Distributed adaptive consensus disturbance rejection for multi-agent systems on directed graphs

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Distributed Adaptive Consensus Disturbance Rejection for Multi-agent Systems on Directed Graphs

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Abstract—This paper considers the distributed consensus disturbance rejection problem for general linear multi-agent systems with deterministic disturbances under directed communication graphs. Based on the relative state information of the neighboring agents, the consensus protocols, which consist of two observers, including a state observer and a separate disturbance observer, are designed to guarantee that the consensus error goes to zero with complete disturbance rejection. Furthermore, the state observer is designed in a fully distributed fashion with adaptive coupling gain, which has the advantage that the consensus controller design is independent of the Laplacian matrix associated with the communication network. The distributed observer-based consensus disturbance rejection protocols are further extended to containment control. Finally, an example is provided to demonstrate the effectiveness of the proposed strategies.

Index Terms—Consensus control, multi-agent systems, disturbance rejection, adaptive control.

I. INTRODUCTION

Over the last decade, cooperative control of multi-agent systems has gained a surge of interest from various research communities, due to its potential applications in cooperative surveillance, formation of unmanned aerial vehicles (UAV), sensor networks, flocking and social networks [1, 2, 3, 4]. In the aforementioned literature, a fundamental theme is consensus control, meaning that the individual agent negotiates with its neighbors through local interaction to reach a global agreement.

Since the fundamental work [5, 6], in which consensus problem was first proposed in [5], and reference [6] established a framework of consensus control for single-integrator systems with different networks, much effort has been made from different aspects, such as consensus with time delay [7, 8, 9], finite-time consensus [10, 11], consensus with connectivity preservation [12, 13], and synchronization of complex networks [14, 15] which can be seen as an extension of consensus problem. The early results focus on the first-, second- and high-order integrator systems [6, 16, 17], and then the recent results have been extended more general linear and nonlinear multi-agent systems [18, 19, 20, 21, 22]. In the above literature, the formulation of consensus can be categorized, based on the number of leaders, into three types of problems: leaderless consensus, leader-following consensus (or consensus tracking), and containment control, which has multiple leaders. For leaderless consensus, the final consensus value depends on the initial condition of each agent, while consensus tracking has a reference (leader) as the common trajectory, and for containment control, the followers move in the convex hull spanned by the multiple leaders. Thus, it is clear that, in leader-follower consensus, the leader can know the agreement value in advance, and the containment control has the advantage in some tasks, for example, the leaders can formulate a safe area for the followers in a hazardous environment.

In this paper, we address consensus tracking control for general linear multi-agent systems with disturbance under directed graphs. Among the works in consensus tracking, the authors in [23] considered consensus tracking for double integrators with a dynamic leader under an undirected communication topology. The consensus tracking problem for high-order integrator systems subject to external disturbances was solved in [24], guaranteeing that consensus tracking errors are ultimately bounded. A unified framework has been introduced in [18] for consensus of linear multi-agent systems and synchronization of complex networks. Along this line, the consensus tracking problem for the linear system has been addressed in [25] with both state feedback and output feedback. In [26], the authors studied consensus for linear systems based the low gain approach. One common feature of the previous literature is that the consensus protocol design depends on the eigenvalues of the Laplacian matrix associated with the whole network, which is not easy to compute in practice if the network is very large with some uncertain connections. In order to remove this limitation, much effort has been made in the references [27, 28, 29], in which the main idea is to introduce an adaptive coupling gain in the controller. However, a drawback is that the adaptive coupling gain is difficult to be applied to directed communication graphs, whose Laplacian matrices are generally asymmetric, resulting in the difficulty in designing Lyapunov functions and adaptive protocols. Recently, a new result is given in

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[30], which has readdressed the adaptive coupling technique via multiplying a new nonlinear function in the controller, accompanied by a new integral-type Lyapunov function in the stability proof. By this new approach, the consensus protocols design can be applied to the directed graph without requiring the eigenvalues of the Laplacian matrix. But it is worth noting that the controllers are designed by multiplying the adaptive coupling gain and the nonlinear function together with the tracking error, resulting in possible bursting in the amplitude of the control input.

On the other hand, disturbances widely exist in practical systems. Thus, it is important to consider the disturbance rejection problem. In literature, disturbance rejection and attenuation have been extensively studied for individual systems [31, 32] and recently for multi-agent systems [33, 34] and complex networks [14, 35]. In [33, 34], the consensus control problems with disturbance have been solved in the H_∞ framework, which can only deal with an $\mathcal{L}_2$-type disturbance. Besides, the idea of H_∞ control is to suppress the disturbance for the worst case, which thus is conservative to some extent. In [14], the robustness of a complex network to Gaussian white noise is investigated and the consensus problem for linear integrators are addressed. In [35], a multiplex distributed proportional-integral approach is proposed to solve the consensus problem of a group of agents with constant disturbances. For a non-vanishing disturbance such as sinusoidal or constant disturbance, a key strategy is to design some input based on the state or output information to cancel the disturbance in the feed-forward sense, which can be realized through disturbance observers based on specified disturbance models. In the framework of disturbance observer-based control, typical results can be found in the references [36, 37, 38], and the method has been applied to various situations, such as industrial robotic manipulators, and maglev suspension systems. Albeit the significant developments in consensus control of linear systems in the last decade, rejection of deterministic disturbances in consensus control did not catch much attention until recently. In [39], the consensus problem was addressed for second-order integrators under an undirected graph. Fairly recently, in [40], the author considered the consensus disturbance rejection problem for the linear agents, but the communication topology is undirected when an adaptive coupling gain is used.

Motivated by the aforementioned results, in this paper, we aim to address the consensus disturbance rejection problem for general linear multi-agent systems on directed graphs. The main contributions of the paper are the following:

1) Firstly, based on the relative state information of the neighboring agents, two new disturbance rejection consensus protocols are proposed for the general directed graph based on two observers, of which, one is the state observer, and the other is the distributed disturbance observer to handle the period disturbance. Both of the protocols guarantee that the consensus errors converge to zero asymptotically. Besides, the disturbance observer gain is designed independently of the state observer, while in [40], the disturbance observer gain and state observer design are coupled with each other through the observability of the disturbance.

2) Secondly, the consensus protocol with the adaptive state observer is fully distributed in the sense that the controller is designed without knowing the global information of the Laplacian matrix associated with the whole network. Based on this new adaptive estimator, the proposed consensus protocol can achieve consensus for directed graphs whose Laplacian matrices are not symmetric.

3) Finally, a distributed adaptive observer is designed to decouple the adaptive coupling gain from the control input, which has the advantage that the high-gain coupling has no direct impact on the magnitude of the control input. Furthermore, a low-pass filter motivated by [41], which can be seen as a damping term, is added to reduce the initial adaptive rate of the coupling gain. In contrast, in [30], the coupling gain exists in the controller by multiplying a nonlinear function, and the coupling gain will increase rapidly if the initial consensus error is large, which may result in very big control input.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are introduced. In Section 3, the consensus tracking problem with disturbance rejection is formulated. The main results are developed in Section 4. Simulation results are provided in Section 5, and Section 6 concludes the paper.

II. MATHEMATICAL PRELIMINARIES

A. Notations

Throughout the paper, the notation is defined as follows. Let $\mathbb{R}^{n \times m}$ denote a set of $n \times m$ real matrices, $\mathbb{R}^n$ represent a set of n-dimensional vectors, and $0_{n \times m}$ represent the matrices with all zeros. Besides, let I_N denote the identity matrix of dimension N, **1** represent a column vector with all entries equal to one, and the symbol $\otimes$ denotes the Kronecker product of the matrices. For a matrix X, $\lambda_{\min}(X)$, $\lambda_{\max}(X)$ denote its minimum and maximum eigenvalue, respectively, and $\sigma_{\min}(X)$, $\sigma_{\max}(X)$ denote its minimum and maximum singular value, respectively.

B. Graph theory

In this paper, the information flow among the multiple agents is depicted by a directed graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} \triangleq \{1, \cdots, N\}$ as the node set, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ as the edge set. In the directed graph $\mathcal{G}$, $(i, j) \in \mathcal{E}$ denotes that the ith agent can receive the information from the jth agent, but not vice versa. And a directed path from node i_1 to node i_s is a sequence of ordered pair of nodes in the form that $(i_1, i_2), \cdots, (i_{s-1}, i_s)$. A directed graph contains a spanning tree if there exists a node (which we call root), so that all incoming edges to root may be deleted, whilst all other nodes are reachable from root along directed paths. In the graph theory, the adjacency matrix $\mathcal{A} = [a_{ij}]$ is adopted to express the information transmission direction among the nodes in the way that $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$, otherwise is zero. And $a_{ii} = 0$ for all nodes due to the fact that there is no self-loop in the topology. More importantly, based on the adjacency matrix, the Laplacian matrix $\mathcal{L} = [\mathcal{L}_{ij}]$ is defined as $\mathcal{L}_{ii} = \sum_{j=1, j \neq i}^{N} a_{ij}$

and $\mathcal{L}_{ij} = -a_{ij}, i \neq j$. And the Laplacian matrix associated with the directed graph $\mathcal{G}$ has the following fundamental property.

