DISTANCE SETS OF WELL-DISTRIBUTED PLANAR SETS FOR POLYGONAL NORMS

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Abstract. Let X be a 2-dimensional normed space, and let BX be the unit ball in X. We discuss the question of how large the set of extremal points of BX may be if X contains a well-distributed set whose distance set Δ satisfies the estimate $|\Delta \cap [0, N]| \le CN^{3/2-\epsilon}$. We also give a necessary and sufficient condition for the existence of a well-distributed set with $|\Delta \cap [0, N]| \le CN$.

§0. INTRODUCTION

The classical Erdős Distance Problem asks for the smallest possible cardinality of

$$\Delta(A) = \Delta_{l_2^2}(A) = \left\{ \|a - a'\|_{l_2^2} : a, a' \in A \right\}$$

if $A \subset \mathbb{R}^2$ has cardinality $N < \infty$ and

$$\|x\|_{l_2^2} = \sqrt{x_1^2 + x_2^2}$$

is the Euclidean distance between the points a and a'. Erdős conjectured that $|\Delta(A)| \gg N/\sqrt{\log N}$ for $N \ge 2$. (We write $U \ll V$, or $V \gg U$, if the functions U, V satisfy the inequality $|U| \le CV$, where C is a constant which may depend on some specified parameters). The best known result to date in two dimensions is due to Katz and Tardos who prove in [KT04] that $|\Delta(A)| \gg N^{.864}$ improving an earlier breakthrough by Solymosi and Tóth [ST01].

More generally, one can examine an arbitrary two-dimensional space X with the unit ball

$$BX = \{x \in \mathbb{R}^2 : \|x\|_X \le 1\}$$

and define the distance set

$$\Delta_X(A) = \{\|a - a'\|_X : a, a' \in A\}.$$

For example, let

$$\|x\|_{l_\infty^2} = \max(|x_1|, |x_2|)$$

then for $N \ge 1$, $A = \{m \in \mathbb{Z}^2 : 0 \le m_1 \le N^{1/2}, 0 \le m_2 \le N^{1/2}\}$ we have $|A| \gg N$, $|\Delta_{l_\infty^2}(A)| \ll N^{1/2}$. This simple example shows that the Erdős Distance Conjecture can not be directly extended for arbitrary two-dimensional spaces. We note, however, that the estimate $|\Delta_X(A)| \gg N^{1/2}$, proved by Erdős [E46] for

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Euclidean norms, extends (with the same proof) to arbitrary 2-dimensional spaces X; see also [I01], [G04].

Also, for a positive integer N we denote

$$\Delta_{X,N}(A) = \{\|a - a'\|_X \leq N : a, a' \in A\}\,.$$

We say that a set $S \subset X$ is well-distributed if there is a constant K such that every closed ball of radius K in X contains a point from S. In other words, for every point $x \in X$ there is a point $y \in S$ such that $\|x - y\|_X \leq K$. Sometimes it is said that S is a K-net for X. Clearly, for any well-distributed set S and $N \geq 2K$ we have

$$|\{x \in S : \|x\|_X \leq N/2\}| \gg N^2 \tag{1}$$

where the constant in $\gg$ depends only on K. Therefore, for any well-distributed set $S \in l_2^2$ we have, by [T02],

$$|\Delta_{l_2^2,N}(S)| \gg N^{1.728},$$

and the Erdős Distance Conjecture implies for large N

$$|\Delta_{l_2^2,N}(S)| \gg N^2/\sqrt{\log N}.$$

On the other hand, for a well-distributed set $S = \mathbb{Z}^2 \subset l_\infty^2$ we have

$$|\Delta_{l_\infty^2,N}(S)| = 2N + 1.$$

Iosevich and the second author [IŁ03] have recently established that a slow growth of $|\Delta_{X,N}(S)|$ for a well-distributed set $S \subset X$ is possible only in the case if BX is a polygon with finitely or infinitely many sides. Let us discuss possible definitions of polygons with infinitely many sides. For a convex set $A \subset X$ by $Ext(A)$ we denote the set of extremal points of A. Namely, $x \in Ext(A)$ if and only if $x \in A$ and for any segment $[y, z]$ the conditions $x \in [y, z] \subset A$ imply $x = y$ or $x = z$. Clearly, $Ext(BX)$ is a closed subset of the unit circle

$$\partial BX = \{x \in X : \|x\|_X = 1\}.$$

Also, it is easy to see that $Ext(BX)$ is finite if and only if BX is a polygon with finitely many sides, and it is natural to consider BX as a polygon with infinitely many sides if $Ext(BX)$ is small. There are different ways to define smallness of $Ext(BX)$ and, thus, polygons with infinitely many sides:
1) in category: $Ext(BX)$ is nowhere dense in ∂BX;
2) in measure: $Ext(BX)$ has a zero linear measure (or a small Hausdorff dimension);
3) in cardinality: $Ext(BX)$ is at most countable.
Clearly, 3) implies 2) and 2) implies 1).

It has been proved in [IŁ03] that the condition

$$\underline{\lim}_{N\to\infty} |\Delta_{X,N}(S)| N^{-3/2} = 0 \tag{0.1}$$

for a well-distributed set S implies that BX is a polygon in a category sense. Following [IŁ03], we prove that, moreover, BX is a polygon in a measure sense.

Theorem 1. *Let S be a well-distributed set.*
(i) Assume that (0.1) holds. Then the one-dimensional Hausdorff measure of $Ext(BX)$ is 0;
(ii) If moreover

$$|\Delta_{X,N}(S)| = O(N^{1+\alpha}) \tag{0.2}$$

for some $\alpha \in (0, 1/2)$ then the Hausdorff dimension of $Ext(BX)$ is at most 2α.

If $|\Delta_{X,N}(S)|$ has an extremally slow rate of growth for some well-distributed set S, namely,

$$|\Delta_{X,N}(S)| = O(N) \tag{0.3}$$

then, as it has been proved in [IŁ03], BX is a polygon with finitely many sides. However, if we weaken (0.3) we cannot claim that BX is a polygon in a cardinality sense.

Theorem 2. *Let $\psi(u)$ be a function $(0, \infty) \to (0, \infty)$ such that $\lim_{u\to\infty} \psi(u) = \infty$. Then there exists a 2-dimensional space X and a well-distributed set $S \subset X$ such that*

$$|\Delta_{X,N}(S)| = o(N\psi(N)) \quad (N \to \infty) \tag{0.4}$$

but $Ext(BX)$ is a perfect set (and therefore is uncountable).

