Distortion Measures for Sparse Signals

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Abstract: *An important issue in Blind Source Separation (BSS) is how to measure the similarity between a true source and its estimate. This is a simple but not completely trivial topic that has been a bit overlooked in the literature. Special problems arise when the source signals are sparse and the BSS problem is degenerate (i.e. we have more sources than observed mixtures). In this paper we review the most popular distortion measures that have been used to assess the performance of different BSS algorithms. We show that the common distortion measures are not suitable for the degenerate blind separation of sparse sources. Finally we propose a class of alternative distortion measures for sparse sources.*

Keywords: *Blind Source Separation, Independent Component Analysis, sparse signals, distortion measures*

INTRODUCTION

In the basic Blind Source Separation (BSS) problem, a set of linear mixtures $x_1(t),...,x_n(t)$ of some unknown source signals $s_1(t),...,s_m(t)$ are observed. The goal of BSS is to determine the source signals if only the mixtures are given. Given the system:

$$\mathbf{x} = \mathbf{A}\mathbf{s} \tag{1}$$

where $\mathbf{s} = [s_1(t),...,s_m(t)]^T$, $\mathbf{x} = [x_1(t),...,x_n(t)]^T$ and $\mathbf{A}$ is the unknown mixing matrix, we want to estimate a demixing matrix $\hat{\mathbf{W}} = \hat{\mathbf{A}}^\dagger$. Here † denotes the Moore-Penrose pseudoinverse. Under the statistically strong but often physically plausible assumption of mutually independent source signals, we can recover the sources by means of Independent Component Analysis (ICA) [1].

SEPARATION ACCURACY MEASURES

The most widely studied BSS situation is the non degenerate case where there are at least as many mixtures as there are sources (i.e., $n \geq m$). When that is the case the accuracy of a BSS algorithm can be assessed from its ability to estimate the mixing matrix. The most widely used measure for assessing the accuracy of the estimated mixing matrix is the Amari performance index P_{err} [2]:

$$P_{err} = \frac{1}{2N} \sum_{i,j=1}^{M} \left(\frac{|p_{ij}|}{\max_k |p_{ik}|} + \frac{|p_{ij}|}{\max_k |p_{kj}|} \right) - 1 \tag{2}$$

where $p_{ij} = \left(\hat{\mathbf{A}}^{\dagger}\mathbf{A}\right)_{ij}$. When the separation is perfect, the Amari index is equal to zero. In the worst case, i.e. when the estimated sources contain the same proportion of each original source signal, the Amari index is equal to $m/2-1$.

However, the most common situation in many applications is the degenerate BSS problem, i.e. $n < m$. This is most likely the case when we try to separate the underlying brain sources from electroencephalographic (EEG) or magnetoencephalographic (MEG) recordings using a reduced set of electrodes. In degenerate demixing, the accuracy of a BSS algorithm cannot be described using only the estimated mixing matrix. In this case it becomes of particular importance to measure how well BSS algorithms estimate the sources with adequate criteria. The most commonly used index to assess the quality of the estimated sources is the Signal to Interference Ratio (SIR) [3]:

$$SIR = \frac{|\langle \hat{s}_i, s_i \rangle|^2}{\|\hat{s}_i\|^2 \, \|s_i\|^2 - |\langle \hat{s}_i, s_i \rangle|^2} \qquad (3)$$

SIR takes into account the fact that, in general, BSS is able to recover the sources only up to (a permutation and) a gain factor α. It is easy to check that if $\hat{s}_i = \alpha s_i$ the SIR is infinite. By contrary, when the estimated source is orthogonal to the true source, the SIR is equal to zero.

MEASURES OF DISTORTION FOR SPARSE SIGNALS

The simplest definition of sparseness states that in a sparse matrix or vector most of elements are zero. In the noisy case this definition is not valid since very small measurement noise makes completely sparse data completely non-sparse. For a review of sparseness measures for noisy signals see [4].

The SIR measure given by Eq. 3 assume that each sample of the true sources and their estimations are equally important. However, in a sparse signal, the most interesting signal features are located in the brief activations of the signal. To assess the quality of the separation of sparse signals (e.g. evoked potentials from background EEG) we propose to use a new similarity measure between an estimated source s_1 and the true source s_2:

$$SIR_{sparse} = \frac{S_{sparse}}{I_{sparse}} \qquad (4)$$

where S_{sparse} and I_{sparse} are calculated following these steps:

1. Estimate the probability density function (pdf) of s_1 and s_2. The pdfs can be approximated by the data histograms of the signals.