Lemma 1: ([42]) The Laplacian matrix $\mathcal{L}$ associated with graph $\mathcal{G}$ has the property that it has a simple eigenvalue at zero with the vector $\mathbf{1}$ as a corresponding eigenvector and all other eigenvalues have positive real parts if and only if the graph $\mathcal{G}$ contains a directed spanning tree.

Assumption 1: The graph $\mathcal{G}$ contains a directed spanning tree with the leader as the root node.

Because the leader is denoted by the root node that has no incoming edges, the Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ has the following structure

$$\mathcal{L} = \begin{bmatrix} 0 & 0_{1 \times N} \\ \mathcal{L}_2 & \mathcal{L}_1 \end{bmatrix}, \tag{1}$$

where $\mathcal{L}_1 \in \mathbb{R}^{N \times N}$ and $\mathcal{L}_2 \in \mathbb{R}^{N \times 1}$. In light of Lemma 1, zero is a simple eigenvalue of the Laplacian matrix $\mathcal{L}$ on the condition that the Assumption 1 is satisfied. Thus, it is not difficult to conclude that $\mathcal{L}_1$ is a nonsingular M-matrix that has a significant property as follows:

Lemma 2: ([30]) For the nonsingular M-matrix $\mathcal{L}_1$, there exists a diagonal matrix $G = \text{diag}(g_1, \cdots, g_N)$ with $g_i > 0, i = 1, \cdots, N$ such that $G\mathcal{L}_1 + \mathcal{L}_1^T G \triangleq \hat{\mathcal{L}}_1 > \lambda_0 I_N > 0$, where λ_0 is the minimum eigenvalues of the matrix $\hat{\mathcal{L}}_1$. And the positive definite matrix G can be computed as $[g_1, \cdots, g_N]^T = (\mathcal{L}_1^T)^{-1} \mathbf{1}$.

III. PROBLEM STATEMENT

Consider a group of $N + 1$ agents with general linear dynamics, consisting of N followers and one leader indexed with 0. The dynamics of the followers are denoted by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i + Ed_i(t), \tag{2}$$

where for agent i, $i = 1, \ldots, N$, $x_i \in \mathbb{R}^n$ is the state vector, $u_i \in \mathbb{R}^m$ is the control input vector. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $E \in \mathbb{R}^{n \times s}$ are constant matrices, $d_i(t) \in \mathbb{R}^s$ is the disturbance that is generated by an exosystem, i.e.

$$\dot{d}_i = S_i d_i,$$

where $S_i \in \mathbb{R}^{s \times s}$ is a known constant matrix that is called exosystem matrix, which can be different for each agent.

The leader has dynamics as $\dot{x}_0(t) = Ax_0(t) + Bu_0(t)$ with $u_0 = 0$.

The control objective of this paper is to design distributed adaptive consensus protocols for each follower with disturbance, such that the state of each follower converges to the state of the leader, that is,

$$\lim_{t \to \infty} (x_i(t) - x_0(t)) = 0,$$

for any initial condition of $x_i(0)$, $i = 1, 2, \cdots, N$.

Assumption 2: The eigenvalues of the exosystem matrix are distinct, and on the imaginary axis.

Assumption 3: The disturbances in the subsystems satisfy the matching condition, i.e., there exists a matrix $F \in \mathbb{R}^{m \times s}$ such that $E = BF$.

Remark 1: The Assumption 2 on the eigenvalues guarantees that the disturbance d_i is the non-vanishing harmonic disturbance including constants and sinusoidal functions, which can also be used as the basis functions to approximate other periodic signals, and this assumption is commonly used in the disturbance rejection and output regulation community [21, 43]. Assumption 3 is the matching condition on disturbances in the system, and sinusoidal functions or constants can be completely rejected. Besides, unmatched disturbances under some conditions can be transformed to the matched ones according to references [40, 43].

IV. MAIN RESULT

A. Observer-Based Consensus Disturbance Rejection with Static Coupling

In this section, based on the relative state of the neighboring agents, the consensus protocols with disturbance rejection are designed as follows

$$u_i = cK\theta_i - F\hat{d}_i \tag{3}$$

where $c > 0 \in \mathbb{R}$ is the feedback coupling gain independent of the Laplacian matrix. θ_i is the state of the following estimator that is only based on the relative state information

$$\dot{\theta}_i = A\theta_i + cBK\theta_i + \rho(1 + \beta_i)BK(e_i - \xi_i), \tag{4}$$

$$\beta_i = (e_i - \xi_i)^T P^{-1}(e_i - \xi_i), \tag{5}$$

where ρ is a positive constant to be determined later. $\xi_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j) + a_{i0}(x_i - x_0)$ is the relative state information, and $e_i = \sum_{j=1}^{N} a_{ij}(\theta_i - \theta_j) + a_{i0}\theta_i$, $K = -B^T P^{-1}$, and $P > 0$ is a positive solution of the following LMI:

$$AP + PA^T - 2cBB^T < 0 \tag{6}$$

and $\hat{d}_i$ is the estimate of the disturbance, with the disturbance observer designed as:

$$\begin{aligned} \hat{d}_i &= z_i + L\xi_i, \\ \dot{z}_i &= S_i z_i + (S_i L - LA)\xi_i - cLBKe_i \end{aligned} \tag{7}$$

where L is the feedback gain to be determined later, and z_i is the virtual state of the disturbance observer.

Remark 2: Only based on the relative state information, a state observer (4) is proposed, which is then adopted to design the disturbance observer and consensus controller in sequence. The main motivation to design the state observer is to provide extra freedom for the controller design through improving the order of the whole system. Furthermore, the disturbance observer (7) is designed in the distributed fashion for that only the relative information, including relative state and relative estimate, is used. By exploring the structure of the exosystem in Assumption 2, the disturbance observer and the state observer design can be decoupled from each other, which is different to the result in [40], where the disturbance observer gain depends on the state observer.

Before moving forward, let $\delta_i = x_i - x_0$, $\tilde{d}_i = \hat{d}_i - d_i$ as the tracking error and disturbance estimation error, respectively.

By virtue of the consensus protocol (3), the closed-loop error dynamics of the ith agent can be obtained as

$$\dot{\delta}_i(t) = A\delta_i(t) + cBK\theta_i - E\tilde{d}_i. \tag{8}$$

Let $\delta(t) = [\delta_1^T(t), \cdots, \delta_N^T(t)]^T$, $\tilde{d} = [\tilde{d}_1^T(t), \cdots, \tilde{d}_N^T(t)]^T$, $\xi(t) = [\xi_1^T(t), \cdots, \xi_N^T(t)]^T$, $e(t) = [e_1^T(t), \cdots, e_N^T(t)]^T$. Then we have the fact that

$$\xi(t) = (\mathcal{L}_1 \otimes I_n)\delta(t), \quad e(t) = (\mathcal{L}_1 \otimes I_n)\theta(t). \tag{9}$$

where $\mathcal{L}_1$ is defined in (1). As $\mathcal{L}_1$ is nonsingular according to Lemma 1, there exists a bijective mapping between vectors $\xi(t), e(t)$ and $\delta(t), \theta(t)$, respectively.

Then we obtain the dynamics of ξ_i, e_i in the compact form as

$$\begin{aligned}
\dot{\xi} &= (I_N \otimes A)\xi + (I_N \otimes cBK)e - (\mathcal{L}_1 \otimes E)\tilde{d}, \\
\dot{e} &= [I_N \otimes (A + cBK)]e + [\rho\mathcal{L}_1(I_N + \hat{\beta}) \otimes BK](e - \xi).
\end{aligned} \tag{10}$$

where $\hat{\beta} = \text{diag}(\beta_1, \cdots, \beta_N)$.

Before giving the main result in this section, a lemma on the disturbance observer is provided as follows.

Lemma 3: For the full column rank matrix E, if the observer gain L is designed to guarantee that LE is positive definite, the estimation error $\tilde{d}_i$ converges to zero exponentially based on the disturbance observer (7). Besides, as a special case, the observer gain can be designed as $L = \mu E^T$ with $\mu > 0$.

