

Distributed Detection by a Large Team of Sensors in Tandem ^{1, 2}

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Abstract. The problem of decentralized binary hypothesis testing by a team consisting of N decision makers (DMs) in tandem is considered. Each DM receives an observation and transmits a binary message to its successor; the last DM has to decide which hypothesis is true. Necessary and sufficient conditions are derived for the probability of error to asymptotically (as $N \rightarrow \infty$) go to zero. The result is generalized for multiple hypotheses and multiple messages. An easily implementable suboptimal decision scheme is also considered; necessary and sufficient conditions for the probability of error to asymptotically go to zero are derived for this case as well. The trade off between the complexity of the decision rules and their performance is examined and numerical results are presented.

¹ This research was supported by the National Science Foundation under grant NSF/IRI-8902755 (under a subcontract from the University of Connecticut) and by the Office of Naval Research under grant ONR/N00014-84-K-0519.

² This paper has been submitted for publication in the IEEE Transactions on Aerospace and Electronic Systems.

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1. INTRODUCTION AND MOTIVATION

Despite the considerable research interest on distributed decision making in a hypothesis testing environment, almost all the results focus on small teams. The explosive combinatorial complexity of the problems in this framework suggests that only simple and very restricted architectures of large organizations can be analyzed. The infinite parallel organization was considered in [T88]. In this paper the infinite tandem architecture, the other limit case, is analyzed.

There exist two main reasons which motivated our research. First, it is desirable to determine under what conditions the probability of error of the infinite tandem team (organization) goes to zero. Then, it is also interesting to examine the behavior of the members of a team when a change in the architecture of the team occurs. It is known that if a new DM is added to a team, *all* the DMs have to modify their decision rules to adapt to the new architecture; this requires that the problem be solved again from the beginning for the new architecture. For this reason, we introduce a suboptimal decision scheme which is considerably more simple to implement, under which, each DM tries *selfishly* to maximize the performance of its own personal decision instead of the global team performance. One can make an argument that this decision scheme is more descriptive of human organizations than the optimal decision scheme, since human DMs are reluctant to sacrifice their individual performance for the good of the organization.

The suboptimal decision scheme can be easily adapted to account for changes in the team architecture; it will not yield optimal performance, but will provide a descriptive flavor of the team dynamics. Moreover, it provides an upper bound on the optimal team probability of error, which is also bounded from below by the probability of error of the centralized team. For all the above reasons, the proposed suboptimal decision scheme is extensively analyzed; conditions for the probability of error of the infinite team to go to zero, when this scheme is implemented, are derived and its performance is compared to the optimal team performance.

The sequential (tandem) decentralized problem is defined as follows. As shown in Figure 1, the team consists of N DMs and there are M hypotheses H_0, \dots, H_{M-1} with known prior probabilities $P(H_i) > 0$. Let y_n the observation of the n th DM, $n = 1, \dots, N$, be a random variable taking values from a set Y . The y_n 's are conditionally independent and identically distributed given any hypothesis, with a known probability distribution $P(y | H_i)$, $i = 1, \dots, M$. Let D be a positive integer. The first DM evaluates a D -valued message u_1 based on its own observation and transmits it to its successor; that is, $u_1 = \gamma_1(y_1)$, where $\gamma_1: Y \rightarrow \{1, \dots, D\}$ is a measurable function. Each subsequent DM evaluates a D -valued message u_n based on its own observation y_n and on the message u_{n-1} from its predecessor and transmits it to its successor; that is, $u_n = \gamma_n(y_n, u_{n-1})$ where the measurable function $\gamma_n: Y \times \{1, \dots, D\} \rightarrow \{1, \dots, D\}$ is called the *decision rule* of DM n . The decision u_N of the final DM N is the team decision and declares one of the hypotheses to be true.

The objective is to choose the decision rules $\gamma_1, \dots, \gamma_N$ which minimize the probability of error of the decision of DM N .