2. Let f_i denote the centered pdf of s_i. Then, we build the following weight functions:

$$w_i(x) = \left(\left(\frac{f_i(x)}{f_i(0)} \right)^{\beta} - \left(\frac{f_i(x)}{f_i(0)} \right)^{\gamma} \right)^{\delta} \qquad (5)$$

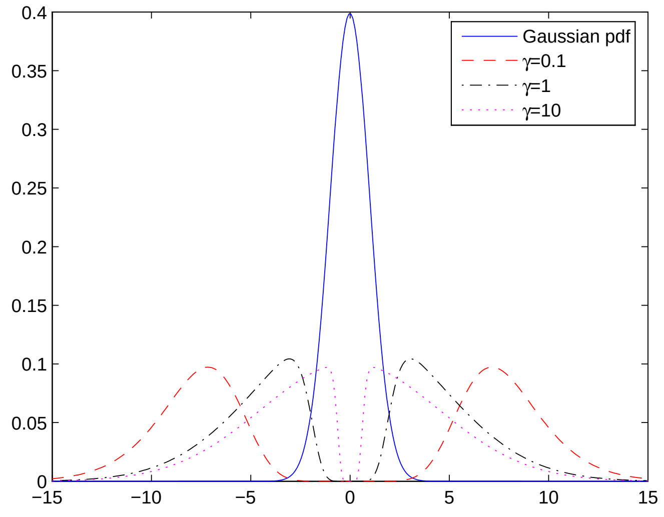

Figure 1: Shape of the weight function for $f_i(x)$ being the Gaussian distribution, $\beta = 0.01$, $\delta = 5$ and different values of γ.

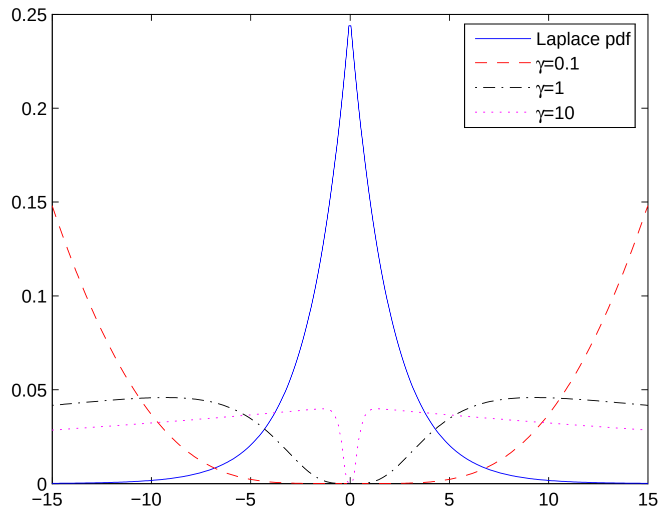

Figure 2: Shape of the weight function for $f_i(x)$ being the Laplace distribution, $\beta = 0.01$, $\delta = 5$ and different values of γ.

for $i = 1, 2$. By choosing suitable shape parameters we can emphasize more or less the tails of the distribution (i.e., the sparse activations of the signals). In Fig. 1 and Fig. 2 we show these weight functions for several values of the shape parameters and for $f_i(x)$ being a Gaussian and a Laplace distribution.

3. Estimate the gain factor α. If the length of signals s_1 and s_2 is T samples then, α is estimated as follows:
$$\alpha = \frac{\sum_{t=1}^{T} w_1(s_1(t))w_2(s_2(t))s_1(t)s_2(t)}{\sum_{t=1}^{T} w_1(s_1(t))w_2(s_2(t))s_2^2(t)} \quad (6)$$
That is, we calculate α using a weighted mean average where we emphasize the time instants when s_1 and s_2 are synchronously sparsely activated.

4. Estimate S_{sparse} as:
$$S_{sparse} = \alpha^2 \sum_{t=1}^{T} w_1(s_1(t))w_2(s_2(t))s_2^2(t) \quad (7)$$

5. Estimate I_{sparse} as:
$$I_{sparse} = \sum_{t=1}^{T} w_1(s_1(t))w_2(s_2(t))e^2(t) \quad (8)$$
where $e(t) = s_1(t) - \alpha s_2(t)$.