Proof: Let $\tilde{d} = [\tilde{d}_1^T(t), \cdots, \tilde{d}_N^T(t)]^T$, it is not difficult to get the estimation dynamics in the compact form as

$$\dot{\tilde{d}} = [I_N \otimes S_i - \mathcal{L}_1 \otimes LE]\tilde{d}. \tag{11}$$

Consider the Lyapunov function candidate

$$V_1(t) = \tilde{d}^T(G \otimes I_s)\tilde{d},$$

where $G = \text{diag}(g_1, \cdots, g_N)$ from Lemma 2. The time derivative of V_1 along the trajectory of (11) is obtained as

$$\begin{aligned}
\dot{V}_1 &= \tilde{d}^T[G \otimes (S_i + S_i^T)]\tilde{d} - \tilde{d}^T[(G\mathcal{L}_1 + \mathcal{L}_1^T G) \otimes LE]\tilde{d} \\
&\leq \tilde{d}^T[G \otimes (S_i + S_i^T)]\tilde{d} - \tilde{d}^T[\lambda_0 I_N \otimes LE]\tilde{d} = -\tilde{d}^T H\tilde{d}
\end{aligned}$$

where $H = -G \otimes (S_i + S_i^T) + \lambda_0 I_N \otimes LE$.

Based on the Assumption 2, it is obvious that all the eigenvalues of matrix H are on the right half side. Consequently, we can obtain that $\dot{V}_1 \leq 0$, and $\dot{V}_1 = 0$ is equivalent to $\tilde{d} = 0$. This completes the proof. ∎

The main result in this section is summarized in the following theorem.

Theorem 1: Suppose the directed topology satisfies Assumption 1 together with Assumption 2 and 3, the gain matrix L is designed such that $-LE$ is Hurwitz, and the gain ρ in the state observer (4) satisfies that $\rho \geq \frac{4cg_{\max}}{\lambda_0}$, where λ_0 is the minimal eigenvalue of the matrix $\hat{\mathcal{L}}_1$, $g_{\max} = \max\{g_1, \cdots, g_N\}$, then the consensus protocol (3) solves the consensus disturbance rejection problem with the state observer (4), and disturbance observer (7).

Proof: Consider the following Lyapunov function candidate

$$V_2 = e^T(I_N \otimes P^{-1})e + \frac{\gamma_1}{2}\sum_{i=1}^{N} g_i(2 + \beta_i)\beta_i.$$

Then, the time derivative of V_2 along the trajectory of (10) is given by

$$\dot{V}_2 = 2e^T(I_N \otimes P^{-1})\dot{e} + \gamma_1\sum_{i=1}^{N} g_i(1 + \beta_i)\dot{\beta}_i.$$

Observe that

$$\begin{aligned}
&2e^T(I_N \otimes P^{-1})\dot{e} \\
&= e^T[I_N \otimes (P^{-1}A + A^T P^{-1} - 2cP^{-1}BB^T P^{-1})]e \\
&\quad - 2e^T[\rho\mathcal{L}_1(I_N + \hat{\beta}) \otimes \Gamma](e - \xi) \\
&\leq -e^T(I_N \otimes H_1)e + e^T(I_N \otimes \frac{1}{2}H_1)e + (e - \xi)^T \cdot \\
&\quad \left[\frac{2\rho^2\sigma_{\max}^2(\mathcal{L}_1)\lambda_{\max}(\Gamma)}{\lambda_{\min}(H_1)}(I_N + \hat{\beta})^2 \otimes \Gamma\right](e - \xi) \\
&= -e^T(I_N \otimes \frac{1}{2}H_1)e + (e - \xi)^T\left[\frac{2\rho^2\sigma_{\max}^2(\mathcal{L}_1)\lambda_{\max}(\Gamma)}{\lambda_{\min}(H_1)}\right. \\
&\quad \left. \cdot (I_N + \hat{\beta})^2 \otimes \Gamma\right](e - \xi),
\end{aligned} \tag{12}$$

where $-H_1 = P^{-1}A + A^T P^{-1} - 2cP^{-1}BB^T P^{-1} < 0$, $\Gamma = P^{-1}BB^T P^{-1}$, and the inequality has used the Young's Inequality.

Based on the dynamics (10), one can obtain

$$\begin{aligned}
&\gamma_1\sum_{i=1}^{N} g_i(1 + \beta_i)\dot{\beta}_i \\
&= 2\gamma_1(e - \xi)^T[G(I_N + \hat{\beta}) \otimes P^{-1}](\dot{e} - \dot{\xi}) \\
&\leq \gamma_1(e - \xi)^T[G(I_N + \hat{\beta}) \otimes (P^{-1}A + A^T P^{-1}) \\
&\quad - \rho(I_N + \hat{\beta})(G\mathcal{L}_1 + \mathcal{L}_1^T G)(I_N + \hat{\beta}) \otimes \Gamma](e - \xi) \\
&\quad + 2\gamma_1(e - \xi)^T[G(I_N + \hat{\beta})\mathcal{L}_1 \otimes P^{-1}E]\tilde{d},
\end{aligned}$$

where the inequality has used that $G\mathcal{L}_1 + \mathcal{L}_1^T G = \hat{\mathcal{L}}_1 \geq \lambda_0 I_N$, then the inequality can be computed as

$$\begin{aligned}
&\leq \gamma_1(e - \xi)^T[G(I_N + \hat{\beta}) \otimes (P^{-1}A + A^T P^{-1}) \\
&\quad - \rho\lambda_0(I_N + \hat{\beta})^2 \otimes \Gamma](e - \xi) \\
&\quad + 2\gamma_1(e - \xi)^T[G(I_N + \hat{\beta})\mathcal{L}_1 \otimes P^{-1}E]\tilde{d} \\
&\leq \gamma_1(e - \xi)^T[G(I_N + \hat{\beta}) \otimes (P^{-1}A + A^T P^{-1} - 2c\Gamma)]\cdot \\
&\quad (e - \xi) - \gamma_1\rho\frac{\lambda_0}{2}(e - \xi)^T[(I_N + \hat{\beta})^2 \otimes \Gamma](e - \xi) \\
&\quad + 2\gamma_1(e - \xi)^T[G(I_N + \hat{\beta})\mathcal{L}_1 \otimes P^{-1}E]\tilde{d},
\end{aligned} \tag{13}$$

where the last inequality is derived by virtue of the fact that $\rho \geq \frac{4cg_{\max}}{\lambda_0}$, and $(I_N + \hat{\beta}) \geq I_N$.

In light of the Young's inequality, it can be obtained that

$$2\gamma_1(e - \xi)^T[G(I_N + \hat{\beta})\mathcal{L}_1 \otimes P^{-1}E]\tilde{d}$$

$$\leq \gamma_1\rho\frac{\lambda_0}{4}(e - \xi)^T[(I_N + \hat{\beta})^2 \otimes \Gamma](e - \xi)$$

$$+ \gamma_1\frac{4}{\lambda_0\rho}\tilde{d}^T[(G\mathcal{L}_1)^T(G\mathcal{L}_1) \otimes F^TF]\tilde{d} \qquad (14)$$

$$\leq \gamma_1\rho\frac{\lambda_0}{4}(e - \xi)^T[(I_N + \hat{\beta})^2 \otimes \Gamma](e - \xi)$$

$$+ \gamma_1\frac{4g_{\max}^2\lambda_{\max}(\mathcal{L}_1^T\mathcal{L}_1)\lambda_{\max}(F^TF)}{\lambda_0\rho\lambda_{\min}(H)}\tilde{d}^TH\tilde{d}.$$

Consider the Lyapunov function candidate for the whole closed-loop systems as

$$V = \gamma_1\gamma_2 V_1 + V_2.$$

Thus, based on the result (12)(13)(14) and Lemma 3, the derivative of the Lyapunov function is

$$\dot{V} \leq \gamma_1(e - \xi)^T[G(I_N + \hat{\beta}) \otimes [P^{-1}A + A^TP^{-1} - 2c\Gamma]\cdot$$

$$(e - \xi) - e^T(I_N \otimes \frac{1}{2}H_1)e + (e - \xi)^T\cdot$$

$$\left[\frac{2\rho^2\sigma_{\max}^2(\mathcal{L}_1)\lambda_{\max}(\Gamma)}{\lambda_{\min}(H_1)}(I_N + \hat{\beta})^2 \otimes \Gamma\right](e - \xi)$$

$$- \gamma_1\frac{\rho\lambda_0}{4}(e - \xi)^T[(I_N + \hat{\beta})^2 \otimes \Gamma](e - \xi) - \gamma_1\gamma_2\cdot$$

$$\tilde{d}^TH\tilde{d} + \gamma_1\frac{4g_{\max}^2\lambda_{\max}(\mathcal{L}_1^T\mathcal{L}_1)\lambda_{\max}(F^TF)}{\lambda_0\rho\lambda_{\min}(H)}\tilde{d}^TH\tilde{d}$$

$$\leq -\gamma_1(e - \xi)^T[G(I_N + \hat{\beta}) \otimes H_1](e - \xi)$$

$$- e^T(I_N \otimes \frac{1}{2}H_1)e - \gamma_1\tilde{d}^TH\tilde{d}.$$

$$(15)$$

where $\gamma_1 \geq \frac{8\rho\sigma_{\max}^2(\mathcal{L}_1)\lambda_{\max}(\Gamma)}{\lambda_0\lambda_{\min}(H_1)}$ and $\gamma_2 \geq 1 + \frac{4g_{\max}^2\lambda_{\max}(\mathcal{L}_1^T\mathcal{L}_1)\lambda_{\max}(F^TF)}{\lambda_0\rho\lambda_{\min}(H)}$.