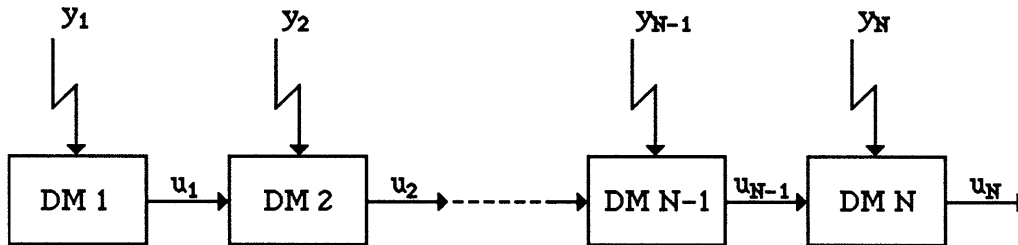


Figure 1. The Team Consisting of N DMs in Tandem

Many results have focused on similar problems with small teams and small number of messages [ET82], [TV87], [R87], [TP89]. As mentioned above, in [T88] asymptotic results were obtained for the parallel team, where all DMs transmit a message to a fusion center which makes the final team decision. We can easily obtain that, given the conditional independence assumption for the observations, the optimal decision rule of each DM (except the first one) is given in the form of likelihood ratios with $DM(M-1)/2$ constant thresholds. These thresholds can be obtained by examining all the solutions of a set of coupled algebraic equations which are usually very hard to solve. Comparison with our results indicates that the parallel architecture asymptotically performs better than the tandem architecture, contradicting the claims in [VT88] where each sensor employed the Neyman-Pearson test. Note also that if the conditional independence assumption fails, the optimal decision rules do not have to be given by likelihood ratio tests and the problems are computationally intractable (NP-hard) even for small values of N and D [TA85].

For the parallel team it is known that the probability of error goes very quickly to zero, for any logical choice of decision rules. We would like to determine whether the probability of error for the tandem team goes asymptotically to zero. In the next section we obtain necessary and sufficient conditions for this for the case of binary hypotheses ($M = 2$) and binary messages ($D = 2$) and in the following section we generalize the result. In section 4 we introduce the suboptimal "selfish" decision scheme which is very easily computable and derive necessary and sufficient conditions for the probability of error to asymptotically go to zero. Finally, in section 5 we compare the performances of the optimal and suboptimal decision schemes and try to establish the trade off between the performance and the computational complexity of the decision rules.

Notation. $\lfloor x \rfloor$ denotes the integer part of x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x .

2. THE OPTIMAL INFINITE TANDEM TEAM

We consider the case of binary hypothesis testing ($M = 2$) and binary communication messages⁴ ($D = 2$). In this case each DM or team of DMs can be expressed by its *Receiver Operating Characteristic* (ROC) curve; this curve usually given in the form of two parametric equations is a representation of the *probability of detection* $P_D (= P(u = 1 | H_1))$ of the DM or team of DMs as a function of the *probability of false alarm* $P_F (= P(u = 1 | H_0))$ and has many interesting properties [V68]. The probability of error of a tandem team consisting of N DMs is defined as follows:

$$P^e(N) = P(H_0) P_F + P(H_1) (1 - P_D) \quad (1)$$

Consider the team which consists of N DMs in tandem and performs binary hypothesis testing. Given some prior probabilities (hence also some $\eta = P(H_0)/P(H_1)$), we generalize the results of [E82], as in [R87], and obtain the *necessary* conditions for the optimal decision rules of the DMs as a set of likelihood ratio tests with constant thresholds:

$$\Lambda(y_1) \underset{u_1=0}{\overset{u_1=1}{\gtrless}} \frac{P(u_N=1 | u_1=1, H_0) - P(u_N=1 | u_1=0, H_0)}{P(u_N=1 | u_1=1, H_1) - P(u_N=1 | u_1=0, H_1)} \eta \equiv \eta_1 \quad (2)$$

$$\Lambda(y_n) \underset{u_n=0}{\overset{u_n=1}{\gtrless}} \frac{P(u_N=1 | u_n=1, H_0) - P(u_N=1 | u_n=0, H_0)}{P(u_N=1 | u_n=1, H_1) - P(u_N=1 | u_n=0, H_1)} \frac{P(u_{n-1}=i | H_0)}{P(u_{n-1}=i | H_1)} \eta \equiv \eta_n^i$$

for $n = 2, 3, \dots, N-1$ and $i = 0, 1$ (3)

and:

$$\Lambda(y_N) \underset{u_N=0}{\overset{u_N=1}{\gtrless}} \frac{P(u_{N-1}=i | H_0)}{P(u_{N-1}=i | H_1)} \eta \equiv \eta_N^i \quad \text{for } i = 0, 1 \quad (4)$$