Also, we find a necessary and sufficient condition for a space X to make (0.3) possible for some well-distributed set $S \subset X$. Take two non-collinear vectors e_1, e_2 in X. They determine coordinates for any $x \in X$, namely, $x = x_1e_1 + x_2e_2$. Then, for any non-degenerate segment $I \subset X$, we can define its slope $Sl(I)$: if the line containing I is given by an equation $u_1x_1 + u_2x_2 + u_0 = 0$, then we set $Sl(I) = -u_1/u_2$. We write $Sl(I) = \infty$ if $u_2 = 0$; it will be convenient for us to consider ∞ as an algebraic number.

Theorem 3. *The following conditions on X are equivalent:*
(i) BX is a polygon with finitely many sides, and there is a coordinate system in X such that the slopes of all sides of BX are algebraic;
(ii) there is a well-distributed set $S \subset X$ such that (0.3) holds.

Corollary 1. *If a norm $\|\cdot\|_X$ on $\mathbb{R}^2$ is so that BX is a polygon with finitely many sides and all angles between its sides are rational multiples of π then there is a well-distributed set $S \subset X$ such that (0.3) holds.*

Corollary 2. *If a norm $\|\cdot\|_X$ on $\mathbb{R}^2$ is defined by a regular polygon BX then there is a well-distributed set $S \subset X$ such that (0.3) holds.*

We remark that a similar algebraicity condition arose in the work of Laczkovich-Ruzsa [LR96] on counting the number of similar copies of a fixed pattern embedded in a point set. (See also [EE94].)

The Falconer conjecture (for the plane) says that if the Hausdorff dimension of a compact $A \subset \mathbb{R}^2$ is greater than 1 then $\Delta(A)$ has positive Lebesgue measure. The best known result is due to Wolff who proved in [W99] that the distance set has positive Lebesgue measure if the Hausdorff dimension of A is greater than $4/3$. One can ask a similar question for an arbitrary two-dimensional normed space X. It turns out that this question is related to distance sets for well-distributed and separated sets. By Theorem 4 from [IŁ04], Theorem 3 and Proposition 1 we get the following.

Corollary 3. *If a norm* $\|\cdot\|_X$ *on* $\mathbb{R}^2$ *is defined by a polygon* BX *with finitely many sides all of which have algebraic slopes then there is a compact* $A \subset X$ *such that the Hausdorff dimension of* A *is* 2 *and Lebesgue measure of* $\Delta_X(A)$ *is* 0.

After this paper was completed, K. Falconer [Fa04] proved (using different methods) that the same result is in fact true without the supposition on the slopes of the sides.

Recall that, by [IŁ03], it is enough to prove the implication $(ii) \to (i)$ in Theorem 3 assuming that BX is a polygon. In that case we prove a stronger result.

Theorem 4. *Let* BX *be a polygon with finitely many sides which does not satisfy the condition (i) of Theorem 3. Then for any well-distributed set* S *we have*

$$|\Delta_{X,N}(S)| \gg N \log N / \log\log N \quad (N \geq 3). \tag{0.5}$$

Comparison of Theorem 4 with Theorem 2 shows that the growth of $|\Delta_{X,N}(S)|$ for well-distributed sets and $N \to \infty$ does not distinguish the spaces X with small and big cardinality of $Ext(BX)$.

As remarked in [IŁ03], the well-distribution assumption on the point set S is essential for results such as Theorems 1–4, as it ensures that the set of directions between pairs of points in S is dense. If this fails, then K can be modified arbitrarily in the "missing" directions without affecting the distance set of S. (See [G04] for further discussion of the general case.) We also note that Solymosi and Vu [SV04] obtain good bounds for Euclidean distance sets of well-distributed sets in 3 or more dimensions, and that their method may extend to other metrics. As noted in [SV04], known examples of sets with small distance sets tend to be lattice-like and therefore well-distributed.

§1. PROOF OF THEOREMS 1 AND 2

Proof of (i). Without loss of generality we may assume that $BX \subset Bl_2^2$ and the set S is well-distributed in X with the constant $K = 1/2$. Also, choose $\delta > 0$ so that

$$\delta Bl_2^2 \subset BX. \tag{1.1}$$

By (0.1), for any $\varepsilon > 0$ there are arbitrary large N_0 such that

$$|\Delta_{X,N_0}(S)| \leq \varepsilon N_0^{3/2}.$$

If $N_0 \geq 8$ then the number of integers $j \geq 0$ with $N_0/2 + 4j \leq N_0 - 2$ is

$$\geq (N_0/2 - 2)/4 \geq N_0/8.$$

Thus, there is at least one j such that $N = N_0/2 + 4j$ satisfies the condition

$$|(\Delta_X(S)| \cap (N-2, N+2)) \leq 8\varepsilon N_0^{3/2}/N_0 \leq 12\varepsilon N^{1/2}. \tag{1.2}$$

So, (1.2) holds for arbitrary large N.

We take any N satisfying (1.2) and an arbitrary $P \in S$. Let Q be the closest point to P in the space X (observe that it exists since S is closed due to (0.1)). Then, by well-distribution of S (recall that $K = 1/2$) we have

$$\|P - Q\|_X \leq 1. \tag{1.3}$$

Without loss of generality, $P = 0$. Denote $M = [2N\delta]$ and consider the rays

$$L_j = \{(r, \theta) : \theta = \theta_j = 2\pi j/M\},$$

where (r, θ) are the polar coordinates in l_2^2. Consider a point R_j, $1 \leq j \leq M$, with the polar coordinates $(r_j, (\theta_{j-1} + \theta_j)/2)$ such that $\|R_j\|_X = N$. By (1.1) we have

$$r_j \geq \delta N.$$

Therefore, the Euclidean distance from R_j to the rays L_{j-1} and L_j is

$$r_j \sin(\pi/M) \geq N\delta \sin(\pi/(2N\delta)) > 1. \tag{1.4}$$

provided that N is large enough. Therefore, the distance from R_j to these rays in X is also greater than 1. Also, the distance from R_j to the circles

$$\Gamma_1 = \{R : \|R\|_X = N - 1\}, \quad \Gamma_2 = \{R : \|R\|_X = N + 1\}$$

in X is equal to 1. Thus, the X-disc of radius $1/2$ with the center at R_j is contained in the open region U_j bounded by L_{j-1}, L_j, Γ_1, and Γ_2. By the supposition on S there is a point $P_j \in U_j \cap S$.

