.4)'\big)$ we need to assume $\psi \in [c, d]$.

Finally in (4.3) the sum must be multiplied by $(d-\psi)^{-1}$ and the term $(d-t)$ must be inserted immediately to the right of the left integral sign in each summand. Similar comments apply to each of the corollaries as well.

REMARK 2. – If in equation (4.7) we assume $K=2$, $p_1(x)\geqslant 0$ and $p_2(x)\geqslant 0$, with $g_i(x)\equiv x$, then (4.8) is a special case of inequality (2.13) of Theorem 2.1 of [2]. In fact, if in (1.3) under (H_4), it is assumed that each α_i satisfies (H_2), then Theorem 4.1 can be established under less restrictive conditions than (4.2). It is here possible to substitute conditions (2.9) of [2]. For this, we also need to substitute conditions (3.11) of [2] for the first two conditions of $(4.4)^+$, to substitute conditions (3.12) for (4.5), (where $\Psi = C$), and finally to substitute $(3.12)_{\text{mon}}$ for $(4.5)^+_{\text{mon}}$.

Similar adjustments can be made to Theorem 4.1 here under Remark 1, which deals with semicycles. Thus, all results of section 2 of [2] generalize to functional equations as described in the previous paragraph.

5. – Comparison of results.

As mentioned in the introduction, considerable work on « distance between zeros of solution » is given by Myškis. Chapter V of [8] is largely devoted to a study of the equation

$$y''(x) + \int_a^b y(u)\, d\alpha(x, u) = 0 \tag{5.1}$$

on $I=[a, b)$, where g and h satisfy (H_1) and α satisfies the monotoneity conditions of (H_2) with some adjustment on the integrability conditions of (H_2). That is to say, Myškis assumes

$$g(x)\leqslant x\,, \qquad x-h(x)\leqslant \Delta_a\,, \qquad 0\leqslant \alpha\big(x, \overline{g(x)}\big)-\alpha\big(x, \underline{h(x)}\big)\leqslant M_a \tag{5.2}$$

and

$$\lim_{\substack{t\to x\\ a\leqslant t<b}} \int_{-\infty}^{+\infty} |\alpha(t, u)-\alpha(x, u)|\, du = 0\,.$$

Theorem 48 of [7, p. 109] is given below.

THEOREM (MYŠKIS). – *Let y be a solution of* (5.1) *on I, where* (5.2) *and other conditions of* (H_2) *mentioned above are assumed. Suppose also that*

$$\Delta_a M_a^{\frac{1}{2}} < (\pi/2) + 2^{\frac{1}{2}} = 2.9850 \ldots , \tag{5.3}$$

and that y satisfies

$$y(x) \leqslant 0 \quad \text{on } [a - \Delta_a, a], \quad y(a) = 0, \quad y'(a) \geqslant 0 \quad \text{and} \quad y(x) \not\equiv 0 \quad \text{on } I . \tag{5.4}$$

Then there is a number c in $[a, b)$ such that $y(x) \equiv 0$ on $[a, c]$ and such that one and only one of the following two cases hold:

(5.5) (i) $y'(x) > 0$ *on* (c, b);

(5.5) (ii) *y' has a smallest zero d in (c, b) for which $(d - x)M_a^{\frac{1}{2}} \geqslant \pi/2$, and $y'(x) > 0$ on (c, d).*

Furthermore, if (5.5) (ii) *holds then one and only one of the following two conditions hold*:

(5.6) (i) $y(x) > 0$ *on* (c, b),

(5.6) (ii) *y has a smallest zero d^* in (d, c) for which $(d^* - d)M_a^{\frac{1}{2}} \geqslant 2^{\frac{1}{2}}$, and consequently it follows from this and* (5.5) (iii) *that $(d^* - c)\, M_a^{\frac{1}{2}} \geqslant (\pi/2) + 2^{\frac{1}{2}}$.*

We see that (5.5) (ii) provides a lower bound on the length of the positive left quarter-cycle $[c, d]$, and the last inequality in (5.6) (ii) does similarly for the positive semicycle $[c, d^*]$. In both cases, the initial function satisfies the general conditions of (4.2) of Theorem 4.1.

Thus, if (H_2) is satisfied by α in equation (5.1), we may apply Theorem 4.1. Inequality (4.3) yields

$$1 < \int_{\Psi}^{d} \int_{\underline{c \vee h(t)}}^{\overline{c \vee g(t)}} [u - c]\, d\alpha(t, u)\, dt , \tag{5.7}$$

where, since $g(x) \leqslant x$, $\Psi = d \wedge \sup\{\psi\} \cup \{t : g(t) < c\} \geqslant c$ may be chosen.