6. Finally, we obtain the Signal to Interference Ratio as $SIR_{sparse} = S_{sparse}/I_{sparse}$

SIMULATIONS

We performed some simple simulations to check the behavior of the proposed measure. The original source signal has the following form:

$$s = g(t_0, f_0, T_0) + b(t); \quad (9)$$

where $g(t_0, f_0, T_0)$ is a gaussian bell of frequency f_0, duration T_0 and latency $t0$; and $b(t)$ is a low amplitude baseline signal. The estimated source signal has the form:

$$\hat{s} = g(t_0 - \Delta t, f_0, T_0) + \frac{|\Delta t|}{2T_0} b(t) + (1 - \frac{|\Delta t|}{2T_0}) b'(t); \quad (10)$$

where L is the length (in samples) of the true and estimated source signals, $b(t)$ is the same baseline signal present in the true source and $b'(t)$ is a random baseline signal independent from $b(t)$. Errors in the estimation of the latency of a sparse source can be critical in some applications like analysis of evoked brain potentials where the typically studied parameters of the evoked waveforms are their amplitudes, durations and latencies. However, it is not so important to estimate correctly the baseline

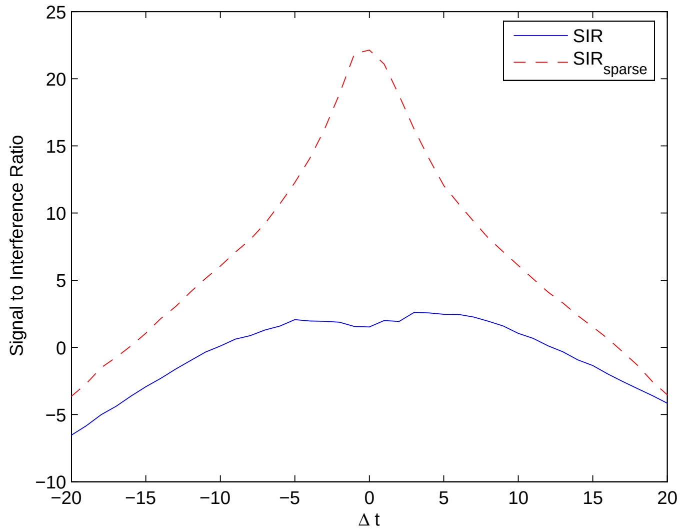

Figure 3: Averaged SIR and SIR_{sparse} for 50 independent realizations of $b'(t)$. Vertical axis are dBs, horizontal axis are data samples.

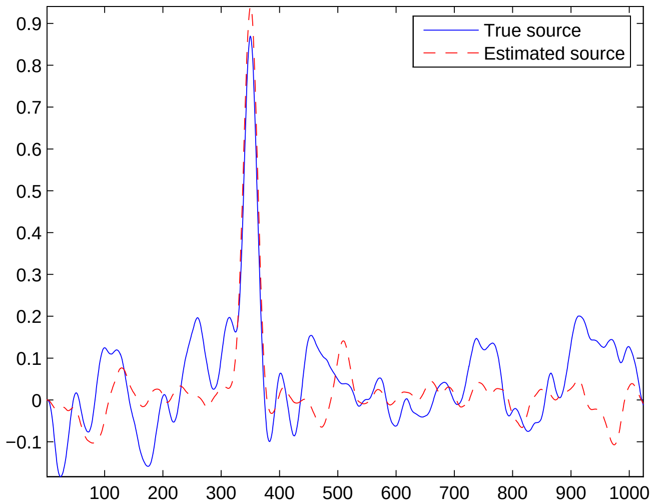

Figure 4: True source and estimated source for $\Delta t = 0$.

signal. Therefore, our SIR_{sparse} measure should have a clear maximum in $\Delta t = 0$ and should rapidly decrease for $|\Delta t| > 0$. In Fig. 3 we can see how SIR_{sparse} has a sharp maximum for $|\Delta t| = 0$ whereas the standard SIR is almost flat for small values of $|\Delta t|$. Furthermore, the best SIR (corresponding to $|\Delta t| = 4$ samples) is as low as $2.5\ dB$ what would make us think that the separation was very inaccurate in all cases. However, this is not true. In Fig. 4 we can see the original true source and the estimated source for $|\Delta t| = 0$. It is clear from Fig. 4 that the main signal structure (i.e. the sparse source activation) was accurately estimated.

CONCLUSION

In this paper we have reviewed the most commonly used measures to assess the accuracy of the Blind Source Separation algorithms. We argued that those measures are not suitable when both the true source and the estimated source are sparse. A new class of separation accuracy indexes for sparse signals have been proposed. Simulations show that the proposed indexes are more correlated with the perceptual quality of the separated sparse sources than classical Signal To Interference measures.

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