MEMORANDUM
RM-4888-NASA
MARCH 1966

DISTORTION AND CROSSTALK OF LINEARLY FILTERED, ANGLE-MODULATED SIGNALS

E. Bedrosian

PREPARED FOR:

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

The RAND *Corporation*
SANTA MONICA · CALIFORNIA

MEMORANDUM
RM-4888-NASA
MARCH 1966

DISTORTION AND CROSSTALK OF LINEARLY FILTERED, ANGLE-MODULATED SIGNALS

E. Bedrosian

The RAND Corporation
1700 MAIN ST. · SANTA MONICA · CALIFORNIA · 90406

PREFACE

This Memorandum was prepared as part of RAND's continuing study of Communications Satellite Technology for the National Aeronautics and Space Administration. It presents a new formulation which can be used to compute the distortion experienced by an FM signal as it passes throught the various circuits encountered in an FM receiver. The nature of the fundamental expansion used is such that it overcomes the major criticism of previous approaches to the problem.

The theoretical results of the Memorandum should add to the understanding of distortion and crosstalk in FM signals, and therefore constitute a significant contribution to communication theory and practice. The results can be used by engineers designing advanced communication systems to achieve optimum performance.

SUMMARY

An important problem in the theory and practice of receiving
angle-modulated signals is the proper design of the filtering elements
which must be employed. It has been known for some time that such
filtering introduces distortion and crosstalk into the signal, but the
formulations for their computation available heretofore have not been
applicable to the level of accuracy required because of an inherent
limitation in their basic character.

In this Memorandum, an angle-modulated signal having an arbitrary
phase function is applied to a symmetrical, narrow, band-pass filter
and the phase of the output is expanded in a series which converges
for all values of the input phase subject to certain conditions on
the input signal and the filter impulse response. The expansion is
unique in that its leading term is the linearly filtered input rather
than the customary quasi-stationary term.

A spectral analysis is performed by assuming a stochastic modula-
ting signal and examining the third- and fifth-order terms of the out-
put phase expansion. The results are presented in the form of signal,
distortion and crosstalk components of the output spectrum and it is
argued that they represent the principal contributions in the case
where the total distortion is small.

To illustrate the method and to present results of direct utility
for an example of current interest, numerical computations are performed
for the case of an FM signal, having a uniform modulation baseband,
which is passed through a single-pole filter. The spectra of the

signal, distortion and crosstalk components of the resulting signal are then presented graphically.

The importance of this example derives from the fact that the limited signal power available from communication satellites and space probes has stimulated the use of FM signals and frequency feedback receivers. Such receivers are constrained to the use of single-pole filters for reasons of stability and therefore incur considerable distortion and crosstalk in comparison with conventional FM receivers. The numerical results presented here will permit the selection of the proper filter bandwidth in order to attain the maximum threshold reduction while maintaining a specified level of crosstalk and distortion.

ACKNOWLEDGMENTS

CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

f steady-state frequency variable

f_o carrier frequency

t, τ time variables

$\varphi(t), \theta(t)$ input and output phase functions; phase rates if dotted

$g(t), G(f)$ impulse response and transfer function of normalized, equivalent, low-pass filter, Eqs. (2)

$u, v, x, y, \nu, \rho, \sigma$ variables of integration

$\Phi(t)$ linearly filtered input phase, Eq. (8); subscript denotes order of integrand, Eq. (A-8)

$\varphi(t, \omega), \theta(t, \omega)$ sample functions of input and output phase; ω identifies sample function in process space Ω

$R(t), W(f)$ autocorrelation and spectral density functions, Eq. (10); subscript identifies process; superscript identifies component

S, D, C superscripts denoting signal, distortion and crosstalk components of output spectral density

$E, \mathcal{F}$ expectation and Fourier transform operators, Eqs. (10), (11), and (12)

S, D_3, D_5 signal and third- and fifth-order distortion terms in output phase expansion, Eqs. (7), (C-1)

A, B lowest and highest baseband frequencies

D rms deviation of instantaneous frequency

f_c filter 3-db half-bandwidth; underlining other frequency variables denotes normalization with respect to f_c

SDR signal-to-distortion ratio

x, ζ, h general variables

a, b defined functions, Eqs. (A-2), (C-19)

R   series remainder, Eq. (A-9)

$\delta^n f(x;h)$ n^{th} variation of f(x) with increment h, Eq. (B-2)

$R(\zeta), N(\zeta), D(\zeta)$ defined functions, Eq. (B-9); superscript denotes order of derivative

I. INTRODUCTION

One of the most intriguing problems relating to the theory of angle (i.e., frequency or phase) modulation is that of obtaining a useful relationship between the properties of the input and output modulation when an angle-modulated signal is passed through a linear network. From the earliest work by Carson and Fry[1] in 1937 to the present, interest has persisted as others have extended results and added new approaches. These are reviewed in detail by Baghdady[2] and by Panter,[3] and it need be noted here only that the results have generally been presented either as infinite series involving Taylor's series expansions of both the modulating signal and the filter transfer function (Carson and Fry,[1] Bloch,[4] Van der Pol,[5] Stumpers,[6] Clavier[7]), or as finite series involving the poles of the polynomial representation and but one derivative of the modulating signal (Hupert,[8,9] Weiner and Leon[10,11]).

It is characteristic of all these expansions that they are in the form of a leading term called the quasi-stationary response and a remainder constituting a correction term. The quasi-stationary response, a term introduced by Carson and Fry, is obtained by substituting the time-dependent, instantaneous frequency of the modulated signal for the steady-state frequency variable in the network transfer function. The quasi-stationary response is appealing intuitively since it is what might be expected if the frequency variations were made slowly enough. It has been useful analytically because the applications of interest are usually those in which the distortion is

sufficiently small that the correction term can be neglected and the quasi-stationary response alone used in further analysis. As might be expected, considerable attention has been directed to questions of convergence and to obtaining bounds on the correction term (Baghdady,[2] Rowe,[12] Hess,[13] Weiner and Leon[11]).

Initial applications of these techniques were made using deterministic signals. Modulation using one[14-20] or two[21] sinusoids leads to a tractable analysis and yields results useful in determining harmonic distortion and the IF bandwidths required in FM receivers. (See Refs. 13, 22-24.) Swept frequency analyses[25,26] have applications to spectrum analyzers, while responses to frequency steps and impulses[5,27-30] are valuable in studies of impulsive interference, such as ignition noise, or in certain modulation techniques, such as frequency- or phase-shift keying.

More recently, analyses have been performed using stochastic modulating signals. These have been adapted to the square-law phase characteristic of the ionosphere by Shaft[31] and Prosin[32] and to the case of echo distortion by Bennett, Curtis and Rice.[33] Generalized analyses applicable to band-pass filters have been made by Rice,[34] Bosse,[35] Medhurst[36] and Magnusson.[37] While these analyses have made significant contributions by considering more realistic modulating signals, they have been criticized because of their quasi-stationary nature (Bedrosian[38]) or their first-order basis (Enloe and Ruthroff[39]) inasmuch as terms of possibly significant contribution may have been discarded.

Recently, the limited amount of signal power available in communication links from earth satellites and space probes has revived interest in the technique of FM with frequency feedback invented by Chaffee.[40] Considerations of stability require that a single-pole IF filter be used and Enloe[41] shows that optimum threshold reduction requires that the IF filter have a 3-db bandwidth comparable to the larger of the peak frequency deviation or the highest baseband frequency used (Ruthroff and Bodtmann,[42] Giger and Chaffee[43]). Thus, it has become all the more important that a satisfactory theoretical analysis be developed.

In contrast with a conventional FM receiver in which the IF amplifier can be designed to have an essentially linear phase characteristic and a uniform passband with steep skirts, an FM receiver employing feedback is forced to operate over an IF characteristic which is quite nonlinear. Although the feedback process itself helps reduce distortion, significant distortion and intermodulation or crosstalk can occur in practical situations and must be taken into account. In fact, at least one manufacturer of high-quality, multi-channel systems finds the intermodulation sufficiently severe to warrant the use of threshold extension by frequency feedback only when the system falls below the conventional FM threshold.

The analysis presented in this Memorandum does not proceed from the expansions customarily used. A signal with an arbitrary angle modulation is applied to a symmetrical, band-pass filter and an expression is obtained for the angle modulation on the output signal. This output phase is then expanded in a series in which the leading

(first-order) term is recognized as the filtered but undistorted
signal component of the output. The subsequent terms constitute
the higher odd-order contributions to the output. A spectral analysis
using stochastic modulation then derives the signal component of the
output spectrum from the autocorrelation of the first-order term,
the first-order distortion from the cross correlation of the first-
and third-order terms, and the first-order crosstalk from the auto-
correlation of the third-order term. The output signal, distortion
and crosstalk spectra are computed numerically for the specific
example of a signal frequency-modulated by a uniform spectrum of
gaussian noise and passed through a single-pole filter.

Questions of convergence and of higher-order contributions due
to neglecting the fifth- and higher-order terms in the expansion are
considered. In general, it is argued that the first-order distortion
and crosstalk computed herein constitute the principal distortion
contributions to the output in the case of small distortion. Conse-
quently, the results should find wide application in cases of practical
interest.

II. INPUT-OUTPUT PHASE RELATIONSHIP

Consider an angle-modulated signal of unit amplitude having a carrier frequency, f_o cps, and a phase modulation, $\varphi(t)$, which, in complex form, can be written

$$\exp\, i[2\pi f_o t + \varphi(t)] \tag{1}$$

Furthermore, let the properties of $\varphi(t)$ and its derivative, the phase rate $\dot{\varphi}(t)$, be such that the signal is narrow band, i.e., most of its energy is restricted to frequencies in the vicinity of f_o.

Take this angle-modulated signal as the input to a symmetrical, band-pass filter centered at f_o and let the filter be sufficiently narrow that its transfer function can be well approximated by a symmetrical function of the linear off-center frequency difference, $f-f_o$, where f is the steady-state frequency variable. It is then readily shown that the effect of band-pass filtering on the original signal is the same as the effect of filtering by the low-pass equivalent filter on the low-pass equivalent signal. That is, f_o may simply be set equal to zero.

Let $G(f)$ and $g(t)$ denote, respectively, the normalized steady-state transfer function and impulse response of the equivalent low-pass filter. Then G and g are a Fourier pair

$$g(t) = \int_{-\infty}^{\infty} df\, G(f) e^{i2\pi f t}, \quad G(f) = \int_{0}^{\infty} dt\, g(t) e^{-i2\pi f t} \tag{2}$$

where $g(t)$ is real and vanishes for $t < 0$, and $G(f)$ is complex with an even real part and an odd imaginary part so that

$$G(f) = G^*(-f) \tag{3}$$

where * denotes the complex conjugate. The normalization chosen is

$$\int_0^\infty du\ g(u) = G(0) = 1 \tag{4}$$

The filter output signal is given by the convolution of the input with the filter impulse response

$$\int_0^\infty du\ g(u)\ \exp\ [i\ \varphi(t-u)] \tag{5}$$

and the output phase by the argument of Eq. (5)

$$\theta(t) = \tan^{-1} \frac{\int_0^\infty du\ g(u)\sin\ \varphi(t-u)}{\int_0^\infty du\ g(u)\cos\ \varphi(t-u)} \tag{6}$$

Expanding as shown in Appendix A then leads to

$$\theta(t) = \Phi(t) - \frac{1}{3!}\int_0^\infty du\ g(u)[\varphi(t-u)-\Phi(t)]^3 + \frac{1}{5!}\int_0^\infty du\ g(u)[\varphi(t-u)-\Phi(t)]^5 \tag{7}$$

$$- \frac{1}{12}\int_0^\infty du\ g(u)[\varphi(t-u)-\Phi(t)]^2 \int_0^\infty du\ g(u)[\varphi(t-u)-\Phi(t)]^3 + O(\varphi^7)$$

where

$$\Phi(t) \equiv \int_0^\infty du\ g(u)\varphi(t-u) \tag{8}$$

corresponds to the linearly filtered input phase and $0(\varphi^7)$ denotes

terms of order 7 and higher in $\varphi(t)$. Necessary and sufficient condi-

tions for Eq. (7) to converge for all φ are that the Taylor's series

for the input signal $\exp[i\varphi(t)]$ converge uniformly in the infinite

interval on t, that the complex output signal given by Eq. (5) not

vanish, and that

$$\int_0^\infty du|g(u)| < \sqrt{2} \tag{9}$$

The output phase rate, $\dot{\theta}(t)$, is then simply the time derivative of

Eq. (7), providing the resulting series also converges uniformly.

It is difficult to state general conditions on the behavior at

the input signal based on the foregoing and specific cases must be

considered individually. However, the convergence condition of Eq. (9)

does insure, in effect, that the filter impulse response and transfer

function are sufficiently "well behaved." The condition is clearly

satisfied by a first-order filter--inspection of impulse response

plots shows it to be satisfied by Bessel filters of all orders and by

low-order Butterworth and Tchebycheff filters as well.

The expansion given by Eq. (7) can also be obtained directly as

a functional Taylor's series,[44] as shown in Appendix B. Though the

approach is more straightforward, it does not appear possible to state

conditions of convergence. At best, it confirms that the contributions

of various orders are completely represented in the expansion.

The form of the input-output relationship given by Eq. (7) is

particularly desirable in that the leading term clearly corresponds to

the undistorted signal component of the output and is simply the

linearly filtered input. The remaining terms then constitute the various higher odd orders of distortion and are interesting in that they highlight the difference between the delayed and filtered inputs, i.e., $\varphi(t-u)-\Phi(t)$, as the expansion parameter. This quantity is generally the least for that delay at which the impulse response is the greatest. For large delays, its growth is easily overcome by the eventual exponential decrease of all physically realizable impulse responses. Also, it is worth noting that the compactness of the expansion is itself of considerable value because it greatly facilitates the ensuing spectral analysis.

III. SPECTRAL ANALYSIS

In communications systems analysis, the effect of the small non-linearities encountered in signal processing is frequently better described in terms of the statistical properties of an ensemble of representative signals than by the explicit nature of a specific signal. Typical of the situations in which an angle-modulated signal can suffer significant distortion due to band-pass filtering is one in which the phase or instantaneous frequency of the carrier is made to vary in accordance with the amplitude of a multi-channel, frequency-division-multiplex speech or data baseband signal. For many purposes, such a composite signal is adequately approximated by a random time function having a gaussian distribution of amplitudes. Furthermore, the total distortion will almost invariably be small enough that the distorted signal is also adequately approximated by such a signal.

Under these conditions, an expression of the spectral density of the output signal in terms of the filter transfer function and the spectral density of the input signal, i.e., a second-moment analysis, can provide useful engineering information. This technique will be applied to the input-output phase relationship given by Eq. (7) using the methods described by Rice,[45] even though the use of a gaussian process cannot be justified except as an adequate approximation to the time process. The convergence criterion developed in Appendix A requires that the Taylor's series expansion for the input signal $\exp[i\varphi(t)]$ converge uniformly in the infinite interval on t. Only bounded functions can be shown to satisfy such a requirement, and while the bound can be arbitrarily large, it can always be exceeded

by a gaussian function. It is hoped that the plausibility of the
results will stimulate efforts at analytical justification of the
procedure.

