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# Distance spectrum of graph compositions*

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**Abstract**

The $D$-eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_p\}$ of a graph $G$ are the eigenvalues of its distance matrix $D$ and form the distance spectrum or $D$-spectrum of $G$ denoted by $spec_D(G)$. In this paper we obtain the $D$-spectrum of the cartesian product of two distance regular graphs. The $D$-spectrum of the lexicographic product $G[H]$ of two graphs $G$ and $H$ when $H$ is regular is also obtained. The $D$-eigenvalues of the Hamming graphs $Ham(d, n)$ of diameter $d$ and order $n^d$ and those of the $C_4$ nanotori, $T_{k,m,C_4}$ are determined.

*Keywords: Distance spectrum, Cartesian product, lexicographic product, Hamming graphs, $C_4$ nanotori.*

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## 1 Introduction

Adjacency matrix of a graph and its spectrum have arisen as a natural tool with which one can study graphs and its structural properties. Also the adjacency spectrum find applications in quantum theory and chemistry [3]. The idea of distance matrix seems a natural generalization, with perhaps more specificity than that of an adjacency matrix. Distance matrix and their spectra have arisen independently from a data communication problem [7] studied by Graham and Pollack in 1971 in which the most important feature is the number of negative eigenvalues of the distance matrix. While the problem of computing the characteristic polynomial of adjacency matrix and its spectrum appears to be solved for many large graphs, the related distance polynomials have received much less attention. The distance matrix is more complex than the ordinary adjacency matrix of a graph since the distance matrix is a complete matrix (dense) while the adjacency matrix often is very sparse. Thus the computation of the characteristic polynomial of the distance matrix is

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computationally a much more intense problem and, in general, there are no simple analytical solutions except for a few trees [6]. For this reason, distance polynomials of only trees have been studied extensively in the mathematical literature [6, 16].

The distance matrix of a graph has numerous applications to chemistry and other branches of science. The distance matrix, contains information on various walks and self-avoiding walks of chemical graphs, is immensely useful in the computation of topological indices such as the Wiener index, is useful in the computation of thermodynamic properties such as pressure and temperature coefficients and it contains more structural information compared to a simple adjacency matrix. In addition to such applications in chemical sciences, distance matrices find applications in music theory, ornithology, molecular biology, psychology, archeology etc. For a survey see [1] and also the papers cited therein.

Let G be a connected graph with vertex set $V(G) = \{u_1, u_2, \ldots, \ldots, u_p\}$. The distance matrix $D = D(G)$ of $G$ is defined so that its $(i, j)$-entry is equal to $d_G(u_i, u_j)$, the distance (= length of the shortest path [2]) between the vertices $u_i$ and $u_j$ of $G$. The eigenvalues of $D(G)$ are said to be the $D$-eigenvalues of $G$ and form the distance spectrum or the $D$-spectrum of $G$ , denoted by $spec_D(G)$.

The characteristic polynomial of the $D$-matrix and the corresponding spectra have been considered in [4, 6, 7, 8]. For some recent works on $D$-spectrum see [9, 10, 11, 12, 13, 18].

For two graphs, the ordinary spectrum of graph compositions is well explored and generalized results of NEPS of graphs are presented in [3]. Such studies for the distance spectrum did not appear in literature yet and hence in this paper we present the following.

Let $G$ and $H$ be two graphs. Let $G + H$ and $G[H]$ denote the cartesian product and lexicographic product of $G$ and $H$ respectively [3].

In this paper we first derive the $D$-spectrum of $G + H$ and $G[H]$. By means of this, the distance spectrum of the Hamming graph and $C_4$ nanotori are obtained. A work of this type is reported here for the first time.

All graphs considered in this paper are simple and we follow [3] for spectral graph theoretic terminology and [2] for distance in graphs. The considerations in the subsequent sections are based on the applications of the following lemmas.