By noting that $V(t) \geq 0$, and $\dot{V}(t) \leq 0$, it can be concluded that $\lim_{t\to\infty} V(t) = V(\infty)$ and $V(\infty) \leq V(t) \leq V(0)$. Thus all the signals in the closed-loop systems including e_i, ξ_i are bounded, that is, $e_i, \xi_i \in \mathbb{L}_\infty$. Integrating both sides of the inequality (15), we can obtain that

$$V(0) - V(\infty) \geq \int_0^\infty \{\gamma_1(e - \xi)^T[G(I_N + \hat{\beta}) \otimes H_1](e - \xi)$$

$$+ e^T(I_N \otimes \frac{1}{2}H_1)e + \gamma_1\tilde{d}^TH\tilde{d}\},$$

$$(16)$$

which implies $\sqrt{g_i(I_N + \beta_i)}(e_i - \xi_i), e_i, \tilde{d}_i \in \mathbb{L}_2$. Then from the dynamics (10), we have $\dot{e}_i, \dot{\xi}_i \in \mathbb{L}_\infty$. By virtue of the Barbalat's Lemma, we can conclude that $\lim_{t\to\infty}(e_i - \xi_i) = 0$, $\lim_{t\to\infty} e_i = 0$, $\lim_{t\to\infty}\tilde{d}_i = 0$. Consequently, the consensus tracking error ξ_i goes to zero asymptotically, meaning that the consensus tracking problem has been solved. ∎

Remark 3: As pointed out in [18], a sufficient condition for the existence of the feedback gain matrix K is that (A, B) is stabilizable which guarantees that the LMI (6) has a positive definite solution $P > 0$. Besides, the static coupling $c > 0$ is an independent parameter that has no relation with the Laplacian matrix of the communication graph, while the dependence relation has been removed to the coupling gain ρ in the state observer (4), which has the advantage that very large coupling ρ has little influence on the controller.

B. Observer-Based Consensus Disturbance Rejection with Dynamic Coupling

As is illustrated in the Theorem 1, the parameter ρ in the state observer (4) depends on the parameters $\lambda_0, g_{\max}$, which are associated with the Laplacian matrix of the whole communication topology. Obviously, the whole network graph is the global information and, therefore, the consensus protocols in the above section are not in the fully distributed fashion. In order to remove this limitation to make the controllers fully distributed, a new adaptive coupling gain is adopted in the state observer design. Based on the relative state of the neighboring agents, a distributed consensus protocol with adaptive state observer for the i-th agent is proposed as follows

$$u_i = cK\theta_i - F\hat{d}_i \qquad (17)$$

where $c \in \mathbb{R}$ is the feedback coupling gain independent of the Laplacian matrix. θ_i is the state of the following adaptive estimator with adaptive couplings as

$$\dot{\theta}_i = A\theta_i + cBK\theta_i + (\alpha_i + \beta_i)BK(e_i - \xi_i), \qquad (18)$$

$$\dot{\alpha}_i = k_i(e_i - \xi_i)^T\Gamma(e_i - \xi_i) - k_i\sigma_i(\alpha_i - b_i(t)), \qquad (19)$$

$$\beta_i = (e_i - \xi_i)^TP^{-1}(e_i - \xi_i), \qquad (20)$$

and $b_i(t), i = 1, \cdots, N$ are the filtered coupling weight estimates that are designed as follows

$$\dot{b}_i(t) = \rho_i(\alpha_i(t) - b_i(t)), i = 1, \cdots, N, \qquad (21)$$

where $k_i > 0, \rho_i > 0$ are the learning rates, $\sigma_i > 0$ is a small design parameter, and ξ, e_i is the same as in Section IV-A, and

$$\Gamma = P^{-1}BB^TP^{-1}, K = -B^TP^{-1},$$

where $P > 0$ is a positive solution of the following LMI:

$$AP + PA^T - 2cBB^T < 0 \qquad (22)$$

and $\hat{d}_i$ is the estimate of the disturbance that keeps the same form with (7) in the above section.

Follow the notations in (8)(9), then we obtain the dynamics of ξ_i, e_i in the compact form as

$$\dot{\xi} = (I_N \otimes A)\xi + (I_N \otimes cBK)e - (\mathcal{L}_1 \otimes E)\tilde{d},$$
$$\dot{e} = [I_N \otimes (A + cBK)]e + [\mathcal{L}_1(\hat{\alpha} + \hat{\beta}) \otimes BK](e - \xi). \qquad (23)$$

where $\hat{\alpha} = \text{diag}(\alpha_1, \cdots, \alpha_N), \hat{\beta} = \text{diag}(\beta_1, \cdots, \beta_N)$.

Before moving forward, the following lemma is significant for the stability analysis.

Lemma 4: Based on adaptive controller (19)(21), we have the fact that $\alpha_i(t) - b_i(t) > 0$ if the initial values satisfy that $\alpha_i(0) - b_i(0) > 0$.

Proof: Based on the adaptive law (19) and (21), we have

$$\dot{\alpha}_i - \dot{b}_i = (e_i - \xi_i)^T\Gamma(e_i - \xi_i) - (k_i\sigma_i + \rho_i)(\alpha_i - b_i).$$

Then it can be obtained that

$$
\begin{aligned}
\alpha_i - b_i &= [\alpha_i(0) - b_i(0)]e^{-(k_i\sigma_i+\rho_i)t} \\
&\quad + \int_0^t e^{-(k_i\sigma_i+\rho_i)(t-\tau)}(e_i - \xi_i)^T\Gamma(e_i - \xi_i)d\tau \\
&\geq [\alpha_i(0) - b_i(0)]e^{-(k_i\sigma_i+\rho_i)t} > 0.
\end{aligned}
$$

This completes the proof. ∎

Based on the above preparations, the main result of this section is given in the following theorem.

Theorem 2: Suppose the communication graph satisfies Assumption 1 together with Assumption 2 and 3, then with the distributed adaptive consensus protocols (17)-(21) and the distributed disturbance observer (7), all the closed-loop signals including ξ_i, θ_i, $\tilde{d}_i$, α_i, b_i are uniformly ultimately bounded and the consensus tracking error converges to zero asymptotically.

Proof: Consider the following Lyapunov function candidate

$$
\begin{aligned}
V_3 &= e^T(I_N \otimes P^{-1})e + \frac{\gamma_1}{2}\sum_{i=1}^N g_i(2\alpha_i + \beta_i)\beta_i \\
&\quad + \frac{\gamma_1}{2}\sum_{i=1}^N \frac{1}{k_i}(\alpha_i - \alpha)^2 + \frac{\gamma_1}{2}\sum_{i=1}^N \frac{\sigma_i}{\rho_i}(b_i - \alpha)^2,
\end{aligned}
$$

where α is a positive scalar to be designed later.

Then, the time derivative of V_3 along the trajectory of (23) is given by

$$
\begin{aligned}
\dot{V}_3 &= 2e^T(I_N \otimes P^{-1})\dot{e} + \gamma_1\Big[\sum_{i=1}^N g_i(\alpha_i + \beta_i)\dot{\beta}_i + \sum_{i=1}^N g_i\dot{\alpha}_i\beta_i\Big] \\
&\quad + \gamma_1\sum_{i=1}^N \frac{1}{k_i}(\alpha_i - \alpha)\dot{\alpha}_i + \gamma_1\sum_{i=1}^N \frac{\sigma_i}{\rho_i}(b_i - \alpha)\dot{b}_i.
\end{aligned}
$$

Similar with the above section, it is obtained that

$$
\begin{aligned}
&2e^T(I_N \otimes P^{-1})\dot{e} \\
&= e^T[I_N \otimes (P^{-1}A + A^TP^{-1} - 2cP^{-1}BB^TP^{-1})]e \\
&\quad - 2e^T[\mathcal{L}_1(\hat{\alpha} + \hat{\beta}) \otimes \Gamma](e - \xi) \\
&\leq -e^T(I_N \otimes \frac{1}{2}H_1)e + (e - \xi)^T \cdot \\
&\quad \Big[\frac{2\sigma_{\max}^2(\mathcal{L}_1)\lambda_{\max}(\Gamma)}{\lambda_{\min}(H_1)}(\hat{\alpha} + \hat{\beta})^2 \otimes \Gamma\Big](e - \xi),
\end{aligned} \tag{24}
$$

where $-H_1 = P^{-1}A + A^TP^{-1} - 2cP^{-1}BB^TP^{-1} < 0$, and the inequality has used the Young's Inequality.