Observe that for any j we have

$$N - 1 < \|P - P_j\|_X < N + 1, \quad N - 2 < \|Q - P_j\|_X < N + 2.$$

Let $U = \{(\|P - P_j\|_X, \|Q - P_j\|_X)\}$. By (1.2),

$$|U| \leq 144\varepsilon^2 N. \tag{1.5}$$

For any $(n_1, n_2) \in U$ we denote

$$J_{n_1,n_2} = \{j : \|P - P_j\|_X = n_1, \quad \|Q - P_j\|_X = n_2\}.$$

By [IŁ03, Lemma 1.4, (i)], if $j_1, j_2, j_3 \in J_{n_1,n_2}$ then one of the points $P_{j_1}, P_{j_2}, P_{j_3}$ must lie on the segment connecting two other points and contained in the circle $\{R : \|P - R\|_X = n_1\}$. This implies that for all $j \in J_{n_1,n_2}$ but at most two indices the intersection of ∂BX with the sector S_j bounded by L_{j-1} and L_j is inside some line segment contained in ∂BX. Therefore, by (1.5), the number of sectors S_j containing an extremal point of BX is at most $288\varepsilon^2 N$. For $R \in \partial BX$ with the polar coordinates (r, θ) denote $\Theta(R) = \theta$. Define the measure on ∂BX in such a way that for any Borel set $V \subset \partial BX$ the measure $\mu_P(V)$ is defined as the Lebesgue measure of $\Theta(V)$. In particular,

$$\mu_p(\partial BX \cap S_j) = \frac{2\pi}{M}.$$

Clearly, μ_p is equivalent to the standard Lebesgue measure on ∂BX. We have proved that

$$\mu_p(Ext(BX)) \leq 288\varepsilon^2 N \frac{2\pi}{M}.$$

But $1/M \le 1/(N\delta)$. Hence,

$$\mu_p(Ext(BX)) \le 2\pi \times 288\varepsilon^2/\delta.$$

As ε can be chosen arbitrarily small, we get $\mu_p(Ext(BX)) = 0$, and this completes the proof of (i).

Proof of (ii) follows the same scheme. Inequality (1.2) should be replaced by

$$|\Delta_X(S) \cap (N-2, N+2)| \le \Delta N^{\alpha},$$

where Δ may depend only on X, S, and α. We define the distance d_p on ∂BX as the distance between the polar coordinates. This metric is equivalent to the X-metric. The set $Ext(BX)$ can be covered by at most $2\Delta^2 N^{2\alpha}$ arcs $\partial BX \cap S_j$ each of them has the d_p-diameter at most $2\pi/(N\delta)$. This implies the required estimate for the Hausdorff dimension of $Ext(BX)$.

Proof of Theorem 2. We select an increasing sequence $\{N_j\}$ of positive integers such that

$$\psi(N) \ge 5^j \quad (N \ge N_j). \tag{1.6}$$

By Λ_j we denote the set of numbers a/q with $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $q \le N_j$. We will construct a ball BX on the Euclidean plane. Moreover, it will be symmetric with respect to the lines $x_1 = x_2$ and $x_1 = -x_2$, and thus it suffices to construct BX in the quadrant $Q = \{x : x_2 \ge |x_1|\}$.

Let D_0 be the square

$$D_0 = \{x : 0 \le x_2 + x_1 \le 1, 0 \le x_2 - x_1 \le 1\}.$$

We will construct a decreasing sequence of polygons D_j; each one will be defined as a result of cutting some angles from the previous one. The sides V_1, V_2 of D_0 with an endpoint at the origin will not be changed. The intersection of the sequence D_j will define the part of our BX in Q. In particular, the points $(\pm 1/2, 1/2)$ will be vertices of all polygons D_j. Therefore, these points as well as the symmetrical points $(\pm 1/2, -1/2)$ will be in ∂BX.

First, we construct D_1 as a result of cutting D_0 by a line $x_2 = u$ for some $u \in (1/2, 1)$. We choose u such that for intersection points x^1 and x^2 of this line with the boundary of D_0 the ratios x_1^j/x_2^j $(j = 1, 2)$ differs from all numbers $\lambda \in \Lambda_1$. Moreover, we take neighborhoods U_j of the points x^j $(j = 1, 2)$ such that

$$\forall y \in U_j \; y_2/y_1 \notin \Lambda_1 \quad (j = 1, 2).$$

In the sequel we shall make other cuts only inside the sets U_1 and U_2. This means that all points x on the boundary of D_1 with $x_1/x_2 \in \Lambda_1$ not belonging to the sides V_1, V_2 as well as their neighborhoods in the boundary of D_1 will remain in all polygons $D_2, D_3, \ldots$, and eventually they will be interior points of some segments in the boundary of BX with a slope -1, 0, or 1,

On the second step, we construct D_2 as a result of cutting D_1 by lines with slopes $-1/2$ and $1/2$ such that for any new vertex x of a polygon D_2 we have $x_2/x_1 \notin \Lambda_2$.

Moreover, we take neighborhoods $U(x)$ of all these points x (each is contained in U_1 or in U_2) such that

$$\forall y \in U(x)\ y_2/y_1 \notin \Lambda_2.$$

Again, we shall make other cuts only inside the sets $U(x)$. This means that all points x on the boundary of D_2 with $x_1/x_2 \in \Lambda_2$ not belonging to the sides V_1, V_2 as well as their neighborhoods in the boundary of D_2 will remain in all polygons $D_3, D_4, \dots,$ and eventually they will be interior points of some segments in the boundary of BX with a slope $a/2$, $a \in \mathbb{Z}$, $|a| \le 2$.

Proceeding in the same way, we shall get a ball BX with the following property: if $x \in \partial BX$ and $x_1/x_2 \in \Lambda_{j+1}$ for some j then x is an interior point of some segment contained in ∂BX with a slope $a/2^j$, $a \in \mathbb{Z}$, $|a| \le 2^j$. This segment is a part of a line $2^j x_2 - ax_1 = b(a,j)$ or a symmetrical line $2^j x_2 - ax_1 = -b(a,j)$. Also, by symmetry, if $x \in \partial BX$ and $x_2/x_1 \in \Lambda_{j+1}$ for some j then $2^j x_1 - ax_2 = b(a,j)$ or $2^j x_1 - ax_2 = -b(a,j)$. In terms of the norm $\|\cdot\|_X$ we conclude that if $x \in X$ and $x_1/x_2 \in \Lambda_{j+1}$ or $x_2/x_1 \in \Lambda_{j+1}$ then $\|x\|_X$ is equal to one of the numbers $|2^j x_1 - ax_2|/|b(a,j)|$ or $|2^j x_2 - ax_1|/|b(a,j)|$, $a \in \mathbb{Z}$, $|a| \le 2^j$. Also, observe that, by our construction, BX is contained in the square $[-1,1]^2$. Therefore,