**Lemma 1.1** ([3]). *Let $G$ be an $r$-regular graph on $p$ vertices with adjacency eigenvalues $r, \lambda_2, \ldots, \lambda_p$. Then $G$ and its complement $\overline{G}$ have the same eigenvectors, and the eigenvalues of $\overline{G}$ are $p - r - 1, -1 - \lambda_2, \ldots, -1 - \lambda_p$.*

**Lemma 1.2** ([5]). *The distance spectrum of the cycle $C_n$ is given by*

| $n$ | *greatest eigenvalue* | *j even* | *j odd* |
|---|---|---|---|
| *even* | $\dfrac{n^2}{4}$ | 0 | $-\operatorname{cosec}^2\left(\dfrac{\pi j}{n}\right)$ |
| *odd* | $\dfrac{n^2-1}{4}$ | $-\dfrac{1}{4}\sec^2\left(\dfrac{\pi j}{2n}\right)$ | $-\dfrac{1}{4}\operatorname{cosec}^2\left(\dfrac{\pi j}{2n}\right)$ |

**Definition 1.3** ([14]). The Hamming graph $Ham(d, n), d \geq 2, n \geq 2$, of diameter $d$ and characteristic $n$ have vertex set consisting of all $d$-tuples of elements taken from an $n$-element set, with two vertices adjacent if and only if they differ in exactly one coordinate. $Ham(d, n)$ is equal to $\underbrace{K_n + K_n + \cdots + K_n}_{d}$, the cartesian product of $K_n$, the complete graph on $n$ vertices, $d$ times. $Ham(3, n)$ is referred to as a cubic lattice graph.

**Lemma 1.4** ([17]). *Let $G$ and $H$ be two connected graphs, and let $u = (u_1, u_2),\ v = (v_1, v_2) \in V(G) \times V(H)$. Let $G + H$ denote their cartesian product. Then*

$$d_{G+H}(u, v) = d_G(u_1, v_1) + d_H(u_2, v_2).$$

## 2 The $D$-spectrum of $G + H$

In this section we derive the $D$-spectrum of the cartesian product of two distance regular graphs.

**Theorem 2.1.** *Let $G$ and $H$ be two distance regular graphs on $p$ and $n$ vertices with distance regularity $k$ and $t$ respectively. Let $spec_D(G) = \{k, \mu_2, \mu_3, \ldots, \mu_p\}$ and $spec_D(H)$ $= \{t, \eta_2, \eta_3, \ldots, \eta_n\}$. Then*

$$spec_D(G + H) = \{nk + pt, n\mu_i, p\eta_j, 0\}$$

*$i = 2, \ldots, p$ , $j = 2, \ldots, n$ and $0$ is with multiplicity $(p - 1)(n - 1)$.*

*Proof.* Let $D_G$ and $D_H$ be the distance matrices of $G$ and $H$ respectively. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$. Then $D_G = [d_{ij}]$ and $D_H = [e_{ij}]$ where $d_{ij} = d_G(u_i, u_j)$ and $e_{ij} = d_H(v_i, v_j)$. Since $G$ and $H$ are distance regular graphs with distance regularities $k$ and $t$ respectively, we have

$$\sum_{j=1}^{p} d_{rj} = k \quad \text{and} \quad \sum_{j=1}^{n} e_{qj} = t \tag{2.1}$$

Also since $G$ is distance regular, the all one column vector of order $p \times 1$ is the eigenvector corresponding to the greatest eigenvalue $k$ of $D_G$. As $D_G$ is real and symmetric, it is diagonalizable and hence admits an orthogonal basis $B_G$ consisting of eigenvectors corresponding to its eigenvalues. Thus if $\mu_i$ is an eigenvalue of $D_G$ which is different from $k$ with an eigenvector $X_i = \left[x_i^1, x_i^2, \ldots, x_i^p\right]^T \in B_G$, then $\sum\limits_{j=1}^{p} x_i^j = 0$.