Let $\hat{b} = \mathrm{diag}(b_1, \cdots, b_N)$. Then, by virtue of the adaptive coupling (19)(20) and the low-pass filter (21), we have

$$
\begin{aligned}
&\gamma_1\Big[\sum_{i=1}^N g_i(\alpha_i + \beta_i)\dot{\beta}_i + \sum_{i=1}^N g_i\dot{\alpha}_i\beta_i\Big] + \gamma_1\sum_{i=1}^N \frac{1}{k_i}(\alpha_i - \alpha)\dot{\alpha}_i \\
&\quad + \gamma_1\sum_{i=1}^N \frac{\sigma_i}{\rho_i}(b_i - \alpha)\dot{b}_i \\
&= 2\gamma_1(e - \xi)^T[G(\hat{\alpha} + \hat{\beta}) \otimes P^{-1}](\dot{e} - \dot{\xi}) + \gamma_1(e - \xi)^T[(G\hat{\beta} \\
&\quad \otimes \Gamma) - (G\hat{\sigma}(\hat{\alpha} - \hat{b}) \otimes P^{-1}) + (\hat{\alpha} - \alpha I_N) \otimes \Gamma](e - \xi) \\
&\quad - \gamma_1\sum_{i=1}^N \sigma_i(\alpha_i - \alpha)(\alpha_i - b_i) + \gamma_1\sum_{i=1}^N \sigma_i(b_i - \alpha)(\alpha_i - b_i) \\
&\leq \gamma_1(e - \xi)^T[G(\hat{\alpha} + \hat{\beta}) \otimes (P^{-1}A + A^TP^{-1}) - (\hat{\alpha} + \hat{\beta}) \\
&\quad (G\mathcal{L}_1 + \mathcal{L}_1^TG)(\hat{\alpha} + \hat{\beta}) \otimes \Gamma + (G\hat{\beta} \otimes \Gamma) + (\hat{\alpha} - \alpha I_N) \otimes \Gamma] \\
&\quad \cdot (e - \xi) + 2\gamma_1(e - \xi)^T[G(\hat{\alpha} + \hat{\beta})\mathcal{L}_1 \otimes P^{-1}E]\tilde{d} \\
&\quad - \gamma_1\sum_{i=1}^N \sigma_i[\alpha_i - b_i]^2 \\
&\leq \gamma_1(e - \xi)^T[G(\hat{\alpha} + \hat{\beta}) \otimes (P^{-1}A + A^TP^{-1}) - \lambda_0(\hat{\alpha} + \hat{\beta})^2 \\
&\quad \otimes \Gamma + (G\hat{\beta} \otimes \Gamma) + (\hat{\alpha} - \alpha I_N) \otimes \Gamma](e - \xi) + \\
&\quad 2\gamma_1(e - \xi)^T[G(\hat{\alpha} + \hat{\beta})\mathcal{L}_1 \otimes P^{-1}E]\tilde{d} - \gamma_1\sum_{i=1}^N \sigma_i[\alpha_i - b_i]^2,
\end{aligned} \tag{25}
$$

where the first inequality has used the result in Lemma 4, and the following fact that

$$
\begin{aligned}
&\gamma_1\sum_{i=1}^N \sigma_i(b_i - \alpha)(\alpha_i - b_i) - \gamma_1\sum_{i=1}^N \sigma_i(\alpha_i - \alpha)(\alpha_i - b_i) \\
&= -\gamma_1\sum_{i=1}^N \sigma_i[\alpha_i - b_i]^2,
\end{aligned}
$$

and the second inequality has used that $G\mathcal{L}_1 + \mathcal{L}_1^TG \geq \lambda_0 I_N$.

Moreover,

$$
\begin{aligned}
&\gamma_1(e - \xi)^T[G\hat{\beta} \otimes \Gamma + (\hat{\alpha} - \alpha I_N) \otimes \Gamma \\
&\quad - \frac{\lambda_0}{2}(\hat{\alpha} + \hat{\beta})^2 \otimes \Gamma](e - \xi) \\
&\leq -\gamma_1(e - \xi)^T[\frac{\lambda_0}{4}(\hat{\alpha} + \hat{\beta})^2 \otimes \Gamma + (\alpha I_N - \frac{1}{\lambda_0}I_N \\
&\quad - \frac{1}{\lambda_0}G^2) \otimes \Gamma](e - \xi) \\
&\leq -\gamma_1(e - \xi)^T[\frac{\lambda_0}{4}(\hat{\alpha} + \hat{\beta})^2 \otimes \Gamma + \bar{\alpha} \otimes \Gamma](e - \xi) \\
&\leq -\gamma_1(e - \xi)^T[2cG(\hat{\alpha} + \hat{\beta}) \otimes \Gamma](e - \xi),
\end{aligned} \tag{26}
$$

where $\alpha \geq \bar{\alpha} + \frac{1}{\lambda_0} + \frac{1}{\lambda_0}g_{\max}^2$, and the constant scalar $\bar{\alpha}$ is determined by the fact that $\frac{\lambda_0}{4}(\hat{\alpha} + \hat{\beta})^2 + \bar{\alpha}I_N \geq \sqrt{\lambda_0\bar{\alpha}}(\hat{\alpha} + \hat{\beta}) \geq 2cG(\hat{\alpha} + \hat{\beta})$ if $\sqrt{\lambda_0\bar{\alpha}}I_N \geq 2cG$, that is, $\bar{\alpha} \geq \frac{4c^2g_{\max}}{\lambda_0}$.

In light of the Young's inequality, it can be obtained that

$$2\gamma_1(e-\xi)^T[G(\hat{\alpha}+\hat{\beta})\mathcal{L}_1 \otimes P^{-1}E]\tilde{d}$$

$$\leq \gamma_1\frac{\lambda_0}{4}(e-\xi)^T[(\hat{\alpha}+\hat{\beta})^2 \otimes \Gamma](e-\xi)$$

$$+ \gamma_1\frac{4}{\lambda_0}\tilde{d}^T[(G\mathcal{L}_1)^T(G\mathcal{L}_1) \otimes F^TF]\tilde{d} \qquad (27)$$

$$\leq \gamma_1\frac{\lambda_0}{4}(e-\xi)^T[(\hat{\alpha}+\hat{\beta})^2 \otimes \Gamma](e-\xi)$$

$$+ \gamma_1\frac{4g_{\max}^2\lambda_{\max}(\mathcal{L}_1^T\mathcal{L}_1)\lambda_{\max}(F^TF)}{\lambda_0\lambda_{\min}(H)}\tilde{d}^TH\tilde{d}.$$

Consider the Lyapunov function candidate for the whole closed-loop systems as

$$V = \gamma_1\gamma_2 V_1 + V_3.$$

Thus, based on the result (24)-(27) and Lemma 3, the derivative of the Lyapunov function is

$$\dot{V} \leq \gamma_1(e-\xi)^T[G(\hat{\alpha}+\hat{\beta}) \otimes [P^{-1}A + A^TP^{-1} - 2c\Gamma](e-\xi)$$

$$- e^T(I_N \otimes \frac{1}{2}H_1)e - \gamma_1\gamma_2\tilde{d}^TH\tilde{d}$$

$$+ (e-\xi)^T[\frac{2\sigma_{\max}^2(\mathcal{L}_1)\lambda_{\max}(\Gamma)}{\lambda_{\min}(H_1)}(\hat{\alpha}+\hat{\beta})^2 \otimes \Gamma](e-\xi)-$$

$$\gamma_1\frac{\lambda_0}{4}(e-\xi)^T[(\hat{\alpha}+\hat{\beta})^2 \otimes \Gamma](e-\xi) - \gamma_1\sum_{i=1}^{N}\rho_i[\alpha_i-b_i]^2$$

$$+ \gamma_1\frac{4g_{\max}^2\lambda_{\max}(\mathcal{L}_1^T\mathcal{L}_1)\lambda_{\max}(F^TF)}{\lambda_0\lambda_{\min}(H)}\tilde{d}^TH\tilde{d}$$

$$\leq -\gamma_1(e-\xi)^T[G(\hat{\alpha}+\hat{\beta}) \otimes H_1](e-\xi)$$

$$- e^T(I_N \otimes \frac{1}{2}H_1)e - \gamma_1\sum_{i=1}^{N}\rho_i[\alpha_i-b_i]^2 - \gamma_1\tilde{d}^TH\tilde{d},$$

$$(28)$$

where $\gamma_1 \geq \frac{8\sigma_{\max}^2(\mathcal{L}_1)\lambda_{\max}(\Gamma)}{\lambda_0\lambda_{\min}(H_1)}$ and $\gamma_2 \geq 1 + \frac{4g_{\max}^2\lambda_{\max}(\mathcal{L}_1^T\mathcal{L}_1)\lambda_{\max}(F^TF)}{\lambda_0\lambda_{\min}(H)}$.