Let $\varphi(t,\omega)$, $\omega\epsilon\Omega$, denote a sample function of a stationary, differentiable, zero-mean, gaussian process having an autocorrelation function $R_\varphi(t)$ and a spectral density $W_\varphi(f)$, where R and W are a Fourier pair

$$R(t) = \int_{-\infty}^{\infty} df\, W(f)e^{i2\pi ft}, \quad W(f) = \int_{-\infty}^{\infty} dt\, R(t)e^{-i2\pi ft} \tag{10}$$

The autocorrelation function is defined by

$$R_\varphi(\tau) = E[\varphi(t,\omega)\varphi(t+\tau,\omega)] \tag{11}$$

where the expectation operator E denotes an ensemble average. The
output spectral density is determined formally from Eq. (7) by using
Eq. (11) to form the autocorrelation function and then by applying
Eq. (10). That is

$$W_\theta(f) = \mathcal{F}E[\theta(t,\omega)\theta(t+\tau,\omega)] \tag{12}$$

where $\mathcal{F}$ denotes the Fourier transform operator and where the operations are applied term-by-term to the pairings which result from using
the expansion for θ. If it is desired to express the results in terms
of the input and output phase rates, the relationships

$$W_{\dot\theta}(f) = (2\pi f)^2 W_\theta(f), \quad W_\varphi(f) = W_{\dot\varphi}(f)/(2\pi f)^2 \tag{13}$$

(Ref. 45, Sec. 3.3) may be used.

To facilitate the identification of the various components of the output spectrum, let the superscripts S, D and C, when applied to the output spectrum $W_\theta(f)$, denote, respectively, the signal, distortion and crosstalk. Then, as shown in detail in Appendix C, a spectral analysis of Eq. (7) yields

$$W_\theta^S(f) = |G(f)|^2 W_\phi(f)$$

$$W_\theta^D(f) = 2W_\phi(f) \int_{-\infty}^{\infty} d\rho \, W_\phi(\rho)$$

$$\times \left\{ \mathrm{Re} \; G(\rho)G(-\rho-f)G(f) - |G(f)|^2 |G(\rho)|^2 \right\}$$

$$W_\theta^C(f) = \frac{1}{6} \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma \, W_\phi(f-\rho-\sigma)W_\phi(\rho)W_\phi(\sigma)$$

$$\times |2G(f-\rho-\sigma)G(\rho)G(\sigma) - G(f-\rho-\sigma)G(\rho+\sigma)$$

$$- G(\rho)G(f-\rho) - G(\sigma)G(f-\sigma) + G(f)|^2$$

(14)

The origins of these components in the expansion of Eq. (7) can be seen by designating the first three terms of the expansion as S, D_3 and D_5, respectively. The leading term is recognized as the signal component and the next two as the third- and fifth-order distortion terms. In this notation, the signal component of the output, $W_\theta^S(f)$, is given by the S×S term in Eq. (12) and is seen to involve only a simple linear operation on the input spectrum $W_\phi(f)$. Also, it is complete in the sense that no further contributions can be made to it by higher-order terms in the expansion.

The distortion and crosstalk components of the output will generally involve operations of all higher orders on the input. Those given in Eq. (14) constitute first-order contributions inasmuch as they are the leading or principal terms of their type. The first-order distortion, $W_\theta^D(f)$, is given by the $S \times D_3$ terms. The $S \times D_5$ and some of the $D_3 \times D_3$ terms give rise to second-order distortion components which are neglected; the first-order crosstalk, $W_\theta^C(f)$, then comes from the balance of the $D_3 \times D_3$ terms.

This distinction between the distortion and crosstalk derives from the presence of the input spectrum, $W_\varphi(f)$, as a multiplicative factor in some of the output spectral terms. These terms, which are designated as distortion, are unique in that their contribution to a given frequency interval in the output is directly related to the amount of input in that same frequency interval; in particular, if there is no input, there will be no output. The contributions of the other components to a given frequency interval in the output are relatively insensitive to the presence or absence of an input in that same frequency interval, since they result from convolutions of the input spectrum and the filter transfer function. By analogy with the intermodulation discernible on an idle telephone channel, these contributions are referred to as crosstalk. Although the crosstalk level is frequently less than the distortion, it is often more objectionable subjectively and therefore constitutes an important system performance criterion.

IV. NUMERICAL EXAMPLE--FM WITH UNIFORM BASEBAND THROUGH A SINGLE-POLE FILTER

As mentioned in the introduction, one of the most important filters to which the foregoing analysis can be applied is the single-pole filter. The normalized equivalent low-pass transfer function of this filter is given by

$$G(f) = \frac{1}{1 + i2\pi f} = \frac{1}{1 + i\underline{f}} \ , \quad \underline{f} = \frac{f}{f_c} \tag{15}$$

where f_c is the 3-db cutoff frequency (which corresponds to the 3-db half-bandwidth of the original band-pass filter). Substituting Eq. (15) into Eqs. (14) yields, after some tedious but straightforward algebra

$$W_\theta^S(f) = \frac{W_\varphi(f)}{1 + \underline{f}^2}$$

$$W_\theta^D(f) = \frac{4\,\underline{f}^2 W_\varphi(f)}{1 + \underline{f}^2}$$

$$\times \int_{-\infty}^{\infty} \frac{d\rho\ \underline{\rho}^2\ W_\varphi(\rho)}{(1+\underline{\rho}^2)[1+(\underline{\rho}+\underline{f})^2]\ [1+(\underline{\rho}-\underline{f})^2]} \tag{16}$$

$$W_\theta^C(f) = \frac{1}{6(1+\underline{f}^2)} \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\ W_\varphi(f-\rho-\sigma)W_\varphi(\rho)W_\varphi(\sigma)(\underline{f}-\underline{\rho}-\underline{\sigma})^2\ \underline{\rho}^2\ \underline{\sigma}^2$$

$$\times \frac{4 + 4\underline{f}^2 + 4\underline{f}(\underline{\rho}+\underline{\sigma})(\underline{f}-\underline{\rho})(\underline{f}-\underline{\sigma}) + (\underline{\rho}+\underline{\sigma})^2(\underline{f}-\underline{\rho})^2(\underline{f}-\underline{\sigma})^2}{[1+(\underline{f}-\underline{\rho}-\underline{\sigma})^2](1+\underline{\sigma}^2)(1+\underline{\rho}^2)[1+(\underline{\rho}+\underline{\sigma})^2]\ [1+(\underline{f}-\underline{\rho})^2]\ [1+(\underline{f}-\underline{\sigma})^2]}$$

where the underlined quantities are normalized to f_c.

An input signal of particular interest is one which is frequency-modulated by a gaussian signal having a uniform spectral density in (A,B) cps, where A $\ll$ B, since such a baseband closely approximates a frequency-division-multiplexed, multi-channel communication signal. The spectral density of the input phase rate can then be written

$$W_{\dot{\varphi}}(f) = \begin{cases} \dfrac{(2\pi\,D)^2}{2(B-A)} & , \quad A \leq |f| \leq B \\[3ex] 0 & , \quad f \quad \text{elsewhere} \end{cases} \tag{17}$$

where D is the rms deviation of instantaneous frequency and where, from Eq. (13), the spectral density of the input phase becomes

$$W_{\varphi}(f) = \begin{cases} \dfrac{D^2}{2(B-A)f^2} & , \quad A \leq |f| \leq B \\[3ex] 0 & , \quad f \quad \text{elsewhere} \end{cases} \tag{18}$$

It is convenient, for simplicity, to set the lowest baseband frequency A equal to zero despite the fact that the rms phase (i.e., the square root of the integral of $W_{\varphi}(f)$ over all f) then does not exist. The expansion for the output phase is, of course, meaningless in that case, but the corresponding expansion for the output phase rate apparently remains valid. However, the only justification given here is that if the spectral density of the output phase rate is computed from Eqs. (13) and (16) using Eq. (18) for $W_{\varphi}(f)$ and if the limiting values are taken as A approaches zero, then the results have

meaning and are identical to those obtained by simply setting A equal
to zero. Consequently, the input phase rate and phase spectra will
be taken as

$$W_{\dot{\varphi}}(f) = \frac{2\pi^2 D^2}{B}$$

$$W_{\varphi}(f) = \frac{D^2}{2B\,f^2} \qquad , \qquad |f| \le B \qquad (19)$$

and zero elsewhere.

Substituting $W_{\varphi}(f)$ from Eq. (19) into Eqs. (16) and using Eq.
(13) to obtain the spectra for the output phase rate then yields

$$\frac{W_{\dot{\theta}}^S(f)}{D^2/f_c} = \frac{2\pi^2}{\underline{B}(1+\underline{f}^2)} \qquad , \qquad |f| \le B$$

$$\frac{W_{\dot{\theta}}^D(f)}{D^4/f_c^3} = \frac{8\pi^2\underline{f}^2}{\underline{B}^2(1+\underline{f}^2)} \int_0^{\underline{B}} \frac{d\underline{\rho}}{(1+\underline{\rho}^2)[1+(\underline{\rho}+\underline{f})^2][1+(\underline{\rho}-\underline{f})^2]} \qquad , \qquad |f| \le B$$

$$\qquad (20)$$

$$\frac{W_{\dot{\theta}}^C(f)}{D^6/f_c^5} = \frac{\pi^2\underline{f}^2}{12\underline{B}^3(1+\underline{f}^2)} \int d\underline{\rho} \int d\underline{\sigma}$$

$$\times \frac{4 + 4\underline{f}^2 + 4\underline{f}(\underline{\rho}+\underline{\sigma})(\underline{f}-\underline{\rho})(\underline{f}-\underline{\sigma}) + (\underline{\rho}+\underline{\sigma})^2(\underline{f}-\underline{\rho})^2(\underline{f}-\underline{\sigma})^2}{[1+(\underline{f}-\underline{\rho}-\underline{\sigma})^2](1+\underline{\sigma}^2)(1+\underline{\rho}^2)[1+(\underline{\rho}+\underline{\sigma})^2][1+(\underline{f}-\underline{\rho})^2][1+(\underline{f}-\underline{\sigma})^2]}$$

where it is understood that the terms vanish in those frequency inter-
vals in which they are not defined and where underlining indicates
normalization to f_c. The region of integration for the crosstalk term

is shown in Fig. 1. Since spectral densities are even functions of frequency, these spectra will be plotted subsequently only for positive frequencies.

The spectral density of the signal component of the output phase rate given by the first of Eqs. (20) is plotted as a function of the relative baseband frequency, f/B, for a number of baseband-to-filter half-bandwidth ratios, B/f_c, in Fig. 2, and in decibels in Fig. 3. As might be expected, the output spectral density is fairly uniform when the filter is wide, i.e., B/f_c small. As the filter is narrowed, the output takes the shape of the filter power response and, indeed, the spectral density is down 3 db at the highest baseband frequency for a filter for which $B/f_c = 1$.

The spectral density of the distortion component of the output phase rate given by the second of Eqs. (20) can be integrated by expanding the integrand in partial fractions. The result is

$$\frac{W_{\dot{\theta}}^D(f)}{D^4/f_c^3} = \frac{8\pi^2}{\underline{B}^2(\underline{f}^2+1)(\underline{f}^2+4)} \left\{ \tan^{-1}\underline{B} + \frac{3\underline{f}}{8(\underline{f}^2+1)} \log\left[\frac{(\underline{B+f})^2+1}{(\underline{B-f})^2+1}\right]\right.$$

$$\left. + \frac{\underline{f}^2-2}{4(\underline{f}^2+1)}\left[\tan^{-1}(\underline{B+f}) + \tan^{-1}(\underline{B-f})\right]\right\}, \quad |f| \leq B$$

(21)

