

Distance measurements as a probe of cosmic acceleration

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ABSTRACT

A major recent development in observational cosmology has been an accurate measurement of the luminosity distance–redshift relation out to redshifts $z = 0.8$ from Type Ia supernova standard candles. The results have been argued as evidence for cosmic acceleration. It is well known that this assertion depends on the assumption that we know the equation of state for all mass–energy other than normal pressureless matter; popular models are based either on the cosmological constant or on the more general quintessence formulation. However, this assertion also depends on a number of other assumptions, implicit in the derivation of the standard cosmological field equations: large-scale isotropy and homogeneity, the flatness of the Universe, and the validity of general relativity on cosmological scales (where it has not been tested). A detailed examination of the effects of these assumptions on the interplay between the luminosity distance–redshift relation and the acceleration of the Universe is not possible unless one can define the precise nature of the failure of any particular assumption. However a simple quantitative investigation is possible and reveals a number of considerations about the relative importance of the different assumptions. In this paper we present such an investigation. We find that the relationship between the distant-redshift relation and the sign of the deceleration parameter is fairly robust and is unaffected if only one of the assumptions that we investigate is invalid so long as the deceleration parameter is not close to zero (it would not be close to zero in the currently favoured $\Omega_\Lambda = 1 - \Omega_{\text{matter}} = 0.7$ or 0.8 Universe, for example). Failures of two or more assumptions in concordance may have stronger effects.

Key words: cosmology: observations – cosmology: theory.

1 INTRODUCTION AND BACKGROUND

One of the most significant cosmological results of recent times has been an accurate determination of the luminosity distance–redshift relation (Schmidt et al. 1998; Perlmutter et al. 1999; Riess 1999; Zehavi & Dekel 1999; Burrows 2000; Riess et al. 2000), from measurements of Type Ia supernova standard candles out to redshifts $z \sim 0.8$. The measurements are highly inconsistent with a flat purely matter-dominated Universe and have consequently been regarded as evidence for cosmic acceleration.

This assertion, however, depends intricately on other assumptions made, and does not necessarily follow from the measurement of the luminosity distance–redshift relation. Suppose that we can describe the expansion of the Universe by a scalefactor $R(t)$ which is a function of cosmic time t . The redshift $z = R(t_0)/R(t) - 1$ is then also a function of t (here t_0 is the current age of the Universe).

The luminosity distance–redshift relation is (e.g. Weinberg 1972)

$$d_L(z) = R(t_0)(1+z)S_k \left[\int_{R(t_0)/(1+z)}^{R(t_0)} \frac{c \, dR}{(dR/dt)R} \right], \quad (1)$$

where c is the speed of light, k is an additional parameter describing the curvature of the Universe ($k = +1$ for a closed Universe, $k = 0$ for a flat Universe, and $k = -1$ for an open Universe), and S is defined so that $S_{+1}(x) = \sin x$, $S_0(x) = x$, and $S_{-1}(x) = \sinh x$. The deceleration parameter is (e.g. Weinberg 1972)

$$q_0 = - \left. \frac{d^2 R}{dt^2} R \right|_{t_0} \left(\frac{dR}{dt} \right)^2. \quad (2)$$

If the Universe is accelerating $q_0 < 0$. It follows from these equations that a function $R(t)$ constrained from a measurement of $d_L(z)$ does not uniquely define a value of q_0 . In fact, it does not even

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uniquely define the *sign* of q_0 . This happens because q_0 depends on the second time derivative of $R(t)$ but $d_L(z)$ only depends on lower order derivatives and their integral. As a simple (albeit unphysical) example, consider a function $R(t)$ that can be expressed as a polynomial in t with coefficients a_n such that $R(t) = \sum a_n t^n$ where the n th term contributes significantly more than the $(n + 1)$ th one. If one then defines another function $R'(t)$ that is identical to $R(t)$ except that the sign of the a_2 coefficient is opposite, then both $R(t)$ and $R'(t)$ generate almost identical $d_L(z)$ relations but deceleration parameters with opposite signs.

