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DISTANCE OF A BLOCH FUNCTION TO THE LITTLE BLOCH SPACE

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Motivated by a formula of P. Jones that gives the distance of a Bloch function to BMOA, the space of bounded mean oscillations, we obtain several formulas for the distance of a Bloch function to the little Bloch space, $\mathcal{B}_0$. Immediate consequences are equivalent expressions for functions in $\mathcal{B}_0$. We also give several examples of distances of specific functions to $\mathcal{B}_0$. We comment on connections between distance to $\mathcal{B}_0$ and the essential norm of some composition operators on the Bloch space, $\mathcal{B}$. Finally we show that the distance formulas in $\mathcal{B}$ have Bloch type spaces analogues.

1. INTRODUCTION

Let U denote the open unit disk and ∂U the unit circle in the complex plane. The *Bloch space* $\mathcal{B}$ of U is the space of holomorphic functions f on U such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in U}\left(1 - |z|^2\right)\left|f'(z)\right| < \infty .$$

It is easy to see that $\|f\|_{\mathcal{B}} = \left|f(0)\right| + \|f\|_{\mathcal{B}}$ defines a norm that makes $\mathcal{B}$ a Banach space that is invariant under Möbius transformations and in fact for all $f \in \mathcal{B}$

$$\|f \circ \omega\, \varphi_a\|_{\mathcal{B}} = \|f\|_{\mathcal{B}} ,$$

where $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$, $a \in U$ and $\omega \in \partial U$.

The *little Bloch space* $\mathcal{B}_0$ of U is the closed subspace of $\mathcal{B}$ consisting of functions f with

$$\lim_{|z|\to 1}\left(1 - |z|^2\right)\left|f'(z)\right| = 0 .$$

Examples of functions in $\mathcal{B}$ include all bounded holomorphic functions on U; but $\mathcal{B}$ contains unbounded functions ($\log(1 - z) \in \mathcal{B}$). Other examples include certain lacunary series.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n},$$

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where (λ_n) is a sequence of integers satisfying

$$\frac{\lambda_{n+1}}{\lambda_n} \geqslant \lambda > 1,$$

λ is a constant and $n \geqslant 1$. Then $f \in \mathcal{B}$ if and only if a_n is a bounded sequence and $f \in \mathcal{B}_0$ if and only if $a_n \to 0$, as $n \to \infty$. See [**1**, **7**, **12**] for more information on $\mathcal{B}$.

The motivation for this paper is a formula of Jones, (see [**1**, Theorem 9] and [**4**, p. 503] for a proof), that gives the distance of a Bloch function to BMOA the space of bounded mean oscillations.

We obtain the following formulas for the distance of a Bloch function to $\mathcal{B}_0$:

Theorem 3.5. *For $f \in \mathcal{B}$, $p \geqslant 2$,*

$$\inf A_p(f) \leqslant \operatorname{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leqslant 6 \inf A_p(f)$$

and

$$\inf A(f) \leqslant \operatorname{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leqslant 6 \inf A(f).$$

See (8), (9) *below for the definitions of the sets $A_p(f)$, $A(f)$ respectively.*

We obtain as a corollary equivalent expressions for functions in $\mathcal{B}_0$:

Corollary 3.6. *For $f \in \mathcal{B}$, the following are equivalent:*

1. *$f \in \mathcal{B}_0$.*
2. *For all $\varepsilon > 0$, $\Omega_\varepsilon(f)$ is a compact subset of U.*
3. *For some $p \geqslant 2$ and all $\varepsilon > 0$, $\int_{\Omega_\varepsilon(f)} 1/(1-|w|^2)^p \, dA(w) < \infty$.*
4. *For any $p \geqslant 2$ and $\varepsilon > 0$, there is a constant c so that*

$$\int_{\Omega_\varepsilon(f)} |g'(z)|^p \, dA(z) \leqslant c \, \|g\|_{\mathcal{B}}^p, \qquad \text{for all } g \in \mathcal{B}.$$

See (1) *below for the definition of the set $\Omega_\varepsilon(f)$.*

We also show that

Theorem 3.9. *For $f \in \mathcal{B}$,*

$$\limsup_{|z|\to 1} |f'(z)| \left(1-|z|^2\right) \leqslant \operatorname{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leqslant 2 \limsup_{|z|\to 1} |f'(z)| \left(1-|z|^2\right).$$

In Section 4 we give several examples of distances of specific functions to $\mathcal{B}_0$ for example,

$$\operatorname{dist}_{\mathcal{B}}\left(\log(1-z), \mathcal{B}_0\right) = 2 = \left\|\log(1-z)\right\|_{\mathcal{B}}$$

and

$$\operatorname{dist}_{\mathcal{B}}(e^{(z+1)/(z-1)}, \mathcal{B}_0) = \frac{2}{e} = \|e^{(z+1)/(z-1)}\|_{\mathcal{B}}.$$

Let const. denote a positive and finite constant which may change from one occurrence to the next but will not depend on the functions involved.

In Section 5 we make a connection between distance to $\mathcal{B}_0$ and the essential norm of some composition operators C_ϕ on $\mathcal{B}$. We show the following:

Corollary 5.1. *Let ϕ be a univalent self-map of U so that $\phi(U)$ has pointwise order of contact 1 with ∂U at the pont 1. Then,*

$$\frac{1}{4}\,\mathrm{dist}_{\mathcal{B}}\big(\log(1-\phi),\mathcal{B}_0\big) \leqslant \|C_\phi\|_e \leqslant \text{const. } \mathrm{dist}_{\mathcal{B}}\big(\log(1-\phi),\mathcal{B}_0\big)\,.$$

In the last section we define the Bloch type spaces $\mathcal{B}_\alpha$ and the little Bloch type spaces $\mathcal{B}_{\alpha,0}$ and we show that the distance formulas in $\mathcal{B}$ have $\mathcal{B}_\alpha$ analogues. The main results in the section are Theorem 6.1 and Corollary 6.4. Finally, we obtain as corollaries equivalent expressions for functions in $\mathcal{B}_{\alpha,0}$ (see Corollary 6.3).

2. Preliminaries

For $f \in \mathcal{B}$ [**12**, Lemma 4.2.8] gives

$$f(z) = f(0) + f'(0)z + \int_U \frac{(1-|w|^2)\,f'(w)}{\overline{w}\,(1-\overline{w}z)^2}\,dA(w), \qquad z \in U\,,$$

where $dA(w)$ is the normalised area measure on U. For any $\varepsilon > 0$ let $\Omega_\varepsilon(f)$ be

$$\Omega_\varepsilon(f) = \Big\{z \in U : \big(1-|z|^2\big)\big|f'(z)\big| \geqslant \varepsilon\Big\}. \tag{1}$$

Then write

$$\begin{aligned} f(z) &= f(0) + f'(0)z + \int_{\Omega_\varepsilon(f)} \frac{(1-|w|^2)\,f'(w)}{\overline{w}\,(1-\overline{w}z)^2}\,dA(w) \\ &\qquad + \int_{U\setminus\Omega_\varepsilon(f)} \frac{(1-|w|^2)\,f'(w)}{\overline{w}\,(1-\overline{w}z)^2}\,dA(w) \\ &= f(0) + f'(0)z + f_1(z) + f_2(z). \end{aligned} \tag{2}$$

The result of Lemma A below is part of the proof of Jones' Theorem that Ghatage and Zheng give in [**4**, p. 512] but we include it for completeness.