which is plotted in Figs. 4 and 5. The distortion increases with baseband frequency in general, going as the square of the frequency for wide filters. A drop-off at the higher frequencies, resembling the skirt of the filter power response, occurs as the filter is narrowed, i.e., as f_c is decreased thereby increasing B/f_c.
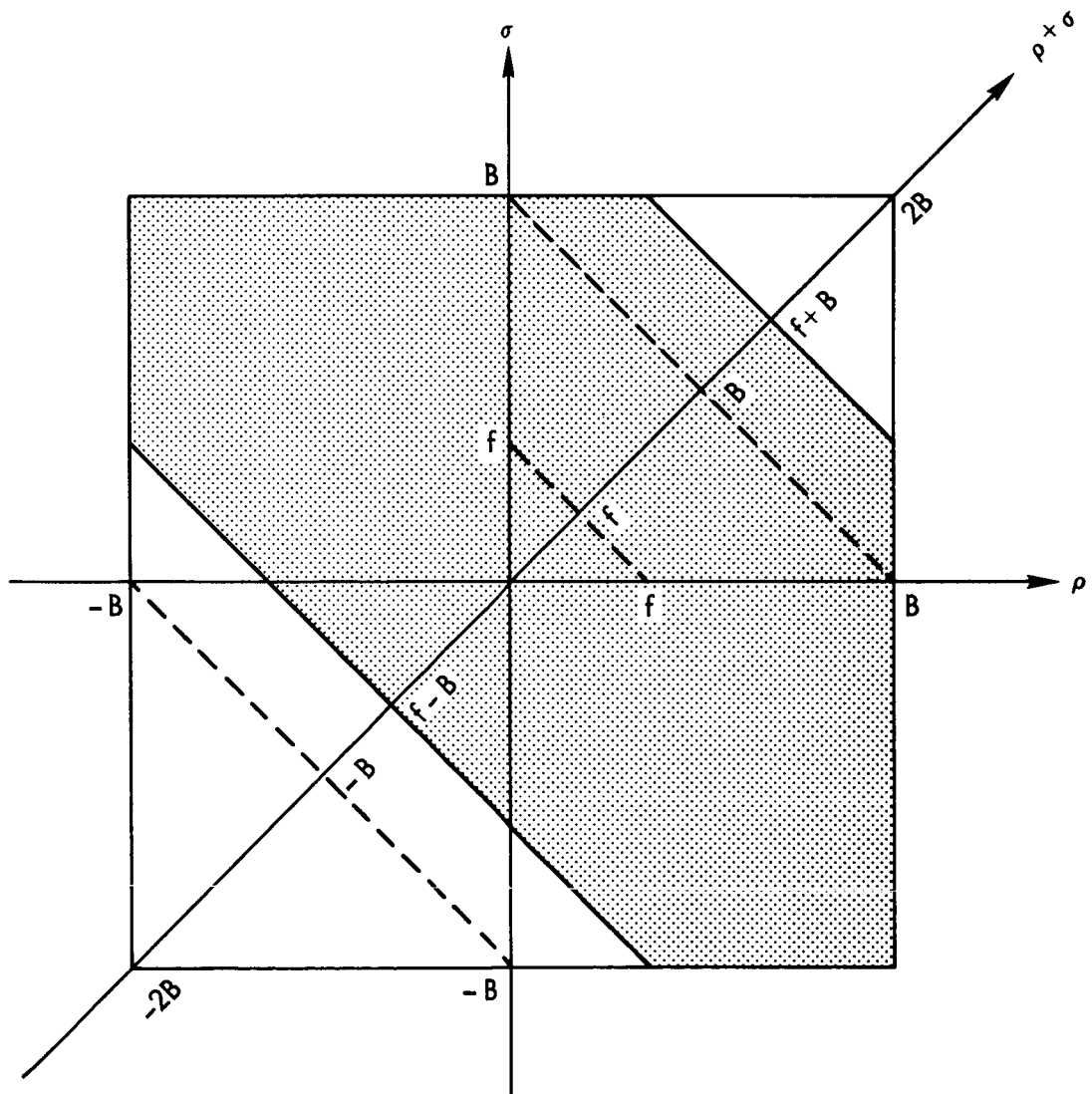
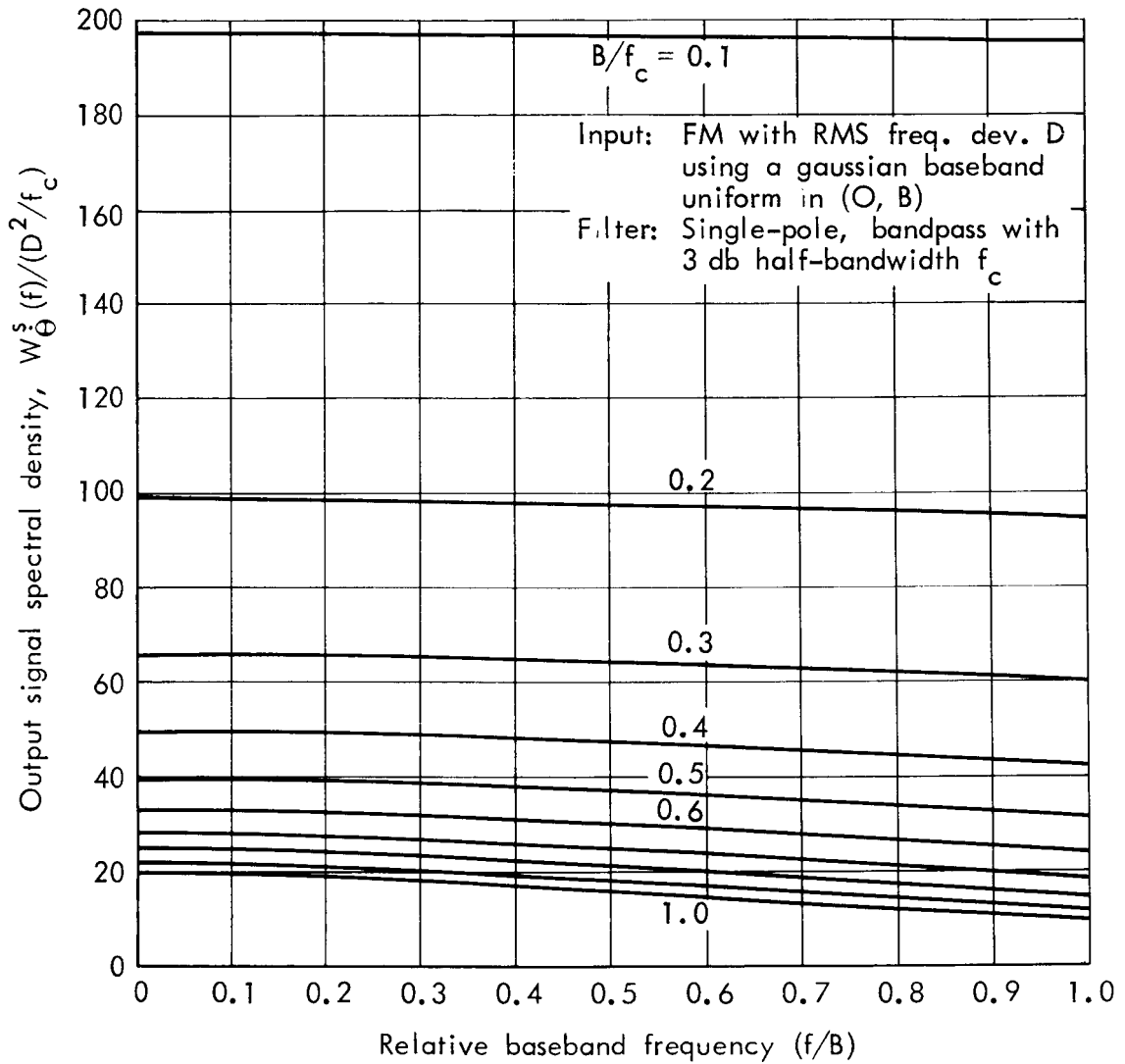Fig. 1—Region of integration for crosstalk computation
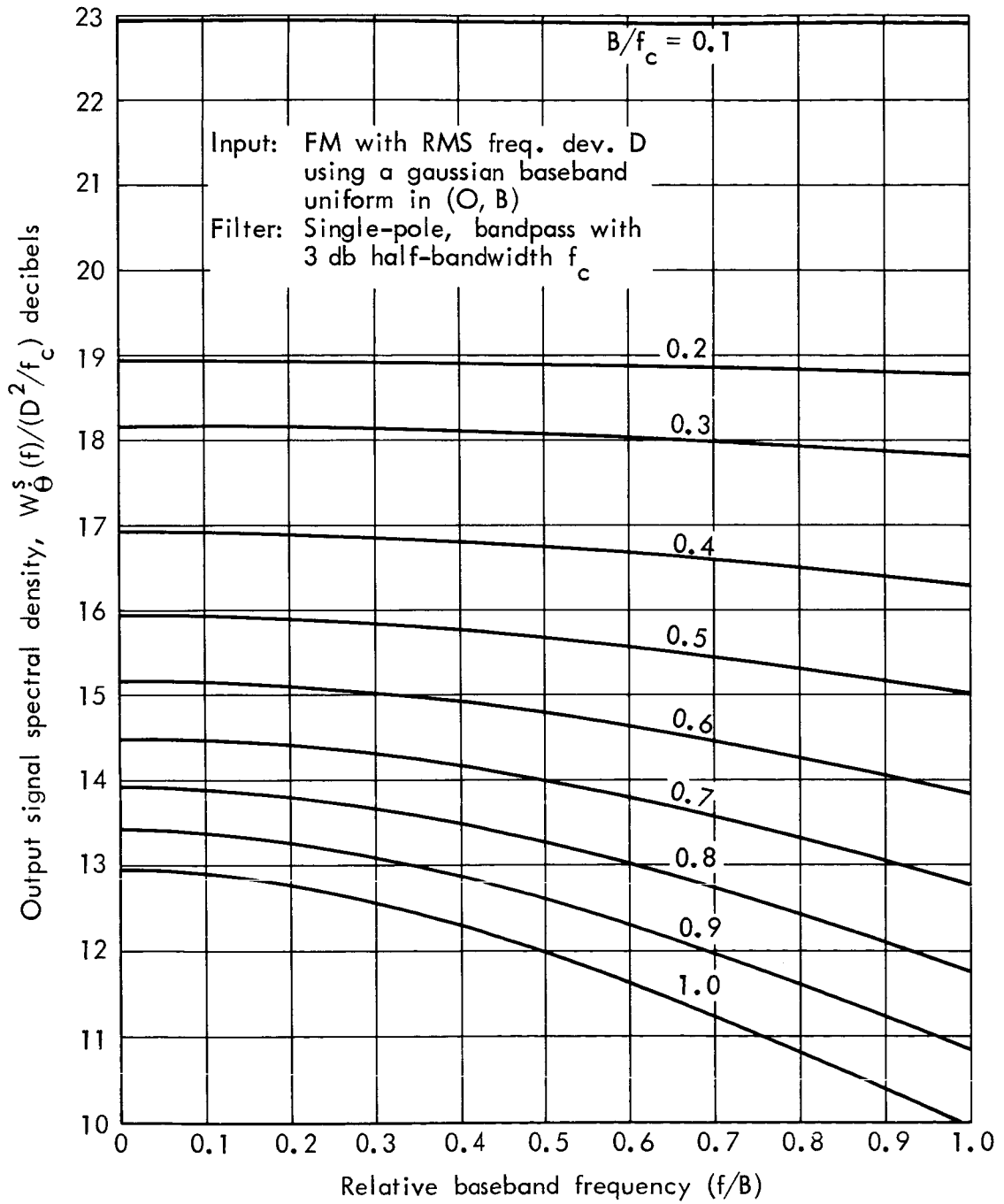
Fig. 2—Output signal spectral density

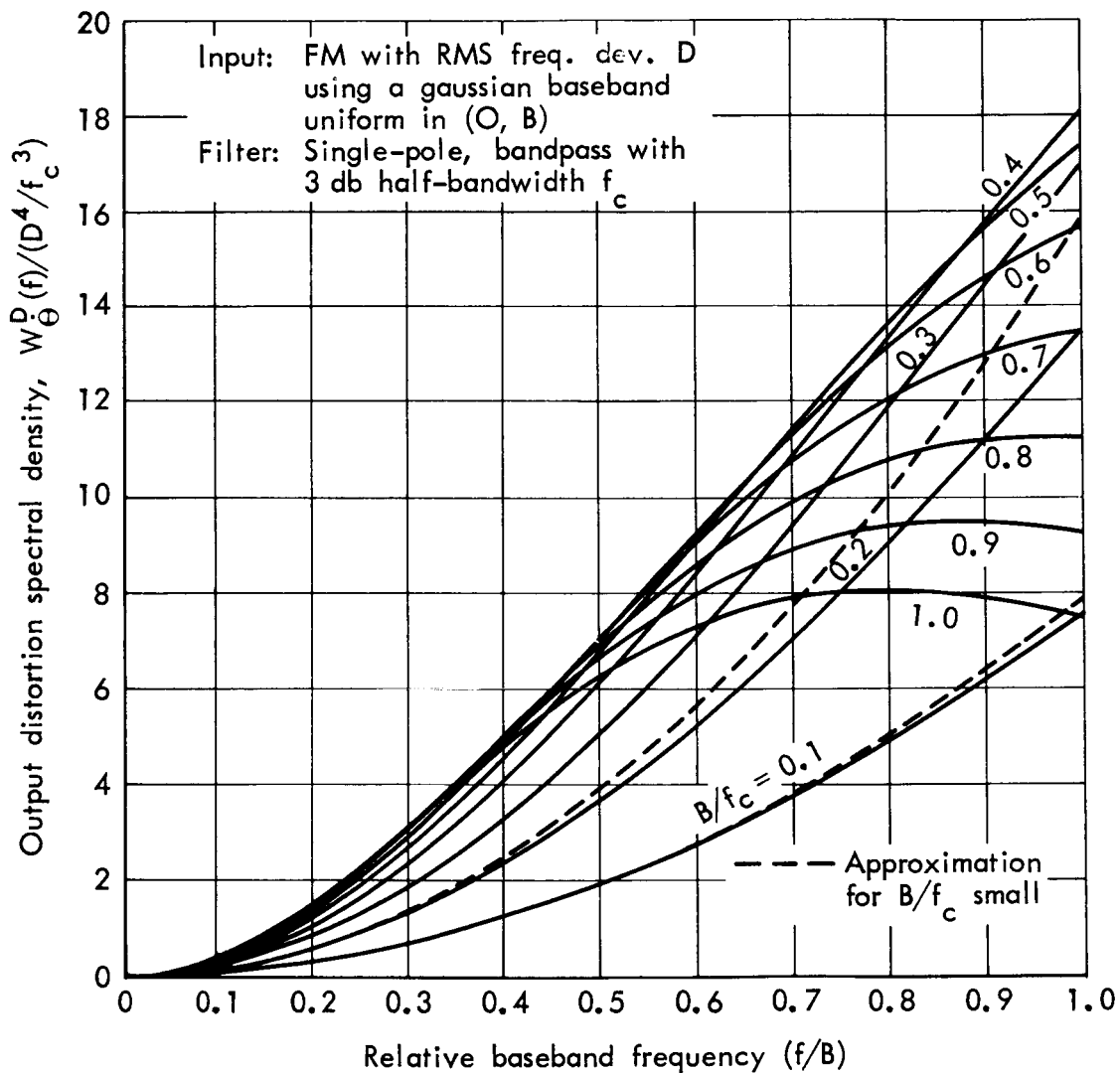Fig.3—Output signal spectral density, decibels
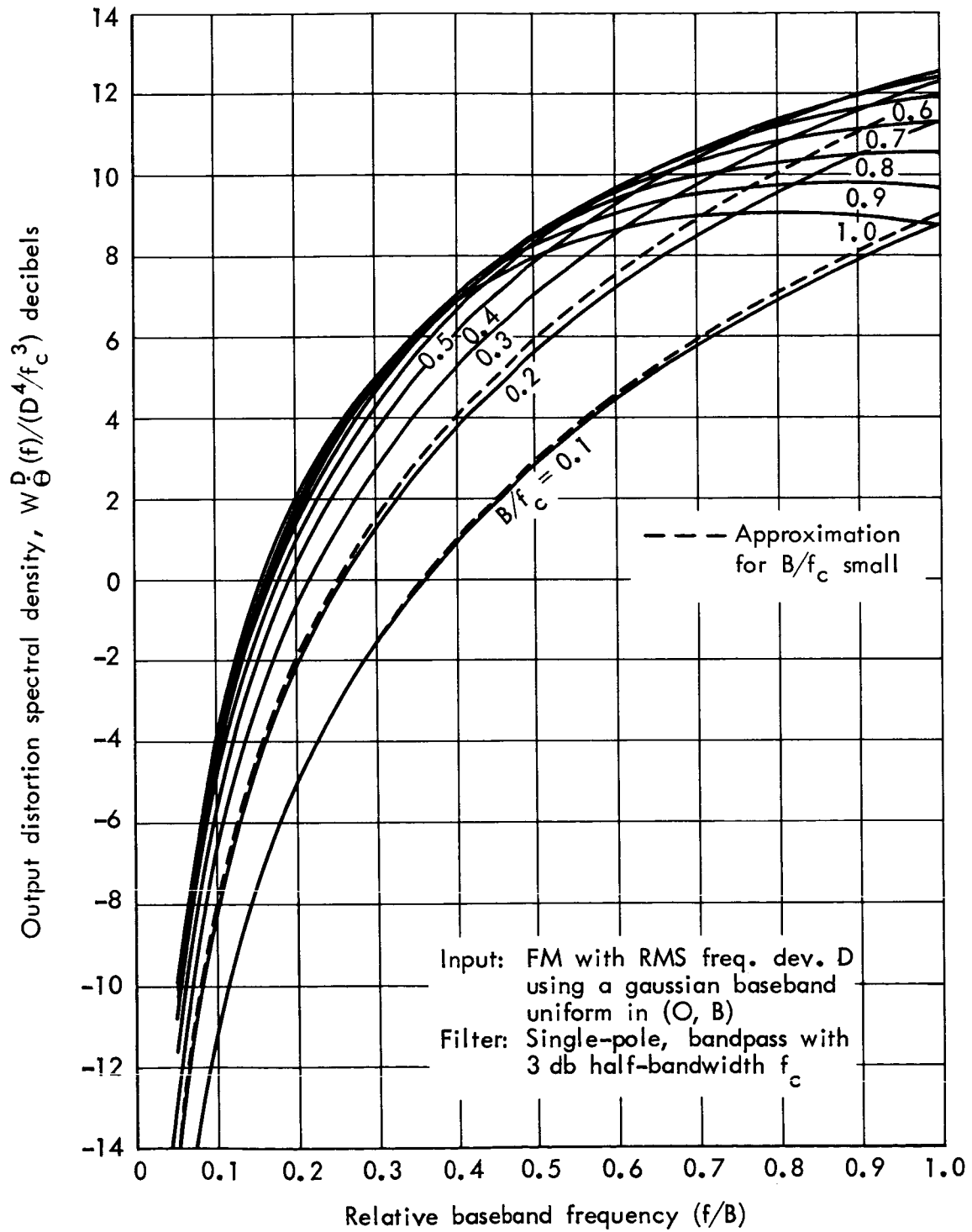
Fig. 4—Output distortion spectral density

Fig. 5—Output distortion spectral density, decibels

In principle, at least one of the integrations for the crosstalk can be performed by the same procedure of expanding by partial fractions. However, the algebra required appears too formidable to warrant the effort. Even if it were accomplished, it is highly likely that the second integration would not be possible analytically. Therefore, numerical results were obtained by machine computation on the IBM 7044, using the technique of iterated integration to evaluate the double integral.[*] The results are plotted in Fig. 6, and in decibels in Fig. 7.