It is conventional to draw a connection between the $d_L(z)$ relation and the sign of the deceleration parameter as follows. One assumes the Friedmann–Robertson–Walker metric (which follows from the requirement of isotropy and homogeneity on the largest scales), Einstein’s theory of general relativity, zero spatial curvature ($k = 0$) and an equation of state so that the only contributions to the stress–energy tensor in general relativity comes from normal matter and the vacuum energy (the cosmological constant). This latter assumption is frequently examined (Caldwell, Dave & Steinhardt 1998; Garnavich et al. 1998; Efstathiou 1999; Maor, Brustein & Steinhardt 2000; Wang et al. 2000) since it has no theoretical basis. If these assumptions are made, the model forces a particular form of $R(t)$ and consequently a particular form of $d_L(z)$ and q_0 in terms of Ω_M and Ω_Λ , the cosmological densities of matter and the vacuum. The deceleration parameter here has the particularly simple form $q_0 = \Omega_M/2 - \Omega_\Lambda$. Given the constraints imposed by the supernova measurements, it is found that the only pairs of Ω_M and Ω_Λ that are allowed force $q_0 < 0$. Therefore the Universe is argued to be accelerating.

However, any of the assumptions may turn out to be invalid. Aside from the much examined assumptions regarding the equation of state turning out to be untrue, it is also possible that $k \neq 0$, that very large-scale inhomogeneities may exist (C el erier 2000), or even that general relativity may turn out to be invalid on the very largest scales (where it has not been directly tested). If any or all of these turn out to be true, we might no longer require an accelerating Universe given the current (or even a far more precise) measurement of the $d_L(z)$ relation; this possibility is investigated in a simple quantitative way in this paper.

2 FRIEDMANN MODELS

The formulation that is normally used to define the cosmological parameters can be conceptually (albeit arbitrarily) divided into the following four steps.

(i) The cosmic geometry is described by the four-dimensional Riemann curvature tensor. If isotropy and homogeneity are assumed, the resulting constraints on the Riemann tensor force the space–time metric to have a specific form, given by the Friedmann–Robertson–Walker metric. Implicit in this form is the parameter k , which takes the value 0 or ± 1 . The existence of this parameter k follows from the assumption that the Ricci scalar (the one-dimensional fully contracted Riemann tensor) is not a function of either time or position, which is required by homogeneity. If this assumption about homogeneity and isotropy is made, then observations of the position l_{peak} of the first Doppler peak in the microwave background spectrum (e.g. de Bernardis et al. 2000) that suggest a value of the total cosmological density Ω_{TOT} close to 1 in units of the critical density then suggest that $k = 0$.

(ii) From the metric, one can then compute explicitly the

curvature tensor and its contractions. From the equations of general relativity, one can then compute the cosmological field equations. Note that general relativity has not been verified observationally to high precision on these cosmological scales (although it is clearly very successful on smaller scales – see section 8 of Weinberg 1972), so its application here is very much an assumption.

(iii) If we then adopt an equation of state that relates density to pressure for every contribution of mass density in the field equations, we can then derive $R(t)$ as a function of these contributions. Some common strategies are to assume that (a) the only contribution comes from normal pressureless matter; (b) part of the contribution comes from normal matter and all the remaining part comes from a cosmological constant; (c) part of the contribution comes from normal matter and all the remaining part comes from some material whose equation of state is defined by the quintessence formulation of Caldwell et al. (1998).

(iv) From measurements of the luminosity distance–redshift relation we constrain combinations of parameters that arise from the considerations in (iii) above. For example, if one has specified exactly two forms of mass density, both of which have known equations of state, then one can constrain the relative mass densities of the two components. [This would be true in case (b) above.] If one makes a further assumption, like requiring a flat universe ($k = 0$), then one has another constraint and in conjunction with the luminosity distance–redshift relation can provide a precise determination of the model parameters. Alternatively any related measurement like the value of l_{peak} will have a similar effect. [Recall from (i) that in the context of these models, the value of l_{peak} that was measured is operationally similar to requiring $k = 0$.] In practice, the constraints on the model parameters in case (b) are tight since the constraints on the mass densities in normal matter and in the cosmological constant are almost orthogonal (see Efstathiou et al. 1999). From the permitted values of the contributions of these mass densities, we can then compute $R(t)$ and hence q_0 , within the errors provided by the measurements.

3 MODIFICATIONS TO FRIEDMANN MODELS

Clearly there are lots of assumptions involved. It is therefore useful to examine each in detail and to see how relaxing each affects the interplay between $d_L(z)$ and q_0 . Assessing the general case is mathematically complicated and unconstrained. However, it is possible to compute the consequences of relaxing the assumptions in simple well-defined ways, and this is done in Table 1. The approach is as follows.

(1) One of the assumptions described in the previous section is relaxed (given in the first column of Table 1) and a modification of the relevant equation is assumed. Each adjustment is described in terms of a parameter ϵ , which is a measure of how big the modification is.

(2) The relevant modified Friedmann equation is derived (the second column of Table 1).

