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Distortion, degeneracy and rezoning in finite elements – A survey

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Abstract. The results obtained from finite element analysis are significantly affected by the quality of elements. In certain applications like shape optimization, crash analysis, metal forming, fluid flow analysis, and large displacement analysis, the finite element mesh is systematically updated in an iterative process. In such situations, in spite of an ideal starting mesh, the quality of elements could deteriorate, causing severly distorted elements. In extreme cases, the elements become degenerate and further progress of analysis is restricted. An understanding of the methods of quantifying element distortion helps in identifying 'bad' geometry and in deciding when to remesh. Knowledge about geometric configurations which cause degeneracy assists in controlling degeneracy during the analysis. This paper contains a survey of available distortion measures and degeneracy conditions for various elements in two and three dimensions. It is a review of the literature in this field in the last two decades. A brief review of rezoning is also included, since it is one of the more popularly used methods to correct a distorted mesh.

Keywords. Distortion; degeneracy; finite element; Jacobian; mapping; rezoning; grid optimization.

1. Introduction

Distortion measures are usually functions of the coordinates of the element. Examples of distortion measures for four-noded quadrilaterals are aspect ratio, skew and tapers. Most of the available literature in defining distortion is for two-dimensional elements. These measures either use terms of the Jacobian matrix or simply define measures in terms of linear dimensions and angles. These are purely a priori, in the sense that they just capture the shape of the element quantitatively. Some authors have also studied the relationship between stiffness matrix terms and element shapes. The parameters used are condition number and trace of the stiffness matrix. It should be noted that the distortion measures only tell the user how distorted the element shape is with respect to a standard element. Whether this is good, bad or not relevant

depends on the problem at hand. The effect of distortion on errors cannot be assessed a priori. However, bad elements are usually sources of trouble especially in regions where the gradient of the solution parameter is high. For example, in elasticity problems, they are the regions of stress concentration.

A degenerate element is one for which the coordinate transformation becomes mathematically invalid. Our discussion is limited to isoparametric elements which use serendipity functions. The element in physical space, x, y, z is mapped to a standard element in ξ, η, ζ coordinate system. For a one to one mapping, the Jacobian of the transformation matrix should not change sign or become zero anywhere in the element domain. The term 'Jacobian' denotes the determinant of the Jacobian matrix. Thus, $J = \det [J]$. The effect of the Jacobian being zero is discussed later in the paper. Degeneracy has to be avoided for a reliable solution. Detecting degeneracy mathematically is a problem of finding the zero of a polynomial. Simple thumb rules are possible only for linear elements and simplified higher order elements. For practical problems in engineering, the Jacobian can be sampled along the boundary and at Gauss points to identify degeneracy quickly. Once the mesh is identified to be bad for further analysis, it is modified using rezoning or adaptive techniques. Rezoning using Laplacian smoothing is the fastest and simplest way of modifying the mesh. Hence it is the most widely used rezoning method with different forms of weights in the rezoning formula. An adaptive strategy based on rezoning uses error measures as weights and iteratively rezones the mesh till convergence. Grid optimization is a more rigorous method of obtaining the optimum mesh for a given problem. The idea of grid optimization is to seek a minimum potential energy configuration with nodal coordinates as design variables along with nodal displacements.

This paper contains mainly two sections. The first section is a survey of various distortion measures proposed in the literature. It also includes the methods of element evaluation used to study the behaviour of elements to known displacement fields. The second section is a review of mathematical literature on nonvanishing of Jacobian in the element. A brief review of rezoning follows this section.

2. Distortion measures and element evaluation

The first step towards defining distortion measures is to parametrize the shape of the finite element in terms of quantities which are independent of coordinate systems. The coordinates of the element itself can constitute a set of parameters, but in that case, the distortion measure will be dependent on the coordinate system. In addition, it will be very difficult to make any intuitive judgement about the shape of the element from its coordinates. Hence in order to capture shape variation, independent of coordinate systems, various parameters have been proposed for linear and quadratic elements. Of these, the most popular are the ones for the four-noded bilinear isoparametric element. There are four shape parameters for this element-aspect ratio, skew, and two tapers along each coordinate direction. Using these four parameters, it can be shown that all types of shape changes can be independently captured. General guidelines are available to limit these values for any analysis. For example, most commercial codes recommend an aspect ratio between 1 and 3 (e.g., I-DEAS). However, distortion of the element is caused by a combination of all these parameters. Hence limits on these quantities independent of each other is not sufficient. This fact is another argument for the need to have a distortion measure which is representative

of the behaviour of the element. Most of these measures are discussed with 2-D examples like bilinear and quadratic elements. Some of the measures like distortion metric can be extended to 3-D elements. The literature available for 3-D distortion measures is very limited.

2.1 *Shape parameters*

Robinson (1985) expressed shape parameters in terms of simple polynomial coefficients with a clear physical meaning. These parameters are evaluated from the elements of the Jacobian matrix. This concept is reviewed here. In the finite element analysis of general structures, quadrilateral elements can be used on curved surfaces. The plane in which the element lies need not necessarily be flat. The stiffness matrix for a warped element is usually based on a flat projected plane. It is thus necessary to consider warpage as one of the parameters. The warped quadrilateral (figure 1) with straight edges is denoted by its corner nodes A, B, C, D. The reference nodes for the projected plane are denoted as 1 to 8. This plane contains the midpoints of each side. The warpage of a quadrilateral is measured by its deviation from a flat projected plane. Each corner node of the warped element will be at a distance h from the corresponding corner point of the projected plane (figure 1). The height h is given by $h = \text{abs}(\overline{OA} \cdot \overline{Z})$. For a flat element $h = 0$. The warpage is measured by the warpage parameter θ (degrees) which is defined as

$$\theta = \sin^{-1}(h/l)$$

where l is half of the smallest side of the element. Shape parameters for four-noded flat (projected) quadrilateral elements can be written from the interpolation functions found in any text book on finite element analysis (e.g. see Chandrupatla & Belegundu 1991),

$$x(or\ y) = \sum_{i=1}^{4} N_i x_i (or\ y_i)$$

where N_i are the standard shape functions for a four-noded quadrilateral element. The above equations show that eight parameters are needed to define a quadrilateral. In this case they are x and y coordinates of the four corner nodes. This form, however hides the significance of shape parameters. An alternative form of shape

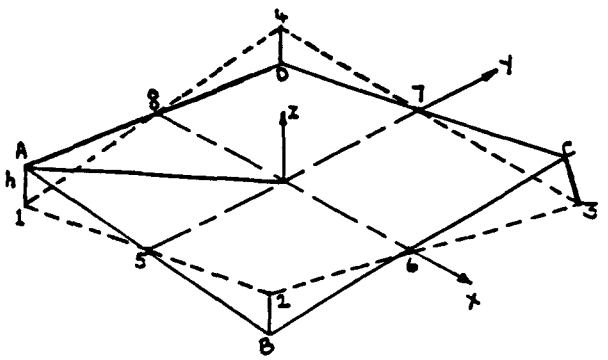

Figure 1. A warped quadrilateral element and its projected plane.