By noting that $V(t) \geq 0$, and $\dot{V}(t) \leq 0$, it can be concluded that $\lim_{t\to\infty} V(t) = V(\infty)$ and $V(\infty) \leq V(t) \leq V(0)$. Thus all the signals in the closed-loop systems including $e_i, \xi_i, \alpha_i, b_i$ are bounded, that is, $e_i, \xi_i, \alpha_i, b_i \in \mathbb{L}_\infty$. Integrating both sides of the inequality (28), we can obtain that

$$V(0) - V(\infty) \geq \int_0^\infty \{\gamma_1(e-\xi)^T[G(\hat{\alpha}+\hat{\beta}) \otimes H_1](e-\xi)$$

$$+ e^T(I_N \otimes \frac{1}{2}H_1)e$$

$$+ \gamma_1\sum_{i=1}^{N}\rho_i[\alpha_i-b_i]^2 + \gamma_1\tilde{d}^TH\tilde{d}\},$$

which implies $\sqrt{g_i(\alpha_i+\beta_i)}(e_i-\xi_i), e_i, (\alpha_i-b_i), \tilde{d}_i \in \mathbb{L}_2$. Then from the dynamics (18)-(23), we have $\dot{e}_i, \dot{\xi}_i, \dot{\alpha}_i, \dot{b}_i \in \mathbb{L}_\infty$. By virtue of the Barbalat's Lemma, we can conclude that $\lim_{t\to\infty}(e_i-\xi_i) = 0$, $\lim_{t\to\infty} e_i = 0$, $\lim_{t\to\infty}(\alpha_i-b_i) = 0$, $\lim_{t\to\infty}\tilde{d}_i = 0$. Consequently, the consensus tracking error ξ_i goes to zero asymptotically, meaning that the consensus tracking problem has been solved. Besides, $\alpha_i - b_i$ goes to zero, meaning that the adaptive coupling α_i has tracked the

low-pass filter state b_i, and goes to a constant asymptotically. ∎

Remark 4: In the controller (17), the constant c is a positive constant parameter that is independent of the eigenvalues of the Laplacian matrix, thus the consensus protocol is designed in the fully distributed fashion. Moreover, the adaptive coupling α_i is removed from the controller to the distributed observer which has no direct impact on the amplitude of the control input. Besides, the part $-\sigma_i(\alpha_i - b_i)$ in (19) can be seen as a damping term added to the adaptive coupling in the state observer, which helps to reduce the initial learn rate of the adaptive coupling gain. In the references [30], the adaptive coupling gains are designed by multiplying the consensus errors, which has the limitation that the bursting phenomenon in the transient process of the adaptive coupling results in large control input, which is not practical in the real applications due to the actuator's finite ability. Besides, the adaptive observer (18) has been designed partially motivated by [44].

Remark 5: The distributed disturbance observer (7) has adopted the idea from the classical disturbance-observer-based control, which has the advantage that the estimation of the disturbance can be seen as an added patch to the nominal controller, that is, we can add the distributed disturbance observer to a multi-agent system that is already achieved consensus by some other controllers. Thus, it is more convenient in the real applications.

Remark 6: In the leader-follower consensus problem, the control input is applied to each agent to achieve the consensus tracking objective. As to the case where the control is exerted to only a fraction of agents, we believe that the pinning control method [45, 46] will be helpful to solve this future work of this paper.

C. Extensions to Containment Control with Multiple Leaders

In this section, the above result is extended to the case with multiple leaders, which is named containment control in the literature. Suppose the group of agents consists of M leaders, labeled as $1, \cdots, M$, and $N - M$ followers that labeled with $M+1, \cdots, N$. The convex hull for the states of multiple leaders set $X_L \triangleq \{x_1, \cdots, x_M\}$ is defined as $\mathrm{co}(X_L) \triangleq \{\sum_{i=1}^{M}\alpha_i x_i | \sum_{i=1}^{M}\alpha_i = 1, \alpha_i \geq 0\}$. Then the containment control is defined as follows.

Definition 1: The containment control problem is solved if a group of followers converge to the convex hull $\mathrm{co}(X_L)$ spanned by the states of multiple leaders.

Similar to the Assumption 1, the communication topology $\mathcal{G}'$ among the agents satisfies the following assumption:

Assumption 4: In the graph $\mathcal{G}'$, there exists at least one leader that has a directed path for each of the $N-M$ followers. Assume that the leaders have no parents, the Laplacian matrix $\mathcal{L}'$ associated with the graph $\mathcal{G}'$ has the following structure:

$$\mathcal{L}' = \begin{bmatrix} 0_{M\times M} & 0_{M\times(N-M)} \\ \mathcal{L}_2' & \mathcal{L}_1' \end{bmatrix}, \qquad (29)$$

where $\mathcal{L}_1' \in \mathbb{R}^{(N-M)\times(N-M)}$ and $\mathcal{L}_2' \in \mathbb{R}^{(N-M)\times M}$.

Lemma 5: ([47]) If the Assumption 4 is satisfied, the matrix $\mathcal{L}_1'$ in the equation (29) is a M-matrix, and each entry of

$-(\mathcal{L}'_1)^{-1}\mathcal{L}_2$ is nonnegative and all row sums of $-(\mathcal{L}'_1)^{-1}\mathcal{L}'_2$ equals to one.

Define $\mathbb{L}, \mathbb{F}$ as the set of leaders and followers, respectively, with $\mathbb{L} = \{1, \cdots, M\}, \mathbb{F} = \{M + 1, \cdots, N\}$. Then, let $\bar{\xi}_i = \sum_{j \in \mathbb{L} \cup \mathbb{F}} a_{ij}(x_i - x_j)$, $\bar{e}_i = \sum_{j \in \mathbb{L} \cup \mathbb{F}} a_{ij}(\theta_i - \theta_j)$, where $\theta_i = 0, i \in \mathbb{L}$. Besides, let $x_L = [x_1^T, \cdots, x_M^T]^T$, $x_F = [x_{M+1}^T, \cdots, x_N^T]^T$, and $\bar{\xi}(t) = [\bar{\xi}_{M+1}^T(t), \cdots, \bar{\xi}_N^T(t)]^T$, $\bar{e}(t) = [\bar{e}_{M+1}^T(t), \cdots, \bar{e}_N^T(t)]^T$. Then one can obtain the fact that

$$\bar{\xi}(t) = (\mathcal{L}'_1 \otimes I_n)x_F + (\mathcal{L}'_2 \otimes I_n)x_L, \quad \bar{e}(t) = (\mathcal{L}'_1 \otimes I_n)\theta(t). \tag{30}$$

Based on the result in Lemma 5, $\bar{\xi}(t) = (\mathcal{L}'_1 \otimes I_n)[x_F + ((\mathcal{L}')_1^{-1}\mathcal{L}'_2 \otimes I_n)x_L] = 0$ equals to $x_F = [-((\mathcal{L}')_1^{-1}\mathcal{L}'_2 \otimes I_n)x_L]$, where $-((\mathcal{L}')_1^{-1}\mathcal{L}'_2 \otimes I_n)x_L$ is the convex hull of the dynamic leaders. Thus the containment control has been transformed to prove that $\bar{\xi} \to 0$ as $t \to \infty$.

Based on the fact (30) and the similar form of controllers in the above section, the dynamics of $\bar{\xi}, \bar{e}$ are obtained as

$$\dot{\bar{\xi}} = (I_N \otimes A)\bar{\xi} + (I_N \otimes cBK)\bar{e} - (\mathcal{L}'_1 \otimes E)\tilde{d},$$
$$\dot{\bar{e}} = [I_N \otimes (A + cBK)]\bar{e} + [\mathcal{L}'_1(\hat{\alpha} + \hat{\beta}) \otimes BK](\bar{e} - \bar{\xi}). \tag{31}$$

where $\hat{\alpha} = \mathrm{diag}(\alpha_{M+1}, \cdots, \alpha_N), \hat{\beta} = \mathrm{diag}(\beta_{M+1}, \cdots, \beta_N)$.

The result on the containment control is summarized in the following theorem.

Theorem 3: Suppose Assumption 2 and 3 hold, and the communication network $\mathcal{G}'$ satisfies Assumption 4, the consensus protocol (17) with state estimator (18) and disturbance observers (7) solves the containment control with disturbance rejection.

Proof: The proof is similar with that in Theorem 2, thus is omitted here. ∎

Remark 7: The containment control in this section with multiple leaders can be seen as the extension of the consensus tracking problem in Theorem 2. Actually, when $M = 1$, the containment control will reduce to the consensus tracking control.