Lemma A. *Given $f \in \mathcal{B}$ and $\varepsilon > 0$, then $f_2 \in \mathcal{B}$ and*

$$\big\|f_2 - f_2(0) - f_2'(0)z\big\|_{\mathcal{B}} \leqslant 6\varepsilon.$$

Proof: Since f_2 is holomorphic on U,

$$f_2'(z) - f_2'(0) = z\int_0^1 f_2''(tz)\,dt$$

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for all $z \in U$. Thus,

$$
\begin{aligned}
\left(1-|z|^2\right)\left|f_2'(z)-f_2'(0)\right| &\leqslant \left(1-|z|^2\right)|z| \int_0^1 \frac{1}{(1-|z|^2 t^2)^2}\, dt \sup_{w\in U}\left(1-|w|^2\right)^2\left|f_2''(w)\right| \\
&\leqslant \left(1-|z|^2\right)|z| \int_0^1 \frac{1}{(1-|z|^2 t)^2}\, dt \sup_{w\in U}\left(1-|w|^2\right)^2\left|f_2''(w)\right| \\
&= |z| \sup_{w\in U}\left(1-|w|^2\right)^2\left|f_2''(w)\right| \\
&\leqslant \sup_{w\in U}\left(1-|w|^2\right)^2\left|f_2''(w)\right|. \qquad (3)
\end{aligned}
$$

Now for each $w \in U$,

$$
\begin{aligned}
\left(1-|w|^2\right)^2\left|f_2''(w)\right| &= \left(1-|w|^2\right)^2\left|\int_{U\setminus\Omega_\varepsilon(f)} \frac{6\overline{u}\left(1-|u|^2\right)f'(u)}{(1-w\overline{u})^4}\, dA(u)\right| \\
&\leqslant 6\left(1-|w|^2\right)^2 \int_{U\setminus\Omega_\varepsilon(f)} \frac{\left(1-|u|^2\right)|f'(u)|}{|1-w\overline{u}|^4}\, dA(u) \\
&\leqslant 6\varepsilon\left(1-|w|^2\right)^2 \int_U \frac{1}{|1-w\overline{u}|^4}\, dA(u) \\
&= 6\varepsilon. \qquad (4)
\end{aligned}
$$

Therefore by (3), (4),

$$\left\|f_2-f_2(0)-f_2'(0)z\right\|_B = \sup_{z\in U}\left(1-|z|^2\right)\left|f_2'(z)-f_2'(0)\right| \leqslant 6\varepsilon. \qquad \square$$

NOTE. Given $f, g \in \mathcal{B}$ and $z \in U$

$$\left(1-|z|^2\right)|f'(z)| \leqslant \|f-g\|_{\mathcal{B}} + \left(1-|z|^2\right)|g'(z)|. \qquad (5)$$

The result of Lemma B below is part of the proof of Theorem 3 in [**4**, p. 512] but we include it for completeness.

LEMMA B. *If $f \in \mathcal{B}_0$ then $\Omega_\varepsilon(f)$ is a compact subset of U for all $\varepsilon > 0$.*

PROOF: Given $f \in \mathcal{B}_0$ and $\varepsilon > 0$, since $\mathcal{B}_0$ is the closure in $\mathcal{B}$ of the polynomials ([**13**, p. 84]), choose a polynomial g so that $\|f-g\|_{\mathcal{B}} < \varepsilon/2$. Then using (5) we obtain

$$\Omega_\varepsilon(f) \subseteq \Omega_{\varepsilon/2}(g) \qquad (6)$$

We shall show that

$$\Omega_{\varepsilon/2}(g) \subseteq D_\varepsilon = \left\{z \in U : \operatorname{dist}(z, \partial U)\, \|g'\|_\infty \geqslant \frac{\varepsilon}{4}\right\}. \qquad (7)$$

Let $z \in \Omega_{\varepsilon/2}(g)$; then

$$\left(1-|z|^2\right)\|g'\|_\infty \geqslant \left(1-|z|^2\right)|g'(z)| \geqslant \frac{\varepsilon}{2}.$$

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So,

$$(1-|z|)\|g'\|_\infty \geqslant \frac{\varepsilon}{4}$$

and (7) follows. The set D_ε is a compact set. Indeed, if $\|g'\|_\infty = 0$ then g is a constant function and $\Omega_{\varepsilon/2}(g) = D_\varepsilon = \emptyset$; and if $\|g'\|_\infty \neq 0$ then

$$D_\varepsilon = \Big\{ z \in U : \operatorname{dist}(z, \partial U) \geqslant \varepsilon / \big(4\, \|g'\|_\infty\big) \Big\},$$

which is clearly a compact subset of U. Therefore by (6) and (7) $\Omega_\varepsilon(f)$ is a compact subset of U as well. ∎

3. Distance formulas

In this section given $f \in \mathcal{B}$, f_1 and f_2 refers to the functions in (2). The distance in the Bloch norm of f to a subset of $\mathcal{B}$, X, is denoted by $\operatorname{dist}_{\mathcal{B}}(f, X)$.

Lemma 3.1. *If $f \in \mathcal{B}$ and there exists a function $g \in \mathcal{B}_0$ so that $\|f-g\|_{\mathcal{B}} \leqslant \alpha$ for some $\alpha > 0$, then $\Omega_\varepsilon(f)$ is a compact set for all $\varepsilon > \alpha$.*

Proof: Fix $\alpha > 0$, let $\varepsilon > \alpha$ then using (5) we obtain $\Omega_\varepsilon(f) \subseteq \Omega_{\varepsilon-\alpha}(g)$. By Lemma B $\Omega_{\varepsilon-\alpha}(g)$ is a compact subset of U therefore so is $\Omega_\varepsilon(f)$. ∎

For $f \in \mathcal{B}$ and $p > 0$, define $A_p(f)$ by

$$A_p(f) = \left\{ \varepsilon > 0 : \frac{\chi_{\Omega_\varepsilon(f)}(z)}{(1-|z|^2)^p}\, dA(z) \text{ is a finite measure} \right\}. \tag{8}$$

And let $A(f)$ be

$$A(f) = \big\{ \varepsilon > 0 : \ \Omega_\varepsilon(f) \text{ is a compact subset of U} \big\}. \tag{9}$$

Proposition 3.2. *For $f \in \mathcal{B}$ and any $p > 0$,*

$$\inf A_p(f) \leqslant \inf\ A(f) \leqslant \operatorname{dist}_{\mathcal{B}}(f, \mathcal{B}_0).$$

Proof: Suppose the right inequality is false; then there exist ε_1 and $\varepsilon_2 > 0$ so that

$$\operatorname{dist}_{\mathcal{B}}(f, \mathcal{B}_0) < \varepsilon_1 < \varepsilon_2 < \inf\ A(f).$$

Therefore there exists a function $g \in \mathcal{B}_0$ so that $\|f-g\|_{\mathcal{B}} < \varepsilon_1$ and $\Omega_{\varepsilon_2}(f)$ is not a compact set. But by Lemma 3.1 $\Omega_{\varepsilon_2}(f)$ must be a compact set, so we arrive at a contradiction. Thus the right inequality holds.

The left inequality follows since $A(f) \subseteq A_p(f)$, for all p. ∎

Proposition 3.3. *Let X be a subspace of $\mathcal{B}_0$ that contains $\{a+bz : \ a, b \in \mathbb{C}\}$ and $f \in \mathcal{B}$. If there is some $p > 0$ such that $f_1 \in X$ for all $\varepsilon \in A_p(f)$, then*

$$\inf A_p(f) \leqslant \operatorname{dist}_{\mathcal{B}}(f, X) \leqslant 6\ \inf A_p(f)\,.$$

PROOF: By Lemma A

$$\|f - f_1 - f(0) - f'(0)z - f_2(0) - f_2'(0)z\|_{\mathcal{B}} \leqslant 6\varepsilon. \tag{10}$$

Since X contains all linear functions, $f_1 \in X$ for all $\varepsilon \in A_p(f)$, (10) gives

$$\mathrm{dist}_{\mathcal{B}}(f, X) \leqslant 6 \inf A_p(f).$$

The left inequality follows from Proposition 3.2. ∎

REMARK. In the proposition above the right inequality holds for any subspace X of $\mathcal{B}$ that contains $\{a + bz : a, b \in \mathbf{C}\}$.