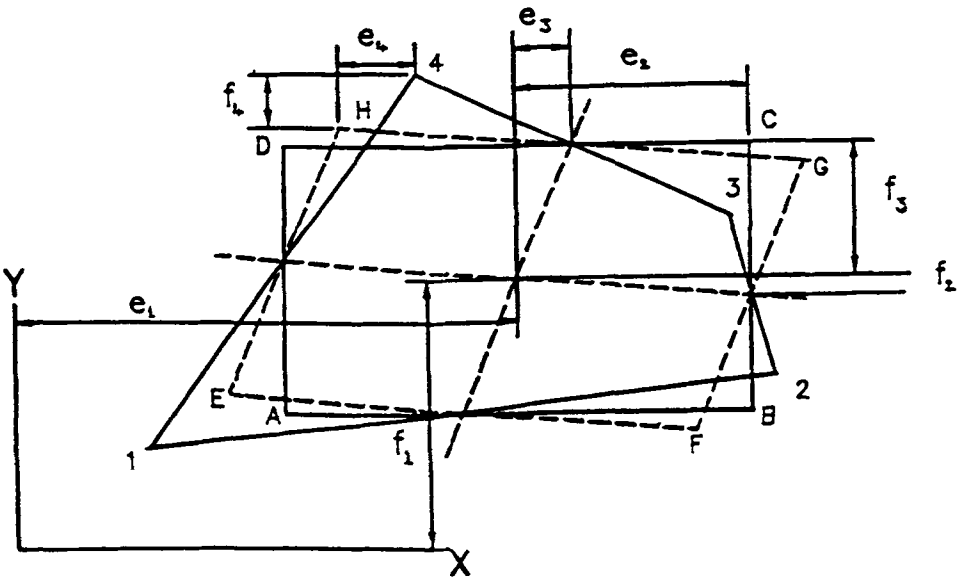

Figure 2. Shape parameters for a 4-noded quadrilateral.

representation is the polynomial form given by

$$x = e_1 + e_2\xi + e_3\eta + e_4\xi\eta,$$
$$y = f_1 + f_2\xi + f_3\eta + f_4\xi\eta, \tag{1}$$

where e and f are coefficients which can be related to the nodal coordinates by substituting the known values of x and y at the corner nodes. For example, e_1 and f_1 are given by

$$e_1 = (1/4)(x_1 + x_2 + x_3 + x_4) \quad \text{and} \quad f_1 = (1/4)(y_1 + y_2 + y_3 + y_4).$$

The physical significance of e_i and f_i are shown in figure 2, where 1–2–3–4 is the quadrilateral element. The rectangle *ABCD* is drawn through midpoints of the sides of the element so that the sides of the rectangle are parallel to the x and y coordinate axes. The dotted lines $E-F-G-H$ form the parallelogram through midpoints of the element sides with its edges parallel to the ξ, η axes. The coefficients e_1 and f_1 define the coordinates of the geometric centre of the element, e_2 and f_3 define the size of the rectangle $A-B-C-D$, e_3 and f_2 give two rotations (skew and rotation of the axes) and e_4 and f_4 give two tapers. If the local axes are defined with the origin at the centre of the element so that $e_1 = f_1 = f_2 = 0$, the required shape parameters can be defined as follows,

$$\text{Aspect ratio} = \text{Max}(e_2/f_3, f_3), \quad \text{Skew} = e_3/f_3,$$
$$\text{Taper along } x = f_4/f_3 \quad \text{and} \quad \text{taper along } y = e_4/e_2.$$

If the transformation from ξ, η to x, y axes is regarded as a continuous deformation, the aspect ratio represents element stretching and skew is equal to the shearing strain at $\xi = 0, \eta = 0$.

The Jacobian matrix for a flat (projected) quadrilateral can also be expressed in terms of the shape parameters. Specifically, the following expression for the determinant of the Jacobian matrix can be easily derived.

$$\det[J] = f_3^2 A(1 + T_x\xi + (T_y - (S/A)\,T_x)\eta),$$

where A is the aspect ratio, S is the skew and T_x and T_y are the tapers along x and y directions, respectively. The parameter f_3 is a half side length of the basic rectangle (see figure 2). The coefficients in the Jacobian are therefore a function of the shape parameters.

2.1a *Shape parameters for an eight-noded quadrilateral*: For an eight-noded quadrilateral, a similar procedure to that for a four-noded quadrilateral can be used, but additional parameters come into the picture. These parameters account for the curved sides and midside nodes. Similar to (1), shape functions for an eight-node quadrilateral can be expressed in polynomial form (Robinson 1988)

$$x = e_1 + e_2\xi + e_3\eta + e_4\xi\eta + e_5\xi^2 + e_6\eta^2 + e_7\xi^2\eta + e_8\xi\eta^2,$$
$$y = f_1 + f_2\xi + f_3\eta + f_4\xi\eta + f_5\xi^2 + f_6\eta^2 + f_7\xi^2\eta + f_8\xi\eta^2,$$

where the coefficients e_i and f_i can be expressed in terms of the coordinates of the element. Referring to figure 3, the shape parameters are defined as given below. In figure 3 the local coordinate system is defined in such a way that $e_1 = f_1 = f_2 = 0$.

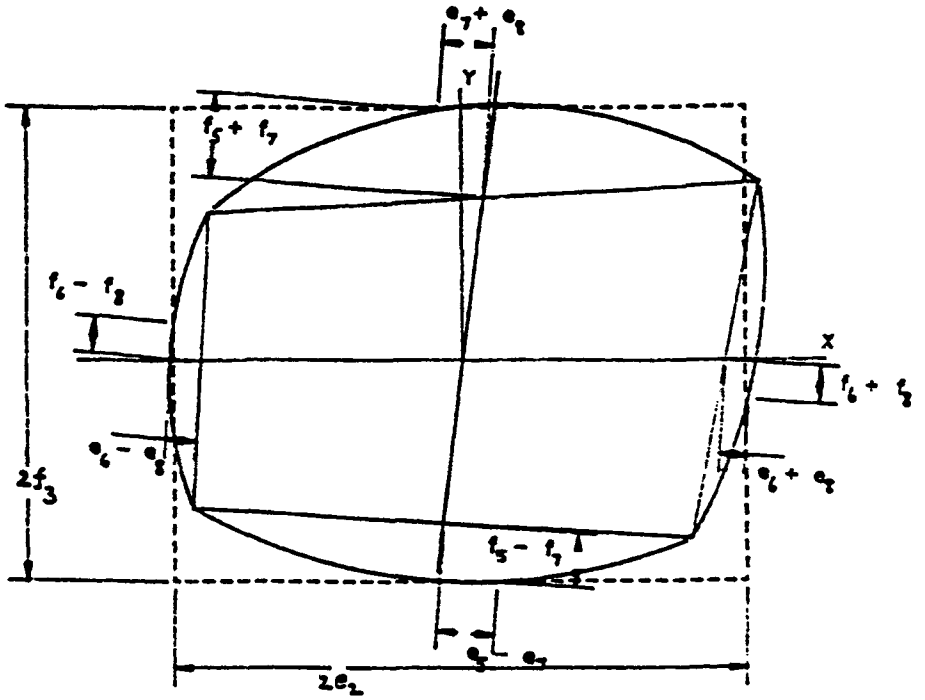

Figure 3. Shape parameters for an 8-noded quadrilateral.

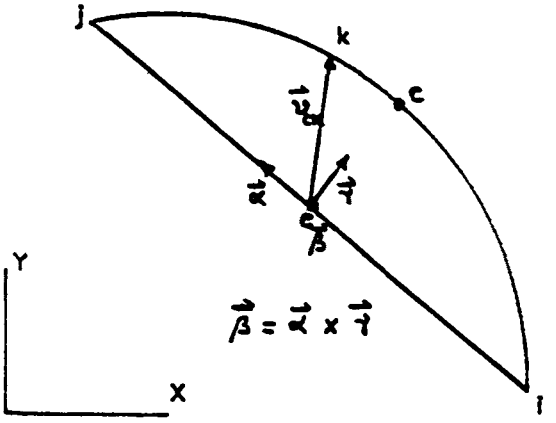

Figure 4. Tangential and normal deviations.

$$\text{Aspect ratio} = e_2/f_3, \text{or } f_3/e_2 \quad \text{(larger of the two)}, \quad \text{Skew} = e_3/f_3 + f_4/e_2,$$
$$\text{taper along } x = f_4/e_3 \quad \text{and} \quad \text{taper along } y = e_4/e_2.$$

The other shape parameters describe the curvature of the sides and offset of midnodes. The offset of the node on a curved boundary from the midpoint of the associated chord is shown in figure 4. The offset measures are defined as the normal and tangential deviations.

$$\text{Tangential deviation, TD} = \alpha/(1/2)\, L$$

where $\alpha = \mathbf{v}_{ck} \cdot \boldsymbol{\alpha}$ is a unit vector along ij.

$$\text{Normal deviation, ND} = \gamma/(1/2)\, L$$

where $\gamma = \boldsymbol{\alpha} \times \boldsymbol{\beta}$. The vector $\boldsymbol{\beta}$ is a unit vector perpendicular to the plane containing the nodes i, j and k. The offset parameters are normalized with respect to half the chord length. When TD is zero, the node on the curved boundary is in the centre at C (see figure 4), and when ND is zero, the actual boundary is straight. A different way of defining distortion parameters for eight-noded quadrilaterals can be found in Hellen (1984). These are briefly described below.