(3) From this equation, we then compute the value of $d_L(z = 1)$ for various fiducial models, listed in Table 2 (note that Model C is consistent with the observations described in Section 1, but Models A and B are not). These represent the case $\epsilon = 0$. We then ask: what values of ϵ generate value of $d_L(z = 1)$ that differs by less than 10 per cent from this fiducial value? These values are presented in Table 3. A measurement of the luminosity distance at

Table 1. Simple modifications to the Friedmann models.

Modification and form	New Friedmann equation ^a
Isotropy and homogeneity	
$\mathcal{R} \sim r^\epsilon$	as standard
$\mathcal{R} \sim t^\epsilon$	as standard
$\mathcal{R} \sim \sin^\epsilon \theta$	$(-\epsilon - \epsilon \cos 2\theta + 2 \sin^2 \theta)\epsilon \sin^{-2+\epsilon} \theta + 4r^2(1 + 3 \sin^\epsilon \theta - \sin^{2\epsilon} \theta)\dot{R}^2 = 32\pi G\rho r^2 R^2$
$\mathcal{R} \sim \sin^\epsilon \phi$	$(2 - \epsilon - \epsilon \cos 2\phi)\epsilon \operatorname{cosec}^2 \theta \sin^{-2+\epsilon} \phi + 4r^2(1 + 3 \sin^\epsilon \phi - \sin^{2\epsilon} \phi)\dot{R}^2 = 32\pi G\rho r^2 R^2$
Density (curvature)	
$\Omega_M + \Omega_\Lambda = 1 + \epsilon$	as standard
General relativity	
$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi GT_{\mu\nu}(T_\alpha^\alpha)^\epsilon$	$H^2 = H_0^2 \left(\frac{R}{R_0}\right)^{-3(1+\epsilon)}$
Equation of state	
$P = \epsilon\rho$ (quintessence ^b)	$H^2 = \Omega_M H_0^2 \left(\frac{R}{R_0}\right)^{-3} + (1 - \Omega_M)H_0^2 \left(\frac{R}{R_0}\right)^{-3(1+\epsilon)}$

The symbols have the following meanings: ϵ is a parameter, defined individually in each case, that describes how we are modifying the model; (t, r, θ, ϕ) are the time, radial, and two angular coordinates used to define the space–time metric $g_{\mu\nu}$; the two-dimensional contraction of the full four-dimensional Riemann curvature tensor is the Ricci tensor $R_{\mu\nu}$, and its contraction to one dimension is the Ricci scalar \mathcal{R} ; $T_{\mu\nu}$ is the Einstein stress–energy tensor, implicit in which are the density ρ and pressure P ; R is the cosmic scalefactor, which depends on t , \dot{R} is its time derivative and $H = \dot{R}/R$; R_0 and H_0 (the Hubble constant) are the present-day values of R and H ; Ω_M is the cosmological density of normal matter obeying the equation of state $P = 0$ in units of the critical density; $\Omega_\Lambda = \Lambda/3H_0^2$ is the cosmological density in the cosmological constant; G is the gravitational constant. Other symbols are defined elsewhere in the text; the speed of light c is set to unity in all the equations in the table.

^a Assuming $k = 0$ and $\Lambda = 0$.

^b In order to define fully a quintessence model we need to state both the equation of state of the quintessent material (parametrized by ϵ here) and the density of this material. For the models we consider here, we fix the density of normal matter to be $\Omega_M = 0.2$ in units of the critical density and the density of the quintessent material to be $\Omega_\epsilon = 0.8$ so that the total density is $\Omega_{\text{TOT}} = 1$. The value of $\Omega_M = 0.2$ is suggested by dividing the value of Ω_{baryon} derived from big bang nucleosynthesis constraints (e.g. Smith, Kawano & Malaney 1993) by the baryon fraction in rich galaxy clusters like Coma (White et al. 1993). It is also the value suggested for $\Omega_{\text{dark matter}}$ by (see Trimble 1987) assigning an amount of dark matter to each luminous galaxy (e.g. using the correlations of Kormendy 1990) and integrating over the galaxy luminosity function (e.g. Ellis et al. 1996). Neither of these methods depend on the presence or absence of a form of mass density that does not behave according to the equation of state $P = 0$, like a cosmological constant. The value of $\Omega_{\text{TOT}} = 1$ is suggested by the measurements of the cosmic microwave background by de Bernardis et al. (2000).

$z = 1$ that is precise to 10 per cent requires observations slightly deeper and more precise than those of Perlmutter et al. (1999), but should be attainable with larger samples of supernovae observed with large telescopes in the near future.