V. SIMULATION

In this section, a simulation example is provided to illustrate the effectiveness of the above theoretical results. Consider a network of six agents, consisting of five followers and one leader labeled with 0, whose dynamics, which is the Hill equation of the spacecraft in the low Earth orbit, are given as follows [18]:

$$A = \begin{bmatrix} 0 & I_3 \\ A_1 & A_2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ I_3 \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3\eta_0^2 & 0 \\ 0 & 0 & -\eta_0^2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 2\eta_0 & 0 \\ -2\eta_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $E = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$. The communication graph is represented by Fig. 1, and only the follower indexed by 1 is available to the leader's information. From the Fig. 1, it is easy to see that the communication topology contains
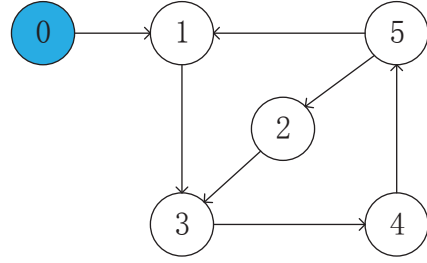

Fig. 1. Communication topology.

a directed spanning tree. In this simulation, the harmonic disturbance is generated by the exo-system with the matrix $S_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}$, where ω_i is the disturbance frequency. And in this simulation, $\eta_0 = 0.001, \omega_1 = 1, \omega_2 = 2, \omega_3 = 1.5, \omega_4 = 0.5, \omega_5 = 2.5$.

A. Case 1:

In this case, the consensus protocols in Theorem 1 is verified. By solving the LMI (6) with the LMI toolbox of MATLAB, the feedback gain $K = -B^T P^{-1}, \Gamma = P^{-1}BB^T P^{-1}$ can be computed as

$$K = \begin{bmatrix} -0.6596 & 0.0013 & 0 & -1.9789 & 0 & 0 \\ -0.0013 & -0.6596 & 0 & 0 & -1.9789 & 0 \\ 0 & 0 & -0.6596 & 0 & 0 & -1.9789 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} 0.4351 & 0 & 0 & 1.3053 & 0.0026 & 0 \\ 0 & 0.4351 & 0 & -0.0026 & 1.3053 & 0 \\ 0 & 0 & 0.4351 & 0 & 0 & 1.3053 \\ 1.3053 & -0.0026 & 0 & 3.9159 & 0 & 0 \\ 0.0026 & 1.3053 & 0 & 0 & 3.9160 & 0 \\ 0 & 0 & 1.3053 & 0 & 0 & 3.9159 \end{bmatrix}.$$

where the matrix P is

$$P = \begin{bmatrix} 2.2740 & -0.0000 & 0 & -0.7580 & -0.0015 & 0 \\ -0.0000 & 2.2740 & 0 & 0.0015 & -0.7580 & 0 \\ 0 & 0 & 2.2740 & 0 & 0 & -0.7580 \\ -0.7580 & 0.0015 & 0 & 0.7580 & -0.0000 & 0 \\ -0.0015 & -0.7580 & 0 & -0.0000 & 0.7580 & 0 \\ 0 & 0 & -0.7580 & 0 & 0 & 0.7580 \end{bmatrix}.$$

The Laplacian matrix $\mathcal{L}_1$ associated with the graph in Fig. 1 is that

$$\mathcal{L}_1 = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{32}$$

Then, based on Lemma 2, matrix $G = \mathrm{diag}(5, 10, 9, 17, 16)$ and $\lambda_0 = 1.7919$. Thus, the static coupling is chosen as $\rho = 32 \geq \frac{4cg_{\max}}{\lambda_0} = 31.2517$, and the matrix $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T$. The initial states of the system x_i, the state observer θ_i, and the disturbance observer z_i are chosen randomly within $[-10, 10]$ based on the $rand$ function in MATLAB. Norm of consensus errors ξ_i are described given in Fig. 2a, which shows clearly that the tracking errors converge to zero, that is, the consensus tracking is achieved.

B. Case 2:

In this case, the consensus protocols with adaptive estimators in Theorem 2 are applied. Compared with the first case, the Laplacian matrix of the communication network is not
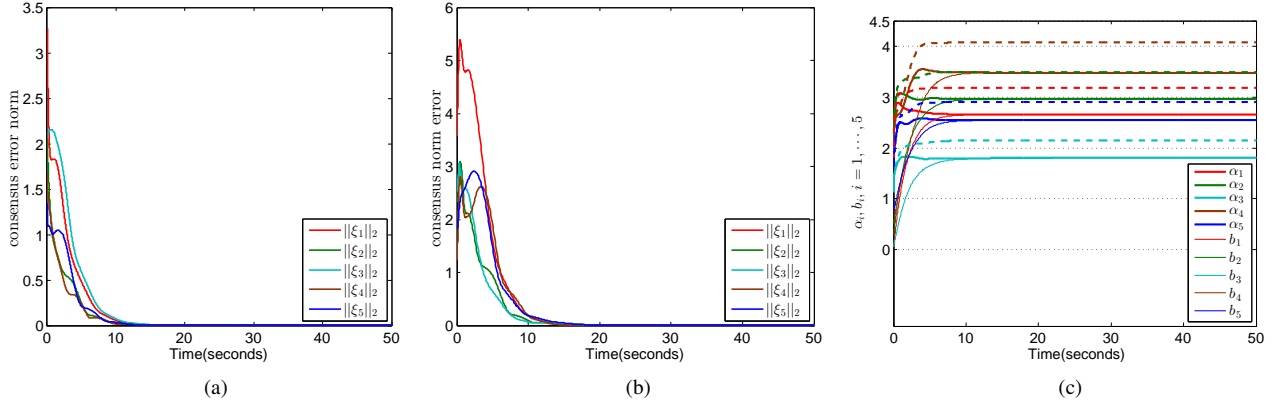
Fig. 2. (a) Case 1: Norm of consensus error $\xi_i, i = 1, \cdots, 5$, (b) Case 2: Norm of consensus error $\xi_i, i = 1, \cdots, 5$, (c) Case 2: The adaptive coupling α_i and filter state b_i.

required. Instead, a new adaptive coupling α_i is added. The initial conditions for the adaptive coupling α_i and the filter b_i are chosen randomly within $[3, 10]$ and $[0, 2]$, respectively, which can guarantee that the initial states satisfy the relation $\alpha_i(t_0) > b_i(t_0), i = 1, \cdots, 5$. For the space limitation, only the norm of consensus errors are shown in Fig.2b, from which it can be seen that consensus tracking is achieved for case 2 with adaptive coupling in the state observer.

In Fig. 2c, the solid thick lines denote the adaptive coupling $\alpha_i, i = 1, \cdots, 5$, and solid thin lines denote the filter state $b_i, i = 1, \cdots, 5$. From the Fig.2c, it can be seen that $\alpha_i, b_i, i = 1, \cdots, 5$ converge to some constant values asymptotically, and $\alpha_i \to b_i, i = 1, \cdots, 5$. Moreover, the dashed lines denote the adaptive coupling gain without the low-pass filter, indicating that the low-pass filter has reduced the learn rate of the coupling gain to make the gains smaller.

VI. CONCLUSION

This paper has addressed the distributed consensus disturbance rejection problem for general linear multi-agent systems with the directed communication topology. Two observer-based consensus protocols are designed with static-coupling state observer and adaptive state estimator, respectively, and the result with the adaptive estimator has been extended to the containment control. The consensus protocol with adaptive state estimator has separated the adaptive coupling gain from the controller to the observer, which isolates the possible burst of the adaptive coupling gain from the control input. In addition, the adaptive consensus protocol is designed in a fully distributed format in the sense that the parameters in the protocol are independent of the global information, such as the Laplacian matrix associated with the whole communication network. Finally, an example has demonstrated the effectiveness of the theoretical results.

REFERENCES

[1] Y. Kuriki and T. Namerikawa, "Consensus-based cooperative formation control with collision avoidance for a multi-UAV system," in *Proc. IEEE Conf. American Control*, Portland, Oregon, USA, 2014, pp. 2077-2082.

[2] W. Yu, G. Chen, Z. Wang, and W. Yang, "Distributed consensus filtering in sensor networks," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 39, no. 6, pp. 1568-1577, Dec. 2009.

[3] W. Dong, "Flocking of multiple mobile robots based on backstepping," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 41, no. 2, pp. 414-424, Apr. 2011.

[4] Y. Cao, "Consensus of multi-agent systems with state constraints: a unified view of opinion dynamics and containment control," in *Proc. IEEE Conf. American Control*, Chicago, IL, USA, 2015, pp. 1439-1444.

[5] M. H. DeGroot, "Reaching a consensus," *J. Amer. Statist. Assoc.*, vol. 69, no. 345, pp. 118-121, 1974.

[6] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520-1533, Sept. 2004.

[7] W. Zhong, G. Liu, and C. Thomas, "Global bounded consensus of multiagent systems with nonidentical nodes and time delays," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 42, no. 5, pp. 1480-1488, Oct. 2012.