Let $\mathbf{r}$ denote the position vector of a point in the x, y coordinate system and $\mathbf{r}_{,\xi}$ and $\mathbf{r}_{,\eta}$ be the tangent vectors at any point ξ, η. Let δ be the angle between the two tangent vectors. The distortion parameters are defined as

$$\text{Aspect ratio} = \frac{|\mathbf{r}_{,\xi}|}{|\mathbf{r}_{,\eta}|} \quad \text{or} \quad \frac{|\mathbf{r}_{,\eta}|}{|\mathbf{r}_{,\xi}|} \quad \text{(larger of the two)},$$
$$\text{skew} = 90^\circ - \delta.$$

Max. chord aspect ratio = (maximum chord length)/(minimum chord length),
Boundary node offset, R = (length ck)/(length ij).

These parameters have been implemented in the BERQUAL program (Hellen 1984). The effect of these parameters on displacements for some structural problems and the discussion of the program can be found in Hellen (1986, 1987).

2.2 *Shape sensitivity analysis*

The basic idea of shape sensitivity studies is to vary the element shape parameters singly and in combination to assess their effect on the accuracy of the results produced, in a prescribed loading situation. This type of study have been carried out by Burrows (1986) and Robinson (1985, 1990) among others. In Burrows' work, a standard 2-D plane stress element is distorted in a predetermined way and subjected to standard loading conditions. Two types of loadings are considered – a constant stress and a constant moment. Theoretical nodal displacements and nodal forces are compared with FE nodal displacements and forces. Another method of element evaluation introduced by Robinson (1987) called continuum region element (CRE) testing is based on the idea that a rectangular continuum is available for which theoretical solutions are known for various loading conditions. Then the region is treated as a single element or a patch of elements. The applied load is in the form of specified displacements which are nonzero. The strain energy of the field due to FE formulation is computed and compared with the theoretical value of the strain energy. Examples on this approach to element testing can be found in the works of Robinson (1976, 1987, 1990). An analytical investigation on aspect ratio sensitivity of elements can be found in Robinson (1991). The elements studied are taken from commercial finite element codes like ABAQUS, MSC/NASTRAN, and ANSYS.

2.3 *Distortion measure based on polynomial order of the element geometry*

The shape parameters defined above for flat elements are easy to evaluate and implement in commercial codes. However there is need to understand their effects on solution errors. Since the error is a result of the combination of distortion and the variation of the unknown solution, a general correlation of the two is impossible. However, solution errors can be computed for assumed polynomial strain fields. This error can then be related to a known element distortion. Analytical study of distortion based on this concept was done by Barlow (1987, 1989). This approach for element evaluation is summarized below.

The basis of distortion measure is the polynomial order of the element geometry. Here, an example of this measure is presented with 8-noded isoparametric elements. The geometry of the element in an x, y coordinate system is written in a form which incorporates the distortion parameters. The form is,

$$[x\ y] = l[\xi\ \eta] + l\mathbf{F}[\boldsymbol{\varepsilon}_x\ \boldsymbol{\varepsilon}_y],$$

where

$$\mathbf{F} = [1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^2, \xi^2\eta, \xi\eta^2], \tag{2}$$

is a (1 × 8) vector of polynomial terms in N, $4l^2$ = area of the element (a scalar which makes the problem nondimensional.) and $\boldsymbol{\varepsilon}_x$ and $\boldsymbol{\varepsilon}_y$ are (8 × 1) vectors of the distortion parameters given by

$$\boldsymbol{\varepsilon}_x^T = [\varepsilon_{x1}, \varepsilon_{x2}, \ldots, \varepsilon_{x8}],$$

$$\boldsymbol{\varepsilon}_y^T = [\varepsilon_{y1}, \varepsilon_{y2}, \ldots, \varepsilon_{y8}].$$

The first right-hand side term in (2) is the undistorted parent element and the remaining terms are the distortions of that element. These can be grouped into polynomial orders of distortion as given below.

The terms ε_{x1} and ε_{y1} represent just an offset from the origin. The terms ε_{x2}, ε_{y2}, ε_{x3} and ε_{y3} represent linear geometric distortions. They are similar to aspect ratio and skew parameters. The next three terms ε_{x4}, ε_{y4}, ε_{x5}, ε_{y5}, ε_{x6} and ε_{y6} are the quadratic distortions. These can be interpreted as anti-symmetric edge curvature, taper and symmetric midside node offset. The last two terms ε_{x7}, ε_{y7}, ε_{x8} and ε_{y8} are the cubic distortions. They represent symmetric edge curvature and anti-symmetric midside node offset. The values of the distortion parameters may be extracted from the nodal geometry by substituting nodal coordinates in (2) and solving for ε_x and ε_y. A local coordinate system must be defined at the geometric centre of the element to avoid 'pseudo' distortion measures due to the orientation of the element. This ensures that the distortions are the same for a given shape independent of the element orientation. For graphical description of this measure, see Barlow (1989).

2.3a *Element evaluation*: For assumed distortion, the element is evaluated based on two considerations – its ability to reproduce the required strain fields and the accuracy of integration of the strains to produce the element stiffness, i.e. numerical integration used to compute K. It is assumed that the nodal loads produce an equilibrium stress field of a given polynomial order. For a constant stress strain relationship, this would mean a similar strain field. Now if the elements are capable of reproducing the individual strain terms, it would also be capable of reproducing any linear combination, including those which represent the equilibrium strain field. Thus the following procedure is used in the evaluation.

A polynomial displacement field is assumed and the exact strains are obtained by differentiation. The same polynomial displacements are applied to the finite element and the strains are determined from the element displacement functions. The difference between two strain fields, the exact and the element representation, provides a measure of the error in the element strain representation. The coefficients of the polynomial terms in the applied displacement field will depend on element orientation. Hence the order of magnitude of the error is taken as the maximum which occurs in any individual polynomial term of a given degree. An example is given below. Consider the effect of distortion ε_{x3} (skew) on a cubic displacement field. Let $u = x^3$. Let the element have only skew parameter i.e. ε_{x3} and other parameters be zero. The $x - y$ coordinate system is defined at the element centre with x-axis along the ξ axis. The applied strain field in terms of ξ, η is given by,

$$\varepsilon_{\text{exact}} = 3\xi^2 + 6\varepsilon_{x3}\xi\eta + 3\varepsilon_{x3}^2\eta^2,$$

while the finite element strain is given by,

$$\varepsilon_{FE} = 1 + 6\varepsilon_{x3}\xi\eta + 3\varepsilon_{x3}^2\eta^2.$$

The error in this case would be $e = 3\xi^2 - 1$. This evaluation procedure can be used for any element. In fact, Barlow (1989) has demonstrated this method for 20-noded brick elements in addition to quadrilateral elements. Based on this study, points at which minimum error occurs in the element domain can be obtained. In the above example, e is zero at Gauss points $(\xi, \eta) = (\pm 1/\sqrt{3}, \pm 1/\sqrt{3})$. These points are called optimal stress points. The procedure to analytically determine optimal points for assumed strain fields can be found in the works of Barlow (1976) and Budkowska (1991). For linear displacement fields the optimal points lie at $\xi = 0$, $\eta = 0$.