The main points of comparison though are that in [2] and this paper we consider the problem of distance between zeros for much larger classes of equations with, in general, less restrictions on g and h, and in the initial and terminal functions. Myškis deals with linear equations, and does not allow α to be monotone decreasing in its second variable. Also we do not assume condition (5.3), used by Myškis to assure that $[c, d^*]$, above, be a «large» semicycle.

When $m(x) \leqslant 0$ on I and

(5.8) p, m, and g are continuous on I, $p(x) \geqslant 0$, $x - \Delta_a \leqslant g(x) \leqslant x$, $p(x) - m(x) \leqslant M_a$ and $\mu = \nu = 1$,

then equation (1.2) becomes a special case of (5.1) under (5.2).

With the conditions of the preceding paragraph, Theorem 3.1 of Norkin [8, p. 129] becomes a special case of the above Theorem of Myškis. Norkin, however, claims his Theorem 3.1 to be valid for equation (1.2) under (5.8), without the restriction of $m(x) \leqslant 0$. He obtains (1.2) under (5.8), without the restriction of $m(x) \leqslant 0$. He obtains essentially the same conclusions of (5.6) without reference to (5.5), but certainly under his proof the conclusions (5.5) and (5.6) may be included.

NORKIN indicates that the proof of this theorem is based on Lemmas 4.1-4.3, found in [8, pp. 89-91], also due to MYŠKIS, which in turn are based on conditions (1.2′), (1.3′) and (1.4′) of [8, p. 88], which assume an identically zero initial function.

It appears to the author that for NORKIN to claim a proof of his Theorem 3.1, without the restriction of $m(x) \leqslant 0$ on I, based on the above mentioned lemmas, it is necessary as well to assume that

$$y(g(x)) \geqslant 0 \qquad \text{holds if } m(x) > 0\,, \qquad \text{for } x \text{ in } (c, d^*)\,.$$

As a result, the generalization of Myškis' theorem is such that the case of $m(x) > 0$ in (1.2) under (5.2) is somewhat hedged.

For the very special case of equation (1.2) under (5.2), with $p(x) \equiv p > 0$, $m(x) \equiv \equiv m > 0$ and $g(x) \equiv x - \Delta < x$, we may apply Remark 1 following Corollary 3, to that corollary to obtain a result for semicycles. From (4.10) we have

$$\begin{aligned}(5.9) \qquad 1 < (d^* - \psi)^{-1} \int_{\psi}^{d^*} (d^* - t)\,[c \vee (t - \Delta) - c]\,p\,dt \\ + (d^* - \psi)^{-1} \int_{\psi}^{d^*} (d^* - t)\Big[c - \Big(\big((c - \Delta) \vee (t - \Delta)\big) \wedge c\Big)\Big]\,m\,dt \\ \leqslant [p(d^* - c - \Delta)^2/4] + [m\Delta^2/2]\,,\end{aligned}$$

provided ψ, satisfying (2.4), is in $[c, d^*]$. (This condition of «ψ in $[c, d^*]$» is not needed for applying (4.10) to the left quartercycle $[c, d]$). The inequality provided by the extremes of inequality (5.9) may in a sense be thought of as an extension of the final inequality in (5.6) (ii), since Myškis' theorem may not handle this equation.

A final comment is that our implicit lower bounds on the «distance between zeros» are obtained in terms of integrals of the coefficients rather than maximum values of such. When both Myškis' theorem and our results apply, it is when the coefficients are «nearly» constant that Myškis' results are better, otherwise they are not necessarily so.

6. – A necessary condition for oscillation.

The Lyapunov type inequality (2.12) may be applied to obtain a necessary condition for the existence of a certain type of oscillatory solution on $I = [a, \infty)$ of (1.2) under (H_3), where, in addition it is assumed that $p(x) \equiv 0$, $m(x) > 0$, $g(x) < x$ and g is incresing on I.

Under these conditions, we use the notation of MYŠKIS [7], (see also [8]), and call a quarter-cycle or semicycle $[c, d]$, *large* or *small* accordingly as $c < g(d)$ or $c \geqslant g(d)$ hold. We shall call a solution oscillatory if there is no largest zero.

LEMMA 6.1. – *Let y be an oscillatory solution of* (1.2) *on $I = [a, \infty)$, where together with* (H_3); $p(x) \equiv 0$, $m(x) > 0$, $g(x) < x$ *and g is increasing on I, are assumed. Assume that y has a first positive, (negative), left quarter-cycle and y is negative, (positive), on the corresponding initial segment, excepting its right endpoint.*

Then all zeros of y and y' to the right of this left quarter-cycle are isolated and interlace each other. Also for each other positive, (negative), left quarter-cycle $[c, d]$ of y, values of γ and δ satisfying (2.2) *may be chosen so that $\gamma \leqslant \delta < c$ and* (2.3), ((2.3)$'$), *holds. Finally, it is necessary that $g(x) \to \infty$ as $x \to \infty$, and the zeros of y have no finite cluster point.*

PROOF. – This lemma follows by an inductive argument on the quarter-cycles and semicycles going from left to right. Several comments in the proof of Corollary 2.2 apply here.