$$\|x\|_X \ge \max(|x_1|, |x_2|). \tag{1.7}$$

Now let us take the lattice $S = \mathbb{Z}^2$ and estimate $|\Delta_{X,N}(S)|$ for $N_j < N \le N_{j+1}$. If $x, y \in S$ and $\|x-y\|_X \le N$, then we have $\|x-y\|_X = |(z_1,z_2)|_X$ where $z_1, z_2 \in \mathbb{Z}$ and, by (1.7), $\max(|z_1|,|z_2|) \le N$. Hence, $(z_1,z_2) = (0,0)$, or $x_1/x_2 \in \Lambda_{j+1}$, or $x_2/x_1 \in \Lambda_{j+1}$. Therefore, $\|x-y\|_X = 0$ or $\|x-y\|_X$ is equal to one of the numbers $|2^j x_1 - ax_2|/|b(a,j)|$ or $|2^j x_2 - ax_1|/|b(a,j)|$, $a \in \mathbb{Z}$, $|a| \le 2^j$. For every a we have

$$|2^j x_1 - ax_2| \le 2^j |x_1| + |a| \times |x_2| \le 2^{j+1} N.$$

Taking the sum over all a we get

$$|\Delta_{X,N}(S)| \le (2^{j+1}+1) 2^{j+1} N + 1 \le 2^{2j+3} N. \tag{1.8}$$

On the other hand, by (1.6),

$$\psi(N) \ge 5^j. \tag{1.9}$$

Comparing (1.8) and (1.9), we get (0.4) and thus complete the proof of the theorem.

§2. PROOF OF THEOREM 3, PART I

In this section we prove that the condition (i) of Theorem 3 implies (ii).

Assume that ∂BX consists of a finite number of line segments with slopes $\beta_1, \beta_2, \dots, \beta_r$, all real and algebraic. Let $\mathbb{F}_{\mathbb{Q}}[\beta_1, \dots, \beta_r]$ be the field extension of $\mathbb{Q}$ generated by $\beta_1, \dots, \beta_r$, and let α_0 be its primitive element, i.e. an algebraic number such that $\mathbb{F}_{\mathbb{Q}}[\beta_1, \dots, \beta_r] = \mathbb{F}_{\mathbb{Q}}[\alpha_0]$. We may assume that α_0 is an algebraic integer: indeed, if α_0 is a root of $P(x) = a_d x^d + \cdots + a_0$, then $\alpha_0' = a_d \alpha_0$ is a root of $a_d^{d-1} P(x/a_d) = x^d + a_{d-1} x^{d-1} + a_{d-2} a_d x^{d-2} + \cdots + a_0 a_d^{d-1}$, hence an algebraic integer, and generates the same extension.

It suffices to prove that there is a well-distributed set $S \subset \mathbb{R}^2$ such that

$$|\{x + \beta y : (x, y) \in S - S, \ |x| + |y| \le R\}| \ll R, \tag{2.1}$$

for each $\beta \in \mathbb{F}_{\mathbb{Q}}[\alpha]$.

Since $\mathbb{F}_{\mathbb{Q}}[\beta_1, \dots, \beta_r] \subset \mathbb{R}$, we have $\alpha_0 \in \mathbb{R}$. Let $\alpha_1, \dots, \alpha_{d-1}$ be the algebraic conjugates of α_0 in $\mathbb{C}$ (of course they need not belong to $\mathbb{F}_{\mathbb{Q}}[\alpha_0]$). Define for $C > 0$

$$T(C) = \{\sum_{j=0}^{d-1} a_j \alpha_0^j : \ a_j \in \mathbb{Z}, |\sum_{j=0}^{d-1} a_j \alpha_k^j| \le C, \ k = 1, \dots, d-1\},$$

and

$$S = T(C) \times T(C),$$

where C will be fixed later.

We first claim that $T(C)$ is well distributed in $\mathbb{R}$ (with the implicit constant dependent on C), and that

$$|T(C) \cap [-R, R]| \ll R. \tag{2.2}$$

Indeed, let $x = (x_0, x_1, \dots, x_{d-1})^T$ solve

$$\sum_{j=0}^{d-1} \alpha_0^j x_j = 1,$$

$$\sum_{j=0}^{d-1} \alpha_k^j x_j = 0, \ k - 1, \dots, d-1.$$

Since the Vandermonde matrix $A = (\alpha_k^j)$ is nonsingular, x is unique. In particular, it follows that x is real-valued; this may be seen by taking complex conjugates and observing that α_k is an algebraic conjugate of α_0 if and only if so is $\bar{\alpha}_k$, hence $\bar{x}$ solves the same system of equations.

To prove the first part of the claim, it suffices to show that there is a constant K_1 such that for any $y \in \mathbb{R}$ there is a $v \in T(C)$ with $|y - v| \le K_1$. Fix y, then we have

$$y = \sum_{j=0}^{d-1} \alpha_0^j y x_j.$$

Let v_j be an integer such that $|v_j - y x_j| \le 1/2$, and let $v = \sum_{j=0}^{d} \alpha_0^j v_j$. Then

$$|y - v| = |\sum_{j=0}^{d-1} \alpha_0^j (y x_j - v_j)| \le \frac{1}{2} \sum_{j=0}^{d-1} |\alpha_0^j| =: K_1,$$

and, for $k = 1, \dots, d-1$,

$$|\sum_{j=0}^{d-1} \alpha_k^j v_j| \le |\sum_{j=0}^{d-1} \alpha_k^j (y x_j - v_j)| + y |\sum_{j=0}^{d-1} \alpha_k^j x_j| \le \frac{1}{2} \sum_{j=0}^{d-1} |\alpha_k^j|.$$

The claim follows if we let $C \geq \frac{1}{2}\sum_{j=0}^{d-1}|\alpha_k^j|$.

We now prove (2.2). It suffices to verify that there is a constant K_2 such that for any $y \in \mathbb{R}$ there are at most K_2 elements of $T(C)$ in $[y-C, y+C]$. Let $a = \sum_{j=0}^{d-1}\alpha_0^j a_j$, then the conditions that $a \in T(C)$ and $|y-a| \leq C$ imply that

$$A\tilde{a} - \tilde{y} \in CQ,$$

where $\tilde{a} = (a_0, \dots, a_{d-1})^T$, $\tilde{y} = (y, 0, \dots, 0)^T$, and $Q = [-1,1]^d$. In other words, $\tilde{a} \in A^{-1}\tilde{y} + CA^{-1}Q$. But it is clear that the number of integer lattice points contained in any translate of $CA^{-1}Q$ is bounded by a constant.