2.4 *Distortion measures based on the Jacobian matrix*

The multiple distortion parameters described above are obtained from the geometry of the element. Although simple to evaluate, their effect on the accuracy of the solution are interdependent. Thus, if a decision about element quality has to be made based on some numerical measure of the element geometry, a combination of all the shape parameters has to be used. In the previous sections, it was shown that the Jacobian can be expressed in terms of these shape parameters. Hence the value of the Jacobian in the element domain reveals very useful information about the element distortion. In this section some measures based on elements of the Jacobian matrix are described.

2.4a *Distortion parameter*: Distortion parameter (DP) was introduced by Nicolas & Citipitioglu (1977). For a general quadrilateral element it is defined as

$$\text{DP} = (4 \det[J]_{\min}/A) \text{ and for three dimensional elements it is defined as}$$

$$\text{DP} = (8 \det[J]_{\min}/V)$$

where A and V are respectively the area and the volume of the element. They are used to nondimensionalize the parameter. Above, $\det[J]_{\min}$ refers to the minimum value of the Jacobian determinant in the element. For a parallelogram shaped four-noded quadrilateral element, DP is 1·0. For a four-noded element which degenerates to a triangle, DP is 0·0. The value of DP becomes negative at some point if the four-noded element becomes concave. Similar interpretations for 3-D and general higher order elements are not as obvious. One of the major disadvantages of this parameter is that it cannot detect extremely degenerate parallelograms. A rectangle, with extremely high aspect ratio would still have a DP = 1·0 and so does a highly skewed parallelogram. Distortion of elements to such shapes are quite common in applications involving large changes in finite element mesh, like crash analysis, large displacement analysis or metal forming. A variation of DP, which is defined as the ratio of maximum value of Jacobian to its minimum value in the domain is sometimes used. This parameter also has all the above mentioned problems.

2.5 *Decomposition of the Jacobian matrix*

Separate measures of element orthogonality, aspect ratio, orientation and volume are computed by algebraic decomposition of the Jacobian matrix in this measure (Kerlick & Klopfer 1982). The decomposition is first done by splitting off that part of the matrix which gives relative orientation of the computational coordinate axes with respect to the physical axes. For example, for a 2-D element, the direction cosines of the ξ axis with respect to the x-axis is given by

$$\cos(\alpha) = (x_{,\xi}/x_{,\xi}^2 + y_{,\xi}^2)^{1/2}.$$

The components of J are expressed in terms of metric tensors, $g_{ij} = x_i x_j + {}^{y}{}_i y_j$. The parameters orthogonality, volume, orientation and aspect ratios are now expressed in terms of the components g_{ij}. This method is described for finite difference analysis in Kerlick's work (Kerlick & Klopfer 1982). These separate measures cannot describe combination of stretching and shearing.

2.6 *Distortion metric*

The distortion metric proposed by Oddy *et al* (1988) is based on the observation that an analogy can be drawn between element distortion and strain. If a body with the shape of the element in the computational space ξ, η, ζ were deformed until it had the shape of the element in physical space, x, y, z, then the elements with large distortions would experience the analogue of large strains. Hence for this "pseudo deformation", Green's strain is measured. If J_{ij} are the elements of Jacobian matrix, in n-dimensional space ($n = 2$ or 3), the elements of J are first normalized to neutralize the effect of size. Thus

$$J'_{ij} = J_{ij}/|J|^{1/n}.$$

Let $C_{ij} = J'_{ki} J'_{kj}$. Using standard tensor notations, the deviatoric strains can be written as

$$e_{ij} = (1/2)\left(C_{ij} - \frac{1}{n} C_{kk} \delta_{ij} \right).$$

The second invariant of this tensor in 2-D or 3-D can be written as

$$J_2 = (1/8)[C_{ij} C_{ij} - (1/n)(C_{kk})^2]$$

Since the factor 8 serves to just scale the above quantity, the distortion metric simply defined as the expression within the brackets. This distortion metric is a function of the fourth power of the elements of a Jacobian matrix. It is also a function of position and not a constant over the entire domain. The authors use the largest value of the metric at Gauss points as the measure of distortion. However there is no basis for this selection. The comparison of various values of distortion metric with errors for selected examples with known theoretical solutions are also discussed in the above reference. A direct relation of a known value of distortion to some level of error a priori is not possible since the error in any analysis is highly dependent on the problem itself. The measures discussed here should therefore be used only as a guideline to detect excessive distortion of the element which is in general not desirable.

The transformation of the element from the parent to a local coordinate system can be thought of as a continuous deformation process in which the final deformed shape is the shape of the element in the physical x, y, z coordinate system. By expressing this "pseudo" deformations as functions of coordinates in the ξ, η, ζ system, and assuming large strains, it can be shown that the above measure is directly related to the strain energy density of this pseudo deformation.

2.7 *Other indicators of element quality*

So far we have discussed distortion measures which are solely dependent on element geometry. There are some indicators, like condition number of stiffness matrix, studied by some authors as an overall indication of the quality of discretization. Distortion is just one of the factors which is reflected in these indicators along with interpolation function used, order of the element, etc.. Notable work in this area has been done (Melosh 1963; Khanna 1965; Fried 1971–73). In Fried (1972), the condition number of the stiffness matrix due to nonuniform meshes is studied. Bounds are derived on the

condition number and expressed in terms of the extremal eigenvalues of stiffness and mass matrices and discretization parameters of the mesh. The condition number is related to the size of the elements, h. Similar to condition number, trace of the element stiffness matrix has been used by Rigby & McNeice (1972) for evaluating their performance. It is suggested that elements with lower trace are likely to be better for general loading cases. Strain energy of the element has also been used instead of stiffness for comparison of two elements (Melosh 1963; Khanna 1965). Khanna's method compares two element stiffness matrices by calculating the difference in strain energies. A modification of this method and comparison of hexahedron elements based on strain energy can be found in the reference by Rigby & McNeice (1972).

The performance of the element for a given loading situation can be directly studied by using elements with known distortion values and comparing quantities like stress errors, strain errors, strain energies, energy norm of the error etc.. For example, in Salmon & Abel (1989), shape distortion effects on nine-noded quadrilateral membrane elements is studied. The shape parameters described above, aspect ratio, skew, and tapers are used to define distortion. Stress and strain errors are plotted for each type of shape distortion. A similar study on quadrilateral elements can be found in a more recent reference (Liu & Elmaraghy 1992). The exact energy error is used as a basis of comparison of meshes with distorted elements of varying degree of distortion. A generalized finite element evaluation procedure for evaluating two- and three-dimensional elements is described (Dow *et al* 1985). This procedure compares the strain energy content and the strain distribution of the finite element model to that of the continuum region it represents for well-defined strain states. Triangular and quadrilateral elements undergoing a series of progressive initial distortions are used for evaluation. From the results of this evaluation, the authors suggest an algorithm to predict maximum strain energy error as a function of initial geometry.

3. Element degeneracy conditions for isoparametric elements

In this section, the determinant of the Jacobian matrix is studied in detail. For different elements, the mathematical conditions which dictate zero of the Jacobian are explored. The term 'Jacobian' denotes the determinant of the Jacobian matrix. Thus, $J = \det[J]$.

Using the strain displacement relationship $\varepsilon = \mathbf{Bq}$, where $\mathbf{q}$ is a vector of nodal displacements, the strain can be expressed in the local ξ, η, ζ coordinate system as

$$\varepsilon = (1/J)\mathbf{G}(\xi, \eta, \xi)q$$

where the elements of the vector $\mathbf{G}$ are functions of ξ, η, ζ. The matrix $\mathbf{B}$ relates strains to nodal displacements. It can be seen from the above equation that if J is zero at some point within the element, it implies that the strains are infinite or that we have an indeterminate formula. If such points occur at Gauss points used for numerical integration, the calculations cannot be continued. Even otherwise, we can expect the accuracy to deteriorate as such a point is approached (Jordan 1970). Mathematically this condition represents a nonunique mapping between the computational space (ξ, η) and the physical space (x, y) in a local region around the point where J is zero. This follows from the inverse function theorem (see appendix A). Since J is a continuous function, the above requirement implies that J should not change sign anywhere in the domain. We note the condition when J is zero at some point in the

element as degeneracy. In this section, the conditions which cause degeneracy are reviewed for various elements.