A function f holomorphic on U belongs to the *minimal Besov space* B_1 if and only if $\int_U |f''(z)|\, dA(z) < \infty$. B_1 is a subspace of $\mathcal{B}_0$, in fact B_1 is a subspace of $A(U)$ the Banach space of functions that are continuous on the closed unit disk and holomorphic on the open unit disk with the supremum norm. See **[2]** and **[12]** for more information on B_1.

PROPOSITION 3.4. *For $f \in \mathcal{B}$ and $p \geqslant 2$,*

$$\inf A_p(f) \leqslant \mathrm{dist}_{\mathcal{B}}(f, B_1) \leqslant 6 \inf A_p(f).$$

PROOF: Let $\varepsilon \in A_p(f)$. We shall show that $f_1 \in B_1$. Recall that

$$f_1(z) = \int_{\Omega_\varepsilon(f)} \frac{(1 - |w|^2)\, f'(w)}{\overline{w}\,(1 - \overline{w}z)^2}\, dA(w).$$

Then,

$$|f_1''(z)| \leqslant \int_{\Omega_\varepsilon(f)} \left| \frac{(1 - |w|^2)\, f'(w)}{\overline{w}\,(1 - \overline{w}z)^4}\ 6\overline{w}^2 \right| dA(w)$$

and

$$\begin{aligned}
\int_U |f_1''(z)|\, dA(z) &\leqslant 6 \int_U \int_{\Omega_\varepsilon(f)} \frac{(1 - |w|^2)\, |f'(w)|}{|1 - \overline{w}z|^4}\, dA(w) dA(z) \\
&= 6 \int_{\Omega_\varepsilon(f)} (1 - |w|^2)\, |f'(w)| \int_U \frac{1}{|1 - \overline{w}z|^4}\, dA(z)\, dA(w) \\
&\leqslant 6 \|f\|_{\mathcal{B}} \int_{\Omega_\varepsilon(f)} \frac{1}{(1 - |w|^2)^2}\, dA(w) \\
&\leqslant 6 \|f\|_{\mathcal{B}} \int_{\Omega_\varepsilon(f)} \frac{1}{(1 - |w|^2)^p}\, dA(w) < \infty, \qquad (11)
\end{aligned}$$

for all $p \geqslant 2$. Therefore (11) and Proposition 3.3 imply the result. ∎

The Besov space B_1 is a subspace of the little Bloch space $\mathcal{B}_0$ that contains all polynomials. Thus, the closure of B_1 in the Bloch norm is $\mathcal{B}_0$, since $\mathcal{B}_0$ is the closure of all polynomials in the Bloch norm. The next theorem follows from Proposition 3.2 and Proposition 3.4. Recall the definitions of $A_p(f)$ and $A(f)$ in (8), (9) respectively.

THEOREM 3.5. *For $f \in \mathcal{B}$, $p \geqslant 2$,*

$$\inf A_p(f) \leqslant \operatorname{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leqslant 6 \inf A_p(f)$$

and

$$\inf A(f) \leqslant \operatorname{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leqslant 6 \inf A(f).$$

REMARK. The best bounds for $\operatorname{dist}_{\mathcal{B}}(f, \mathcal{B}_0)$ are $\inf A(f)$ from below, $6 \inf A_2(f)$ from above, as $A(f) \subseteq A_p(f) \subseteq A_q(f) \subseteq A_2(f)$ for all p, q with $2 \leqslant p \leqslant q$.

The following theorem of Arazy, Fisher and Peetre is from [**2**, p. 132]. We give a different proof of (2) implies (1).

THEOREM C. *Let μ be a positive measure on U and let $0 < p < \infty$. Then,*

(1) $$\int_U \frac{d\mu(z)}{(1-|z|^2)^p} < \infty$$

if and only if there is a constant c with

(2) $$\int_U |f'(z)|^p \, d\mu(z) \leqslant c \, \|f\|_{\mathcal{B}}^p, \text{ for all } f \in \mathcal{B}.$$

PROOF: For any $f \in \mathcal{B}$

$$\int_U |f'(z)|^p \, d\mu(z) = \int_U \frac{|f'(z)|^p (1-|z|^2)^p}{(1-|z|^2)^p} \, d\mu(z)$$
$$\leqslant \|f\|_{\mathcal{B}}^p \int_U \frac{1}{(1-|z|^2)^p} \, d\mu(z).$$

This proves the easy implication, (1) implies (2). Ramey and Ullrich proved in [8] that there exists functions $f, g \in \mathcal{B}$ so that

$$|f'(z)| + |g'(z)| \geqslant \frac{\text{const.}}{(1-|z|^2)}. \tag{12}$$

Therefore it is easy to see that

$$\text{const.} \left(|f'(z)|^p + |g'(z)|^p\right) \geqslant \frac{1}{(1-|z|^2)^p}.$$

Integrating the above with respect to $d\mu(z)$ shows that (2) implies (1). ∎

The following is an immediate consequence of Theorem 3.5 and Theorem C.

COROLLARY 3.6. *For $f \in \mathcal{B}$, the following are equivalent:*

1. $f \in \mathcal{B}_0$.
2. *For all $\varepsilon > 0$ $\Omega_\varepsilon(f)$ is a compact subset of U.*
3. *For some $p \geqslant 2$ and all $\varepsilon > 0$,*

$$\int_{\Omega_\varepsilon(f)} \frac{1}{(1-|w|^2)^p} \, dA(w) < \infty.$$

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4. *For any $p \geqslant 2$ and $\varepsilon > 0$, there is a constant c so that*

$$\int_{\Omega_\varepsilon(f)} |g'(z)|^p \, dA(z) \leqslant c \, \|g\|_{\mathcal{B}}^p, \quad \text{for all } g \in \mathcal{B}.$$

Let $D(0,\alpha)$ denote the disk centred at 0 of radius α. A *nontangential approach region* Ω_α $(0 < \alpha < 1)$ in U with vertex $\zeta \in \partial U$ is the convex hull of $D(0,\alpha) \cup \{\zeta\}$ minus the point ζ.

For any region G in the complex plane let ∂G denote the boundary of the region. For an open subset G of U with $\zeta \in \partial G \cap \partial U$ we say that it has *pointwise order of contact (at most) b $(b > 0)$* with ∂U at ζ if

$$\frac{1 - |z|}{|\zeta - z|^b} \geqslant \text{const.} \tag{13}$$

as z approaches ζ within G. So if ϕ is a holomorphic self-map of U such that $\phi(U) = \Omega_\alpha$ $(0 < \alpha < 1)$ with vertex ζ then $\phi(U)$ has pointwise order of contact 1 with ∂U at ζ; if $\phi(U)$ is a disk inside U whose boundary makes tangential contact with ∂U at the point ζ then $\phi(U)$ has pointwise order of contact 2 with ∂U at the point ζ.