It remains to prove (2.1). Observe first that if $x, x' \in T(C)$, then $x - x' \in T(2C)$. Thus, in view of (2.2), it is enough to prove that for any two algebraic integers $\beta, \gamma \in \mathbb{Z}_{\mathbb{Q}}[\alpha]$ there is a $C_1 = C_1(\beta, \gamma)$ such that if $x, y \in T(2C)$, then $x\beta + y\gamma \in T(C_1)$. By the triangle inequality, it suffices to prove this with $y = 0$. Let $x \in T(C)$, then $x = \sum_{j=0}^{d-1}\alpha_0^j x_j$ for some $x_j \in \mathbb{Z}$. We also write $\beta = \sum_{j=0}^{d-1}\alpha_0^j b_j$, with $b_j \in \mathbb{Z}$. Then $\beta y = \sum_{i,j=0}^{d-1}\alpha_0^{i+j} x_i b_j$. We thus need to verify that

$$|\sum_{i,j=0}^{d-1}\alpha_k^{i+j} x_i b_j| \leq C_1$$

for $k = 1, \dots, d-1$. But the left side is equal to

$$|\sum_{i=0}^{d-1}\alpha_k^i x_i| \cdot |\sum_{j=0}^{d-1}\alpha_k^j b_j|,$$

which is bounded by $C_1(\beta) = C\max_k |\sum_{j=0}^{d-1}\alpha_k^j b_j|$.

Example. Let BX be a symmetric convex octagon whose sides have slopes $0, -1, \infty, \sqrt{2}$. Let also $T(C) = \{i + j\sqrt{2} : \ |i - j\sqrt{2}| \leq C\}$, and $S = T(10) \times T(10)$. It is easy to see that $T(C)$ is well distributed and that (2.2) holds. Let $x, y \in S$, then $x - y = (i + j\sqrt{2}, k + l\sqrt{2})$, where $i + j\sqrt{2}, k + l\sqrt{2} \in T(20)$. Depending on where $x - y$ is located, the distance from x to y will be one of the following numbers:

$$c_1|i + j\sqrt{2}|,$$

$$c_2|k + l\sqrt{2}|,$$

$$c_3|(i+k) + (j+l)\sqrt{2}|,$$

$$c_4|(i + j\sqrt{2})\sqrt{2} - (k + l\sqrt{2})| = c_4|(2j - k) + (i - l)\sqrt{2}|.$$

Clearly, the first three belong to $T(20\max(c_1, c_2, c_3))$. For the fourth one, we have

$$c_4|(2j-k) - (i-l)\sqrt{2}| = c_4| - (i - j\sqrt{2})\sqrt{2} - (k - l\sqrt{2})|$$

$$\leq 20c_4(1 + \sqrt{2}).$$

Hence all distances between points in S belong to $T(C)$ for some C large enough, and in particular satisfy the cardinality estimate (2.2).

§3. ADDITIVE PROPERTIES OF MULTIDIMENSIONAL SETS AND SETS WITH SPECIFIC ADDITIVE RESTRICTIONS

Let Y be a linear space over $\mathbb{R}$ or over $\mathbb{Q}$. For $A, B \subset Y$ and $\alpha \in \mathbb{R}$ or $\mathbb{Q}$ we denote

$$A + B = \{a + b : a \in A, b \in B\}, \ \alpha A = \{\alpha a : a \in A\}.$$

We say that a set $A \subset Y$ is a d-dimensional if A is contained in some d-dimensional affine subspace of Y, but in no $d-1$-dimensional affine subspace of Y. We will denote the dimension of a set A by d_A.

The following result is due to Ruzsa [Ru94, Corollary 1.1].

Lemma 3.1. *Let $A, B \subset \mathbb{R}^d$, $|A| \leq |B|$, and assume that $A + B$ is d-dimensional. Then*

$$|A + B| \geq |B| + d|A| - d(d+1)/2. \tag{3.1}$$

The special case of Lemma 3.1 with $A = B$ was proved earlier by Freiman [F73, p. 24]). In this case we also have the following corollary.

Corollary 3.1. *Let $A \subset \mathbb{R}^d$, and assume that $|A + A| \leq K|A|$, $K \leq |A|^{1/2}$. Then the dimension of A does not exceed K.*

Proof. Let $|A| = N \geq 1$, then $d_A \leq N - 1$. Suppose that $d_A > K$. The function $f(x) = (x+1)N - x(x+1)/2$ is increasing for $x \leq N - 1/2$, hence by (3.1) we have

$$KN \geq f(d_A) > f(K) = (K+1)N - \frac{K(K+1)}{2},$$

i.e. $K(K+1) > 2N$, which is not possible if $K^2 \leq N$.

We observe that Lemma 3.1, and hence also Corollary 3.1, extends to the case when A, B are subsets of a linear space Y over $\mathbb{Q}$. Assume that Y is d-dimensional, and take a basis $\{e_1, \dots . e_d\}$ in Y. Consider the space $\mathbb{R}^d$ with a basis $\{e'_1, \dots . e'_d\}$. We can arrange a mapping $\Phi : Y \to Y'$ by

$$\Phi(\sum_{j=1}^{d} \alpha_j e_j) = \sum_{j=1}^{d} \alpha_j e'_j.$$

It is easy to see that Φ is Freiman's isomorphism of any order and, in particular, of order 2: this means that for any $y_1, , y_2, z_1, z_2$ from Y the condition

$$y_1 + y_2 \neq z_1 + z_2$$

implies

$$\Phi(y_1) + \Phi(y_2) \neq \Phi(z_1) + \Phi(z_2).$$

Therefore, if A, B are finite subsets of Y and $A' = \Phi(A), B' = \Phi(B)$, then $|A+B| = |A' + B'|$, and we get the required inequality for $|A + B|$.

The following is a special case of [N96, Theorem 7.8].

Lemma 3.2. *If $N \in \mathbb{N}$, $K > 1$, $A \subset Y$, and $B \subset Y$ satisfy*

$$\min(|A|, |B|) \geq N, \quad |A + B| \leq KN, \tag{3.2}$$

we have

$$|A + A| \leq K^2|A|.$$

Corollary 3.2. *If $N \in \mathbb{N}$, $K > 1$, and if $A, B \subset Y$ satisfy (3.2) for some K with $K^2(2K^2 + 1) < N$, then $d_{A+B} \leq K$. In particular, $d_A \leq K$ and $d_B \leq K$.*

Proof. By Lemma 3.2, we have $|A + A| \leq K^2 N$, hence Corollary 3.1 implies that

$$d_A \leq K^2,$$

and similarly for B. Hence $d_{A+B} \leq d_A + d_B \leq 2K^2$. By Lemma 3.1, we have

$$KN \geq |A + B| \geq (1 + d_{A+B})N - \frac{d_{A+B}(d_{A+B} + 1)}{2}$$

$$\geq d_{A+B}N + N - K^2(2K^2 + 1) \geq d_{A+B}N,$$

which proves the first inequality. To complete the proof, observe that $d_{A+B} \geq \max(d_A, d_B)$.