3.1 *Triangular element*

For a three-noded linear triangular (CST) element, the magnitude of J is equal to twice the area of the triangle. Thus degeneracy for this element occurs when the triangle collapses to a straight line.

Next consider a 6-noded higher order triangular element. In general, all three sides can be curved. A general relationship which describes degeneracy is too complicated to obtain and hence special cases are considered. A triangle with all three straight sides and one with two straight sides cover the majority of the applications. For such a triangle the "quarter-point" rule was first derived by Mitchell *et al* (1971).

(1) $J > 0$ for all points in the triangular element if $(1/4) < k_1, k_2, k_3 < (3/4)$
(2) $J = 0$ at some point in the triangular element if k_i takes a value in either of the intervals $(0, 1/4)$ or $(3/4, 1)$ for any value of i where $i = 1, 2, 3$.

Similar rules are also derived by Jordan (1970) using vector analysis. In addition, if the triangle has at least two straight edges and one curved side, as shown in figure 5, it is shown that the node 5 on the curved side should be in the shaded region so that $J > 0$. This region is formed by the two lines which are parallel to the straight sides and pass through their quarter points.

If all the three sides are curved, a general rule to detect degeneracy is not available. But some specific rules can be obtained. Jordan (1970) provides a simple geometric check. Refer to figure 5. Draw the line from node 1 to node 6 and extend it by 1/3 of its length to point 6′. Similarly with 2 to 5 to get 5′. Rotate CCW around node 3 from 6′ to 5′. If the rotation angle is 180° or more, J will vanish somewhere (the converse need not hold). This geometric interpretation is simply a statement of a scalar triple product formula for J obtained from vector analysis. For details see (Jordan 1970).

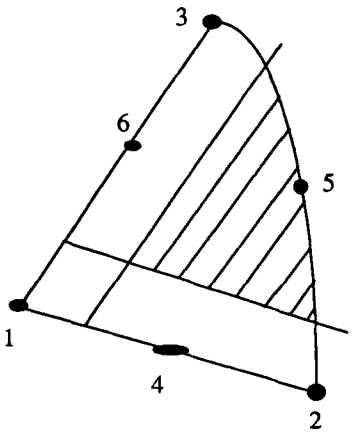

Figure 5. Valid region for midside node.

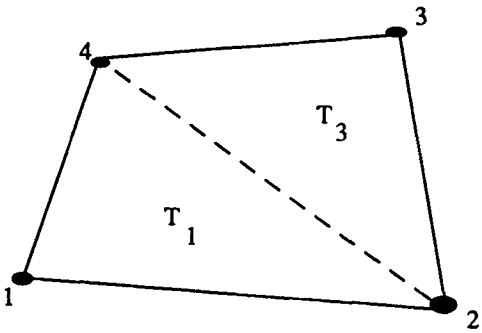

Figure 6. 4-noded quadrilateral element.

3.2 *Four-noded quadrilateral element*

The four-noded quadrilateral element is shown in figure 6. The shape functions are the serendipity functions most commonly used for this element (see Chandrupatla & Belegundu 1991). Using these functions, the Jacobian J can be expressed as

$$J = a + b\zeta + c\eta,$$

where a, b, c are functions of the nodal coordinates. Thus, J is a bilinear function of ξ, η and attains its minimum at the corners. The determinant of Jacobian has an interesting geometric interpretation for four-noded quadrilaterals (Okabe 1981). Refer to figure 6. Let T_i be the area of the partial triangle formed by the vertex i and two adjacent nodes. Then the Jacobian at node i can be written as

$$K_i = |J(\xi_i, \eta_i)| = (1/2)\, T_i, \quad i = 1, 2, 3, 4 \quad \text{and} \quad K_0 = |J(0,0)| = (1/4)A,$$

where A is the area of the original quadrilateral. The coefficients a, b, c can be expressed in terms of K_i as

$$a = K_0,$$
$$b = \tfrac{1}{2}(K_2 - K_1),$$
$$c = \tfrac{1}{2}(2K_0 - K_1 - K_2).$$

Since J at the corners represent the area of the triangle described above, $J = 0$ indicates that the quadrilateral degenerates to a triangle with the node at which $J = 0$ being on the line joining its adjacent nodes. Hence a 4-noded quadrilateral element, which is not convex, has $J = 0$ somewhere in its domain. The converse is also true. Thus, a four-noded quadrilateral element which has $J = 0$ somewhere in the domain is nonconvex in shape.

3.3 *Eight-noded quadrilateral*

Detecting degeneracy and conditions that cause degeneracy in higher order elements is more complex. The most commonly used higher order quadrilateral element is an 8-noded quadrilateral (see figure 7).

The simplest form of this element is when its sides are straight lines. For this element, the restrictions on midside nodes were studied by Steinmueller (1974). The

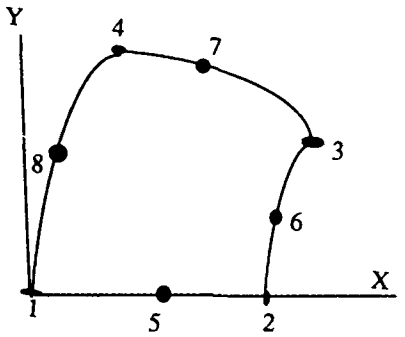

Figure 7. 8-noded quadrilateral element.

rules for positioning midside nodes is derived below. The approach given here is slightly simplified. In figure 7, assume that the origin of the coordinate system is at node 1 and the x-axis is along 1–2. Thus coordinates of 2 are $(x_2, 0)$. Let $(a, 0)$ be the position of midside node 5. The x-coordinate of a point on 1–2 can be written as

$$x = N_2 x_2 + N_5 a = \tfrac{1}{2}(1+\xi)\xi x_2 + (1-\xi^2)a.$$

For invertible mapping i.e. $J > 0$, we need that $J = (\partial x/\partial \xi) > 0$ from which we can show that

$$(1/4)x_2 < a < (3/4)x_2.$$

When the midside node is at any of these limiting positions, the Jacobian is zero at that point. Such elements are called 'quarter point elements'. They are widely used in fracture mechanics, since it was discovered that they possess the appropriate $r^{1/2}$ singularity required in those problems. Numerical difficulties that may arise in 3-D quarter point elements due to negative Jacobian and aspect ratio are discussed by Peono and others (Peano 1987; Peano & Pasini 1982).

A geometrical check similar to the one described for a 6-noded triangle can also be used for 8-noded quadrilateral elements (Jordan 1970). However, for an element with all four curved sides, simple rules like quarter point rule, are not easily derived. A comprehensive mathematical analysis of a general 8-noded quadrilateral is presented (Field 1983). In this reference, algorithms have been developed to determine invertible transformations. The outline of the approach is given below.