Like the distortion, the crosstalk tends to zero at the lower end of the baseband and, in general, increases with frequency within the baseband. Also, the effect of the filter power response is again evident for narrow filters. Typical of a third-order spectral density, the crosstalk spectrum extends to three times the highest baseband frequency and falls smoothly to zero. The signal and distortion spectra both vanish abruptly beyond the highest baseband frequencies since they both contain the input signal spectrum as a multiplicative factor.

Useful approximations to these spectra can be obtained by assuming the highest baseband frequency B sufficiently small in comparison with the filter 3-db half-bandwidth, f_c, so that variations of the denominator terms in Eqs. (16) over the regions of integration can be neglected. Simple integration then yields

---

[*]The Romberg approximation to the definite integral was employed with a relative difference between successive approximations of 10^{-3}.

Fig. 6—Output crosstalk spectral density

Input: FM with RMS freq. dev. D using a gaussian baseband uniform in (O, B)

Filter: Single-pole, bandpass with 3 db half-bandwidth f_c

Relative baseband frequency (f/B)

Output crosstalk spectral density, $W_\theta^c(f)/(D^6/f_c^5)$

0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

$B/f_c = 1.0$

Approximation for B/f_c small

Fig. 7—Output crosstalk spectral density, decibels

Input: FM with RMS freq. dev. D
using a gaussian baseband
uniform in (O, B)

Filter: Single-pole, bandpass with
3 db half-bandwidth f_c

Relative baseband frequency (f/B)

Output crosstalk spectral density, $W_\theta^\xi(f)/(D^6/f_c^5)$ decibels

$$\frac{W_{\dot\theta}^{S}(f)}{D^2/f_c} \cong \frac{2\pi^2}{\underline{B}} \quad , \qquad |f| \leq B$$

$$\frac{W_{\dot\theta}^{D}(f)}{D^4/f_c^3} \cong \frac{8\pi^2 \underline{f}^2}{\underline{B}} \quad , \qquad |f| \leq B \tag{22}$$

$$\frac{W_{\dot\theta}^{C}(f)}{D^6/f_c^5} \cong \begin{cases} \dfrac{\pi^2 \underline{f}^2}{3 \underline{B}^3}(3 \underline{B}^2 - \underline{f}^2) & , \quad |f| \leq B \\[3mm] \dfrac{\pi^2 \underline{f}^2}{6 \underline{B}^3}(3 \underline{B} - |f|)^2 & , \quad B \leq |f| \leq 3 B \end{cases}$$

These spectra exceed the correct values slightly but do not differ significantly for values of B/f_c up to about 0.2. They are shown in Figs. 4 to 7 as dashed lines for $B/f_c = 0.1$ and 0.2.

The ratios of signal-to-distortion and signal-to-crosstalk spectral densities are shown in Figs. 8 and 9. These spectral ratios are quite similar in shape and decrease steadily across the baseband. The signal-to-crosstalk spectral ratio exceeds the signal-to-distortion spectral ratio by a substantial amount for cases of practical interest, being about 10 db greater even when the rms frequency deviation equals the filter 3-db half-bandwidth, i.e., $D/f_c = 1$. The approximations of Eqs. (22) can also be used to obtain simple formulations of these ratios valid for B/f_c small. These become

$$\frac{W_{\dot\theta}^{S}(f)}{W_{\dot\theta}^{D}(f)} = \frac{1}{4 \underline{D}^2 \underline{f}^2} \quad , \qquad |f| \leq B$$

$$\frac{W_{\dot\theta}^{S}(f)}{W_{\dot\theta}^{C}(f)} = \frac{6 \underline{B}^2}{\underline{f}^2 \underline{D}^4 (3 \underline{B}^2 - \underline{f}^2)} \quad , \qquad |f| \leq B \tag{23}$$
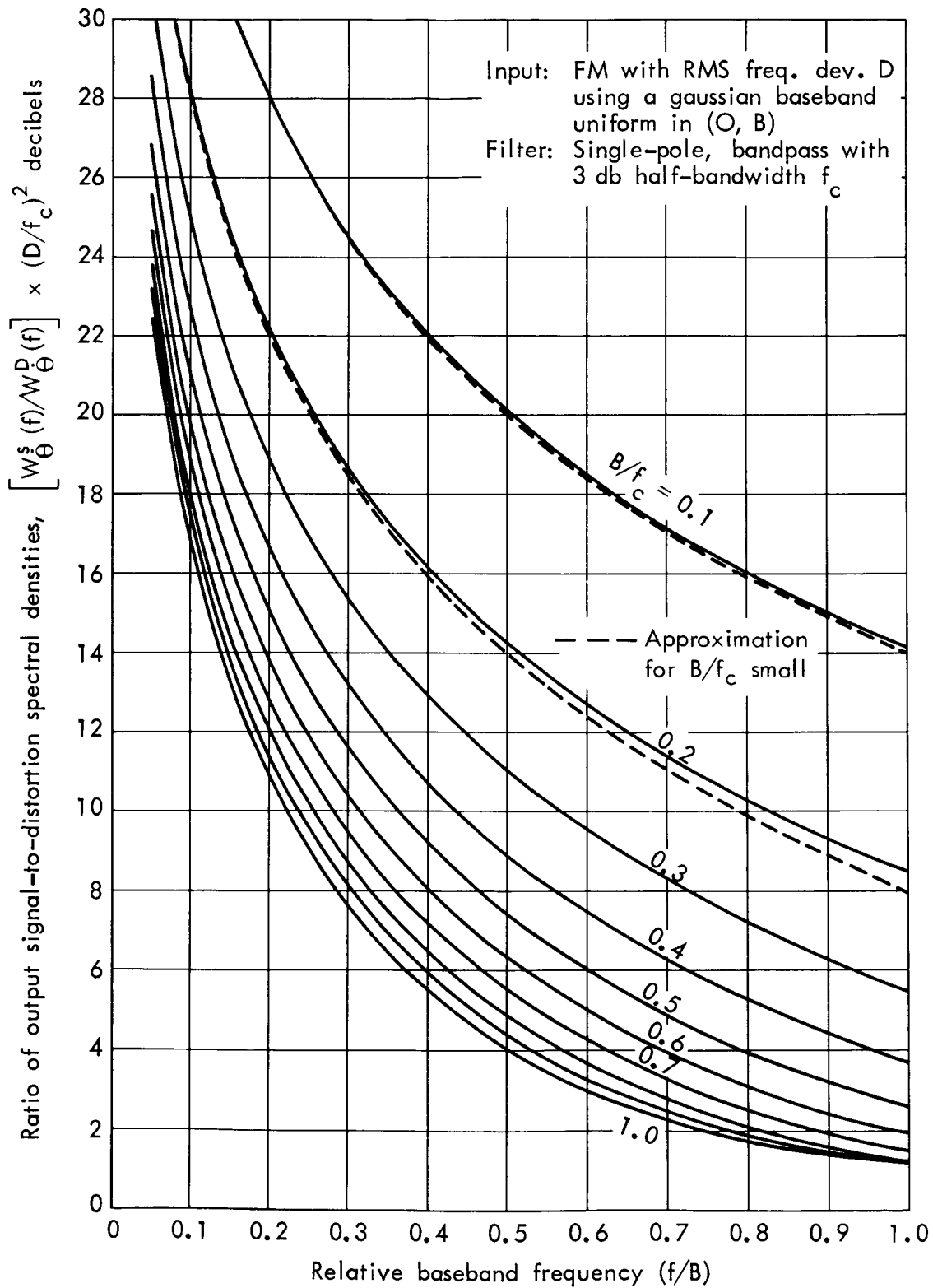
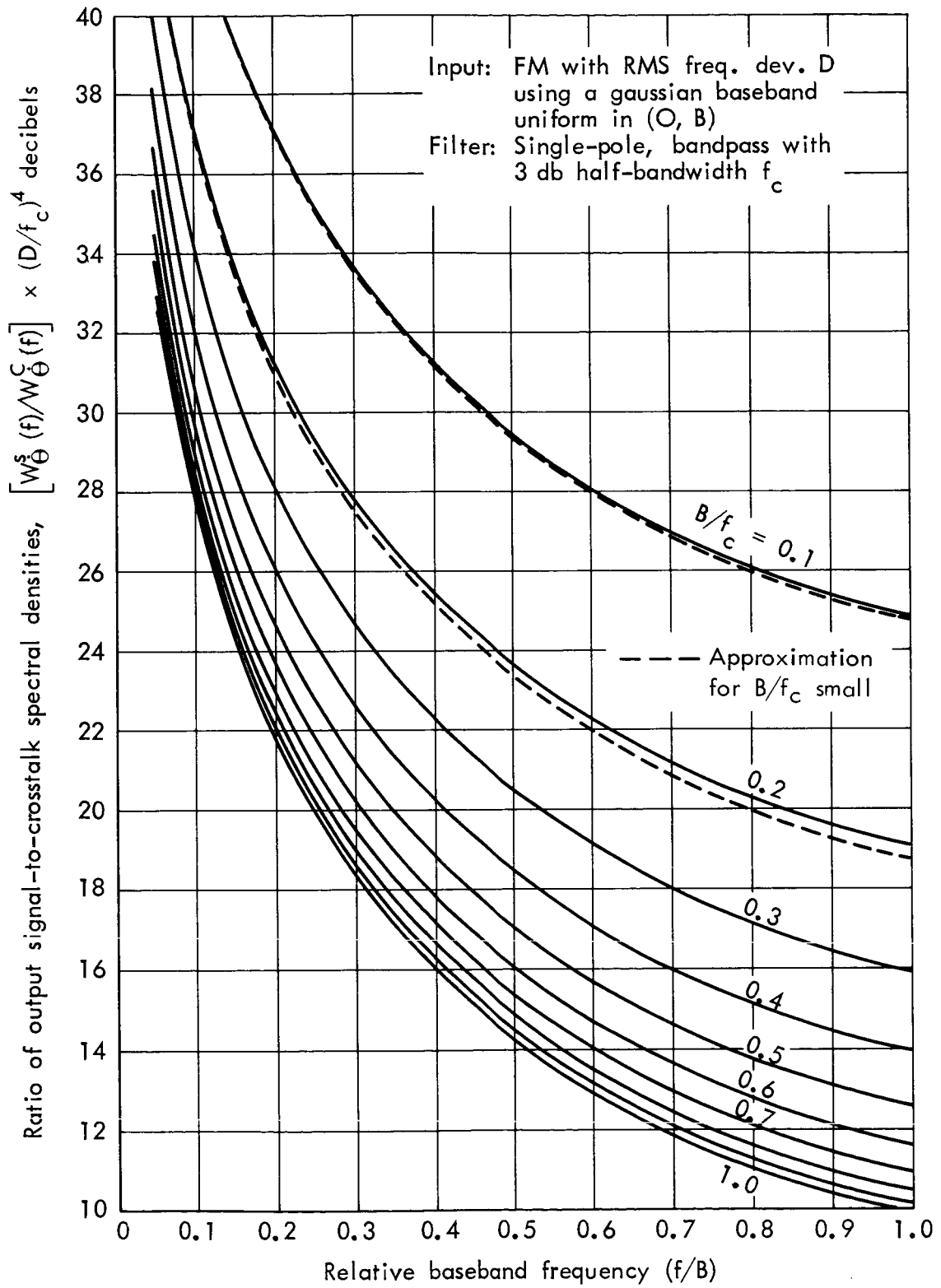Fig. 8—Ratio of output signal-to-distortion
spectral densities, decibels

Fig. 9—Ratio of output signal-to-crosstalk spectral densities, decibels

and are shown as dashed lines on Figs. 8 and 9 for the two lower

values of B/f_c.

Finally, the ratio of total signal to total distortion, SDR, is

plotted against the highest normalized baseband frequency, B/f_c, in

Fig. 10. The total signal is simply the integral of the first of

Eqs. (20). The total distortion was computed numerically from Eq. (21)

on the RAND JOSS* computer using a two-point gaussian integration with

20 intervals. The approximations of Eq. (22) may also be integrated

from -B to B to yield

$$SDR \simeq \frac{3}{4\, \underline{D}^2 \underline{B}^2} \tag{24}$$

which is valid for B/f_c small and is shown on Fig. 10 as a dashed

line. It may be conjectured that if this ratio is large, then the

decreasing sizes of distortion and crosstalk relative to signal will

insure that the higher-order contributions of distortion and cross-

talk can be safely neglected. It is on this basis that the foregoing

theory is represented as an accurate measure of the distortion and

crosstalk with the only condition being that the total distortion be

small.