In [**3**, p. 2191] Bourdon, Cima, and Matheson introduced the notion of mean order of contact. An open subset G of U has *mean order of contact (at most) b $(b > 0)$* with ∂U if

$$\int_0^{2\pi} \chi_G(re^{i\theta}) \, d\theta = O(1 - r)^{1/b} \tag{14}$$

as $r \to 1^-$. The integral on the left side of (14) represents the angular measure of G intersected with the circle $\{z \in U : |z| = r\}$.

Recall the definition of $\Omega_\varepsilon(f)$ in (1) for any $f \in \mathcal{B}$.

Proposition 3.7. *Let $f \in \mathcal{B}$ so that for all $\varepsilon > 0$ the mean order of contact of $\Omega_\varepsilon(f)$ with ∂U is $\alpha_\varepsilon < 1$. Then $f \in \mathcal{B}_0$.*

Proof: For a fixed $\varepsilon > 0$, (14) gives

$$\begin{aligned}\int_{\Omega_\varepsilon(f)} \frac{1}{(1 - |w|^2)^2} \, dA(w) &= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \chi_{\Omega_\varepsilon(f)}(re^{i\theta}) d\theta \, \frac{r}{(1 - r^2)^2} \, dr \\ &\leqslant \text{const.} \int_0^1 \frac{(1 - r)^{1/\alpha_\varepsilon}}{(1 - r)^2} \, dr < \infty\end{aligned}$$

since $(1/\alpha_\varepsilon) - 2 > -1$. Thus by Corollary 3.6 $f \in \mathcal{B}_0$. ∎

Proposition 3.8. *Let $f \in \mathcal{B}$ be so that for all $\varepsilon > 0$ the mean order of contact of $\Omega_\varepsilon(f)$ with ∂U is 1. Then for all $\varepsilon > 0$ and all $\beta > 0$,*

$$\int_{\Omega_\varepsilon(f)} \frac{1}{(1 - |w|^2)^{2-\beta}} \, dA(w) < \infty.$$

We omit the proof as it is similar to the one of Proposition 3.7.

REMARK. For the function $f(z) = \log(1-z)$ the mean order of contact of $\Omega_\varepsilon(f)$ with ∂U is 1 for all $\varepsilon > 0$ (see the remark after example 4.1 for a proof). This shows that the conclusion of the above proposition is valid for the function $\log(1-z)$. Therefore $p \geqslant 2$ in Theorem 3.5 is best possible since $\inf A_p(f) = 0$ for all $p < 2$. Similarly in Corollary 3.6 $p \geqslant 2$ is best possible. If $g \in \mathcal{B}_0$ then by condition (2) of Corollary 3.6

$$\int_{\Omega_\varepsilon(g)} \frac{1}{(1-|w|^2)^p}\, dA(w) < \infty$$

for all p. But the converse is valid only for $p \geqslant 2$.

It is well known that the Bloch space $\mathcal{B}$ can be thought of as the area version of the space of bounded mean oscillations $BMOA$. Thus, motivated by a formula for the distance of a BMOA function to VMOA, the space of vanishing mean oscillations, given by Stegenga and Stephenson in [**10**] we prove the following theorem. In the proof we use a modified version of an argument of Montes-Rodriguez, given in [**6**, p. 346].

THEOREM 3.9. *For $f \in \mathcal{B}$,*

$$\limsup_{|z|\to 1} |f'(z)|\,(1-|z|^2) \leqslant \operatorname{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leqslant 2 \limsup_{|z|\to 1} |f'(z)|\,(1-|z|^2)\,.$$

PROOF: For a given $f \in \mathcal{B}$ and $g \in \mathcal{B}_0$ (5) gives

$$\limsup_{|z|\to 1} (1-|z|^2)\,|f'(z)| \leqslant \|f-g\|_{\mathcal{B}}\,,$$

from which the left inequality follows.

For the right inequality fix $f \in \mathcal{B}$, $M > 0$, $0 < \rho < 1$ so that for all $z \in U$ with $|z| \geqslant 1-\rho$, $|f'(z)|\,(1-|z|^2) \leqslant M$. We shall show that $\operatorname{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leqslant 2M$. Also fix for the moment r with $1 > r > 1-\rho$, let $f_r(z) = f(rz)$ for $z \in U$. Then,

$$\begin{aligned} \|f-f_r\|_{\mathcal{B}} &= \sup_{z\in U} |(f-f_r)'(z)|\,(1-|z|^2) \\ &\leqslant \sup_{|z|\leqslant(1-\rho)/r} |(f-f_r)'(z)|\,(1-|z|^2) + \sup_{|z|>(1-\rho)/r} |(f-f_r)'(z)|\,(1-|z|^2) \\ &= I + II\,. \end{aligned} \tag{15}$$

Then

$$\begin{aligned} II &\leqslant \sup_{|z|>(1-\rho)/r} |f'(z)|\,(1-|z|^2) + \sup_{|z|>(1-\rho)/r} |f_r'(z)|\,(1-|z|^2) \\ &\leqslant \sup_{|z|>1-\rho} |f'(z)|\,(1-|z|^2) + \sup_{|z|>1-\rho} |f'(z)|\,(1-|z|^2) \leqslant 2M. \end{aligned} \tag{16}$$

Also

$$\begin{aligned} I &= \sup_{|z|\leqslant(1-\rho)/r} |(f-f_r)'(z)|\,(1-|z|^2) \\ &\leqslant \sup_{|z|\leqslant(1-\rho)/r} |f'(z)-f'(rz)|\,(1-|z|^2) + (1-r) \sup_{|z|\leqslant(1-\rho)/r} |f'(rz)|\,(1-|z|^2) \\ &\leqslant A + (1-r)\|f\|_{\mathcal{B}}. \end{aligned} \tag{17}$$

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By taking the line integral of f'' from rz to z we get

$$
\begin{aligned}
A &= \sup_{|z|\leqslant(1-\rho)/r} \left|f'(z)-f'(rz)\right|\left(1-|z|^2\right) \\
&\leqslant (1-r) \sup_{|z|\leqslant(1-\rho)/r} \left|f''\big(\xi(z)\big)\right|\,|z|\,\left(1-|z|^2\right), \qquad (18)
\end{aligned}
$$

where $\xi(z)$ is a point in the closed disk of radius$(1-\rho)/r$. Using the Maximum Modulus Theorem and Cauchy's Estimates for f' on the circle centred at $\xi(z)$ with radius $R=\big(1-(1-\rho)/r\big)/2$ in (18), we obtain

$$
\begin{aligned}
A &\leqslant \frac{1-r}{R} \max_{|z|=R+(1-\rho)/r} \left|f'(z)\right| \sup_{|z|\leqslant(1-\rho)/r} |z|\left(1-|z|^2\right) \\
&\leqslant \frac{(1-r)}{R}\|f\|_{\mathcal{B}}\, \frac{(1-\rho)/r}{1-((1-\rho)/r+R)^2}, \qquad (19)
\end{aligned}
$$

since

$$
\max_{|z|=(1-\rho)/r+R} \left|f'(z)\right| \leqslant \|f\|_{\mathcal{B}}\, \frac{1}{1-((1-\rho)/(r)+R)^2}.
$$

Therefore by (15), (16), (17),

$$
\begin{aligned}
\operatorname{dist}_{\mathcal{B}}(f,\mathcal{B}_0) &\leqslant \|f-f_r\|_{\mathcal{B}} \\
&\leqslant A+(1-r)\|f\|_{\mathcal{B}}+2M. \qquad (20)
\end{aligned}
$$

Notice that

$$
\lim_{r\to 1} \frac{1}{R}\, \frac{(1-\rho)/r}{1-((1-\rho)/(r)+R)^2} = \frac{8\,(1-\rho)}{\rho^2\,(4-\rho)} \qquad (21)
$$

thus letting $r\to 1$ in (19) and in (20), we obtain

$$
\operatorname{dist}_{\mathcal{B}}(f,\mathcal{B}_0) \leqslant 2M,
$$

which proves the right inequality. ∎

Recall the Koebe Distortion Theorem (see for example [**9**, p. 156]) which asserts that if ψ is a univalent function on U then for any $z\in U$,

$$
\frac{1}{4}\left|\psi'(z)\right|\left(1-|z|^2\right) \leqslant \operatorname{dist}\big(\psi(z),\partial\psi(U)\big) \leqslant \left|\psi'(z)\right|\left(1-|z|^2\right), \qquad (22)
$$

where $\operatorname{dist}\big(\psi(z),\partial\psi(U)\big)$ is the Euclidean distance from $\psi(z)$ to $\partial\psi(U)$. By (22), an immediate corollary of Theorem 3.9 is as follows.