Lemma 3.3. *Let $K > 0$, A and B be finite nonempty subsets of $\mathbb{R}$, $\alpha \in \mathbb{R} \setminus \{0\}$. Also, suppose that the following conditions are satisfied*

$$|A - \alpha B| \leq K|B|. \tag{3.3}$$

Then there is a set $B' \subset B$ such that

$$|A - \alpha B'| \leq K|B'|, \tag{3.4}$$

$$|B'| \geq |A|/K, \tag{3.5}$$

and for any $b_1, b_2 \in B'$ the number $\alpha(b_1 - b_2)$ is a linear combination of differences $a_1 - a_2$, $a_1, a_2 \in A$, with integer coefficients.

Proof. Let us construct a graph H on B. We join $b_1, b_2 \in B$ (not necessary distinct) by an edge if there are $a_1, a_2 \in A$ such that $a_1 - \alpha b_1 = a_2 - \alpha b_2$. Let $B_1, \ldots, B_s$ be the components of connectedness of the graph H. Thus, for any $j = 1, \ldots, s$ and for any $b_1, b_2 \in B_s$ there is a path connecting b_1 and b_2 and consisting of edges of H (a one-point path for $b_1 = b_2$ is allowed). This implies that $\alpha(b_1 - b_2)$ is a sum of differences $a_1 - a_2$ for some pairs $(a_1, a_2) \in A \times A$. Also, denoting

$$S = A - \alpha B, \quad S_j = A - \alpha B_j,$$

we see that, by the choice of $B_1, \ldots, B_s$, the sets S_j $(j = 1, \ldots, s)$ are disjoint.

Since

$$|B| = \sum_{j=1}^{s} |B_j|, \quad |S| = \sum_{j=1}^{s} |S_j|,$$

there is some j such that

$$|S_j|/|B_j| \le |S|/|B|,$$

and, by (3.3),

$$|S_j| \le K|B_j|.$$

On the other hand,

$$|S_j| = |A - \alpha B_j| \ge |A|.$$

Hence,

$$|B_j| \ge |S_j|/K \ge |A|/K.$$

So, the set $B' = B_j$ satisfies (3.4) and (3.5), and Lemma 3.3 follows.

Lemma 3.4. *Let $K > 0$, A and B be finite nonempty subsets of $\mathbb{R}$, $\alpha_1, \alpha_2 \in \mathbb{R}\backslash\{0\}$. Also, suppose that the conditions*

$$|A - \alpha_1 B| \le K|B|, \tag{3.6}$$

$$|A - \alpha_2 B| \le K|A|, \tag{3.7}$$

are satisfied. Then there are nonempty sets $A' \subset A$ and $B' \subset B$ such that

$$|A - \alpha_1 B'| \le K|B'|, \tag{3.8}$$

$$|A' - \alpha_2 B'| \le K|A'|, \tag{3.9}$$

$$|A'| \ge |A|/K^2, \tag{3.10}$$

and for any $a_1', a_2' \in A'$ the difference $a_1' - a_2'$ is a linear combination of numbers $\frac{\alpha_2}{\alpha_1}(a_1 - a_2)$, $a_1, a_2 \in A$, with integer coefficients.

Proof. By (3.6), we can use Lemma 3.3 for $\alpha = \alpha_1$, and we get (3.8) and (3.5). Further, we use Lemma 3.3 again for B', A (thus, in the reverse order), and we get (3.9) and also

$$|A'| \ge |B'|/K.$$

Combining the last inequality with (3.5) we obtain (3.10). The proof of the lemma is complete.

Replacing (3.8) by a weaker inequality

$$|A' - \alpha_1 B'| \le K|B'|$$

and iterating Lemma 3.4, we get the following.

Lemma 3.5. *Let $K > 0$, A and B be finite nonempty subsets of $\mathbb{R}$, $\alpha_1, \alpha_2 \in \mathbb{R}\setminus\{0\}$. Also, suppose that the conditions (3.6) and (3.7) are satisfied. Then there are nonempty sets $A_j \subset A$ and $B_j \subset B$ ($j = 0, 1, \ldots,$) such that $A_0 = A$, $B_0 = B$, $A_j \subset A_{j-1}$, $B_j \subset B_{j-1}$ for $j \geq 1$,*

$$|A_j - \alpha_2 B_j| \leq K|A_j| \quad (j \geq 1),$$

$$|A_j| \geq |A|/K^{2j},$$

and for any $a_1, a_2 \in A_j$ the difference $a_1 - a_2$ is a linear combination of numbers $\frac{\alpha_2^j}{\alpha_1^j}(a_1' - a_2')$, $a_1', a_2' \in A$, with integer coefficients.

Now we are in position to come to the main object of our constructions: to show that under the assumptions of Lemma 3.5, providing that the number α_1/α_2 is transcendental, we can conclude that the dimension of the set A over $\mathbb{Q}$ cannot be too small.

Corollary 3.6. *Let $K > 0$, A and B be finite nonempty subsets of $\mathbb{R}$, $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ such that α_1/α_2 is transcendental. Also, suppose that the conditions (3.6) and (3.7) are satisfied. Then, if for some $d \in \mathbb{N}$ the inequality*

$$|A| > K^{2d} \tag{3.11}$$

holds, then the dimension of A over $\mathbb{Q}$ is greater than d.

Proof. By Lemma 3.5 and (3.11), we have $|A_d| \geq 2$. Take distinct $a_1, a_2 \in A_d$. Then also $a_1, a_2 \in A_j$ for $j = 0, 1, \ldots, d$, and, by Lemma 3.6, the difference $a_1 - a_2$ is a linear combination of numbers $\frac{\alpha_2^j}{\alpha_1^j}(a_1' - a_2')$, $a_1', a_2' \in A$, with integer coefficients. Therefore, all numbers $b_j = \frac{\alpha_1^j}{\alpha_2^j}(a_1 - a_2)$ belong to the linear span of $a_1' - a_2'$, $a_1', a_2' \in A$, over $\mathbb{Q}$. But, since α_1/α_2 is transcendental, the numbers b_j ($j = 0, \ldots, d$) are linearly independent over $\mathbb{Q}$. Therefore, the dimension of the linear span of $a_1' - a_2'$, $a_1', a_2' \in A$, over $\mathbb{Q}$ is at least $d + 1$, as required.