[8] C. Wang, Z. Zuo, Z. Lin, and Z. Ding, "A truncated prediction approach to consensus control of lipschitz nonlinear multi-agent systems with input delay," *IEEE Trans. Control Netw. Syst.*, DOI: 10.1109/TCNS.2016.2545860, Mar. 2016.

[9] C. Wang, Z. Zuo, Z. Lin, and Z. Ding, "Consensus Control of a Class of Lipschitz Nonlinear Systems With Input Delay," *IEEE Trans. Circuits Syst. I Regul. Pap.*, vol. 62, no. 11, pp. 2730-2738, Oct. 2015.

[10] S. Li, and X. Wang, "Finite-time consensus and collision avoidance control algorithms for multiple AUVs," *Automatica*, vol. 49, no. 11, pp. 3359-3367, Nov. 2013.

[11] Y. Zhao, Z. Duan, G. Wen, and G. Chen, "Distributed finite-time tracking of multiple non-identical second-order nonlinear systems with settling time estimation," *Automatica*, vol. 64, pp. 86-93, Feb. 2016.

[12] A. Ajorlou, A. Momeni, and A. G. Aghdam, "A class of bounded distributed control strategies for connectivity preservation in multi-agent systems," *IEEE Trans. Autom.*

Control, vol. 55, no. 12, pp. 2828-2833, Sept. 2010.

[13] Z. Chen, M. Fan, and H. Zhang, "How Much Control is Enough for Network Connectivity Preservation and Collision Avoidance?," *IEEE Trans. Cybern.*, vol. 45, no. 8, pp. 1647-1656, Aug. 2015.

[14] A. Buscarino, L. V. Gambuzza, M. Porfiri, L. Fortuna, and M. Frasca, "Robustness to noise in synchronization of complex networks," *Scientific reports*, Jun. 2013.

[15] P. DeLellis, M. Di Bernardo, and D. Liuzza, "Convergence and synchronization in heterogeneous networks of smooth and piecewise smooth systems," *Automatica*, vol. 56, pp. 1-11, Jun. 2015.

[16] Z. Meng, Z. Lin, and W. Ren, "Robust cooperative tracking for multiple non-identical second-order nonlinear systems," *Automatica*, vol. 49, no. 8, pp. 2363-2372, Aug. 2013.

[17] W. Yu, G. Chen, W. Ren, J. Kurths, and W. Zheng, "Distributed higher order consensus protocols in multiagent dynamical systems," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 8, pp. 1924-1932, Aug. 2011.

[18] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 57, no. 1, pp. 213-224, Jan. 2010.

[19] G. Wen, Y. Zhao, Z. Duan, W. Yu, and G. Chen, "Containment of higher-order multi-leader multiagent," *IEEE Trans. Autom. Control*, Aug. 2015. DOI: 10.1109/TAC.2015.2465071.

[20] J. Sun and Z. Geng, "Adaptive output feedback consensus tracking for linear multi-agent systems with unknown dynamics," *Int. J. Control*, vol. 88, no. 9, pp. 1735-1745, Mar. 2015.

[21] Z. Ding, "Adaptive consensus output regulation of a class of nonlinear systems with unknown high-frequency gain," *Automatica*, vol. 51, pp. 348-355, Jan. 2015.

[22] T. Yang, Z. Meng, G. Shi, Y. Hong, and K. Johansson, "Network Synchronization with Nonlinear Dynamics and Switching Interactions," *IEEE Trans. Autom. Control*, Nov. 2015. DOI: 10.1109/TAC.2015.2497907.

[23] Y. Cao and W. Ren, "Distributed coordinated tracking with reduced interaction via a variable structure approach," *IEEE Trans. Autom. Control*, vol. 57, no. 1, pp. 33-48, Apr. 2012.

[24] S. Su, Z. Lin, and A. Garcia, "Distributed Synchronization Control of Multiagent Systems With Unknown Nonlinearities," *IEEE Trans. Cybern.*, Feb. 2015, DOI: 10.1109/TCYB.2015.2402192.

[25] H. Zhang, F. L. Lewis, and Z. Qu, "Optimal design for synchronization of cooperative systems: state feedback, observer and output feedback," *IEEE Trans. Autom. Control*, vol. 56, no. 8, pp. 1948-1952, Apr. 2011.

[26] J. Seo, H. Shim, and J. Back, "Consensus of high-order linear systems using dynamic output feedback compensator: low gain approach," *Automatica* vol. 45, no. 11, pp. 2659-2664, Nov. 2009.

[27] Z. Li, X. Liu, W. Ren, and L. Xie, "Distributed tracking control for linear multiagent systems with a leader of bounded unknown input," *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 518-523, Feb. 2013.

[28] W. Yu, W. Ren, W. Zheng, G. Chen, and J. Lu, "Distributed control gains design for consensus in multiagent systems with second-order nonlinear dynamics," *Automatica*, vol. 49, no. 7, pp. 2107-2115, Jul. 2013.

[29] J. Sun and Z. Geng, "Adaptive consensus tracking for linear multi-agent systems with heterogeneous unknown nonlinear dynamics," *Int. J. Robust. Nonlin. Control*, vol. 26, no. 1, pp. 154-173, Jan. 2015.

[30] Z. Li, G. Wen, Z. Duan, and W. Ren, "Designing fully distributed consensus protocols for linear multiagent systems with directed graphs," *IEEE Trans. Autom. Control*, vol. 60, no. 4, pp. 1152-1157, Apr. 2015.

[31] Z. Ding, "Global stabilization and disturbance suppression of a class of nonlinear systems with uncertain internal model," *Automatica*, vol. 39, no. 3, pp. 471-479, Mar. 2003.

[32] Z. Ding, "Asymptotic rejection of general periodic disturbances in output-feedback nonlinear systems," *IEEE Trans. Autom. Control*, vol. 51, no. 2, pp. 303-308, Feb. 2006.

[33] Z. Li, X. Liu, M. Fu, and L. Xie, "Global H_∞ consensus of multi-agent systems with Lipschitz non-linear dynamics," *IET Control Theory Appl.*, vol. 6, no. 13, pp. 2041-2048, Sept. 2012.

[34] G. Wen, G. Hu, W. Yu, and G. Chen, "Distributed H_∞ Consensus of Higher Order Multiagent Systems With Switching Topologies," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 61, no. 5, pp. 359-363, May 2014.

[35] D. Lombana, and M. Di Bernardo, "Multiplex PI control for consensus in networks of heterogeneous linear agents," *Automatica*, vol. 67, pp. 310-320, May 2016.

[36] L. Guo and W. Chen, "Disturbance attenuation and rejection for systems with nonlinearity via DOBC approach," *Int. J. Robust. Nonlin. Control*, vol. 15, no. 3, pp. 109-125, Dec. 2005.

[37] S. Li, J. Yang, W. Chen, and X. Chen, *Disturbance Observer-Based Control: Methods and Applications*. London, U.K.: CRC press, 2014.

[38] W. Chen, J. Yang, L. Guo, and S. Li, "Disturbance observer-based control and related methods: an overview," *IEEE Trans. Ind. Electron.*, Sept. 2015, DOI: 10.1109/TIE.2015.2478397.

[39] H. Yang, Z. Zhang, and S. Zhang, "Consensus of second-order multi-agent systems with exogenous disturbances," *Int. J. Robust. Nonlin. Control*, vol. 21, no. 9, pp. 945-956, Aug. 2011.

[40] Z. Ding, "Consensus Disturbance Rejection With Disturbance Observers," *IEEE Trans. Ind. Electron.*, vol. 62, no. 9, pp. 5829-5837, Sept. 2015.

[41] T. Yucelen and W. M. Haddad, "Low-frequency learning and fast adaptation in model reference adaptive control," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 1080-1085, Apr. 2013.

[42] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5,
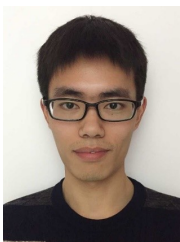
pp. 655-661, May 2005.

[43] A. Isidori, *Nonlinear Control Systems*. Berlin, Germany: Springer-Verlag, 1995.

[44] Y. Lv, Z. Li, Z. Duan, and J. Chen, "Fully Distributed Adaptive Output Feedback Protocols for Linear Multi-Agent Systems with Directed Graphs: A Sequential Observer Design Approach," Nov. 2015, arXiv preprint arXiv:1511.01297.

[45] W. Yu, G. Chen, and J. Lü, "On pinning synchronization of complex dynamical networks," *Automatica*, vol. 45, no. 2, pp. 429-435, 2009.

[46] P. DeLellis, M. di Bernardo, and M. Porfiri, "Pinning control of complex networks via edge snapping," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 21, no.3, 2011.

[47] Y. Cao, W. Ren, and M. Egerstedt, "Distributed containment control with multiple stationary or dynamic leaders in fixed and switching directed networks," *Automatica*, vol. 48, no. 8, pp. 1586-1597, Aug. 2012.

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