Corollary 3.10. *For any univalent function $f\in\mathcal{B}$,*

$$
\limsup_{|z|\to 1} \operatorname{dist}\big(f(z),\partial f(U)\big) \leqslant \operatorname{dist}_{\mathcal{B}}(f,\mathcal{B}_0) \leqslant 8\,\limsup_{|z|\to 1} \operatorname{dist}\big(f(z),\partial f(U)\big).
$$

4. Examples

It is easy to see that for any constant c and any $f \in \mathcal{B}$

$$\mathrm{dist}_{\mathcal{B}}(f + c, \mathcal{B}_0) = \mathrm{dist}_{\mathcal{B}}(f, \mathcal{B}_0) = \inf_{g \in \mathcal{B}_0} \|f - g\|_{\mathcal{B}} .$$

Also because of the Möbius invariance of $\mathcal{B}$

$$\mathrm{dist}_{\mathcal{B}}(f \circ \omega\, \varphi_a, \mathcal{B}_0) = \mathrm{dist}_{\mathcal{B}}(f, \mathcal{B}_0) , \tag{23}$$

for all $a \in U$, $\omega \in \partial U$.

Example 4.1. The function $f(z) = \log(1 - z)$ belongs to the Bloch space but not to the little Bloch space. We shall show that

$$\mathrm{dist}\big(\log(1 - z), \mathcal{B}_0\big) = 2 = \big\|\log(1 - z)\big\|_{\mathcal{B}} . \tag{24}$$

First

$$\frac{1 - |z|^2}{|1 - z|} \leqslant \frac{1 - |z|^2}{1 - |z|} \leqslant 2$$

for all $z \in U$, therefore $\big\|\log(1 - z)\big\|_{\mathcal{B}} \leqslant 2$; on the other hand

$$\big\|\log(1 - z)\big\|_{\mathcal{B}} \geqslant \frac{1 - (1 - (1/n))^2}{|1 - (1 - (1/n))|}$$

or $\big\|\log(1 - z)\big\|_{\mathcal{B}} \geqslant 2 - (1/n)$ for all nonnegative integers n and so $\big\|\log(1 - z)\big\|_{\mathcal{B}} = 2$. By (1) the set

$$\Omega_\varepsilon(f) = \left\{ z \in U : \ \frac{1 - |z|^2}{|1 - z|} \geqslant \varepsilon \right\} \tag{25}$$

behaves like a nontangential approach region in U that touches the unit circle only at the point 1; so $\Omega_\varepsilon(f)$ is never a compact subset of U unless it is the empty set, thus trivially compact. If $\Omega_\varepsilon(f) \neq \emptyset$ then $\varepsilon < 2$ since for any $z_0 \in \Omega_\varepsilon(f)$,

$$\varepsilon \leqslant \frac{1 - |z_0|^2}{|1 - z_0|} \leqslant 1 + |z_0| < 2 ;$$

on the other hand if $\varepsilon < 2$ then $\Omega_\varepsilon(f) \neq \emptyset$, since it is easy to see that $z_n = 1 - (1/n) \in \Omega_\varepsilon(f)$ for all n large enough. Therefore $\Omega_\varepsilon(f) = \emptyset$ if and only if $\varepsilon \geqslant 2$ if and only if $\Omega_\varepsilon(f)$ is a compact subset of U. Thus

$$\inf A\big(\log(1 - z)\big) = 2 = \big\|\log(1 - z)\big\|_{\mathcal{B}} . \tag{26}$$

Since

$$\mathrm{dist}_{\mathcal{B}}\big(\log(1 - z), \mathcal{B}_0\big) \leqslant \big\|\log(1 - z)\big\|_{\mathcal{B}}$$

and by Theorem 3.5

$$\inf A\big(\log(1-z)\big) \leqslant \operatorname{dist}_{\mathcal{B}}(\log(1-z), \mathcal{B}_0)\,,$$

using (26) we obtain (24) and by (23) for all $a \in U$, $\omega \in \partial U$

$$\operatorname{dist}_{\mathcal{B}}\left(\log\left(1-\omega\frac{a-z}{1-\overline{a}z}\right), \mathcal{B}_0\right) = 2\,.$$

Similarly to the proof of (24) one can show that

$$\operatorname{dist}_{\mathcal{B}}\big(\log(k'), \mathcal{B}_0\big) = 6 = \big\|\log(k')\big\|_{\mathcal{B}}\,,$$

where k is the Koebe function $k(z) = z/((1-z)^2)$ and that

$$\operatorname{dist}_{\mathcal{B}}\left(\log\left(\frac{z}{1-z^2}\right)', \mathcal{B}_0\right) = 4 = \left\|\log\left(\frac{z}{1-z^2}'\right)\right\|_{\mathcal{B}}\,.$$

REMARK. By (25) for the function $f(z) = \log(1-z)$ and each $\varepsilon > 0$, $\Omega_\varepsilon(f)$ has pointwise order of contact 1 with ∂U at the point 1. In [**3**, p. 2192] is shown that for domains whose boundary touches the unit circle at exactly one point ζ, if G has pointwise order of contact 1 with ∂U at ζ then it has mean order of contact 1 with ∂U. Therefore for all $\varepsilon > 0$ $\Omega_\varepsilon(f)$ has mean order of contact 1 with ∂U.

For the rest of this section let ϕ be a univalent self-map of the unit disk; then $\phi \in \mathcal{B}_0$ as for example $\phi(U)$ has a finite area and all such functions lie in $\mathcal{B}_0$. Below we shall describe the distance of $\log(1-\phi)$ to $\mathcal{B}_0$ for several ϕ. First notice that if $\phi(U) \subseteq U \backslash D(1,r)$, where $D(1,r)$ is the disk centred at 1 of radius r, then $\log(1-\phi) \in \mathcal{B}_0$.