Let $u = (u_1, u_2),\ v = (v_1, v_2) \in V(G) \times V(H)$. Then by Lemma 1.4

$$d_{G+H}(u, v) = d_G(u_1, v_1) + d_H(u_2, v_2).$$

By a suitable ordering of vertices in $G + H$ and by virtue of Lemma 1.4, its $D$-matrix, $C$ can be written in the form

$$C = \begin{bmatrix} d_{11}+e_{11} & \cdots & d_{11}+e_{1n} & \cdots & \cdots & d_{1p}+e_{11} & \cdots & d_{1p}+e_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{11}+e_{n1} & \cdots & d_{11}+e_{nn} & \cdots & \cdots & d_{1p}+e_{n1} & \cdots & d_{1p}+e_{nn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{p1}+e_{11} & \cdots & d_{p1}+e_{1n} & \cdots & \cdots & d_{pp}+e_{11} & \cdots & d_{pp}+e_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{p1}+e_{1n} & \vdots & d_{p1}+e_{nn} & \cdots & \cdots & d_{pp}+e_{n1} & \cdots & d_{pp}+e_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} d(u_1,u_1)\cdot J_n + D_H & d(u_1,u_2)\cdot J_n + D_H & \cdots & \cdots & d(u_1,u_p)\cdot J_n + D_H \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d(u_p,u_1)\cdot J_n + D_H & d(u_p,u_2)\cdot J_n + D_H & \cdots & \cdots & d(u_p,u_p)\cdot J_n + D_H \end{bmatrix}$$

$$= D_G \otimes J_n + J_p \otimes D_H$$

where $\otimes$ denotes the tensor product of matrices.
Now we find the eigenvalues of $C$ by considering eigenvectors associated with them. The following relation for matrices is well known [15]. For the matrices $A, B, C$ and $D$

$$(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$$

whenever the products $AC$ and $BD$ exist.

Let $\mathbf{1}_G$ denote the all one eigenvector corresponding to the eigenvalue $k$ of $G$ and $\mathbf{1}_H$ the all one eigenvector corresponding to the eigenvalue $t$ of $H$. Then

$$D_G \cdot \mathbf{1}_G = k\mathbf{1}_G \quad \text{and} \quad D_H \cdot \mathbf{1}_H = t\mathbf{1}_H$$

Therefore

$$\begin{aligned} C \cdot (\mathbf{1}_G \otimes \mathbf{1}_H) &= (D_G \otimes J_n + J_p \otimes D_H) \cdot (\mathbf{1}_G \otimes \mathbf{1}_H) \\ &= (D_G \cdot \mathbf{1}_G) \otimes (J_n \mathbf{1}_H) + (J_p \mathbf{1}_G) \otimes (D_H \cdot \mathbf{1}_H) \\ &= k\mathbf{1}_G \otimes n\mathbf{1}_H + p\mathbf{1}_G \otimes t\mathbf{1}_H \\ &= nk \cdot (\mathbf{1}_G \otimes \mathbf{1}_H) + pt \cdot (\mathbf{1}_G \otimes \mathbf{1}_H) \\ &= (nk + pt) \cdot (\mathbf{1}_G \otimes \mathbf{1}_H) \end{aligned}$$

showing that $\mathbf{1}_G \otimes \mathbf{1}_H$ is the eigenvector corresponding to the eigenvalue $nk + pt$ of $C$.

Let $X_i$ be the eigenvector corresponding to the eigenvalue $\mu_i$ of $D_G$. Then $X_i \otimes \mathbf{1}_H$ is the eigenvector corresponding to the eigenvalue $n\mu_i$ of $C$. For

$$\begin{aligned} C \cdot (X_i \otimes \mathbf{1}_H) &= (D_G \otimes J_n + J_p \otimes D_H) \cdot (X_i \otimes \mathbf{1}_H) \\ &= (D_G \cdot X_i) \otimes (J_n \mathbf{1}_H) + (J_p X_i) \otimes (D_H \cdot \mathbf{1}_H) \\ &= \mu_i X_i \otimes n\mathbf{1}_H + 0 \otimes t\mathbf{1}_H \\ &= n\mu_i (X_i \otimes \mathbf{1}_H) \end{aligned}$$

Similarly if $Z_j$ is an eigenvector corresponding to the eigenvalue $\eta_j$ of $D_H$, then $\mathbf{1}_G \otimes Z_j$ is an eigenvector corresponding to the eigenvalue $p\eta_j$ of $C$.