(4) We also compute the values of ϵ that leave the sign of the deceleration parameter within 10 per cent of its fiducial value and that leave the sign of the deceleration parameter unchanged (the third and fourth columns of Table 3).

If the values of ϵ allowed in the second column of Table 3 all satisfy the condition given in the fourth column of Table 1, then we can conclude that a measurement of $d_L(z = 1)$ precise to 10 per cent does indeed constrain the sign of q_0 (for example, this is true of the case labelled ‘General Relativity’) for the relevant fiducial model.

An example of this calculation follows. Consider the last row in the ‘density (curvature)’ section of the table i.e. the line labelled ‘C2’. Given the Friedmann equation, equation (1) now becomes

$$d_L(z) = \frac{c(1+z)}{\sqrt{|\epsilon|}H_0} S_k \left[\sqrt{|\epsilon|} x_z(\epsilon) \right] \quad (3)$$

where $k = +1$ if $\epsilon > 0$, $k = -1$ if $\epsilon < 0$ and $k = 0$ if $\epsilon = 0$, and

$$x_z(\epsilon) = \int_0^z \frac{dz'}{\sqrt{(1+z')^2(1+0.2z') + (0.8+\epsilon)z'(2+z')}}. \quad (4)$$

If $\epsilon = 0$ (fiducial model C), then $d_L(z = 1) = 1.65c/H_0$. For a value within 10 per cent of this [$1.48c/H_0 < d_L(z = 1) < 1.81c/H_0$], equation (3) requires $-0.50 < \epsilon < 0.18$. Equation (2)

Table 2. Fiducial models.

Model	Parameters	k	$d_L(z = 1)/cH_0^{-1}$	q_0
A	$\Omega_M = 1$	0	1.17	0.50
B	$\Omega_M = 0.2$	-1	1.41	1.10
C	$\Omega_M = 0.2, \Omega_\Lambda = 0.8$	0	1.65	-0.70

now becomes

$$q_0 = -0.7 - \epsilon. \quad (5)$$

For the fiducial model $q_0 = -0.7$ and is negative. Values of ϵ that result in a value of q_0 within 10 per cent of this value are $-0.07 < \epsilon < 0.07$, from equation (5). To keep q_0 negative, equation (5) requires $\epsilon > -0.70$. Therefore any changes in ϵ that are consistent with a measurement of $d_L(z = 1)$ precise to 10 per cent do not change the sign of q_0 , at least for this fiducial model and form of perturbation.

Such numbers, although interesting, are of limited use since the form of the modification that we use is so specific. We would, however, make the following four comments.

(i) The choice of the fiducial model is important. For fiducial models like Model B which have q_0 close to zero, a small change in $d_L(z = 1)$ could be consistent with a model in which the sign of q_0 changes. However, for model C, which is consistent with current observation, the fiducial value of q_0 is far enough away from zero

Table 3. Effects of modifications on observables.

Modification and form	Values of ϵ that are consistent with a measurement of $d_L(z = 1)$ that is accurate to 10 per cent	Values of ϵ that are consistent with the fiducial value of q_0 to within 10 per cent	Values of ϵ that keep unchanged the sign of q_0
Isotropy and homogeneity			
$\mathcal{R} \sim r^\epsilon$	any ϵ	any ϵ	any ϵ
$\mathcal{R} \sim t^\epsilon$	any ϵ	any ϵ	any ϵ
$\mathcal{R} \sim \sin^\epsilon \theta$	depends on (r, θ)	depends on (r, θ)	depends on (r, θ)
$\mathcal{R} \sim \sin^\epsilon \phi$	depends on (r, θ, ϕ)	depends on (r, θ, ϕ)	depends on (r, θ, ϕ)
Density (curvature)			
$\Omega_M + \Omega_\Lambda = 1 + \epsilon$	A: $-0.45 < \epsilon < 0.63$ B: $-0.30 < \epsilon < 0.41$ C1 ^a : $-0.18 < \epsilon < 0.54$ C2 ^a : $-0.50 < \epsilon < 0.18$	$-0.10 < \epsilon < 0.10$ $-0.02 < \epsilon < 0.02$ $-0.14 < \epsilon < 0.14$ $-0.07 < \epsilon < 0.07$	$\epsilon > -1.00$ $\epsilon > -0.20$ $\epsilon < 1.40$ $\epsilon > -0.70$
General relativity			
$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}(T^\alpha)^\epsilon$	A: $-0.19 < \epsilon < 0.22$ B: $-0.89 < \epsilon < 0.52$ C: $-0.39 < \epsilon < 0.36$	$-0.03 < \epsilon < 0.03$ $-0.03 < \epsilon < 0.03$ $-0.23 < \epsilon < 0.23$	$\epsilon > -0.33$ $\epsilon > -0.33$ $\epsilon < 2.33$
Equation of state			
$P = \epsilon\rho^b$	A: $-0.295 < \epsilon < 0.27$ B: $-0.79 < \epsilon < -0.21$ C: $-1.41 < \epsilon < -0.65$	$-0.04 < \epsilon < 0.04$ $-0.34 < \epsilon < -0.33$ $-1.06 < \epsilon < -0.94$	$\epsilon > -0.42$ $\epsilon > -0.42$ $\epsilon < -0.42$