EXAMPLE 4.2. Let ϕ be such that $\phi(U) = \Omega_\alpha$ a nontangential approach region in U with vertex 1, for some $0 < \alpha < 1$. We shall show that

$$\frac{1}{4}\big\|\log(1-\phi)\big\|_{\mathcal{B}} \leqslant \operatorname{dist}_{\mathcal{B}}\big(\log(1-\phi), \mathcal{B}_0\big) \leqslant \big\|\log(1-\phi)\big\|_{\mathcal{B}}\,. \tag{27}$$

Using the Schwarz–Pick Lemma it is easy to see that $\log(1-\phi) \in \mathcal{B}$. Then by (1) and (22) for all $\varepsilon > 0$ the set

$$\Omega_\varepsilon\big(\log(1-\phi)\big) = \left\{z \in U: \quad \frac{|\phi'(z)|\,(1-|z|^2)}{|1-\phi(z)|} \geqslant \varepsilon\right\}$$

contains the set

$$G = \left\{z \in U: \quad \frac{\operatorname{dist}(\phi(z), \partial\phi(U))}{|1-\phi(z)|} \geqslant \varepsilon\right\}.$$

Then $\phi(G)$ behaves like a nontangential approach region contained in $\phi(U)$. Therefore, unless $G = \emptyset$, it is not a compact subset of U. If $\Omega_\varepsilon\big(\log(1-\phi)\big)$ is a compact subset of U then so is G and therefore $G = \emptyset$ and for all $z \in U$ we have

$$\frac{\operatorname{dist}(\phi(z), \partial\phi(U))}{|1-\phi(z)|} < \varepsilon\,.$$

By (22) $\|\log(1-\phi)\|_{\mathcal{B}}/4 < \varepsilon$ so we obtain

$$\frac{1}{4}\,\|\log(1-\phi)\|_{\mathcal{B}} \leqslant \inf A\big(\log(1-\phi)\big)\,.$$

Thus Theorem 3.5 gives (27).

Next we give an example where equality holds in the right inequality of (27) for a function ϕ with $\phi(U)$ inside a nontangential approach region.

EXAMPLE 4.3. For $0<\alpha<1$ define

$$\phi_\alpha(z) = \frac{\sigma(z)^\alpha - 1}{\sigma(z)^\alpha + 1}$$

where

$$\sigma(z) = \frac{1+z}{1-z}, \quad z \in U\,.$$

The map ϕ_α is a holomorhic self-map of U whose image is a lens-shaped region thus it is called a *"lens map"* ([**9**, p. 27]). We shall show that

$$\inf A\big(\log(1-\phi_\alpha)\big) = \operatorname{dist}\big(\log(1-\phi_\alpha), \mathcal{B}_0\big) = \|\log(1-\phi_\alpha)\|_{\mathcal{B}} = 2\alpha\,. \tag{28}$$

By (1) for each $\varepsilon > 0$

$$\begin{aligned} \Omega_\varepsilon\big(\log(1-\phi_\alpha)\big) &= \left\{ z\in U: \ \frac{|\phi_\alpha'(z)|\,(1-|z|^2)}{|1-\phi_\alpha(z)|} \geqslant \varepsilon \right\} \\ &= \left\{ z\in U: \ \frac{2\,\alpha\,(1-|z|^2)}{|1-z^2|\,|1+((1-z)/(1+z))^\alpha|} \geqslant \varepsilon \right\}. \end{aligned} \tag{29}$$

Fix $\varepsilon < 2$, let $z_n = 1 - 1/n$; then by (29) we can easily see that $z_n \in \Omega_\varepsilon\big(\log(1-\phi_\alpha)\big)$ if and only if

$$\frac{2\,\alpha}{1+(1/(2n-1))^\alpha} \geqslant \varepsilon\,,$$

which is true for all n large enough thus $\Omega_\varepsilon\big(\log(1-\phi_\alpha)\big)$ is not a compact subset of U. This shows that

$$2\,\alpha \leqslant \inf A\big(\log(1-\phi_\alpha)\big)\,. \tag{30}$$

For each $0<\alpha<1$

$$\begin{aligned} \|\log(1-\phi_\alpha)\|_{\mathcal{B}} &= \sup_{z\in U} \frac{2\,\alpha\,(1-|z|^2)}{|1-z^2|\,|1+((1-z)/(1+z))^\alpha|} \\ &\leqslant 2\,\alpha\,\sup_{z\in U}\left|\frac{1}{1+((1-z)/(1+z))^\alpha}\right| \\ &\leqslant 2\,\alpha \end{aligned} \tag{31}$$

since

$$1+\big((1-z)/(1+z)\big)^\alpha$$

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maps U onto the sector

$$\left\{w : \left|\arg(w-1)\right| < (\alpha\pi)/2\right\}.$$

Thus Theorem 3.5, (30) and (31) give (28).

REMARK. A formula like in (27) is not true for all functions of the form $\log(1-\phi)$ where $\phi(U)$ has mean order of contact b $(0 < b \leqslant 1)$ with ∂U, as there are such ϕ with $\log(1-\phi) \in \mathcal{B}_0$. Below we give examples of this.

(i) For each nonnegative integer n let

$$\Delta_n = \left\{z \in U : \ \left|\arg(1-z)\right| < \theta_n, \quad \rho_n < |z| \leqslant \rho_{n+1}\right\},$$

where $\rho_n = 1 - (1/2^n)$, $0 < \theta_n < \pi/7$ such that $\Sigma_{n=1}^{\infty}\theta_n < \infty$. Let $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$. It follows that Δ, as it clearly has pointwise order of contact 1 with ∂U at the point 1, has mean order of contact 1 (see the remark after example 4.1). We would like to give a direct proof of this for our domain Δ using the definition of mean order of contact; since

$$\int_0^{2\pi} \chi_\Delta(re^{i\theta})\, d\theta \leqslant \Sigma_{n=1}^{\infty} \int_0^{2\pi} \chi_{\Delta_n}(re^{i\theta})\, d\theta \leqslant \Sigma_{n=1}^{\infty} 2\psi\, r, \tag{32}$$

where $\psi = \arg(z)$, z is a point, on the first quadrant, on the boundary of the arc $\Delta \cap \left\{z \in U : \ |z| = r\right\}$. By the Law of Cosines and the Law of Sines

$$\tan\psi = \frac{1-r}{r}\tan\theta_n$$

therefore (32) gives

$$\begin{aligned}\int_0^{2\pi} \chi_\Delta(re^{i\theta})\, d\theta &\leqslant \Sigma_{n=1}^{\infty} 2r\tan^{-1}\Big(\frac{1-r}{r}\ \tan\theta_n\Big) \\ &\leqslant \ \text{const.}\ \Sigma_{n=1}^{\infty}\theta_n\, (1-r) \leqslant \ \text{const.}\ (1-r)\,.\end{aligned}$$

Thus (14) shows that Δ has mean order of contact with ∂U 1. Let ϕ be the Riemann map from U onto Δ. It is easy to see, using (22), that $\log(1-\phi) \in \mathcal{B}_0$, thus (27) is not valid for such a ϕ.

(ii) If ϕ is such that $\phi(U)$ has mean order of contact b $(0 < b < 1)$ with ∂U and $\phi(U)$ lies inside a polygon inscribed in the unit circle then $\log(1-\phi) \in \mathcal{B}_0$; this follows from Corollary 5.2 in [**3**] and Theorem 5.3 in [**5**]. Thus (27) is not valid for such a ϕ.

EXAMPLE 4.4. If $\phi(U)$ is a polygon inscribed in the unit circle and one of the vertices of the polygon is the point 1 then it is easy to see that (27) is valid, as in the proof of the formula in the case where $\phi(U) = \Omega_\alpha$ $(0 < \alpha < 1)$.