In addition to these eigenvalues we can see that 0 appears to be an eigenvalue with multiplicity $(p-1)(n-1)$. For let $R_p^i,\ i = 2, 3, \ldots, p$ be the $(p-1)$ linearly independent eigenvectors corresponding to the eigenvalue 0 of $J_p$ and $T_n^j,\ j = 2, 3, \ldots, n-1$ be the $(n-1)$ linearly independent eigenvectors corresponding to the eigenvalue 0 of $J_n$. Then the $(p-1)(n-1)$ vectors $R_p^i \otimes T_n^j$ are linearly independent and are the eigenvectors corresponding to 0 of $C$. For

$$\begin{aligned}
C \cdot \left(R_p^i \otimes T_n^j\right) &= \left(D_G \otimes J_n + J_p \otimes D_H\right) \cdot \left(R_p^i \otimes T_n^j\right) \\
&= \left(D_G \cdot R_p^i\right) \otimes \left(J_n \cdot T_n^j\right) + \left(J_p R_p^i\right) \otimes \left(D_H \cdot T_n^j\right) \\
&= \left(D_G \cdot R_p^i\right) \otimes 0 + 0 \otimes \left(D_H \cdot T_n^j\right) \\
&= 0
\end{aligned}$$

Now the $pn$ vectors $X_i \otimes \mathbf{1}_H, \mathbf{1}_G \otimes Z_j$ and $R_p^i \otimes T_n^j$ are linearly independent and as $C$ has a basis consisting of linearly independent eigenvectors, the theorem follows. □

### 2.1 The $D$-spectrum of $Ham(d, n)$

In [14], the ordinary spectrum of the cubic lattice graph is obtained. In this section we use Theorem 2.1 to obtain the $D$-spectrum of $Ham(d, n)$.

**Theorem 2.2.** *Let $Ham(d, n)$ be the Hamming graph of characteristic $n$. Then the $D$-eigenvalues of $Ham(d, n)$ are $dn^{d-1}(n-1)$, $0$ and $-n^{d-1}$ with multiplicities $1,\ n^d - d(n-1) - 1$ and $d(n-1)$ respectively.*

*Proof.* The graph $K_n$ is distance regular with distance regularity $n-1$. Now the proof follows by repeated application of Theorem 2.1 and from the ordinary spectrum of $K_n$ [3]. □

### 2.2 The $D$-spectrum of the $C_4$ nanotori, $T_{k,m,C_4}$

The graph $C_k + C_m$ where both $k$ and $m$ are odd is defined as the $C_4$ nanotori, $T_{k,m,C_4}$.

**Theorem 2.3.** *The distance spectrum of the $C_4$ nanotori, $T_{k,m,C_4}$ consists of the following numbers*

$$\frac{(m+k)(mk-1)}{4},\ -\frac{m}{4}\sec^2\left(\frac{\pi j}{2k}\right),\ -\frac{m}{4}\operatorname{cosec}^2\left(\frac{\pi r}{2k}\right),$$
$$-\frac{k}{4}\sec^2\left(\frac{\pi t}{2m}\right),\ -\frac{k}{4}\operatorname{cosec}^2\left(\frac{\pi l}{2m}\right)$$

*where $j \in \{1, 2, \ldots, k-1\}$ and even, $r \in \{1, 2, \ldots, k-1\}$ and odd $t \in \{1, 2, \ldots, m-1\}$ and even and $l \in \{1, 2, \ldots, m-1\}$ and odd together with $0$ of multiplicity $(m-1)(k-1)$.*

*Proof.* The cycle $C_{2n+1}$ is distance regular with distance regularity $n(n+1)$. Now the proof follows from Theorem 2.1 and Lemma 1.2. □

# 3 The $D$-spectrum of $G[H]$

In this section we obtain the distance spectrum of the lexicographic product $G[H]$ of two graphs $G$ and $H$. The following definition of the lexicographic product of $G$ and $H$ is from [3].