⁴ The two messages which each DM is able to transmit can be denoted m_1 and m_2 . Without loss of generality assume that:

$$\frac{P(u_n=m_1 | H_0)}{P(u_n=m_1 | H_1)} \geq \frac{P(u_n=m_2 | H_0)}{P(u_n=m_2 | H_1)}$$

Suppose that DM n receives an observation y_n ; if DM n 's optimal decision is $u_n=m_i$ when he receives $u_{n-1}=m_j$, then it can be shown that his optimal decision is again $u_n=m_i$ when he receives $u_{n-1}=m_i$ (for $n = 2, 3, \dots, N$). Moreover, if DM N 's optimal decision is $u_N=1$ (0) when he receives $u_{N-1}=m_1$ (m_2), then it can be shown that his optimal decision is again $u_N=1$ (0) when he receives $u_{N-1}=m_2$ (m_1). Then backtracking it can be seen that m_1 is interpreted as 0 and m_2 is interpreted as 1.

We will now demonstrate that the optimal probability of error of the infinite tandem team does not necessarily have to be zero. For this, we will present a special ROC curve for the individual DM and we will calculate inductively the ROC curve of the team consisting of N such DMs in tandem; this in general can not be done analytically. We will then take the limit of the team ROC curve as $N \rightarrow \infty$ and show that the team probability of error is bounded away from zero.

Therefore, consider the DM whose ROC curve is presented in Figure 2, for some $\alpha > 1$ (the underlying probability distributions are presented in Table 1). An example of a DM with such a ROC curve is a person, who observes the outcomes of the tosses of a coin known to be α to 1 biased in favor of the most likely outcome, and tries to decide whether the coin is biased in favor of 'heads' or in favor of 'tails'. We start by obtaining the ROC curve of a team which consists of N such DMs.

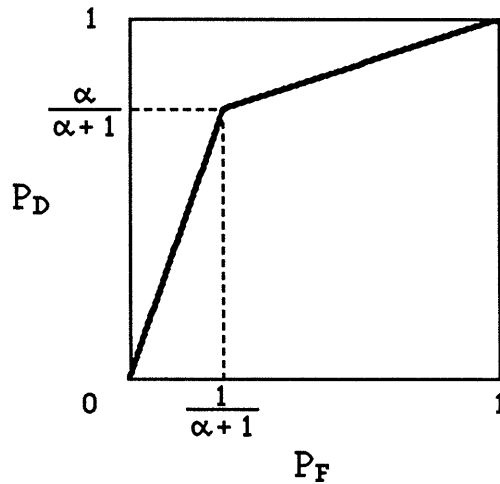


Figure 2. A Special ROC Curve

LEMMA 1. Consider the tandem team which consists of $N \geq 2$ identical DMs; suppose that the ROC curve of each DM is given in Figure 2. Then the team ROC curve consists of the following $2N$ points:

$$(P_F^0, P_D^0) = (0,0) \tag{5a}$$

$$(P_F^{2j-1}, P_D^{2j-1}) = \left[\left(\frac{1}{\alpha+1} \right)^{N+2-2j} \sum_{i=0}^{j-1} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i, \left(\frac{\alpha}{\alpha+1} \right)^{N+2-2j} \sum_{i=0}^{j-1} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right] \tag{5b}$$

for $j = 1, 2, \dots, \lfloor \frac{N}{2} \rfloor$

y H	0	1
H ₀	$\frac{\alpha}{\alpha+1}$	$\frac{1}{\alpha+1}$
H ₁	$\frac{1}{\alpha+1}$	$\frac{\alpha}{\alpha+1}$

Table 1. Probability Distributions for DM of Figure 2

$$(P_F^{2j}, P_D^{2j}) = \left[\left(\frac{1}{\alpha+1} \right)^{N-1-2j} \left[1 - \frac{\alpha}{\alpha+1} \sum_{i=0}^j \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right], \right. \\ \left. \left(\frac{\alpha}{\alpha+1} \right)^{N-1-2j} \left[1 - \frac{1}{\alpha+1} \sum_{i=0}^j \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right] \right] \\ \text{for } j = 1, 2, \dots, \left\lfloor \frac{N}{2} \right\rfloor - 1 \quad (5c)$$

$$(P_F^{N-1+j}, P_D^{N-1+j}) = (1 - P_D^{N-j}, 1 - P_F^{N-j}) \quad \text{for } j = 1, 2, \dots, N \quad (5d)$$

The proof is by induction on N . We consider the cases of N odd and N even separately and then combine the results to obtain (5). The detailed proof is presented in the appendix. Note that since the ROC curve of the individual DM is symmetric, the team ROC curve is also symmetric.