Corollary 3.7. *If A is a subset of $\mathbb{R}$, $2 \leq |A| < \infty$, α is a transcendental real number, then*

$$|A - \alpha A| \gg |A| \log |A| / \log \log |A|.$$

Proof. Suppose that the conclusion fails, then for any $\epsilon > 0$ we may find arbitrarily large N and $A \subset \mathbb{R}$ with $|A| = N$ such that

$$|A - \alpha A| \leq KN, \ K = \epsilon \frac{\log N}{\log \log N}.$$

By Corollary 3.2, we have $d_A \leq K$. On the other hand, (3.6) holds with $B = A$, $\alpha_1 = \alpha$, and, since $A - \alpha^{-1}A = -\alpha^{-1}(A - \alpha A)$, (3.7) holds with $B = A$ and $\alpha_2 = \alpha^{-1}$. Corollary 3.7 then implies that

$$N \leq K^{2K}.$$

Taking logarithms of both sides, and assuming that $2\epsilon < 1$, we obtain

$$\log N \leq 2\epsilon \frac{\log N}{\log \log N}(\log(2\epsilon) + \log \log N - \log \log \log N) \leq 2\epsilon \log N,$$

which is not possible if N was chosen large enough.

Remark 1. On the other hand, if $\alpha \in \mathbb{R}$ is an algebraic number, then one can use our construction from §2 to show that for any $N \in \mathbb{N}$ there is a set $A \subset \mathbb{R}$, $|A| = N$, such that

$$|A - \alpha A| \leq C|A|,$$

where C depends only on α.

Remark 2. We do not know whether the bound in Corollary 3.7 is optimal. However, for any transcendental number α we can construct a set A such that

$$|A - \alpha A| \leq C_\epsilon |A|^{1+\epsilon}$$

for any $\epsilon > 0$[1]. Namely, let

$$A = \{\sum_{i=1}^{m} a_i \alpha^i : \ a_i = 1, ..., n\},$$

then $|A| = n^m$ and

$$A + \alpha A \subset \{\sum_{i=1}^{m+1} a_i \alpha^i : \ a_i = 1, ..., 2n\},$$

which has cardinality $\leq (2n)^{m+1}$. Let us take $n = 2^m$, $N = n^m = 2^{m^2}$, in which case $(2n)^{m+1} = 2^{(m+1)^2} \ll N \exp(C\sqrt{\log N})$, less than $C_\epsilon N^{1+\epsilon}$ for any $\epsilon > 0$, as claimed.

Finally, we state a lemma due to J. Bourgain[B99, Lemma 2.1]. For our purposes, we need a slightly more precise formulation than that given in [B99]; the required modifications are described below.

Lemma 3.8. *Let $N \geq 2$, A, B be finite subsets of $\mathbb{R}$ and $G \subset A \times B$ such that*

$$|A|, |B| \leq N, \tag{3.12}$$

$$|S| \leq N \quad \textit{where} \quad S = \{a + b : (a, b) \in G\}, \tag{3.13}$$

$$|G| \geq \delta N^2. \tag{3.14}$$

Then there exist $A' \subset A$, $B' \subset B$ satisfying the conditions

$$|(A' \times B') \cap G| \gg \delta^5 N^2 (\log N)^{-C_1}, \tag{3.15}$$

$$|A' - B'| \ll N^{-1} (\log N)^{C_2} \delta^{-13} |(A' \times B') \cap G|. \tag{3.16}$$

[1] We thank Ben Green for pointing out this example.

In [B], the bounds (3.15) and (3.16) involved factors of the form $N^{\gamma+}$ and $N^{\gamma-}$, where $N^{\gamma+}$ ($N^{\gamma-}$) means $\le C(\varepsilon)N^{\gamma+\varepsilon}$ for all $\varepsilon > 0$ and some $C(\varepsilon) > 0$ (resp., $\ge c(\varepsilon)N^{\gamma-\varepsilon}$ for all $\varepsilon > 0$, $c(\varepsilon) > 0$). We need a slightly stronger statement, namely that the same bounds hold with the factors in question obeying the inequalities $\ll N^{\gamma}(\log N)^{C}$ or $\gg N^{\gamma}(\log N)^{-C}$, respectively, for some appropriate choice of a constant C. A careful examination of the proof in [B99] shows that it remains valid with this new meaning of the notation $N^{\gamma+}$ and $N^{\gamma-}$, and that one may in fact take $C_1 = 5$, $C_2 = 10$. We further note that although Bourgain states his lemma for $A, B \subset \mathbb{Z}^d$, the same proof works for $A, B \subset \mathbb{R}$ if the exponential sum inequality [B99,(2.7)] is replaced by

$$|G| < \int_S \chi_A * \chi_B \le |S|^{1/2} \|\chi_A * \chi_B\|_2;$$

we then observe that

$$\|\chi_A * \chi_B\|_2^2 = |\{(a, a', b, b') \in A \times A \times B \times B : a + b = a' + b'\}|$$

$$= |\{(a, a', b, b') \in A \times A \times B \times B : a - b' = a' - b\}| = \|\chi_A * \chi_{-B}\|_2^2,$$

and proceed further as in [B99]. A similar modification should be made in [B99,(2.36)].

§4. PROOF OF THEOREM 4

In this section we prove Theorem 4; note that this also proves the implication (ii)⇒(i) of Theorem 3.

Suppose that BX is a polygon with finitely many sides for which the conclusion of the theorem fails, i.e. that there is a well distributed set S such that for any $\epsilon > 0$ there is an increasing sequence of positive integers $N_1, N_2, \dots \to \infty$ with

$$|\Delta_{X,N_j}(S)| < \epsilon N_j \psi(N_j), \tag{4.1}$$

where

$$\psi(N) = \log N / \log\log N.$$

Without loss of generality we may assume that ∂BX contains a vertical line segment and a horizontal line segment, and that $c_1 B l_2^2 \subset BX \subset B l_2^2$. Let also $c_2 \in (0, 1/10)$ be a small constant such that all sides of BX have length at least $8c_2$.