**Definition 3.1.** Let $G$ and $H$ be two graphs on vertex sets $V(G) = \{u_1, u_2, \ldots, \ldots, u_p\}$ and $V(H) = \{v_1, v_2, \ldots, \ldots, v_n\}$ respectively. Then their lexicographic product $G[H]$ is a graph defined by $V(G[H]) = V(G) \times V(H)$, the cartesian product of $V(G)$ and $V(H)$ in which $u = (u_1, v_1)$ be adjacent to $v = (u_2, v_2)$ if and only if either

1. $u_1$ be adjacent to $v_1$ in $G$ or
2. $u_1 = v_1$ and $u_2$ be adjacent to $v_2$ in $G$.

## Distance in $G[H]$

We prove the following lemma on distance in lexicographic product of graphs.

**Lemma 3.2.** *Let G and H be two connected graphs with atleast two vertices and let $u = (u_1, v_1)$, $v = (u_2, v_2) \in V(G) \times V(H)$. Then*

$$d_{G[H]}(u, v) = \begin{cases} d_G(u_1, u_2) \text{ if } u_1 \neq u_2 \\ 1 \text{ if } u_1 = u_2 \text{ and } v_1 \text{ adjacent to } v_2 \\ 2 \text{ if } u_1 = u_2 \text{ and } v_1 \text{ not adjacent to } v_2 \end{cases}$$

*Proof.* We show that in the corresponding composition there exist a path between $u$ and $v$ of length as given in the lemma. Let $d_G(u_1, u_2) = t$ and $u_1 = s_0, s_1, \ldots, s_t = u_2$ be the shortest $u_1 - u_2$ path in $G$.
Let $u = (u_1, v_1)$, $v = (u_2, v_2) \in V(G) \times V(H)$ and $u_1 \neq u_2$. Since the successive ordered pairs in any $u - v$ path can change both the coordinates and also as $u_2$ is reachable from $u_1$ by not less that $t$ steps, any $u - v$ path in $G[H]$ is of length atleast $t$.

Now the following $u - v$ path in $G[H]$ is of length $t$.

$P : u = (s_0, v_1), (s_1, v_2), (s_2, v_2), \ldots, (s_t, v_2) = v$. Thus $d_{G[H]}(u, v) = d_G(u_1, u_2)$ if $u_1 \neq u_2$.

Now suppose $u_1 = u_2$ and $v_1$ be adjacent to $v_2$. Then by the definition of $G[H]$, we have $d_{G[H]}(u, v) = 1$.

Now suppose $u_1 = u_2$ and $v_1$ is not adjacent to $v_2$. Let $s_1$ be adjacent to $u_1$ in $G$. Then $u$ is not adjacent to $v$ and $u = (u_1, v_1), (s_1, v_2), (u_1, v_2) = v$ is a $u - v$ path of length 2. Thus $d_{G[H]}(u, v) = 2$. Hence the Lemma. □

**Theorem 3.3.** *Let $G$ be a graph with $D$-matrix $D_G$ and $H$, an $r$-regular graph with an adjacency matrix $A$. Let $spec_D(G) = \{\mu_1, \mu_2, \ldots, \mu_p\}$ and the ordinary spectrum of $H$ be $\{r, \lambda_2, \lambda_3, \ldots, \lambda_n\}$. Then*

$$spec_D G[H] = \begin{pmatrix} n\mu_i + 2n - r - 2 & -(\lambda_j + 2) \\ 1 & p \end{pmatrix}, \ i = 1 \text{ to } p \text{ and } j = 2 \text{ to } n - 1$$