EXAMPLE 4.5. If $\phi(U)$ is a disk tangent to the unit circle at the point 1 then it is easy to see that ϕ has the form $\psi \circ M$ where $\psi(z) = \lambda\, z + (1-\lambda)$ $(0 < \lambda < 1)$ and M is a

Möbius transformation. Using the Möbius invariance of the Bloch seminorm and (24) we obtain

$$\begin{aligned}\operatorname{dist}_{\mathcal{B}}(\log(1-\phi),\mathcal{B}_0) &= \operatorname{dist}_{\mathcal{B}}(\log(1-\psi),\mathcal{B}_0)\\ &= \operatorname{dist}_{\mathcal{B}}(\log\lambda(1-z),\mathcal{B}_0)\\ &= \operatorname{dist}_{\mathcal{B}}(\log(1-z),\mathcal{B}_0) = 2\,.\end{aligned}$$

EXAMPLE 4.6. The function $S(z) = e^{(z+1)/(z-1)}$ belongs to the Bloch space but not to the little Bloch space; actually $S \in H^\infty \setminus \mathcal{B}_0$. We shall show that

$$\operatorname{dist}_{\mathcal{B}}(S,\mathcal{B}_0) = \frac{2}{e} = \|S\|_{\mathcal{B}}\,. \tag{33}$$

For $\alpha > 0$, the function S maps the circle

$$C_\alpha = \left\{z \in U : \frac{1-|z|^2}{|1-z|^2} = \alpha\right\}$$

(centre $\alpha/(1+\alpha)$, radius $1/(1+\alpha)$ internally tangent to the unit circle at the point 1) to the circle $\{\varsigma \in U :\ |\varsigma| = e^{-\alpha}\}$. By (1)

$$\begin{aligned}\Omega_\varepsilon(S) &= \left\{z \in U :\ \left(1-|z|^2\right)|S'(z)| \geqslant \varepsilon\right\}\\ &= \left\{z \in U :\ \frac{2|S(z)|\,(1-|z|^2)}{|1-z|^2} \geqslant \varepsilon\right\}.\end{aligned} \tag{34}$$

On C_α the left hand side of the inequality in (34) equals $2\alpha\, e^{-\alpha}$. Thus,

$$\Omega_\varepsilon(S) = \left\{z \in U : 2\alpha e^{-\alpha} \geqslant \varepsilon,\quad \alpha = \frac{1-|z|^2}{|1-z|^2}\right\}.$$

Also,

$$\|S\|_{\mathcal{B}} = \sup_{\alpha>0} 2e^{-\alpha}\alpha = \frac{2}{e}\,. \tag{35}$$

The set $\Omega_\varepsilon(S)$ is either the empty set, when $\varepsilon > 2/e$ or a circle internally tangent to the unit circle at $z = 1$ when $\varepsilon = 2/e$, or it is the area between two circles both internally tangent to the unit circle at $z = 1$ when $\varepsilon < 2/e$. Therefore $\Omega_\varepsilon(f)$ is never a compact subset of U unless it is the empty set thus trivially compact. Thus, Theorem 3.5 and (35) show that

$$\begin{aligned}\inf A(S) = \|S\|_{\mathcal{B}} = \frac{2}{e} &\leqslant \operatorname{dist}_{\mathcal{B}}(S,\mathcal{B}_0)\\ &= \operatorname{dist}_{\mathcal{B}}(S-S(0),\mathcal{B}_0)\\ &\leqslant \left\|S-S(0)\right\|_{\mathcal{B}} = \|S\|_{\mathcal{B}} = \frac{2}{e}\,.\end{aligned} \tag{36}$$

Therefore (33) follows.

5. Distance to $\mathcal{B}_0$ and the essential norm of a composition operator

If ϕ is a holomorphic self-map of U, then the *composition operator* C_ϕ

$$C_\phi f = f \circ \phi$$

maps holomorphic functions f to holomorphic functions. It is a bounded operator on $\mathcal{B}$, and if $\phi \in \mathcal{B}_0$ it is bounded on $\mathcal{B}_0$ as well ([**2**, Theorem 12]). The essential norm of C_ϕ, $\|C_\phi\|_e$, is the distance in the operator norm from C_ϕ to the compact operators. In [**6**, Theorem 2.1, Proposition 2.2] Montes-Rodriguez showed that for ϕ a univalent self-map of U

$$\|C_\phi\|_e = \limsup_{|z|\to 1} \frac{|\phi'(z)|\,(1-|z|^2)}{1-|\phi(z)|^2}\,. \tag{37}$$

Corollary 5.1. *Let ϕ be a univalent self-map of U so that $\phi(U)$ has pointwise order of contact 1 with ∂U at the point 1. Then,*

$$\frac{1}{4}\,\mathrm{dist}_{\mathcal{B}}(\log(1-\phi), \mathcal{B}_0) \leqslant \|C_\phi\|_e \leqslant \text{const.}\ \mathrm{dist}_{\mathcal{B}}\bigl(\log(1-\phi), \mathcal{B}_0\bigr)\,.$$

Proof: By (13), (37) and Theorem 3.9

$$\begin{aligned}
\mathrm{dist}_{\mathcal{B}}\bigl(\log(1-\phi), \mathcal{B}_0\bigr) &\leqslant 2\,\limsup_{|z|\to 1} \frac{|\phi'(z)|\,(1-|z|^2)}{|1-\phi(z)|} \\
&\leqslant 4\,\limsup_{|z|\to 1} \frac{|\phi'(z)|\,(1-|z|^2)}{1-|\phi(z)|^2} = 4\,\|C_\phi\|_e \\
&\leqslant \text{const.}\ \limsup_{|z|\to 1} \frac{|\phi'(z)|\,(1-|z|^2)}{|1-\phi(z)|} \\
&\leqslant \text{const.}\ \mathrm{dist}_{\mathcal{B}}(\log(1-\phi), \mathcal{B}_0)\,.
\end{aligned}$$

Therefore the result follows. ∎

6. Distance in Bloch type spaces

In [**13**] Zhu defined the *Bloch type spaces* of holomorphic functions on U, which are generalisations of the Bloch space $\mathcal{B}$. For each $\alpha > 0$, $\mathcal{B}_\alpha$ denotes the space of holomorphic functions f on U for which

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z\in U} \bigl|f'(z)\bigr|\bigl(1-|z|^2\bigr)^\alpha < \infty\,.$$

The *little Bloch type spaces* are generalisations of the little Bloch space $\mathcal{B}_0$. For each $\alpha > 0$ let $\mathcal{B}_{\alpha,0}$ denote the space of all functions in $\mathcal{B}_\alpha$ so that

$$\lim_{|z|\to 1} \bigl|f'(z)\bigr|\bigl(1-|z|^2\bigr)^\alpha = 0\,.$$

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Both $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha,0}$ are Banach spaces with norm

$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \|f\|_{\mathcal{B}_\alpha},$$

for all $f \in \mathcal{B}_\alpha$. As Zhu notes these spaces are not new. When $0 < \alpha < 1$ then $\mathcal{B}_\alpha$ can be identified with the holomorphic Lipschitz space $\mathrm{Lip}_{1-\alpha}$, the space of all holomorphic functions on U with $|f(z) - f(w)| \leqslant c\,|z-w|^{1-\alpha}$, for some constant $c > 0$ (depending on f) and all z, $w \in U$. And when $\alpha > 1$, $\mathcal{B}_\alpha$ can be identified with the space of holomorhic functions f with

$$\sup_{z\in U}\left(1-|z|^2\right)^{\alpha-1}|f(z)| < \infty.$$

Below we shall describe how the distance formulas in $\mathcal{B}$ have Bloch type spaces analogues.

One can define analogues of $\Omega_\varepsilon(f)$, $A_p(f)$ and $A(f)$ for Bloch type spaces, just by replacing, on the definition of these sets, (see (1), (8), (9)), the term $1-|z|^2$ with $\left(1-|z|^2\right)^\alpha$. Call these new sets $\Omega_{\varepsilon,\alpha}(f)$, $A_{p,\alpha}(f)$ and $A_\alpha(f)$ respectively. Then by Corollary 4 in [**13**] the analogue of (2) holds in $\mathcal{B}_\alpha$ (change powers 2 to $1+\alpha$). Thus Lemma A (with 6 replaced with a constant, const., that depends on α) holds. Lemma B follows from Proposition 2 in [**13**], which says that $\mathcal{B}_{\alpha,0}$ is the closure of all polynomials in the $\mathcal{B}_\alpha$ norm. Lemma 3.1, Proposition 3.2, Proposition 3.3 (with 6 replaced with a constant, const., that depends on α) hold. We omit the details as the proofs are similar to the corresponding results in $\mathcal{B}$.