^a For the fiducial model C, modifications that are introduced can be considered as modifications to the behaviour of the matter component (line C1) or the cosmological constant component (line C2). Results are presented for both.

^b Note that the value of $\epsilon = 0$ is not ‘permitted’ for models B and C. This is because in neither case can the fiducial model be described as an unmodified version of the model under consideration here. In fact, fiducial model C is exactly equivalent to a modified model with $\epsilon = -1$; this value of ϵ is of course at the centre of the permitted range. The reason that models A, B, and C give different constraints on ϵ in the second and third columns is because different fiducial models have been adopted. In the fourth column, where the limiting ϵ does not depend on the fiducial model, this limiting value is the same in all three cases.

that all of the modifications considered here do not change the sign of q_0 without changing $d_L(z = 1)$ at the 10 per cent level.

(ii) Small dependences of the Ricci scalar on any metric coordinates introduce small changes in the Friedmann equation. Larger dependences that are power laws in either time or the radial position coordinate also have negligible effect. Larger dependences having other functional forms will have a bigger effect but these will probably need to be fairly finely tuned if they are going to affect the relationship between $d_L(z)$ and q_0 and not have any appreciable signature in either the galaxy distribution on large scales or the cosmic microwave background.

(iii) Invalidating general relativity in the way defined in Table 1 does not seem to have a strong effect on the interplay between $d_L(z = 1)$ and q_0 . Values of ϵ large enough to change the sign of q_0 would cause a very substantial change in $d_L(z = 1)$ [except for Model B, but see point (i)].

(iv) Modifying the equation of state according to the quintessence formulation has some effect on the interplay between $d_L(z = 1)$ and q_0 . Maor et al. (2000) consider the case of a time-varying equation of state, which is outside the bounds of the simple mathematical treatment presented here, and find that relaxing the assumption of a constant equation of state can have very profound implications for using $d_L(z)$ to determine q_0 and more generally the fate of the Universe.

In these calculations, we have concentrated on measurements of $d_L(z)$ at $z = 1$. In principle, measurements at higher redshifts would help since $\partial d_L(z)/\partial q_0$ increases as z increases. (Formally, this is true only if the underlying cosmology is known a priori; see also the discussion on this point in Maor et al. 2000.) But one would

then have additional systematic effects that would need to be understood, like K -corrections of Type Ia supernovae, complications as a result of dust extinction (Riess et al. 2000) and possibly even gravitational lensing by matter along our line of sight (Metcalf & Silk 1999). More precise measurements at $z = 1$ and lower redshifts would also help, although these may be fundamentally limited by intrinsic scatter in the peak luminosity of Type Ia supernovae (see the references in the first paragraph of Section 1).

4 SUMMARY

To summarize, the relationship between $d_L(z = 1)$ and the sign of q_0 is fairly robust given the analysis presented here unless the value of q_0 is close to zero. For the $\Omega_\Lambda = 1 - \Omega_M = 0.8$ model currently favoured by observation, the value of q_0 is far enough away from zero that this concern is not valid. This analysis is, however, limited, in that it only considers modifications to the Friedmann models of a very specific type, and those individually. The joint effects of two or more modifications might be more dramatic, but at this stage there are too many unconstrained ways to formulate these effects for a detailed analysis to be productive. In the long term, measurements like those of the cosmic microwave background (Efstathiou et al. 1999), gravitational lensing statistics (e.g. Waga & Frieman 2000) and galaxy number counts (Newman & Davis 2000) will provide other constraints.

In order to address formally the question as to whether or not the Universe is accelerating, it will be necessary to have measurements which probe the second time derivative of $R(t)$. However other, less

direct, measurements like the ones listed in the previous paragraph and the supernova measurements will provide powerful hints if the association between the luminosity distance–redshift relation and the sign of q_0 really does turn out to be only weakly model-dependant.

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