Let U be the master element domain. The coordinates of any point (x, y) can be expressed using the shape function N_i as

$$x = \sum_{i=1}^{8} N_i x_i \quad \text{and} \quad y = \sum_{i=1}^{8} N_i y_i.$$

The Jacobian of the transformation matrix can be obtained from the above equation. The general form of the Jacobian can be written as

$$J(\xi, \eta) = k_0 \eta^3 + A(\xi)\eta^2 + B(\xi)\eta + C(\xi),$$

where k_0 is a constant which depends on the element geometry, A, B and C are the polynomials in ξ. A and B are quadratic and C is cubic. Thus, for a known value of ξ, J is a cubic polynomial in η. Parallel expression can be written expressing J as a

cubic polynomial in ξ with variable coefficients in η,

$$J(\xi,\eta) = v_0\xi^3 + \alpha(\eta)\xi^2 + \beta(\eta)\xi + \gamma(\eta).$$

Since J is a continuous function on U, it must have a minimum and maximum value on U. The idea is to establish the sign of these values wherever they occur in the interior of U. The critical points of $J(\xi,\eta)$ satisfy

$$\partial J/\partial\xi = \partial J/\partial\eta = 0, \quad \text{at} \quad (\xi^*,\eta^*).$$

Thus

$$\partial J/\partial\xi = F(\xi,\eta) = A'(\xi)\eta^2 + B'(\xi)\eta + C'(\xi) = 0,$$
$$\partial J/\partial\eta = G(\xi,\eta) = 3k_0\eta^2 + 2A(\xi)\eta + B(\xi) = 0. \tag{3}$$

The theory of resultant polynomials is used to identify critical points in U. A resultant of two polynomials is a polynomial whose coefficients depend upon the coefficients of the two given polynomials. The construction of resultants of two polynomials is described by Householder (1968). The ordinary resultant is a polynomial which has only the constant term. For the above two polynomials, the ordinary resultant is given by,

$$D(\xi) = \begin{bmatrix} A'(\xi) & B'(\xi) & C'(\xi) & 0 \\ 0 & A'(\xi) & B'(\xi) & C'(\xi) \\ 0 & 3k_0 & 2A(\xi) & B(\xi) \\ 3k_0 & 2A(\xi) & B(\xi) & 0 \end{bmatrix}.$$

From the theorem on resultants (Householder 1968), the necessary and sufficient condition for the two polynomials F and G above to have a common divisor of some degree greater than zero, is that the ordinary resultant $D(\xi) = 0$. A similar polynomial $\delta(\eta)$ can be constructed by writing the polynomial equations in ξ. If both D and δ are not identically zero, then the critical points of $J(\xi,\eta)$ on U are isolated. In such a case, the equation $D(\xi) = 0$ will have at most 7 real roots. For each of these roots, from the theorem of resultants, there will be atleast one η^* such that equation (3) is satisfied. The idea is to find these roots if they lie in U. Sturm sequence can be used to find the number of roots of $D(\xi)$ that lie in $[0,1]$.

The method described above is for a general case. Simplifications are possible for certain cases. The general algorithm for detecting invertible transformation can be summarized as below.

Step 1: Check if $J > 0$ on the boundary. This is numerically straightforward since $J(0,\eta)$, $J(1,\eta)$, $J(\xi i,0)$ and $J(\xi,1)$ are at most cubic polynomials in one variable. To show that any cubic polynomial $P(r) > 0$, show that $P(0)P(1) > 0$ and that $P'(r)$ has no roots in $[0,1]$. If $P(0)P(1) > 0$ and $P'(c) = 0$ for $c = [0,1]$, then show that $P(c) > 0$.

Step 2: If $J > 0$ on the boundary and if the element is a semiquadrilateral, i.e. one which has three straight edges and one curved side, then J is positive everywhere in $[0,1]$. Stop. Otherwise go to step 3.

Step 3: If $J > 0$ on the boundary of U, and if the function J reduces to $J(\xi,\eta) = a_0 + a_1\xi + a_2\eta + a_3\xi\eta$ then J is nonzero everywhere. Stop. Otherwise go to step 4.

Step 4: If $J > 0$, and $k_0 = 0$, evaluate $F(\xi, -B(\xi)/2A(\xi))$. If $F \cong 0$, then J is nonzero everywhere. Else use Sturm sequence to find if the roots lie in $[0, 1]$ and if they do, check if J at those points is positive and stop. If $k_0 \neq 0$, go to step 5.

Step 5: If $J > 0$ on the boundary, and if $k_0 \neq 0$, then use the method described above, i.e. solve D and then (6) to find the roots.

The above algorithm gives necessary conditions for evaluating the sign of J on U. The following are the sufficiency conditions for the nonvanishing of J. For proof of these conditions see (Field 1983).

If $J > 0$ on the boundary of U,

(A) if $3k_0 + A(\xi) \leqslant 0, 0 < 0, 0 < \xi < 1$, then $J(\xi, \eta) > 0$ on U,
(B) $3k_0 + A(\xi) > 0$ and $3k_0 + 2A(\xi) + B(\xi) < 0$, $0 < \xi < 1$, then $J(\xi, \eta) > 0$ on U,
(C) if $k_0 \leqslant 0$, then each of the following conditions imply $J > 0$ on U.

$$B(\xi) + C(\xi) > 0, \quad 0 < \xi < 1,$$

$$\phi = \{3k_0 + 2A(\xi) + B(\xi) > 0\} \cap \{B(\xi) + C(\xi) < 0\}.$$

(D) if $k_0 > 0$, then each of the following imply $J > 0$ on U,

$$A(\xi) > 0, B(\xi) > 0, \quad 0 < \xi < 1,$$

$$A(\xi) \geqslant 0, B(\xi) \leqslant 0, B(\xi) + C(\xi) \geqslant 0, \quad 0 < \xi < 1.$$

Global inversion of bilinear and quadratic isoparametric mapping is also studied in detail by Frey *et al* (1978).

3.4 *Other higher order quadrilateral elements*

A general algorithm similar to the one derived for quadrilateral element is too unwieldy for higher order quadrilateral elements. Even for simple cases of these elements, the solution becomes very complex. Sometimes, graphical methods are used to find the conditions of degeneracy. For example, consider a 12-noded quadrilateral element (cubic) with all straight sides as shown in figure 8. For invertible mapping,

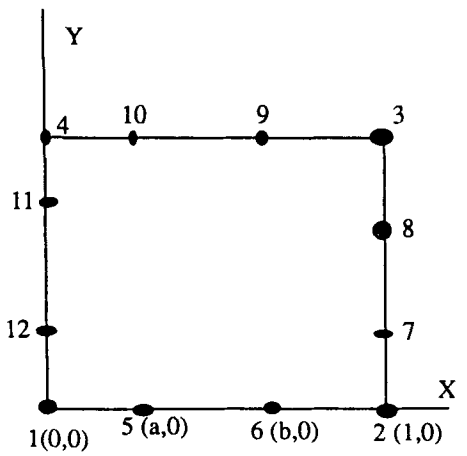

Figure 8. 12-noded quadrilateral element.

it can be shown that the following condition has to be satisfied.

$$(9/2)(2-10\xi-9\xi^2)a+(9/2)(-1+8\xi-9\xi^2)b+(1/2)(2-18\xi+27\xi^2)>0.$$

This inequality represents a region bounded by the envelope of an infinite number of straight lines, represented by the parameter, $\xi, 0 \leqslant \xi \leqslant 1$. By using various values of ξ and drawing these straight lines, we get a closed curve. Admissible values of a and b can be obtained from the interior of this curve.

Similar analysis for cubic isoparametric transformation of a nine-noded triangle is discussed in Mitchell (1979) and Woodford *et al* (1978). A triangle with two straight sides and one cubic side is chosen for this analysis. The effect of certain interpolating polynomials of degree four and five on triangular elements with one curved side is studied in Stephenson & Manohar (1979).