We use Lemma 1 to obtain the ROC curve of the infinite team which consists of identical DMs whose ROC curve is given in Figure 2.

LEMMA 2. *The limit of (P_F^{N-1}, P_D^{N-1}) as N goes to infinity is:*

$$\lim_{N \rightarrow \infty} (P_F^{N-1}, P_D^{N-1}) = \left(\frac{1}{\alpha^2 + \alpha + 1}, \frac{\alpha^2}{\alpha^2 + \alpha + 1} \right) \quad (6)$$

LEMMA 3. *The ROC curve of an infinite tandem team of identical DMs whose individual ROC curve is given in Figure 2 consists of the following points:*

$$\left(\left(\frac{1}{\alpha+1} \right)^i \frac{1}{\alpha^2 + \alpha + 1}, \left(\frac{\alpha}{\alpha+1} \right)^i \frac{\alpha^2}{\alpha^2 + \alpha + 1} \right) \quad (7a)$$

$$\left(1 - \left(\frac{\alpha}{\alpha+1} \right)^i \frac{\alpha^2}{\alpha^2 + \alpha + 1}, 1 - \left(\frac{1}{\alpha+1} \right)^i \frac{1}{\alpha^2 + \alpha + 1} \right) \quad (7b)$$

for $i = 0, 1, 2, \dots$

To prove Lemmas 2 and 3, we consider the cases of N odd and even, take the limits and show that they are equal. The detailed proofs are presented in the appendix. From above the Lemmas, we derive the following two interesting corollaries.

COROLLARY 1. *Consider the infinite tandem team of identical DMs whose ROC curve is given in Figure 2. Suppose that the prior probabilities of both hypotheses are equal. Then the optimal probability of error for the team is:*

$$P^{e(\infty)} = \frac{0.5\alpha + 1}{\alpha^2 + \alpha + 1} \quad (8)$$

Proof. From basic detection theory [V68] it is known that the optimal operating point is the point(s) of the ROC curve at which the slope of the tangent to the ROC curve is $\eta = P(H_0)/P(H_1) = 1$. If the ROC curve is not differentiable (and thus a tangent to it does not exist everywhere) the optimal operating point is the point(s) of the curve which first intersects a line of slope η moving down the P_D axis. From Lemma 3 we can see that any point in the line segment between (P_F^{N-1}, P_D^{N-1}) and (P_F^N, P_D^N) is such a point. Then, substitute for the limit of (P_F^{N-1}, P_D^{N-1}) from (3) into (1), the definition of the probability of error, and the corollary is proved. **Q.E.D.**

REMARK: If $\eta = 1$, the above infinite tandem team improves on the performance of an *individual* DM by a factor of:

$$\frac{P^{e(\infty)}}{P^e(1)} = \frac{(0.5\alpha + 1)(\alpha + 1)}{\alpha^2 + \alpha + 1}$$

Thus the probability of error of the infinite tandem team will be between 50% and 100% of the probability of error of an individual DM. Furthermore as $|\log(\eta)|$ increases the improvement on the performance is even less significant. These facts should suggest the *inefficiencies* of the tandem

architecture. As the number of DMs in an organization increases the team should expand more in a parallel than in tandem since the contribution of DMs which are positioned at the low levels in the chain of command seems to be insignificant.

CORROLARY 2. *The probability of error of an infinite tandem team does not have to be zero.*

Since the probability of error of an infinite tandem team could be bounded away from zero, we present necessary and sufficient conditions on the ROC curve of the individual DM for this to happen.