Let M be a sufficiently large number which may depend on ϵ; all other constants in the proof will be independent of ϵ. Let $T = N_{j_0}$ for some j_0 large enough so that $T > M$, and let $N = c_2 T$. Suppose that one of the two vertical sides of BX is the line segment $\{(x_1, x_2) : \ x_1 = v_1, |x_2 - v_2| \le r\}$, where $v_1 > 0$. Let also $Q = Int\,(N \cdot BX)$, $v = (v_1, v_2)$, and

$$A = \{x_1 : \ (x_1, x_2) \in S \cap Q \text{ for some } x_2\},$$

$$Q' = Q + (T - 2N)v.$$

Observe that both Q and Q' have Euclidean diameter $\le 2N$, and that

$$Q' \subset \{(x_1, x_2) : (T - 3N)v_1 < x_1 < (T - N)v_1\},$$

so that

$$\|x - x'\|_X \geq (1 - 4c_2)T > T/2, \ x \in Q, x' \in Q'.$$

By our choice of c_2 we have $c_2 \leq r/4$, so that

$$T/2 \cdot r \geq 2N.$$

Hence all X-distances between points in Q and Q' are measured using the vertical segments of ∂BX, i.e.

$$\|x - x'\|_X = |x_1 - x_1'|/v_1, \ x = (x_1, x_2) \in Q, x' = (x_1', x_2') \in Q_t.$$

Next, we claim that

$$|\{\|x - x'\|_X : \ x \in S \cap Q, x' \in S \cap Q'\}| < K_0 \epsilon N \psi(N), \tag{4.2}$$

where K_0 is a constant depending only on c_2. Indeed, we have

$$\{\|x - x'\|_X : \ x \in Q, x' \in Q'\} \subset [0, T],$$

hence the failure of (4.2) would imply that

$$|\Delta_{X,T}(S)| \geq K_0 \epsilon N \psi(N) \geq \epsilon T \psi(T),$$

if K_0 is large enough (at the last step we used that $\psi(N) \gg \psi(c_2^{-1}N) = \psi(T)$). But this contradicts (4.1).

It follows that if we define

$$A' = \{x_1' : \ (x_1', x_2') \in S \cap Q' \text{ for some } x_2'\},$$

then we can estimate the cardinality of the difference set $A - A'$ using (4.2):

$$|A - A'| < K_0 \epsilon N \psi(N). \tag{4.3}$$

On the other hand, since S is well distributed, we must have

$$|A|, |A'| \gg N. \tag{4.4}$$

Hence by Corollary 3.2 we have

$$d_A \ll \epsilon \psi(N). \tag{4.5}$$

We may now repeat the same argument with the vertical side of ∂BX replaced by its other sides. In particular, using the horizontal segment in ∂BX instead, we obtain the following. Let

$$B = \{x_2 : \ (x_1, x_2) \in S \cap Q \text{ for some } x_1\},$$

then there is a set $B' \subset \mathbb{R}$ such that

$$|B|, |B'| \gg N, \tag{4.6}$$

$$(4.7) \qquad |B - B'| < K_0 \epsilon N \psi(N),$$

$$(4.8) \qquad d_B \ll \epsilon \psi(N).$$

Furthermore, assume that ∂BX contains a segment of a line $x_1 + \alpha x_2 = \beta$, then

$$(4.9) \qquad |\{x_1 + \alpha x_2 : \ (x_1, x_2) \in S \cap Q\}| \le K_0 \epsilon N \psi(N);$$

this estimate is an easier analogue of (4.3) obtained by counting distances between points in Q and just one point in the appropriate analogue of Q'.

Suppose that ∂BX contains segments of lines $x_1 + \alpha_1 x_2 = C_1$, $x_2 + \alpha_2 x_2 = C_2$ (i.e. with slopes $-1/\alpha_1$, $-1/\alpha_2$), where α_1, α_2 are neither 0 nor ∞, and that the ratio α_1/α_2 is transcendental. Let $G = (A \times B) \cap S$, then $|G| \ge c_4 N^2$ since S is well distributed. By (4.4), (4.6), and (4.9) with $\alpha = \alpha_1$, the assumptions of Lemma 3.8 are satisfied with N replaced by $K_0 \epsilon N \psi(N)$ and $\delta = c_4 (K_0 \epsilon \psi(N))^{-2}$. We conclude that there are subsets $A_1 \subset A$ and $B_1 \subset B$ such that

$$(4.10) \qquad |(A_1 \times B_1) \cap G| \gg N^2 \epsilon^c (\log N)^{-c},$$

$$(4.11) \qquad |A_1 - \alpha_1 B_1| \ll N^{-1} \epsilon^{-c} (\log N)^c |(A_1 \times B_1) \cap G|.$$

Here and below, c denotes a constant which may change from line to line but is always independent of N. We also simplified the right sides of (4.10) and (4.11) by noting that $\psi(N) \le \log N$.

Similarly, applying Lemma 3.8 with G replaced by $(A_1 \times B_1) \cap G$ and α_1 replaced by α_2, we find subsets $A_2 \subset A_1$ and $B_2 \subset B_1$ such that

$$(4.12) \qquad |(A_2 \times B_2) \cap G| \gg N^2 \epsilon^c (\log N)^{-c},$$

$$(4.13) \qquad |A_2 - \alpha_2 B_2| \ll N^{-1} \epsilon^{-c} (\log N)^c |(A_2 \times B_2) \cap G|.$$

Clearly, (4.11) also holds with A_1, B_1 replaced by A_2, B_2.

Thus A_2, B_2 satisfy the assumptions (3.14), (3.15) of Corollary 3.7, with $K = \epsilon^{-c} (\log N)^c$. By (4.4), (4.5) and Corollary 3.7, we must have for some constants c, K_2,

$$cN \le |A_2| < (\epsilon^{-1} \log N)^{K_2 \epsilon \log N / \log \log N},$$

hence

$$\log c + \log N \le \frac{K_2 \epsilon \log N}{\log \log N} (\log \log N - \log \epsilon) \le 2 K_2 \epsilon \log N,$$

a contradiction if ϵ was chosen small enough. This proves that if (0.5) fails, then the ratio between any two slopes, other than 0 or ∞, of sides of BX is algebraic.

To conclude the proof of the theorem, we first observe that if BX is a rectangle, then there is nothing to prove. If BX is a hexagon with slopes $0, \infty, \alpha$, we may always find a coordinate system as in Theorem 3 (i); namely, if we let

$$(4.14) \qquad x_1' = x_1, \ x_2' = \alpha x_2,$$

then the slopes 0 and ∞ remain unchanged, and lines $\alpha x_1 - x_2 = C$ with slope α are mapped to lines $x_1' - x_2' = C/\alpha$ with slope 1. Finally, suppose that BX is a polygon with slopes $0, \infty, \alpha_1, \alpha_2, \dots, \alpha_l$, and apply the linear transformation (4.14) with $\alpha = \alpha_1$. Then the sides of ∂BX with slope α_1 is mapped to line segments with slope 1; moreover, since the ratios α_j/α_1, $j = 2, 3, \dots, l$, remain unchanged in the new coordinates, and since we have proved that these ratios are algebraic, all remaining sides of ∂BX are mapped to line segments with algebraic slopes.

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