3.5 *Isoparametric solids – tetrahedral elements*

For a 4-noded tetrahedral element, which is the simplest solid element, determinant of Jacobian, J, is a constant and proportional to the volume of the element. The relationship is given by, $|J| = 6V_e$, where V_e is the elemental volume. Hence a zero-Jacobian would imply degeneration of tetrahedra to a triangle. For a higher order tetrahedral element, the Jacobian check algorithm described for 8-noded quadrilaterals can be extended easily (Field 1981). Consider a 10-noded tetrahedron element as an example. The Jacobian is of the form

$$J(\xi,\eta,\zeta)=k_1\xi^3+A(\eta,\zeta)\xi^2+B(\eta,\zeta)\xi+C(\eta,\zeta),$$

where k_1 is a constant and A, B and C are given by

$$A(\eta,\zeta)=A_{10}\eta+A_{01}\zeta+A_{00},$$

$$B(\eta,\zeta)=B_{20}\eta^2+B_{11}\eta\zeta+B_{02}\zeta^2+B_{10}\eta+B_{01}\zeta+B_{00},$$

and

$$C(\eta,\zeta)=C_{30}\eta^3+C_{21}\eta^2\zeta+C_{12}\eta\zeta^2+C_{03}\zeta^3+C_{20}\eta^2+C_{11}\eta + C_{02}\zeta^2+C_{10}\eta+C_{01}\zeta+C_{00}.$$

The constants in the above equation are found from nodal coordinates. The first step is to verify that J is positive on the boundary of the element. Since J is a function of two variables on each face of the element, the algorithm described for two variable functions under quadratic elements can be used. The next step is to check the sign of J on the interior of the element. Similar to results of 2-D, simplified results can be obtained under certain conditions. The following is an example. Let

$$L_1(\xi,\eta,\zeta)=3k_1\xi+A_{10}\eta+A_{01}\zeta+A_{00},$$

$$L_2(\xi,\eta,\zeta)=B_{20}\xi+3C_{30}\eta+C_{21}\zeta+C_{20},$$

$$L_3=(\xi,\eta,\zeta)=B_{02}\xi+C_{12}\eta+C_{03}\zeta+C_{02}.$$

If $J>0$ on the boundary of the element and if one of the following conditions is satisfied, then $J>0$ everywhere.

$$(1)\ L_1(0,0,0)\leqslant 0,\quad L_1(0,0,1)\leqslant 0,\quad L_1(0,1,0)\leqslant 0 \quad \text{and} \quad L_1(1,0,0)\leqslant 0$$

or

$$(2)\ L_2(0,0,0) \leqslant 0, \quad L_2(0,0,1) \leqslant 0, \quad L_2(0,1,0) \leqslant 0 \quad \text{and} \quad L_2(1,0,0) \leqslant 0$$

or

$$(3)\ L_3(0,0,0) \leqslant 0, \quad L_3(0,0,1) \leqslant 0, \quad L_3(0,1,0) \leqslant 0 \quad \text{and} \quad L_3(1,0,0) \leqslant 0$$

For a 10-noded tetrahedron with three straight edges, each straight edge having a bisecting midside node and all three straight edges intersecting at (0,0,0), the Jacobian check algorithm has been described by Field (1981).

3.6 *Isoparametric solid elements – 8-noded brick element*

For the 4-noded quadrilateral element, a positive J at the corners is an indication of convexity and positive J in the interior of the element. However *a* 8-noded brick need not be convex to have a positive Jacobian. Neither is a positive Jacobian at its corner nodes, a sufficient condition to assure that the associated transformation has positive Jacobian. For example, consider an element with coordinates (0, 1/4, 1), (1, 0, 1), (1, 1/4, 1), (0, 0, 0), (0, 3/4, 1), (1, 1, 1), (1, 3/4, 0) and (0, 1, 0). The element is not convex, but its Jacobian is positive. Similarly an element with coordinates (0, 0, 1), (4, 0, − 1), (1, 0, 0), (0, 0, 0), (0, 1, 1), (1, 1, 0), (1, − 1, − 1) and (0, 1, 0) has positive J at its corners, but $J(1/2, 0, 0) = 0$. The conditions to ensure positive J for the element whenever J is positive on its boundary has been derived (Field 1981). The procedure is exactly similar to the one above for tetrahedral elements. First, the non-negativity of the Jacobian has to be ensured on the boundary of the element. On each face the Jacobian is a polynomial in two variables and on each edge it is a quadratic in one variable. After verifying that it does not vanish somewhere on the edges of a face, verifying that it does not vanish on the interior of a face requires solution of

$$\frac{\partial J}{\partial \xi} = \frac{\partial J}{\partial \eta} = \frac{\partial J}{\partial \zeta} = 0.$$

Each of these derivatives is linear in the corresponding variable and on each face, one of the three variables is zero. The condition which implies a positive J throughout the element for a positive J on the entire boundary can be obtained using a similar approach as the tetrahedron. For details, see Field (1981). Degeneracy of 8-noded brick elements is also discussed by Peano (1987).

Alternatives to isoparametric transformation have also been considered. For example, Mitchell *et al* (1971) proposed a technique whereby interpolating functions are obtained directly in terms of x and y for the triangle and quadrilateral with arbitrary midside points. Wachspress (1971) developed a basis of wedge functions for convex polygon boundaries. Other notable efforts in this direction can be found in the works by Gordon & Hall (1973), McLeod (1976, 1978) and McLeod & Mitchell (1975). In spite of other alternatives, isoparametric transformations are the most widely used in displacement formulation of finite element analysis.

4. Rezoning and grid optimization methods

Rezoning, by definition, is simply relocation of the nodes based on some criteria. The criteria used could be as simple as improving the geometry of the element so that

they are less distorted or it could be based on the feedback from the finite element solution. In general, the process of rezoning involves two questions – when to rezone and how to rezone. The first question is generally answered by using well known measures of element quality which are purely geometrical in nature. These measures have been discussed in the previous chapter. The error measures based on stress jumps, or energy norms can also be used to decide when to rezone. The algorithm to rezone the mesh is usually based on some kind of nodal averaging which is a variation of the "Laplacian smoothing".

The problem of obtaining an adequate finite element mesh for a given problem can also be formulated and solved as an optimization problem. An optimized mesh has been shown to be twice as efficient as an evenly divided mesh in terms of the number of degrees of freedom to produce the same accuracy (McNiece & Marcal 1973; Turcke & McNiece 1974). Details on grid optimization and other methods are reviewed in this section. This review of rezoning methods is by no means complete. However key references have been discussed as a guideline.

4.1 *Laplacian smoothing*

This is the most popular method of grid smoothing, originally conceived by Winslow (1967). If a node n is shared by elements $e_1, e_2, e_3, \ldots, e_n$, then its coordinates (x_n, y_n) are modified by

$$x_n = \sum_{i=1}^{N} x_{ci} w_i \Big/ \sum_{i=1}^{N} w_i \quad \text{and} \quad y_n = \sum_{i=1}^{N} y_{ci} w_i \Big/ \sum_{i=1}^{N} w_i,$$

where x_{ci} and y_{ci} are the coordinates of the centroid of the element e_i and w_i can be considered as weights appropriately defined for averaging. If refined elements are needed in a certain region of the mesh, large weights have to be given in those regions in the corresponding elements. The solution to the above equation is a rough approximation of the discrete Laplacian. Thus the above formula is strictly valid for orthogonal meshes. Since the solution of Laplace's equation is known to have an averaging property, the above formula is used for rezoning any arbitrary mesh which is distorted. Instead of the nodal coordinates, as described above, the coordinates of the geometric centre of the element which is connected to the node are used. The weights are chosen as some parameter which is to be averaged over the domain. In the simplest form, we can use $w_i = 1{\cdot}0$. The distortion measure is another choice, but its usefulness in reducing solution error has to be explored. Note that the above formula is used iteratively. Successive use of the formula over a number of iterations converges it to a more uniform mesh.

4.2 *Grid optimization methods*

Grid optimization is a systematic way of obtaining finite element meshes that yield the required accuracy for the minimum effort. During early investigation of this problem (Felippa 1977), nodal coordinates were used as design variables in the appropriate energy functional. The resulting equations are highly nonlinear and need lot of computational effort. Hence attempts to find true optima are not very practical. Methods to obtain near-optimum grids based on the application of a solution-based criterion were proposed by Shephard *et al* (1980) and Turcke & McNeice (1974).

There are two key issues to be addressed in the application of these criteria methods – the criteria to be used, and the method of grid enrichment to be employed. The method of grid enrichment used might increase the degrees of freedom in the structure or just redistribute the nodes.