For example, from Fig. 10, the signal-to-distortion ratio equals

10 db for the filter and input spectrum being considered if the filter

half-bandwidth f_c equals 2 Mc, the rms frequency deviation D equals

1.4 Mc and the highest baseband frequency B equals 1 Mc, i.e., D/f_c =

0.7 ⇒ - 1.5 db and B/f_c = 0.5. Then, from Fig. 8, the signal-to-distortion

spectral ratio at the upper end of the baseband (the worst channel) is

2.6 + 3 = 5.6 db and, from Fig. 9, the corresponding signal-to-crosstalk

---

*JOSS is the trademark and service mark of The RAND Corporation
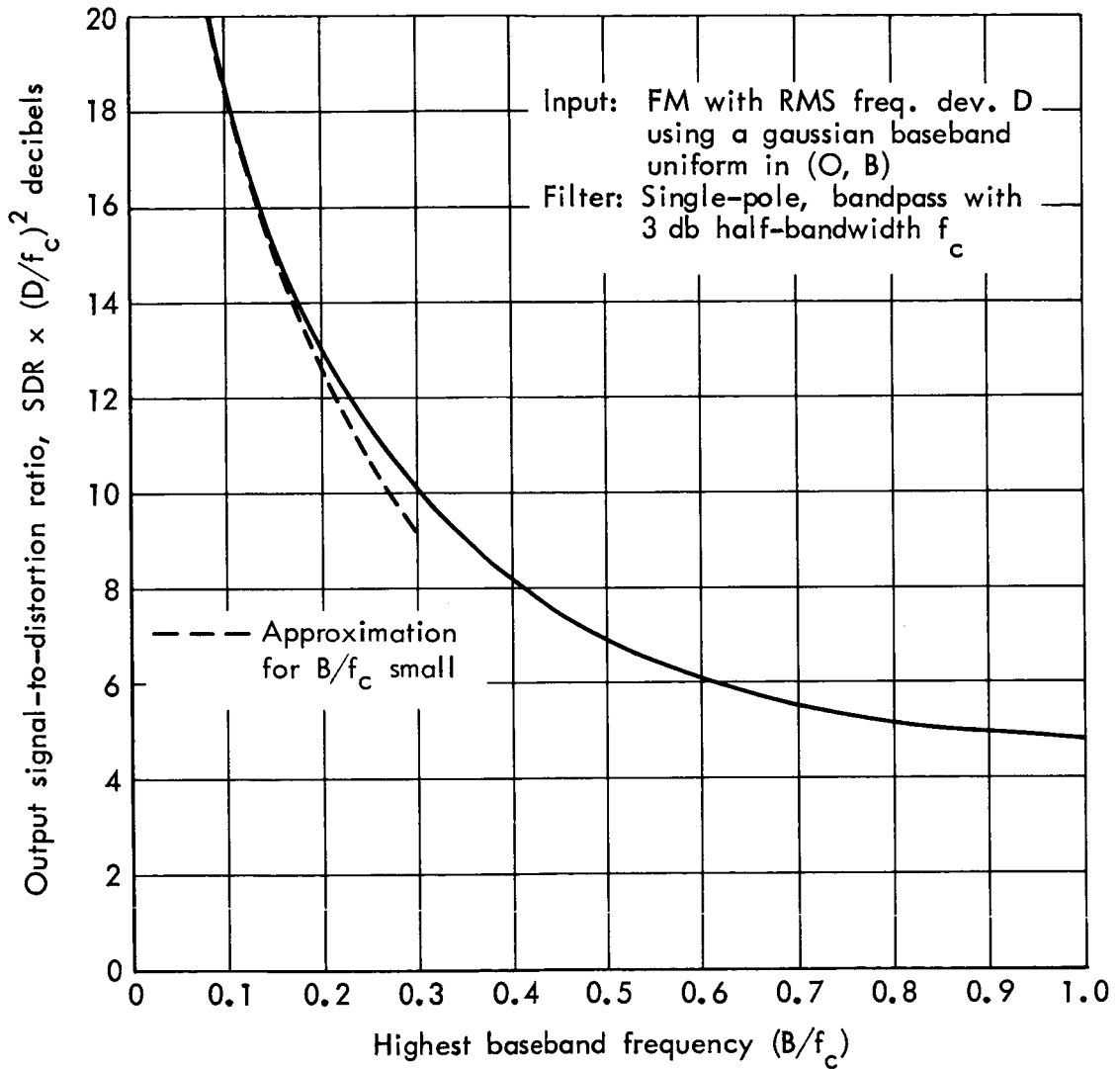for its on-line time-shared computer program and services using that
program.

Fig. 10—Output signal-to-distortion ratio, decibels

spectral ratio is 12.6 + 6 = 18.6 db. This is, of course, a somewhat extreme example of a virtually unusable channel but serves nevertheless to indicate the wide range of applicability of the results.

As a more practical example, consider an FM signal, modulated by a uniform baseband extending to 2.4 Mc and having an rms deviation of 20 Mc in the RF channel. Assume a frequency feedback receiver having a feedback factor of 20 db, uniform across the baseband, and a single-pole IF filter with a 3-db half-bandwidth of 4 Mc. The rms frequency deviation in the IF channel is then 2 Mc and the system parameters become B = 2.4 Mc, D = 2 Mc and f_c = 4 Mc, yielding the ratios B/f_c = 0.6 and D/f_c = 0.5 $\Rightarrow$ - 3 db. Recalling that the distortion in a linear feedback network is reduced by the feedback factor, it follows from Fig. 10 that the signal-to-distortion ratio SDR = 6.1 + 6 + 20 = 32.1 db. The signal-to-distortion spectral ratio at the upper end of the baseband becomes 1.9 + 6 + 20 = 27.9 db from Fig. 8 and the corresponding signal-to-crosstalk spectral ratio is 11.6 + 12 + 20 = 43.6 db from Fig. 9.

It is seen that the distortion and crosstalk remain substantial in terms of the criteria for high-fidelity telephony. Further improvements can be obtained only by increasing the feedback, which poses severe practical problems in such wide-band circuitry, or by widening the IF filter, which vitiates the threshold reducing advantage of the feedback process.

Appendix A

EXPANSION OF OUTPUT PHASE

To expand the output phase given by Eq. (6) use the series

$$\tan^{-1} x = \frac{x}{1+x^2}\left[1+\frac{2}{3}\left(\frac{x^2}{1+x^2}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{x^2}{1+x^2}\right)^2 + \dots\right] \tag{A-1}$$

which converges for $x^2 < \infty$, and take x as the ratio of integrals in Eq. (6)

$$x = \frac{a}{b} = \frac{\displaystyle\int_0^\infty du\ g(u)\sin\varphi(t-u)}{\displaystyle\int_0^\infty du\ g(u)\cos\varphi(t-u)} \tag{A-2}$$

Then the terms of interest in Eq. (A-1) are of the form

$$\frac{x}{1+x^2} = \frac{ab}{a^2+b^2} = \frac{\displaystyle\int_0^\infty du\ g(u)\sin\varphi(t-u)\int_0^\infty du\ g(u)\cos\varphi(t-u)}{\left[\int_0^\infty du\ g(u)\sin\varphi(t-u)\right]^2 +\left[\int_0^\infty du\ g(u)\cos\varphi(t-u)\right]^2} \tag{A-3}$$

and

$$\frac{x^2}{1+x^2} = \frac{a^2}{a^2+b^2} = \frac{\left[\int_0^\infty du\ g(u)\sin\varphi(t-u)\right]^2}{\left[\int_0^\infty du\ g(u)\sin\varphi(t-u)\right]^2 +\left[\int_0^\infty du\ g(u)\cos\varphi(t-u)\right]^2} \tag{A-4}$$

The sine and cosine functions can be expanded into power series which converge for $\varphi^2(t) < \infty$. Let these series, or their equivalent,

the Taylor's series expansion for the input signal $\exp[i\varphi(t)]$, converge

uniformly on the infinite interval on t. Then, since

$$\int_0^\infty du\,|g(u)| < \infty \qquad (A-5)$$

for all stable networks, the series may be integrated term by term.[46]

The numerators in Eqs. (A-3) and (A-4) become

$$ab = \left[\Phi_1 - \frac{1}{3!}\,\Phi_3 + \frac{1}{5!}\,\Phi_5 - \cdots\right]\left[1 - \frac{1}{2!}\,\Phi_2 + \frac{1}{4!}\,\Phi_4 - \cdots\right]$$

$$= \Phi_1 - \frac{1}{3!}\,\Phi_3 - \frac{1}{2!}\,\Phi_1\Phi_2 + \frac{1}{4!}\,\Phi_1\Phi_4 + \frac{1}{2!3!}\,\Phi_2\Phi_3 + \frac{1}{5!}\,\Phi_5 + 0(\varphi^7) \qquad (A-6)$$

and

$$a^2 = \left[\Phi_1 - \frac{1}{3!}\,\Phi_3 + \cdots\right]^2$$

$$= \Phi_1^2 - \frac{2}{3!}\,\Phi_1\Phi_3 + 0(\varphi^6) \qquad (A-7)$$

where the notation Φ_n is a generalization of Eq. (8)

$$\Phi_n = \int_0^\infty du\; g(u)\; \varphi^n(t-u) \qquad (A-8)$$

while the denominators are

$$a^2 + b^2 = \left[\Phi_1 - \frac{1}{3!}\,\Phi_3 + \cdots\right]^2 + \left[1 - \frac{1}{2!}\,\Phi_2 + \frac{1}{4!}\,\Phi_4 - \cdots\right]^2$$

$$= \Phi_1^2 - \frac{2}{3!}\,\Phi_1\Phi_3 + \cdots + 1 - \frac{2}{2!}\,\Phi_2 + \frac{1}{(2!)^2}\,\Phi_2^2 + \frac{2}{4!}\,\Phi_4 + \cdots$$

$$= 1 + \Phi_1^2 - \Phi_2 - \frac{1}{3}\,\Phi_1\Phi_3 + \frac{1}{4}\,\Phi_2^2 + \frac{1}{12}\,\Phi_4 + 0(\varphi^6)$$

$$= 1 + R \qquad (A-9)$$

The reciprocal of the denominators can be expressed in powers of R by using the expansion

$$(1+R)^{-1} = 1 - R + R^2 - \ldots, \qquad R^2 < 1 \qquad \text{(A-10)}$$

which requires that the sum $a^2 + b^2$ lie in the open interval $(0,2)$. It is not possible to bound the sum away from the value zero since certain functions can cause a and b to vanish simultaneously for some values of t. Thus, a necessary condition for inverting the denominator is that $a + ib$, i.e., the output signal given by Eq. (5), not vanish. An example of a function which achieves this at $t = 0$ is $\varphi(t) = 2\pi k \int_0^{-t} g(s)ds + \lambda$, where k is an integer not zero and λ is an arbitrary constant.[*] It will be assumed in the following that such functions are sufficiently singular that they can be avoided or, as will be the case when a spectral analysis is performed using an ensemble of functions, that their effect can be neglected.

To bound the denominator from above, note that it can be written

$$a^2 + b^2 = \int_0^\infty du \int_0^\infty dv \, g(u)g(v)\cos[\varphi(t-u)-\varphi(t-v)]$$

$$\leq \int_0^\infty du \int_0^\infty dv \, |g(u)| \, |g(v)| = \left[\int_0^\infty du|g(u)|\right]^2 \qquad \text{(A-11)}$$

from which it follows that a sufficient condition for inverting the denominator is given by

$$\int_0^\infty du|g(u)| \; < \sqrt{2} \qquad \text{(A-12)}$$

---

[*] Private communication from O. A. Gross of The RAND Corporation.

This condition requires that the filter impulse response be "well behaved." While the looseness of the bound used may have made the condition excessively restrictive for general use, it is satisfactory for the cases of immediate interest. For example, it is clearly satisfied by any non-negative impulse response, such as that of a first-order filter, by virtue of the normalization of Eq. (4). Although it does not appear feasible to apply Eq. (A-12) analytically to the impulse responses of higher-order filters, it can be used graphically on plots similar to those given by Henderson and Kautz.[47] It is apparent by inspection of Fig. 9, Ref. 47, that Bessel filters of all orders satisfy Eq. (A-12). The low-order Butterworth filters, and to a lesser extent Tchebycheff filters, also appear to have sufficiently small negative-going regions in their impulse responses to satisfy the condition.

Returning to Eq. (A-9), expanding in powers of R according to Eq. (A-10) then yields

$$(a^2+b^2)^{-1} = 1 - \left[\phi_1^2 - \phi_2 - \frac{1}{3} \phi_1 \phi_3 + \frac{1}{4} \phi_2^2 + \frac{1}{12} \phi_4 - \cdots \right]$$

$$+ \left[\phi_1^4 - 2\phi_1^2 \phi_2 + \phi_2^2 + \cdots \right] + \cdots$$

$$= 1 - \phi_1^2 + \phi_2 + \frac{1}{3} \phi_1 \phi_3 + \frac{3}{4} \phi_2^2 - \frac{1}{12} \phi_4 + \phi_1^4 - 2\phi_1^2 \phi_2 + 0(\varphi^6) \quad \text{(A-13)}$$

Substituting Eqs. (A-6) and (A-13) into Eq. (A-3) gives the leading term in the expansion of Eq. (A-1)

$$\frac{x}{1+x^2} = (ab)(a^2+b^2)^{-1}$$

$$= \left[\Phi_1 - \frac{1}{6}\Phi_3 - \frac{1}{2}\Phi_1\Phi_2 + \frac{1}{24}\Phi_1\Phi_4 + \frac{1}{12}\Phi_2\Phi_3 + \frac{1}{120}\Phi_5 + \cdots\right]$$

$$\times \left[1 - \Phi_1^2 + \Phi_2 + \frac{1}{3}\Phi_1\Phi_3 + \frac{3}{4}\Phi_2^2 - \frac{1}{12}\Phi_4 + \Phi_1^4 - 2\Phi_1^2\Phi_2 + \cdots\right]$$

$$= \Phi_1 - \Phi_1^3 + \Phi_1\Phi_2 + \frac{1}{3}\Phi_1^2\Phi_3 + \frac{3}{4}\Phi_1\Phi_2^2 - \frac{1}{12}\Phi_1\Phi_4 + \Phi_1^5 - 2\Phi_1^3\Phi_2 - \frac{1}{6}\Phi_3 + \frac{1}{6}\Phi_1^2\Phi_3 - \frac{1}{6}\Phi_2\Phi_3$$

$$- \frac{1}{2}\Phi_1\Phi_2 + \frac{1}{2}\Phi_1^3\Phi_2 - \frac{1}{2}\Phi_1\Phi_2^2 + \frac{1}{24}\Phi_1\Phi_4 + \frac{1}{12}\Phi_2\Phi_3 + \frac{1}{120}\Phi_5 + \cdots$$

$$= \Phi_1 - \Phi_1^3 + \frac{1}{2}\Phi_1\Phi_2 - \frac{1}{6}\Phi_3 + \frac{1}{2}\Phi_1^2\Phi_3 + \frac{1}{4}\Phi_1\Phi_2^2 - \frac{1}{24}\Phi_1\Phi_4$$

$$+ \Phi_1^5 - \frac{3}{2}\Phi_1^3\Phi_2 - \frac{1}{12}\Phi_2\Phi_3 + \frac{1}{120}\Phi_5 + O(\varphi^7) \tag{A-14}$$