Without loss of generality, assume $[c, d]$ is the first left quartercycle and that it is positive. Then it is necessary that $y''(x) < 0$ hold for values of x immediately to the left of d, so that $y''(d) \leqslant 0$.

If $y''(d) < 0$, then d is isolated from other zeros of y'.

If $y''(d) = 0$, then since $m(d) > 0$ we have $y(g(d)) = 0$ and $g(d) = c$. Hence $y''(x) > 0$ for all $x > d$ and y has no larger zeros, a contradiction.

Also, if $z > c$ is the first zero of y greater than c, then $y'(z) < 0$. Indeed $y'(x) < 0$ holds on $(d, z]$. For if $\omega \in (d, z]$ is the smallest zero of y', then $y''(x) > 0$ is necessary for values of x immediately to the left of ω. Consequently $y''(\omega) \geqslant 0$, so $y(g(\omega)) \geqslant 0$ and $c \leqslant g(\omega)$ holds. Again, as above we may conclude $y''(x) > 0$ for almost all $x > \omega$, a contradiction.

Now, for the next larger, negative, left quarter cycle $(c^*, d^*]$ we may pick $\delta = d \vee g(c^*)$ and $\gamma = c \vee g(c^*)$, where $c^* = z$, c and d are as above. If follows that $g(d^*) \in (c, c^*)$.

The argument now proceeds inductively, and since $g(x) < x$, and g is continuous we may argue that the zeros of y have no finite cluster point. We omit the remainder of the proof.

LEMMA 6.2. – *Assume the first paragraph of* Lemma 6.1. *Let $[z_0, z_1]$ and $[z_1, z_2]$ be two consecutive semicycles of y, and $\zeta_1 \in (z_0, z_1)$, $\zeta_2 \in (z_1, z_2)$ be the zeros of y'.*

Then $g^{-1}(z_0) \in (\zeta_1, \zeta_2)$, $g^{-1}(z_1) > \zeta_2$ and $g(\zeta_2) \in (z_0, z_1)$ all hold.

Also, if semicycle $[z_0, z_1]$ is small, then

$$|y'(g^{-1}(z_0))|^{1-\nu} < \int_{g(\psi_2)}^{\psi_2} [g(\psi_2) - g(s)]\, m(s)\, ds \tag{6.1}$$

holds, where $\psi_2 \equiv g^{-1}(z_1)$.

PROOF. – The first paragraph of conclusions follow by an inductive argument as in Lemma 6.1.

The computations (6.1) follow from inequality (2.12) by applying these comments and the previous lemma. We have, since $[z_0, z_1]$ is small, that

$$\begin{aligned}|y'(g^{-1}(z_0))|^{1-\nu} &< \int_{g^{-1}(z_0)}^{\zeta_2} [z_1 - g(s)]^\nu m(s)\, ds \\ &\leqslant \int_{z_1}^{g^{-1}(z_1)} [z_1 - g(s)]^\nu m(s)\, ds \\ &= \int_{g(\psi_2)}^{\psi_2} [g(\psi_2) - g(s)]^\nu m(s)\, ds\,.\end{aligned}$$

THEOREM 4.3. – *Let y be an oscillatory solution of* (1.2) *on $I = [a, \infty)$, where together with* (H_3); $p(x) \equiv 0$, $m(x) > 0$, $g(x) < x$ *and g is increasing on I, are assumed. Let (z_n) denote the sequence of consecutive zeros of y, where if y is positive, (negative), on (z_1, z_2) then y is negative, (positive), on $(g(z_1), z_1)$.*

Then, if each semicycle $[z_n, z_{n+1}]$ is small, it follows that:

$$\limsup_{x\to\infty} |y'(x)|^{1-\nu} \leqslant \limsup_{x\to\infty} \int_{g(x)}^{x} [g(x) - g(s)]^\nu m(s)\, ds \qquad \text{if } 0 < \nu < 1; \tag{6.2}$$

and

$$[\limsup_{x\to\infty} |y'(x)|^{\nu-1}]^{-1} \leqslant \limsup_{x\to\infty} \int_{g(x)}^{x} [g(x) - g(s)]^\nu m(s)\, ds \qquad \text{if } \nu > 1. \tag{6.3}$$

PROOF. – First, define $\psi_{n+1}\, g^{-1}(z_n)$, for each natural n. It is seen that if the semicycle is positive, then

$$y'(\psi_{n+1}) = \max\{y'(x) : z_n \leqslant x \leqslant z_{n+1}\}\,, \tag{6.4}$$

and «max» is changed to min, if $[z_n, z_{n+1}]$ is negative.