*Proof.* Using Lemma 3.2 and by a suitable ordering of vertices of $G[H]$, its $D$-matrix $F$, can be written in the form

$$F=\left[\begin{array}{cccccccccccc}
 & & & d_{12} & \cdots & d_{12} & d_{13} & \cdots & d_{13} & \cdots & \cdots & d_{1p}\\
 & A+2\overline{A} & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
 & & & d_{12} & d_{12} & d_{12} & d_{13} & \cdots & d_{13} & \cdots & \cdots & d_{1p}\\
d_{21} & \cdots & d_{21} & & & & \cdots & \cdots & \cdots & d_{2p} & \cdots & d_{2p}\\
\vdots & \vdots & \vdots & & A+2\overline{A} & & & & & \vdots & \vdots & \vdots\\
d_{21} & \cdots & d_{21} & & & & & & & d_{2p} & \cdots & d_{2p}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
d_{p1} & \cdots & d_{p1} & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & A+2\overline{A} & \\
d_{p1} & \cdots & d_{p1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & &
\end{array}\right]$$

$$= D_G\otimes J_n + I_p\otimes (A+2\overline{A})$$

where $\overline{A}$ denote the adjacency matrix of $\overline{G}$.

Since $H$ is $r$-regular, the all one column vector $\mathbf{1}$ of order $n\times 1$ is an eigenvector of $A$ with an eigenvalue $r$. Then by Lemma 1.1, the all one vector $\mathbf{1}$ is an eigenvector of $A+2\overline{A}$ with an eigenvalue $2n-r-2$. Similarly if $\lambda_j$ is any other eigenvalue of $A$ with eigenvector $Y_j$, then $Y_j$ is an eigenvector of $A+2\overline{A}$ with eigenvalue $-(\lambda_j+2)$ and that $Y_j$ is orthogonal to $\mathbf{1}$.

Let $X_i=\left[\begin{array}{cccc} x_1^i & x_2^i & \ldots & x_p^i\end{array}\right]^T$ be an eigenvector corresponding to the eigenvalue $\mu_i$ of $D_G$. Therefore

$$D_G\cdot X_i=\mu_i X_i$$

Now

$$\begin{aligned}
F\cdot(X_i\otimes\mathbf{1}_n) &= \left(D_G\otimes J_n+I_p\otimes\left(A+2\overline{A}\right)\right)(X_i\otimes\mathbf{1}_n)\\
&=(D_G\cdot X_i)\otimes(J_n\cdot\mathbf{1}_n)+(I_p\cdot X_i)\otimes\left(A+2\overline{A}\right)\cdot\mathbf{1}_n\\
&=\mu_i X_i\otimes n\mathbf{1}_n+X_i\otimes(2n-r-2)\,\mathbf{1}_n\\
&=n\mu_i\,(X_i\otimes\mathbf{1}_n)+(2n-r-2)\,(X_i\otimes\mathbf{1}_n)\\
&=(n\mu_i+2n-r-2)\,(X_i\otimes\mathbf{1}_n)
\end{aligned}$$

Therefore $n\mu_i+2n-r-2$ is an eigenvalue of $F$ with eigenvector $X_i\otimes\mathbf{1}_n$.
As $Y_j$ is orthogonal to $\mathbf{1}$, we have $J_n\cdot Y_j=0$ for each $j=2,3,\ldots,n$.

Let $\{Z_k\}\,,k=1,2,\ldots,p$ be the family of $p$ linearly independent eigenvectors associated with the eigenvalue 1 of $I_p$. Then for each $j=2,3,\ldots,n$, the $p$ vectors $Z_k\otimes Y_j$ are eigenvectors of $F$ with eigenvalue $-(\lambda_j+2)$. For

$$\begin{aligned}
F\cdot(Z_k\otimes Y_j) &= \left(D_G\otimes J_n+I_p\otimes\left(A+2\overline{A}\right)\right)(Z_k\otimes Y_j)\\
&=(D_G\cdot Z_k)\otimes(J_n\cdot Y_j)+(I_p\cdot Z_k)\otimes\left(A+2\overline{A}\right)\cdot Y_j\\
&=0+Z_k\otimes-(\lambda_j+2)\,Y_j\\
&=-(\lambda_j+2)\cdot(Z_k\otimes Y_j)
\end{aligned}$$

Also the $pn$ vectors $X_i \otimes \mathbf{1}_n$ and $Z_k \otimes Y_j$ are linearly independent. As the eigenvectors belonging to different eigenvalues are linearly independent and as $F$ has a basis consisting entirely of eigenvectors, the theorem follows. $\square$

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