Similarly, substituting Eqs. (A-6), (A-7), and (A-13) into Eqs. (A-3) and (A-4) yields the second term in Eq. (A-1)

$$\frac{2}{3}\frac{x}{1+x^2}\frac{x^2}{1+x^2} = \frac{2}{3}(ab)a^2(a^2+b^2)^{-2}$$

$$= \frac{2}{3}\left[\Phi_1 - \frac{1}{6}\Phi_3 - \frac{1}{2}\Phi_1\Phi_2 + \cdots\right]\left[\Phi_1^2 - \frac{1}{3}\Phi_1\Phi_3 + \cdots\right]\left[1 - \Phi_1^2 + \Phi_2 + \cdots\right]^2$$

$$= \frac{2}{3}\left[\Phi_1^3 - \frac{1}{2}\Phi_1^2\Phi_3 - \frac{1}{2}\Phi_1^3\Phi_2 + \cdots\right]\left[1 - 2\Phi_1^2 + 2\Phi_2 + \cdots\right]$$

$$= \frac{2}{3}\left[\Phi_1^3 - 2\Phi_1^5 + 2\Phi_1^3\Phi_2 - \frac{1}{2}\Phi_1^2\Phi_3 - \frac{1}{2}\Phi_1^3\Phi_2 + \cdots\right]$$

$$= \frac{2}{3}\Phi_1^3 - \frac{4}{3}\Phi_1^5 + \Phi_1^3\Phi_2 - \frac{1}{3}\Phi_1^2\Phi_3 + O(\varphi^7) \tag{A-15}$$

The third term in Eq. (A-1) is

$$\frac{8}{15}\frac{x}{1+x^2}\left(\frac{x^2}{1+x^2}\right)^2 = \frac{8}{15}(ab)a^4(a^2+b^2)^{-3}$$

$$= \frac{8}{15}\Phi_1^5 + O(\varphi^7) \tag{A-16}$$

Summation of Eqs. (A-14), (A-15), and (A-16) produces, finally

$$\theta(t) = \Phi_1 - \frac{1}{3}\Phi_1^3 + \frac{1}{2}\Phi_1\Phi_2 - \frac{1}{6}\Phi_3 + \frac{1}{6}\Phi_1^2\Phi_3 + \frac{1}{4}\Phi_1\Phi_2^2 - \frac{1}{24}\Phi_1\Phi_4$$

$$+ \frac{1}{5}\Phi_1^5 - \frac{1}{2}\Phi_1^3\Phi_2 - \frac{1}{12}\Phi_2\Phi_3 + \frac{1}{120}\Phi_5 + \cdots$$

$$= \Phi_1 - \frac{1}{6}\left[\Phi_3 - 3\Phi_2\Phi_1 + 3\Phi_1\Phi_1^2 - \Phi_1^3\right]$$

$$+ \frac{1}{120}\left[\Phi_5 - 5\Phi_4\Phi_1 + 10\Phi_3\Phi_1^2 - 10\Phi_2\Phi_1^3 + 5\Phi_1\Phi_1^4 - \Phi_1^5\right]$$

$$- \frac{1}{12}\left[\Phi_2 - 2\Phi_1\Phi_1 + \Phi_1^2\right]\left[\Phi_3 - 3\Phi_2\Phi_1 + 3\Phi_1\Phi_1^2 - \Phi_1^3\right] + \cdots$$

$$= \Phi(t) - \frac{1}{3!}\int_0^\infty du\, g(u)[\varphi(t-u)-\Phi(t)]^3 + \frac{1}{5!}\int_0^\infty du\, g(u)[\varphi(t-u)-\Phi(t)]^5$$

$$- \frac{1}{12}\int_0^\infty du\, g(u)[\varphi(t-u)-\Phi(t)]^2 \int_0^\infty du\, g(u)[\varphi(t-u)-\Phi(t)]^3 + O(\varphi^7)$$

$$(A\text{-}17)$$

which is the desired result. The algebraic manipulations required between these steps are not obvious and, if desired, the result is most easily verified by expanding Eq. (A-17) and comparing it with the preceding lines.

Appendix B

TAYLOR'S SERIES EXPANSION

The output phase function given by Eq. (6) is vector-valued, i.e., each input phase function $\varphi(t,\omega)$, $\omega \in \Omega$, defines a value of the output phase function $\theta(t,\omega)$ which is a point or vector in $\mathcal{U}$. Such vector-valued functions are treated by Hille and Phillips[44] who show that under certain conditions the Taylor's series expansion of a vector-valued function $f(x)$ is given (Thms. 3.16.2 or 3.17.1) by

$$f(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n(x;h) \tag{B-1}$$

where (Eq. (3.16.1)) $\delta^n f(x;h)$, the nth variation of $f(x)$ with increment h, is given by

$$\delta^n f(x;h) = \left[\frac{d^n}{d\zeta^n} f(x+\zeta h)\right]_{\zeta=0} \tag{B-2}$$

It is also shown (Thm. 26.3.5) that $\delta^n f(x;h)$ is a homogeneous polynomial of degree n in h and that, in particular (Thm. 26.3.2), $\delta f(x;h)$ is linear in h.

Assuming differentiability, it is possible to obtain the expansion for $\theta(t)$ given by Eq. (7) by applying the above to Eq. (6). The expansion will be evaluated at the point $x = 0$ and h will be taken as the input phase φ, i.e.

$$h_u \equiv \varphi(t-u) \tag{B-3}$$

where the subscript has been introduced to identify the variable of integration. In these terms, the expansion becomes

$$f(h) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n f(0;h) \tag{B-4}$$

where

$$\delta^n f(0;h) = \frac{d^n}{d\zeta^n} \tan^{-1} \left. \frac{\int_0^\infty du\ g(u)\sin \zeta h_u}{\int_0^\infty du\ g(u)\cos \zeta h_u} \right|_{\zeta=0} \tag{B-5}$$

The zero-order variation is found directly from Eq. (B-5) for n=0

$$\delta^0 f(0;h) = 0 \tag{B-6}$$

The first derivative is given by

$$\frac{d}{d\zeta} \tan^{-1} \frac{a}{b}$$

$$= \frac{ba' - ab'}{a^2 + b^2}$$

$$= \frac{\int_0^\infty du\ g(u)\cos \zeta h_u \int_0^\infty du\ g(u)h_u\cos \zeta h_u + \int_0^\infty du\ g(u)\sin \zeta h_u \int_0^\infty du\ g(u)h_u\sin \zeta h_u}{\left[\int_0^\infty du\ g(u)\sin \zeta h_u\right]^2 + \left[\int_0^\infty du\ g(u)\cos \zeta h_u\right]^2}$$

$$= \frac{\int_0^\infty du \int_0^\infty dv\ g(u)g(v)h_u\cos \zeta(h_u-h_v)}{\int_0^\infty du \int_0^\infty dv\ g(u)g(v)\cos \zeta(h_u-h_v)} \tag{B-7}$$

so the first variation becomes

$$\delta^1 f(0;h) = \int_0^\infty du \, g(u)h_u \tag{B-8}$$

where the normalization of Eq. (4) has been used.

Further derivatives of Eq. (B-7) are required to find the higher variations. These can be found systematically by writing Eq. (B-7) as

$$R^{(0)}(\zeta) = \frac{N^{(0)}(\zeta)}{D^{(0)}(\zeta)} \tag{B-9}$$

or

$$R^{(0)}(\zeta)D^{(0)}(\zeta) = N^{(0)}(\zeta) \tag{B-10}$$

Differentiating the product then yields higher orders of R implicitly. Thus, the rth derivative of Eq. (B-10) is

$$\sum_{s=0}^{r} \binom{r}{s} R^{(s)}(\zeta)D^{(r-s)}(\zeta) = N^{(r)}(\zeta) \tag{B-11}$$

which, when evaluated at zero and used recursively, yields the nth variation according to

$$\delta^n f(0;h) = R^{(n-1)}(0) \tag{B-12}$$

The expansion of Eq. (B-11) will involve various derivatives of N and D. From their definition by Eqs. (B-7) and (B-9), it is seen that

$$
N^{(m)}(0) = \begin{cases} 0 & ; \quad m \text{ odd} \\ \\ (-1)^{m/2} \displaystyle\int_0^\infty du \int_0^\infty dv \; g(u)g(v)h_u(h_u - h_v)^m; & m \text{ even} \end{cases}
$$

(B-13)

and

$$
D^{(m)}(0) = \begin{cases} 0 & ; \quad m \text{ odd} \\ \\ (-1)^{m/2} \displaystyle\int_0^\infty du \int_0^\infty dv \; g(u)g(v) \; (h_u - h_v)^m; & m \text{ even} \end{cases}
$$

(B-14)

As a result, note that when r is odd in Eq. (B-11) and the derivatives are evaluated at $\zeta = 0$, then all the even derivatives of R are paired with odd derivatives of D, which vanish, and the odd derivative of N on the right causes the sum of all products involving odd derivatives of R to vanish. Commencing recursively from $R^{(1)}(0)$, which clearly vanishes, it then follows that all odd derivatives of R vanish, and from Eq. (B-12), that all even variations therefore vanish, i.e.

$$
\delta^n f(0;h) = 0, \quad n \text{ even} \tag{B-15}
$$

In these terms, the first variation is found again by setting r=0 in Eq. (B-11) which yields, at $\zeta = 0$

$$
R^{(0)}(0) = N^{(0)}(0) = \int_0^\infty du \; g(u)h_u \tag{B-16}
$$

wherein, from Eq. (4)

$$
D^{(0)}(0) = 1 \tag{B-17}
$$

Then, from Eq. (B-12)

$$\delta^1 f(0;h) = \int_0^\infty du\ g(u)h_u \tag{B-18}$$

which, of course, agrees with Eq. (B-8).

To find the third variation, set n=2 in Eq. (B-11) at $\zeta=0$ and drop the vanishing odd derivatives of R. Then

$$R^{(2)}(0) + R^{(0)}(0)D^{(2)}(0) = N^{(2)}(0) \tag{B-19}$$

and

$$\delta^3 f(0;h) = R^{(2)}(0) = -\int_0^\infty du \int_0^\infty dv\ g(u)g(v)h_u(h_u - h_v)^2$$

$$+ \int_0^\infty dw\ g(w)h_w \int_0^\infty du \int_0^\infty dv\ g(u)g(v)\ (h_u - h_v)^2$$

$$= -\int_0^\infty du\ g(u)\left[h_u - \int_0^\infty dv\ g(v)h_v \right]^3 \tag{B-20}$$

where algebraic manipulations similar to those leading to Eq. (A-17) have been performed.

The fifth variation follows from setting n=4 in Eq. (B-11)

$$R^{(4)}(0) + 6R^{(2)}(0)D^{(2)}(0) + R^{(0)}(0)D^{(4)}(0) = N^{(4)}(0)$$

so

$$R^{(4)}(0) = N^{(4)}(0) - 6\left[N^{(2)}(0) - N^{(0)}(0)D^{(2)}(0) \right]D^{(2)}(0) - N^{(0)}(0)D^{(4)}(0) \tag{B-21}$$

Then

$$\delta^5 f(0;h) = \int_0^\infty du \int_0^\infty dv \ g(u)g(v)h_u(h_u-h_v)^4$$

$$- 6\int_0^\infty du \int_0^\infty dv \ g(u)g(v)h_u(h_u-h_v)^2 \int_0^\infty du \int_0^\infty dv \ g(u)g(v)(h_u-h_v)^2$$

$$+6\int_0^\infty du \ g(u)h_u \left[\int_0^\infty du \int_0^\infty dv \ g(u)g(v)(h_u-h_v)^2\right]^2$$

$$-\int_0^\infty du \ g(u)h_u \int_0^\infty du \int_0^\infty dv \ g(u)g(v)(h_u-h_v)^4$$

$$= \int_0^\infty du \ g(u) \left[h_u - \int_0^\infty dx \ g(x)h_x\right]^5$$

$$-10\int_0^\infty du \ g(u) \left[h_u - \int_0^\infty dx \ g(x)h_x\right]^2 \int_0^\infty du \ g(u) \left[h_u - \int_0^\infty dx \ g(x)h_x\right]^3 \tag{B-22}$$

where, again, a considerable number of algebraic manipulations have been performed but not shown.

The final expansion is obtained by substituting Eqs. (B-6), (B-8), (B-15), (B-20) and (B-22) in Eq. (B-1) and using Eqs. (B-3) and (A-8) to return to the original notation. Then

$$\theta(t) = \Phi(t) - \frac{1}{3!} \int_0^\infty du \ g(u)[\varphi(t-u)-\Phi(t)]^3 + \frac{1}{5!} \int_0^\infty du \ g(u)[\varphi(t-u)-\Phi(t)]^5$$

$$- \frac{1}{12} \int_0^\infty du \ g(u)[\varphi(t-u)-\Phi(t)]^2 \int_0^\infty du \ g(u)[\varphi(t-u)-\Phi(t)]^3 + \ldots \tag{B-23}$$

which is the desired result.

Appendix C

SPECTRAL ANALYSIS

As suggested in Section III, let the output phase of Eq. (7) be written

$$\theta(t) = S(t) + D_3(t) + D_5(t) + \ldots \tag{C-1}$$

where S, D_3 and D_5 denote the signal and third- and fifth-order components of the output, respectively. The signal component of the output phase is then obtained immediately from Eq. (12) by considering the S×S term and introducing the autocorrelation function of Eq. (11) and the Fourier transform of Eq. (10) as needed. Thus

$$W_\theta^S(f) = \mathscr{F}E\left[\int_0^\infty du\ g(u)\varphi(t-u,\omega)\int_0^\infty dv\ g(v)\varphi(t+\tau-v,\omega)\right]$$

$$= \int_0^\infty du \int_0^\infty dv\ g(u)g(v)\mathscr{F}E[\varphi(t-u,\omega)\varphi(t+\tau-v,\omega)]$$

$$= \int_0^\infty du \int_0^\infty dv\ g(u)g(v)\mathscr{F}R_\varphi(\tau+u-v)$$

$$= \int_0^\infty du \int_0^\infty dv\ g(u)g(v)\int_{-\infty}^\infty d\tau R_\varphi(\tau+u-v)e^{-i2\pi f\tau}$$

$$= \int_0^\infty du \int_0^\infty dv\ g(u)g(v)\int_{-\infty}^\infty d\tau R_\varphi(\tau)e^{-i2\pi f(\tau-u+v)}$$

$$= \int_0^\infty du\ g(u)e^{i2\pi fu}\int_0^\infty dv\ g(v)e^{-i2\pi fv}\int_{-\infty}^\infty d\tau R_\varphi(\tau)e^{-i2\pi f\tau}$$

$$= G(-f)G(f)W_\varphi(f) \tag{C-2}$$

where the Fourier transforms of Eqs. (2) and (10) are used in the last step. Finally, from the property of G given by Eq. (3)

$$W_\theta^S(f) = |G(f)|^2 W_\phi(f) \tag{C-3}$$

which is the first of Eqs. (14). It has been derived in some detail to illustrate the sequences of steps which are typical of those to follow.