Let A and B be two quantities that depend on a holomorphic function f on U. We say that A is *equivalent* to B, we write $A \sim B$, if

$$\text{const.}\, A \leqslant B \leqslant \text{const.}\, A.$$

We shall show that the analogue of Theorem 3.5 holds for $p > 2$.

Theorem 6.1. *For $f \in \mathcal{B}_\alpha$ $(\alpha > 0)$, $p > 2$,*

$$\operatorname{dist}_{\mathcal{B}_\alpha}(f, \mathcal{B}_{\alpha,0}) \sim \inf A_{p,\alpha}(f) \sim \inf A_\alpha(f).$$

Proof: By the $\mathcal{B}_\alpha$ versions of Propositions 3.2 and 3.3 it suffices to show that for all $\varepsilon \in A_{p,\alpha}(f)$, $f_1 \in \mathcal{B}_{\alpha,0}$. The function f_1 in the $\mathcal{B}_\alpha$ setting is

$$f_1(z) = \int_{\Omega_\varepsilon(f)} \frac{(1-|w|^2)^\alpha f'(w)}{\overline{w}\,(1-\overline{w}z)^{1+\alpha}}\, dA(w).$$

Thus for all $\delta > 0$

$$\begin{aligned}\left(1-|z|^2\right)^\alpha |f_1'(z)| &\leqslant \left(1-|z|^2\right)^\alpha \|f\|_{\mathcal{B}_\alpha} \int_{\Omega_\varepsilon(f)} \frac{1}{(1-|w|\,|z|)^{\alpha+2}}\, dA(w)\\ &\leqslant \text{const.}\left(1-|z|\right)^\delta \int_{\Omega_\varepsilon(f)} \frac{1}{(1-|w|^2)^{2+\delta}}\, dA(w).\end{aligned}$$

It follows that $f_1 \in \mathcal{B}_{\alpha,0}$ for all $\varepsilon \in A_{p,\alpha}(f)$, $p = 2 + \delta$ $(\delta > 0)$. Therefore the result follows. ∎

The analogue of Theorem C holds. In [**11**, Theorem 2.1.1] is the $\mathcal{B}_\alpha$ analogue of (12) (replace the term $1 - |z|^2$ in (12) with $\left(1 - |z|^2\right)^\alpha$). Thus a corollary of the proof of Theorem C is as follows.

THEOREM 6.2. *Let μ be a positive measure on U, let $0 < p < \infty$ and $\alpha > 0$. Then,*

$$\int_U \frac{d\mu(z)}{(1 - |z|^2)^{\alpha p}} < \infty$$

if and only if there is a constant c with

$$\int_U \left|f'(z)\right|^p d\mu(z) \leqslant c \, \|f\|^p_{\mathcal{B}_\alpha}, \qquad \text{for all } f \in \mathcal{B}_\alpha.$$

Therefore we obtain the analogue of Corollary 3.6 for $p > 2$:

COROLLARY 6.3. *For $f \in \mathcal{B}_\alpha$ $(\alpha > 0)$, the following are equivalent:*

1. $f \in \mathcal{B}_{\alpha,0}$.
2. *For all $\varepsilon > 0$ $\Omega_{\varepsilon,\alpha}(f)$ is a compact subset of U.*
3. *For some $p > 2$ and all $\varepsilon > 0$,*

$$\int_{\Omega_{\varepsilon,\alpha}(f)} 1/\left(\left(1 - |w|^2\right)^p\right) dA(w) < \infty.$$

4. *For any $p > 2$ and $\varepsilon > 0$, there is a constant c so that*

$$\int_{\Omega_{\varepsilon,\alpha}(f)} \left|g'(z)\right|^p dA(z) \leqslant c \, \|g\|^p_{\mathcal{B}_\alpha}, \qquad \text{for all } g \in \mathcal{B}_\alpha.$$

The analogue of Theorem 3.9 for all spaces $\mathcal{B}_\alpha$ holds. The left inequality holds as the analogue of (5) is easily seen to be true in $\mathcal{B}_\alpha$. For the right inequality everything works through with the appropriate change of each $1 - |z|^2$ term to $\left(1 - |z|^2\right)^\alpha$ and $\mathcal{B}$, $\mathcal{B}_0$ replaced with $\mathcal{B}_\alpha$, $\mathcal{B}_{\alpha,0}$ respectively. In (19) $1/\left(1 - \left(R + (1 - \rho)/r\right)^2\right)$ needs to be replaced by its α power then the corresponding limit in (21) equals $\left(2^{2\alpha+1}(1 - \rho)\right)/\left(\rho^{\alpha+1}(4 - \rho)^\alpha\right)$. Therefore we obtain the following corollary.

COROLLARY 6.4. *For $f \in \mathcal{B}_\alpha$ $(\alpha > 0)$,*

$$\limsup_{|z| \to 1} \left|f'(z)\right| \left(1 - |z|^2\right)^\alpha \leqslant \operatorname{dist}_{\mathcal{B}_\alpha}(f, \mathcal{B}_{\alpha,0}) \leqslant 2 \limsup_{|z| \to 1} \left|f'(z)\right| \left(1 - |z|^2\right)^\alpha.$$

References

[1] J.M. Anderson, 'Bloch functions: the basic theory', in *Operators and Function Theory*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci **153** (Reidel, Dordrecht, 1985), **pp.** 1-17.

[2] J. Arazy, S.D. Fisher and J. Peetre, 'Möbius invariant function spaces', *J. Reine Angew. Math.* **363** (1985), 110–145.

[3] P.S. Bourdon, J.A. Cima and A.L. Matheson, 'Compact composition operators on BMOA', *Trans. Amer. Math. Soc.* **351** (1999), 2183–2196.

[4] P.G. Ghatage and D. Zheng, 'Analytic functions of bounded mean oscillation and the Bloch space', *Integral Equations Operator Theory* **17** (1993), 501–515.

[5] S. Makhmutov and M. Tjani, 'Composition operators on some Möbius invariant Banach spaces', *Bull. Austral. Math. Soc.* **62** (2000), 1–19.

[6] A. Montes-Rodriguez, 'The essential norm of a composition operator on Bloch spaces', *Pacific J. Math.* **188** (1999), 339–351.

[7] C. Pommerenke, *Boundary behaviour of conformal maps* (Springer–Verlag, Berlin, Heidelberg, 1992).

[8] W. Ramey and D. Ullrich, 'Bounded mean oscillations of Bloch pull-backs', *Math. Ann.* **291** (1991), 590–606.

[9] J.H. Shapiro, *Composition operators and classical fnction theory* (Springer–Verlag, New York, 1993).

[10] D.A. Stegenga and K. Stephenson, 'Sharp geometric estimates of the distance to VMOA', *Contemp. Math.* **137** (1992), 421–432.

[11] J. Xiao, *Holomorphic Q classes*, Lecture Notes in Mathematics **1767** (Springer-Verlag, Berlin, 2001).

[12] K. Zhu, *Operator theory on function spaces* (Marcel Dekker, New York, 1990).

[13] K. Zhu, 'Bloch type spaces of analytic functions', *Rocky Mountain J. Math.* **23** (1993), 1143–1177.

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