The requirements for optimum grids were studied by Turcke & McNeice (1974). These are based on the optimum grids obtained for various two dimensional problems by direct optimization using nodal coordinates as design variables along with nodal displacements. The optimization problem is described below.

The total potential energy obtained from finite element formulation can be written as

$$\Pi = \tfrac{1}{2}\mathbf{a}^T\mathbf{K}\mathbf{a} - a^T F,$$

where **a** is a column vector of unknown nodal displacements, **K** is the stiffness matrix and **F** is the column vector of nodal point forces. If we consider nodal coordinates x_i as design variables along with displacements, then minimum potential energy implies that

$$(Ka - F) = 0,$$

$$\tfrac{1}{2}a^T\frac{\partial K}{\partial x_i}a - \frac{\partial F}{\partial x_i}a^T = 0.$$

Thus the feasible set of nodal co-ordinates and nodal displacements in the above system of nonlinear equations is one that ensures that the potential energy functional is stationary. A sufficient condition for the solution of the above equation to be a minimum is that the Hessian of Π be positive definite. Using the above relations, the best possible approximate solution for the given finite element mesh is obtained. The above optimization problem is solved using Rosenbrock's direct search technique. The authors (Turcke & McNiece 1974) provide specific guidelines for obtaining optimum grids. They suggest that the element edges have to be aligned along isoenergetics i.e. along contours of constant strain energy density. These contours can be obtained from the initial analysis on a course grid.

Furthermore, Shephard *et al* (1980) developed an interactive approach to synthesis of near-optimal meshes. The criteria used is variation of strain energy density in the element based on an argument that the best finite element solution is the one that best approximates the total strain energy. Hence this solution is associated with the best possible approximation to the integrand of the strain energy, i.e. SED. Key nodes are placed on the boundary of the mesh using above criteria and the new mesh is generated.

A variation of strain energy density has been used in Melosh & Marcal (1977) for mesh enrichment. The authors study the effect of gradual introduction of an additional degree of freedom on the potential energy. This effect is measured by 'specific energy difference', which is estimated by taking the difference between the SED at any point and that at the centroid of the element. A criteria based on a measure of the interpolation error associated with the finite element model is used by Kikuchi (1986). In this reference, the nodes are redistributed by an iterative scheme. More details on this method can be found from Diaz *et al* (1983). Mesh modification based on minimization of an interpolation error estimate is presented by Demcowicz & Oden (1986). The examples in this reference are taken from fluid flow problems. A more recent reference on mesh optimization problem (Martinez & Samartin 1991) presents closed form

solutions to determine optimal mesh for 1-D problems. Some practical examples of the use of rezoning and adaptive remeshing can be found in the works of Jung-Ho Cheng & Kikuchi (1986), Jung-Ho Cheng (1988), Liefooghe & Fleury (1990), Yon & Yang (1990) and Yang *et al* (1989).

4.3 *General method for adaptive grid design*

The optimal grid-design problem is defined by Kikuchi (1986)

$$\text{Minimize (Maximum } E_e\text{)},$$

where E_e is an error measure of eth finite element. The necessary condition for the above optimization problem is that (Babuska *et al* 1983),

$$E_e = \text{constant}, \quad e = 1 \text{ to no. of elements.}$$

For stationary heat conduction problems and elastostatic problems, the smoothing scheme based on the Laplacian smoothing works well by taking the weights as E_e/A_e where A_e are element areas (Kikuchi 1986). For one-dimensional problems, the necessary condition is satisfied by repeated application. For two-dimensional problems, repeated application may not yield the necessary optimum, but it always reduces the maximum value of error at the beginning and converges to a fixed grid which is very close to the optimal one. If the initial mesh has only rectangular elements and if the necessary condition is satisfied, the application of Laplacian smoothing does not alter the mesh topology. If the mesh has elements of irregular shape, the smoothing scheme moves the grids even if the necessary condition is satisfied.

4.4 *Other approaches*

Some authors have approached the problem purely from geometric considerations, i.e. optimization to minimize mesh distortion. For example, in a recent reference on mesh smoothing (Parthasarathy & Kodiyalam 1990), a constrained optimization problem is solved starting from a valid initial mesh. The objective function is chosen to be the RMS value of the aspect ratio of the elements. Kennon & Dulckravich (1986) use a linear combination of orthogonality and mesh smoothness measures as cost functions for optimization. Smoothness measure at any node is defined as the sum of the squares of the differences in area between adjacent elements connected to that node. The orthogonality measure at node i is defined as the sum of the squares of the dot products of the adjacent vectors which emanate from the node i. Minimizing the weighted sum of these measures ensures a smooth and orthogonal grid. This approach has been used in 3-D problems using volumes as measures in Carcaillet *et al* (1986). Similar work can also be found in Hayes *et al* (1986).

The problem of rezoning and grid optimization has been a subject of study for more than two decades now. A comprehensive review of the subject can be found in Shephard (1979). More recent papers have concentrated on applications of these methods to practical problems in engineering like flow analysis, metal forming, shape optimization etc. For iterative solution methods like large displacement analysis, the variables like displacement and stresses have to be transferred to the new positions of nodal points. A method to do this by inversion of isoparametric mapping is described in Crawford *et al* (1989).

5. Conclusions

It should be noted that solution errors are a result of the combination of factors – element distortion, the nature of the problem, degree of polynomial used for interpolation, and boundary conditions. Hence, a priori knowledge of the way distortion affects accuracy of analysis is impossible to predict. However, large distortions in general cause large errors. The distortion measures described in this review can be used to detect such distortions. Empirical rules for acceptable levels of distortion are quite common, though these are merely heuristic. Quantifying distortion using a reasonably good measure is very useful especially for rezoning or redefining the nodes to improve mesh quality.

Through the analysis of isoparametric elements for a nonvanishing Jacobian, it was observed that certain shapes of elements are 'forbidden' to maintain positive Jacobian in the element. The methods described here can be used to identify degeneracy.

For rezoning, weighted averaging at the nodes is the most popular method because of simplicity and ease of implementation. This scheme has certain disadvantages. The smoothness and uniformity of the rezoned mesh depends on the weighing factor used. With simple averaging, invalid geometry might result near boundaries for some problems. Though optimization seems to be a better approach, the computational effort can be very large for complex problems. Hence developing better rezoning techniques which yield less distorted and more accurate elements with computational ease is a subject of research.

Appendix A

Theorem on inverse transformation

Let $x = f(u, v)$, $y = g(u, v)$ define a continuously differentiable transformation for all pairs (u, v) in some neighbourhood of a point (u_0, v_0). Let $x_0 = f(u_0, v_0)$, $y_0 = g(u_0, v_0)$, and suppose that the Jacobian is not zero at (u_0, v_0). Then there exist positive numbers a, b, α, β and functions $F(x, y)$, $G(x, y)$ defined when $|x - x_0| < a$, $|y - y_0| < b$ such that the following assertions are true:

Let R be the rectangular region in the xy-plane defined by the inequalities $|x - x_0| < a, |y - y_0| < b$ and let S be the rectangular region in the uv-plane defined by the inequalities $|u - u_0| < \alpha, |v - v_0| < \beta$. Then

(1) To any (x, y) in R corresponds a unique (u, v) in S such that $x = f(u, v)$, $y = g(u, v)$, and this unique pair is given by

$$u = F(x, y), \quad v = G(x, y).$$

(2) The functions F and G are continuous and have continuous partial derivatives given by

$$\frac{\partial F}{\partial x} = \frac{1}{J}\frac{\partial g}{\partial v}, \quad \frac{\partial F}{\partial y} = -\frac{1}{J}\frac{\partial f}{\partial v}, \quad \frac{\partial G}{\partial x} = -\frac{1}{J}\frac{\partial g}{\partial u} \quad \text{and} \quad \frac{\partial G}{\partial y} = \frac{1}{J}\frac{\partial f}{\partial u},$$

where $J = \dfrac{\partial(f, g)}{\partial(u, v)}$ and u, v are expressed in terms of x, y.

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