There are two $S \times D_3$ terms in Eq. (C-1). From Eq. (12) the distortion component of the output phase then becomes

$$W_\theta^D(f) = \frac{1}{3!} \mathcal{F} E \left\{ \int_0^\infty du \ g(u)\phi(t-u,\omega) \int_0^\infty dv \ g(v)[\Phi(t+\tau,\omega)-\phi(t+\tau-v,\omega)]^3 \right.$$

$$\left. + \int_0^\infty du \ g(u)\phi(t+\tau-u,\omega) \int_0^\infty dv \ g(v)[\Phi(t,\omega)-\phi(t-v,\omega)]^3 \right\}$$

$$= \frac{1}{3!} \int_0^\infty du \int_0^\infty dv \ g(u)g(v) \ \mathcal{F} E \left\{ \phi(t-\tau-u,\omega)[\Phi(t,\omega)-\phi(t-v,\omega)]^3 \right.$$

$$\tag{C-4}$$

$$\left. + \phi(t+\tau-u,\omega)[\Phi(t,\omega)-\phi(t-v,\omega)]^3 \right\}$$

Thus, it is necessary to consider expectations of the form $E\{\phi(t\pm\tau-u,\omega) \ [\Phi(t,\omega)-\phi(t-v,\omega)]^3\}$.

Now, if F(t) is a zero-mean gaussian variate, then[48] (Eq. (42b))

$$E\left[F(t_1)F(t_2)\ldots F(t_{2n})\right] = \sum_{\substack{\text{all} \\ \text{pairs}}} E\left[F(t_i)F(t_j)\right]E\left[F(t_k)F(t_\ell)\right]\ldots \tag{C-5}$$

where the sum is taken over all the $1 \cdot 3 \cdot 5 \cdots (2n-1)$ different ways the $2n$ time points $t_1, t_2, \ldots t_{2n}$ can be divided into n pairs. Then, since φ and $\Phi-\varphi$ are such quantities, it follows that

$$E\left\{\varphi(t\pm\tau-u,\omega)[\Phi(t,\omega)-\varphi(t-v,\omega)]^3\right\}$$

$$= 3 \ E\left\{\varphi(t\pm\tau-u,\omega)[\Phi(t,\omega)-\varphi(t-v,\omega)]\right\}E\left\{[\Phi(t,\omega)-\varphi(t-v,\omega)]^2\right\}$$

(C-6)

Using Eq. (8) to eliminate Φ, the first expected value in Eq. (C-6) becomes

$$E\left\{\varphi(t\pm\tau-u,\omega)[\Phi(t,\omega)-\varphi(t-v,\omega)]\right\}$$

$$= E\left\{\varphi(t\pm\tau-u,\omega)\left[\int_0^\infty dx \ g(x)\varphi(t-x,\omega)-\varphi(t-v,\omega)\right]\right\}$$

$$= \int_0^\infty dx \ g(x)E\left\{\varphi(t\pm\tau-u,\omega)[\varphi(t-x,\omega)-\varphi(t-v,\omega)]\right\}$$

$$= \int_0^\infty dx \ g(x)R_\varphi(\pm\tau+x-u)-R_\varphi(\pm\tau+v-u)$$

(C-7)

where Eq. (11) is used to introduce the autocorrelation function R_φ. Similarly, the second expected value in Eq. (C-6) becomes

$$E\{[\Phi(t,\omega)-\varphi(t-v,\omega)]^2\}$$

$$= E\left\{\left[\int_0^\infty dx \; g(x)\varphi(t-x,\omega)-\varphi(t-v,\omega)\right]\left[\int_0^\infty dy \; g(y)\varphi(t-y,\omega)-\varphi(t-v,\omega)\right]\right\}$$

$$= \int_0^\infty dx \int_0^\infty dy \; g(x)g(y)E[\varphi(t-x,\omega)\varphi(t-y,\omega)]$$

$$- 2\int_0^\infty dx \; g(x)E[\varphi(t-x,\omega)\varphi(t-v,\omega)] + E[\varphi^2(t-v,\omega)]$$

$$= \int_0^\infty dx \int_0^\infty dy \; g(x)g(y)R_\varphi(x-y) - 2\int_0^\infty dx \; g(x)R_\varphi(v-x) + R_\varphi(0) \qquad (C-8)$$

From Eq. (10), the third term above is

$$R_\varphi(0) = \int_{-\infty}^\infty d\rho \; W_\varphi(\rho) \qquad (C-9)$$

and the second term becomes

$$\int_0^\infty dx \; g(x)R_\varphi(v-x) = \int_0^\infty dx \; g(x) \int_{-\infty}^\infty d\rho \; W_\varphi(\rho)e^{i2\pi\rho(v-x)}$$

$$= \int_{-\infty}^\infty d\rho \; W_\varphi(\rho)G(\rho)e^{i2\pi\rho v} \qquad (C-10)$$

The leading term in Eq. (C-8) amounts to another integration on Eq. (C-10), or

$$\int_0^\infty dx \int_0^\infty dy \; g(x)g(y)R_\varphi(x-y) = \int_0^\infty dx \; g(x) \int_{-\infty}^\infty d\rho \; W_\varphi(\rho)G(\rho)e^{i2\pi\rho x}$$

$$= \int_{-\infty}^\infty d\rho \; W_\varphi(\rho)G(\rho)G(-\rho) = \int_{-\infty}^\infty d\rho \; W_\varphi(\rho)|G(\rho)|^2 \qquad (C-11)$$

Substituting Eqs. (C-9), (C-10), and (C-11) in Eq. (C-8) then yields

$$E\left\{[\Phi(t,\omega)-\varphi(t-v,\omega)]^2\right\} = \int_{-\infty}^{\infty}d\rho \ W_\varphi(\rho)\left[|G(\rho)|^2-2G(\rho)e^{i2\pi\rho v} + 1\right] \qquad (C-12)$$

and further substituting Eqs. (C-7) and (C-12) in (C-6) yields

$$E\left\{\varphi(t\pm\tau-u,\omega) \ [\Phi(t,\omega)-\varphi(t-v,\omega)]^3\right\}$$

$$= 3 \left\{ \int_0^{\infty} dx \ g(x)R_\varphi(\pm\tau+x-u)-R_\varphi(\pm\tau+v-u)\right\}$$

$$(C-13)$$

$$\times \left\{ \int_{-\infty}^{\infty} d\rho \ W_\varphi(\rho)\left[|G(\rho)|^2-2G(\rho)e^{i2\pi\rho v} + 1\right] \right\}$$

When the Fourier transform operation of Eq. (C-4) is performed on Eq. (C-13), it is clear that only the first factor is involved. Its two forms are

$$\mathscr{F}\left\{ \int_0^{\infty}dx \ g(x)R_\varphi(\tau+x-u)-R_\varphi(\tau+v-u)\right\}$$

$$= \int_0^{\infty}dx \ g(x) \int_{-\infty}^{\infty}d\tau e^{-i2\pi f\tau}\left[R_\varphi(\tau+x-u)-R_\varphi(\tau+v-u)\right]$$

$$= \left[\int_0^{\infty}dx \ g(x)e^{i2\pi fx}- e^{i2\pi fv}\right]e^{-i2\pi fu} \int_{-\infty}^{\infty} d\tau e^{-i2\pi f\tau} \ R_\varphi(\tau)$$

$$= \left[G(-f)-e^{i2\pi fv}\right]e^{-i2\pi fu} \ W_\varphi(f) \qquad (C-14)$$

and, noting that R_φ is an even function

$$\mathscr{F}\left\{\int_0^\infty dx\ g(x)R_\varphi(-\tau+x-u)-R_\varphi(-\tau+v-u)\right\}$$

$$\text{(C-15)}$$

$$= \left[G(f)\ -e^{-i2\pi fv}\right]e^{i2\pi fu}\ W_\varphi(f)$$

Substituting Eqs. (C-13), (C-14), and (C-15) in (C-4) then yields

$$W_\theta^D(f)\ =\frac{1}{2}\int_0^\infty du\int_0^\infty dv\ g(u)g(v)W_\varphi(f)$$

$$\times\ \left\{\left[G(-f)-e^{i2\pi fv}\right]e^{-i2\pi fu}+\left[G(f)-e^{-i2\pi fv}\right]e^{i2\pi fu}\right\}\qquad\text{(C-16)}$$

$$\times\ \left\{\int_{-\infty}^\infty d\rho\ W_\varphi(\rho)\left[|G(\rho)|^2-\ 2G(\rho)e^{i2\pi\rho v}+1\right]\right\}$$

The integrations on u follow trivially.

Consider next the product of the second line of Eq. (C-16) with those functions of ρ in the third line which do not contain v, viz., $|G(\rho)|^2+1$. The bracketed quantities, $[G(\pm f)-\exp(\mp i2\pi fv)]$, then clearly vanish when integrated on v. Thus, the only term remaining is

$$W_\theta^D(f) = - W_\varphi(f) \int_{-\infty}^{\infty} d\rho \; W_\varphi(\rho) G(\rho) \int_0^\infty dv \; g(v) e^{i2\pi\rho v}$$

$$\times \left\{ \left[G(-f) - e^{i2\pi fv} \right] G(f) + \left[G(f) - e^{-i2\pi fv} \right] G(-f) \right\}$$

$$= - W_\varphi(f) \int_{-\infty}^{\infty} d\rho \; W_\varphi(\rho) G(\rho)$$

$$\times \left\{ \left[G(-\rho)G(-f) - G(-\rho-f) \right] G(f) + \left[G(-\rho)G(f) - G(-\rho+f) \right] G(-f) \right\}$$

$$= - W_\varphi(f) \int_{-\infty}^{\infty} d\rho \; W_\varphi(\rho) G(\rho)$$

$$(\text{C-17})$$

$$\times \left[2|G(f)|^2 G(-\rho) - G(-\rho-f)G(f) - G(-\rho+f)G(-f) \right]$$

The spectrum W is an even function so a change in sign on the third term permits the product of the G functions to be written with the negative of the arguments of the similar product in the center term. Then, by the symmetry property of G given by Eq. (3), the sum of the last two terms in Eq. (C-17) is seen to equal twice the real part of either one. Thus, Eq. (C-17) becomes

$$W_\theta^D(f) = 2W_\varphi(f) \int_{-\infty}^{\infty} d\rho \; W_\varphi(\rho) \left\{ \text{Re } G(\rho)G(-\rho-f)G(f) - |G(f)|^2 |G(\rho)|^2 \right\} \quad (\text{C-18})$$

which is the second of Eqs. (14).

The terms of next higher order in Eq. (12) arise from the $S \times D_5$ and the $D_3 \times D_3$ terms of Eq. (C-1). The first of these can be recognized immediately as a second-order contribution to the distortion given by Eq. (C-18). This follows from the fact that when the expected value of the resulting sixth-order product of gaussian variates is factored into the various groups of three pairs according to Eq. (C-5), then only one of the pairs will be a function of the covariance variable τ. When the Fourier transform of the product of the three expected values is taken, only that term will be involved and the result will yield a multiplicative factor $W_\varphi(f)$. In this respect, it is completely analogous to the steps leading to Eq. (C-14).

The $D_3 \times D_3$ term also yields a second-order distortion contribution as well as the desired first-order crosstalk. To show this, note from Eq. (C-1) that

$$D_3 \times D_3 = \frac{1}{(3!)^2} \mathscr{F} E \left\{ \int_0^\infty du \ g(u) [\varphi(t-u,\omega)-\Phi(t,\omega)]^3 \right.$$

$$\left. \times \int_0^\infty dv \ g(v) [\varphi(t+\tau-v,\omega)-\Phi(t+\tau,\omega)]^3 \right\}$$

$$= \frac{1}{36} \mathscr{F} \int_0^\infty du \int_0^\infty dv \ g(u)g(v) \ E\left\{ [\varphi(t-u,\omega)-\Phi(t,\omega)]^3 \right.$$

$$\left. \times [\varphi(t+\tau-v,\omega)-\Phi(t+\tau,\omega)]^3 \right\} \qquad \text{(C-19)}$$

For brevity, let a and b denote, respectively, the first and second bracketed terms in Eq. (C-19). Then, according to Eq. (C-5), the 15 different pairings of $E(a^3 b^3)$ yield the two unique terms

$$E(a^3b^3) = 9E(a^2)E(b^2)E(ab) + 6E^3(ab) \tag{C-20}$$

Of these, only $E(ab)$ is a function of the covariance variable τ, so the first term in Eq. (C-20) contributes a second-order term to the distortion given by Eq. (C-18) exactly as did the SxD_5 terms. Using only the second term of Eq. (C-20) in Eq. (C-19) then yields

$$W_\theta^C(f) = \frac{1}{6} \, \mathcal{F} \int_0^\infty du \int_0^\infty dv \; g(u)g(v) \; E^3\{[\varphi(t-u,\omega)-\Phi(t,\omega)]$$

$$\times [\varphi(t+\tau-v,\omega)-\Phi(t+\tau,\omega)]\} \tag{C-21}$$

and it remains to perform the indicated operations.