By (6.1) we thus have

$$|y'(\psi_{n-1})|^{1-\nu} < \int_{g(\psi_n)}^{\psi_n} [g(\psi_n) - g(s)]^\nu m(s)\, ds\,, \tag{6.5}$$

for each natural n.

Thus, if $0 < \nu < 1$ we have

$$\begin{aligned}\limsup_{x\to\infty} |y'(x)|^{1-\nu} &= \limsup_{h\to\infty} |y'(\psi_{n-1})|^{1-\nu} \\ &\leqslant \limsup_{h\to\infty} \int_{g(\psi_n)}^{\psi_n} [g(\psi_n) - g(s)]^\nu m(s)\, ds \\ &\leqslant \limsup_{x\to\infty} \int_{g(x)}^{x} [g(x) - g(s)]^\nu m(s)\, ds\,.\end{aligned} \tag{6.6}$$

For $\nu > 1$, multiply both sides of (6.5) by $|y'(\psi_{n-1})|^{\nu-1}$ and repeat computations similar to (6.6), noting that «lim sup of product $\leqslant$ product of lim sups» when the terms of the product are positive, this provides the theorem.

For the case of $\nu = 1$ we have the following result.

THEOREM 4.4. – *Assume the first paragraph of* Theorem 4.3, *then, for* $\nu = 1$, *it follows that*

$$1 \leqslant \limsup_{x\to\infty} \int_{g(x)}^{x} [g(x) - g(s)]\, m(s)\, ds \tag{6.7}$$

holds if infinitely many of the semicycles $[z_n, z_{n+1}]$ *are small.*

PROOF. – The proof is again similar to the case of $0 < \nu < 1$, except now (6.5) becomes

$$1 < \int_{g(\psi_n)}^{\psi_n} [g(\psi_n) - g(s)]\, m(s)\, ds\,, \tag{6.8}$$

provided $[z_{n-2}, z_{n-1}]$ is small. Since $g(x) \to \infty$ as $x \to \infty$, the last inequality of (4.6) again applies.

REMARK 1. – In [4], [5] and [6] are given extensions of a result dealing with

$$1 < \limsup_{x\to\infty} \int_{g(x)}^{x} [g(x) - g(s)]\, m(s)\, ds \tag{6.9}$$

as a sufficient condition yielding all bounded solutions of (1.2) under (H_3) with $p(x) \equiv 0$, $g(x) \leqslant x$ and g increasing on I, and $\nu = 1$, to be oscillatory.

They leave as an open question whether or not (6.9) is sharp. That is, can the constant of 1 on the left be decreased and still leave (6.9) as a sufficient condition?

We wish to point out the similarity of (6.7) and (6.9), but also note that (6.7) does not establish the sharpness, since (6.7) follows on the assumption that infinitely many of the semicycles are assumed to be small.

REFERENCES

[1] A. ÈL'BERT, *A generalization of a certain inequality of Ljapunov to differential equations with retarded argument*, (Russian), Ann. Univ. Sci. Budapest, Eötvös Sect. Math., **12** (1968), pp. 107-111.

[2] S. B. ELIASON, *Lyapunov type inequalities for certain second order functional differential equations*, SIAM J. Appl. Math., **27** (1974), pp. 180-199.

[3] G. B. GUSTAFSON, *Bounded oscillations of linear and nonlinear delay-differential equations of even order*, J. Math. Anal. Appl., **46** (1974), pp. 175-189.

[4] G. LADAS, G. S. LADDE - J. S. PAPADAKIS, *Oscillations of functional-differential equations generated by delays*, J. Diff. Eq., **12** (1972), pp. 385-395.

[5] G. LADAS - V. LAKSHMIKANTHAM - J. S. PAPADAKIS, *Oscillations of higher-order retarded differential equations benerated by the retarded argument*, in *Delay and Functional Differential Equations and Their Applications*, Academic Press, K. Schmitt ed., New York, 1972, pp. 219-231.

[6] G. S. LADDE, *Oscillations of nonlinear functional differential equations generated by retarded actions*, Dept. of Math., Univ. Rhode Island, Kingston, R.I., Academic Press, K. Schmitt ed., New York, 1972, pp. 355-365.

[7] A. D. MYSKIS, *Linear Differential Equations with Retarded Arguments*, (German), Deutscher Verlag Der Wissenschaften, Berlin, 1955.

[8] S. B. NORKIN, *Differential Equations of the Second Order with Retarded Argument*, Vol. **31**, Translations of Mathematical Monographs, Amer. Math. Soc., Providence, R.I., 1972.