Noting Eq. (8) for Φ, the expected value in Eq. (C-21) is given by

$$E(ab) = E\left\{\left[\varphi(t-u,\omega)-\int_0^\infty dx \; g(x)\varphi(t-x,\omega)\right]\left[\varphi(t+\tau-v,\omega)-\int_0^\infty dx \; g(x)\varphi(t+\tau-x,\omega)\right]\right\}$$

$$= E[\varphi(t-u,\omega)\varphi(t+\tau-v,\omega)] - \int_0^\infty dx \; g(x) \; \{E[\varphi(t+\tau-v,\omega)\varphi(t-x,\omega)]$$

$$+ E[\varphi(t-u,\omega)\varphi(t+\tau-x,\omega)]\} + \int_0^\infty dx \int_0^\infty dy \; g(x)g(y)E[\varphi(t-x,\omega)\varphi(t+\tau-y,\omega)]$$

$$= R_\varphi(\tau+u-v) - \int_0^\infty dx \; g(x)[R_\varphi(\tau+x-v)+R(\tau+u-x)] + \int_0^\infty dx \int_0^\infty dy \; g(x)g(y)R_\varphi(\tau+x-y) \tag{C-22}$$

where the definition of R given by Eq. (11) is used in the last step. From Eq. (10), the first term in Eq. (C-22) is simply

$$T_1 = \int_{-\infty}^\infty d\rho \; W_\varphi(\rho)e^{i2\pi\rho(\tau+u-v)} \tag{C-23}$$

and the second term, noting Eq. (2) for G, is

$$T_2 = \int_0^\infty dx \; g(x) \int_{-\infty}^\infty d\rho \; W_\phi(\rho) \left[e^{i2\pi\rho(\tau+x-v)} + e^{i2\pi\rho(\tau+u-x)} \right]$$

$$= \int_{-\infty}^\infty d\rho \; W_\phi(\rho) \; e^{i2\pi\rho\tau} \left[e^{-i2\pi\rho v} \; G(-\rho) + e^{i2\pi\rho u} \; G(\rho) \right] \qquad (C-24)$$

The third term follows similarly as

$$T_3 = \int_0^\infty dx \int_0^\infty dy \; g(x)g(y) \int_{-\infty}^\infty d\rho \; W_\phi(\rho) \; e^{i2\pi\rho(\tau+x-y)}$$

$$= \int_{-\infty}^\infty d\rho \; W_\phi(\rho) \; e^{i2\pi\rho\tau} \; G(\rho)G(-\rho) \qquad (C-25)$$

Substituting these into Eq. (C-22) then yields

$$E(ab) = \int_{-\infty}^\infty d\rho \; W_\phi(\rho) e^{i2\pi\rho\tau} \left[G(\rho) - e^{-i2\pi\rho v} \right] \left[G(-\rho) - e^{i2\pi\rho u} \right] \qquad (C-26)$$

Finally, substituting back into Eq. (C-21) in the cube gives

$$W_\theta^C(f) = \frac{1}{6} \, \mathscr{F} \int_0^\infty du \int_0^\infty dv \; g(u)g(v)$$

$$\times \left\{ \int_{-\infty}^\infty dv \; W_\varphi(v) e^{i2\pi v \tau} \left[G(v) - e^{-i2\pi vv} \right] \left[G(-v) - e^{i2\pi vu} \right] \right\} \{\rho\} \{\sigma\}$$

$$= \frac{1}{6} \, \mathscr{F} \int_{-\infty}^\infty dv \int_{-\infty}^\infty d\rho \int_{-\infty}^\infty d\sigma \; W_\varphi(v) W_\varphi(\rho) W_\varphi(\sigma) e^{i2\pi(v+\rho+\sigma)\tau}$$

$$\times \left\{ \int_0^\infty du \; g(u) \left[G(-v) - e^{i2\pi vu} \right] \left[G(-\rho) - e^{i2\pi \rho u} \right] \left[G(-\sigma) - e^{i2\pi \sigma u} \right] \right\}$$

(C-27)

$$\times \left\{ \int_0^\infty dv \; g(v) \left[G(v) - e^{-i2\pi vv} \right] \left[G(\rho) - e^{-i2\pi \rho v} \right] \left[G(\sigma) - e^{-i2\pi \sigma v} \right] \right\}$$

The integrals on u and v are readily determined by expanding the integrands and applying Eq. (2) to each term. Collecting terms involving v yields

$$- 2G(v)G(\rho)G(\sigma) + G(v)G(\rho+\sigma) + G(\rho)G(v+\sigma)$$

(C-28)

$$+ G(\sigma)G(v+\rho) - G(v+\rho+\sigma)$$

The corresponding result in u is similar but with negative arguments. These quantities are complex conjugates (c.f. Eq. (3)), so their product equals the square of their magnitude and Eq. (C-27) becomes

$$W_\theta^C(f) = \frac{1}{6} \ \mathcal{F} \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma \ W_\varphi(\nu) W_\varphi(\rho) W_\varphi(\sigma) \ e^{i2\pi(\nu+\rho+\sigma)\tau}$$

$$\times \ \left| 2G(\nu)G(\rho)G(\sigma) - G(\nu)G(\rho+\sigma) - G(\rho)G(\nu+\sigma) \right. \quad\quad (C-29)$$

$$\left. - G(\sigma)G(\nu+\rho) + G(\nu+\rho+\sigma) \right|^2$$

Applying the Fourier transform to the exponential then yields a delta function $\delta(f-\nu-\rho-\sigma)$ thereby permitting a simple integration on ν. Finally, therefore

$$W_\theta^C(f) = \frac{1}{6} \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma \ W_\varphi(f-\rho-\sigma) W_\varphi(\rho) W_\varphi(\sigma)$$

$$\times \ \left| 2G(f-\rho-\sigma)G(\rho)G(\sigma) - G(f-\rho-\sigma)G(\rho+\sigma) \right. \quad\quad (C-30)$$

$$\left. - G(\rho)G(f-\rho) - G(\sigma)G(f-\sigma) + G(f) \right|^2$$

which is the third of Eqs. (14).

REFERENCES

1. Carson, J. R. and T. C. Fry, "Variable Frequency Electric Circuit Theory with Application to the Theory of Frequency Modulation," BSTJ, Vol. 16, No. 4, October 1937, pp. 513-540.

2. Baghdady, E. J., "Theory of Low Distortion Reproduction of FM Signals in Linear Systems, " Trans. IEEE, Vol. CT-5, No. 3, September 1958, pp. 202-214; "On the Response of a Linear System to an FM Signal," Trans. IEEE, Vol. CT-6, No. 4, December 1959, pp. 387-388.

3. Panter, P. F., Modulation, Noise and Spectral Analysis, McGraw-Hill Book Company, Inc., New York, 1965, Chaps. 8-10.

4. Bloch, A., "Modulation Theory," J. IEE, Vol. 91, Part III, January 1944, pp. 31-42.

5. Van der Pol, B., "The Fundamental Principles of Frequency Modulation," J. IEE, Vol. 93, Part III, May 1946, pp. 153-158.

6. Stumpers, F.L.H.M., "Distortion of Frequency Modulated Signals in Electrical Networks," Communication News, Vol. 9, No. 3, April 1948, pp. 82-92.

7. Clavier, A. G., "Application of Fourier Transforms to Variable-Frequency Circuit Analysis," Proc. IRE, Vol. 37, No. 11, November 1949, pp. 1287-1290.

8. Hupert, J. J., "Transient Response of Narrow Band Networks to Angle-Modulated Signals," Proc. N.E.C., Vol. 18, 1962, pp. 458-468.

9. Hupert, J. J., "Response of Linear Networks to Modulated Waveforms," Proc. IEEE, Vol. 53, No. 9, September 1965, pp. 1267-1268.

10. Weiner, D. D. and B. J. Leon, "On the Quasi-Stationary Response of Linear Time-Invariant Filters to Arbitrary FM Signals," Trans. IEEE, Vol. CT-11, No. 2, June 1964, pp. 308-309.

11. Weiner, D. D. and B. J. Leon, "The Quasi-Stationary Response of Linear Systems to Modulated Waveforms," Proc. IEEE, Vol. 53, No. 6, June 1965, pp. 564-575.

12. Rowe, H. E., "Distortion of Angle-Modulated Waves by Linear Networks," Trans. IEEE, Vol. CT-9, No. 3, September 1962, pp. 286-290.

13. Hess, D. T., "Transmission of F.M. Signals through Linear Filters," _Proc. N.E.C._, Vol. 18, 1962, pp. 469-476.

14. Frantz, W. J., "The Transmission of a Frequency-Modulated Wave through a Network," _Proc. IRE_, Vol. 34, No. 3, March 1946, pp. 114-125.

15. Gladwin, A. S., "The Distortion of Frequency Modulated Waves by Transmission Networks," _Proc. IRE_, Vol. 35, No. 12, December 1947, pp. 1436-1445.

16. Zinn, M. K., "Transient Response of a FM Receiver," _BSTJ_, Vol. 27, No. 4, October 1948, pp. 715-731.

17. Hupert, J. J., "Normalized Phase and Gain Derivatives as an Aid in Evaluation of FM Distortion," _Proc. IRE_, Vol. 42, No. 2, February 1954, pp. 438-446.

18. Medhurst, R. G., "Harmonic Distortion of Frequency-Modulated Waves by Linear Networks," _Proc. IEE_, Vol. 101, Part III, No. 71, May 1954, pp. 171-181.

19. Turbovich, I. T., "The Effect of Frequency and Amplitude Modulated Oscillations on Linear Systems," _Radiotekhnika_, Vol. 15, No. 1, 1960, pp. 30-34.

20. Mikheyev, N. G., "Response of an Oscillatory Circuit to Phase-Modulated Oscillations for Small Modulation Indices," _Radio-tekhniki e Elektronika_, No. 11, 1964, pp. 80-82.

21. Hupert, J. J., "A Method of Evaluation of the Quasi-Stationary Distortion of FM Signals in Tuned Interstages," _Proc. N.E.C._, Vol. 8, 1952, pp. 445-461.

22. Kantor, L. Ya., "Choice of Interstage Coupling Network in Receivers with Frequency Modulation," _Elektrosvyaz_, No. 4, 1957, pp. 29-32.

23. Kononovich, L. M., "The Bandwidth Required in the I.F. Stage of F.M. Receivers," _Elektrosvyaz_, No. 11, 1959, pp. 24-33.

24. Rychka, V. L., "The Passage of Frequency Modulated Oscillations Through an Aperiodic Amplifier," _Elektrosvyaz_, No. 2, 1961, pp. 3-11.

25. Marique, J., "The Response of RLC Resonant Circuits to EMF of Sawtooth Varying Frequency," _Proc. IRE_, Vol. 40, No. 8, August 1952, pp. 945-950.

26. Turbovich, I. T., "Dynamic Frequency Characteristics of Selective Systems," _Radiotekhnika_, Vol. 12, No. 11, 1957, pp. 39-49.

27. Gold, B., "Frequency Transients in Idealized Linear Systems," IRE Conv. Rec., Part 5, 1953, pp. 95-101.

28. McCoy, R. E., "FM Transient Response of Band-Pass Circuits," Proc. IRE, Vol. 42, No. 3, March 1954, pp. 574-579.

29. Gumowski, I., "Transient Response in FM," Proc. IRE, Vol. 42, No. 5, May 1954, pp. 819-822; see also M. Nadler and I. Gumowski, "Discussion of 'Transient Response in FM,' by I. Gumowski," Proc. IRE, Vol. 43, No. 6, June 1955, pp. 750-751 and D. A. Linden, "Transient Response in FM," Proc. IRE, Vol. 45, No. 7, July 1957, pp. 1017-1018.

30. Dennis, F. L. and D. A. Linden, "Computation of FM Transient Response," Proc. IRE, Vol. 47, No. 9, September 1959, pp. 323-324.

31. Shaft, P. D., "Distortion of Multitone FM Signals Due to Phase Nonlinearity," Trans. IEEE, Vol. SET-9, No. 1, March 1963, pp. 25-35.

32. Prosin, A. V., "Theory of Passing Wideband Signals in Earth-Satellite Communications Systems," Radiotekhnika e Elektronika, No. 11, 1963, pp. 1710-1720.

33. Bennett, W. R., H. E. Curtis and S. O. Rice, "Interchannel Interference in FM and PM Systems under Noise Loading Conditions," BSTJ, Vol. 34, No. 3, May 1955, pp. 601-636.

34. Rice, S. D., "Distortion Produced in a Noise Modulated FM Signal by Nonlinear Attenuation and Phase Shift," BSTJ, Vol. 36, No. 4, July 1957, pp. 879-889.

35. Bosse, G., "Die Verzerrung frequenz modulierter Schwingungen beim Durchgang durch lineare Netzwerke," Frequenz, Vol. 12, October 1958, pp. 6-14.

36. Medhurst, R. G., "Explicit Form of F.M. Distortion Products with White-Noise Modulation," Proc. IEE, Monograph No. 352E, Vol. 107, Part c, January 1960, pp. 120-126; see also R. G. Medhurst and J. H. Roberts, "Explicit Form of F.M. Distortion Products with White-Noise Modulation-Extension and Correction," Proc. IEE, Monograph No. 352E, Vol. 107, Part c, September 1960, pp. 367-369.

37. Magnusson, R. I., "Intermodulation Noise in Linear F.M. Systems," Proc. IEE, Monograph No. 459E, Vol. 109, Part c, No. 15, March 1962, pp. 32-44.

38. Bedrosian, E. and P. D. Shaft, "Comment on Distortion of Multitone FM Signals Due to Phase Nonlinearity," Trans. IEEE, Vol. SET-10, No. 3, September 1964, p. 125.

39. Enloe, L. H. and C. L. Ruthroff, "A Common Error in FM Distortion Theory," Proc. IEEE, Vol. 51, No. 5, May 1963, p. 846; see also A. S. Gladwin, R. G. Medhurst, L. H. Enloe and C. L. Ruthroff, "A Common Error in FM Distortion Theory, Proc. IEEE, Vol. 52, No. 2, February 1964, pp. 186-189 and R. I. Magnusson, L. H. Enloe and C. L. Ruthroff, "A Common Error in FM Distortion Theory," Proc. IEEE, Vol. 52, No. 9, September 1964, pp. 1082-1084.

40. Chaffee, J. G., "The Application of Negative Feedback to Frequency Modulation Systems," BSTJ, Vol. 18, No. 3, July 1939, pp. 404-437.

41. Enloe, L. H., "Decreasing the Threshold in FM by Frequency Feedback," Proc. IRE, Vol. 50, No. 1, January 1962, pp. 18-30.

42. Ruthroff, C. L. and W. F. Bodtmann, "Design and Performance of a Broad-Band FM Demodulator with Frequency Compression," Proc. IRE, Vol. 50, No. 12, December 1962, pp. 2436-2445.

43. Giger, A. J. and J. G. Chaffee, "The FM Demodulator with Negative Feedback," BSTJ, Vol. 42, No. 4, Part 1, July 1963, pp. 1109-1135.

44. Hille, E. and R. S. Phillips, Functional Analysis and Semi-Groups, rev. ed., American Mathematical Society Colloquium Publications, Vol. XXXI, American Mathematical Society, Providence, Rhode Island, 1957, Chaps. 3 and 26.

45. Rice, S. O., "Mathematical Analysis of Random Noise," BSTJ, Vol. 23, No. 3, July 1944, pp. 282-332 and Vol. 24, No. 1, January 1945, pp. 46-156; see also N. Wax (ed.), Selected Papers on Noise and Stochastic Processes, Dover Publications, New York, 1945.

46. Bromwich, T. J. I'a., An Introduction to the Theory of Infinite Series, 2d ed. rev., MacMillan and Company, London, 1942, p. 499.

47. Henderson, K. W. and W. H. Kautz, "Transient Responses of Conventional Filters," Trans. IRE, Vol. CT-5, No. 4, December 1958, pp. 333-347; see also G. A. Beck, "Comments on 'Transient Responses of Conventional Filters,'" Trans. IRE, Vol. CT-8, No. 2, June 1961, pp. 166-167 and E. van Vollenhoven and H. A. Reuver, "Comments on 'Transient Responses of Conventional Filters,'" Trans. IRE, Vol. CT-8, No. 4, December 1961, p. 500.

48. Wang, M. C. and G. E. Uhlenbeck, "On the Theory of the Brownian Motion II," Rev. Mod. Phys., Vol. 17, Nos. 2 and 3, April-July 1945, pp. 323-342; see also N. Wax (ed.), Selected Papers on Noise and Stochastic Processes, Dover Publications, New York, 1945.