PROPOSITION 1. *Consider a tandem team which consists of N identical DMs and performs binary hypothesis testing. Then, as $N \rightarrow \infty$, the team will achieve zero probability of error w.p.1, for any prior probabilities, if and only if either the initial slope of the ROC curve of the individual DM is infinite or its final slope is zero.*

Proof. To prove the *only if* part, suppose that the initial slope of the ROC curve of the individual DM is $m_0 < \infty$ and that its final slope is $m_1 > 0$. Denote: $\alpha = \max\{m_0, 1/m_1\}$. Then because of the concavity of the ROC curve, each DM of the team is *worse* than a DM having a two piece-wise ROC curve whose first piece has slope α and whose second piece has slope $1/\alpha$ (Figure 2). By the Lemma 3, an infinite tandem team consisting of such (better) DMs will have a non-zero probability of error. Therefore the infinite tandem team consisting of the original (worse) DMs will also have non-zero probability of error.

To prove the *if* part, suppose without loss of generality (since the names of the two hypotheses can be interchanged) that $m_0 = \infty$. The proof consists of proposing decision rules (not necessarily optimal) for the DMs of the team which will result in a zero asymptotic probability of error; that is, it will be shown that given any δ such that $1 > \delta > 0$, if the proposed decision rules are employed, the probability of error can be made less than δ as $N \rightarrow \infty$. The detailed proof is presented in the appendix. **Q.E.D.**

This is a useful result because it offers a convenient test involving only the individual team member, in order to determine whether the infinite team probability of error is bounded away from zero or not. Moreover, after the proof was completed, it was communicated to us by J.N. Tsitsiklis that similar results had been established in the context of automata with finite memory [C69], [HC70]. There, a finite Markov chain was considered, with the additional restriction that the transition probabilities be time invariant; each state of this chain corresponded to a discrete hypothesis and conditions for the convergence of the chain to a unique (true) state were obtained.

3. INTERPRETATIONS AND GENERALIZATIONS

Let $\Lambda(\cdot)$ denote the likelihood ratio of $y \in Y$ the observation of each individual DM and $\Lambda(Y)$ denote the range of its possible values. Then, Proposition 1 can also be stated in the following equivalent form:

PROPOSITION 1a. *The infinite tandem team probability of error is bounded away from zero, for any prior probability, if and only if there exists some $B > 0$ such that:*

$$|\log[\Lambda(Y)]| < B .$$

We would like to elaborate on these necessary and sufficient conditions. For this consider a single DM who receives measurements serially and who has limited memory so that it can only recall the last one received. Then the conditions imply that if this DM is willing to wait long enough before making its decision, it can achieve any desirable level of performance. On the other hand, if the conditions do not hold the likelihood ratio of an individual DM is bounded. This implies that there exists some DM (N^*+1) of the team whose measurement does not influence its decision, because the decision u_{N^*} is so reliable that it is propagated as u_{N^*+1} regardless of y_{N^*+1} ; since all the DMs are identical, u_{N^*} will be more reliable than the measurement of each subsequent DM and, will eventually become the final team decision as well. Thus only the measurements of the first N^* DMs influence the team decision and since a finite number of measurements results in a non-zero probability of error even in the centralized case, the optimal team probability of error will be bounded away from zero.

It is worthwhile to compare the conditions of Proposition 1 to the conditions in [T88] for the DMs of the infinite parallel team to have identical optimal decision rules. The conditions are "opposite" in the sense that the conditions in [T88] require that a single DM is not able to make a perfectly accurate decision, while our conditions require that a single DM is able to make a decision satisfying any given level of accuracy.

Consider the infinite tandem team; suppose that it performs M -ary hypothesis testing and that the communication messages between DMs are also M -ary. Then define the following likelihood ratios:

$$\Lambda_{ij}(y) \equiv \frac{P(y | H_i)}{P(y | H_j)} \quad \text{for } i, j = 0, 1, \dots, M-1 \text{ and } i > j.$$

Let $\Lambda_{ij}(Y)$ denote the range of the possible values of $\Lambda_{ij}(\cdot)$. The generalization of Proposition 1 for the case of M -ary hypotheses is:

PROPOSITION 2. *Consider the infinite tandem team which performs M -ary hypothesis testing and employs M -ary messages. The team probability of error is bounded away from zero, for any prior probabilities, if and only if there exists some $B_{ij} > 0$ such that:*

$$|\log[\Lambda_{ij}(y)]| < B_{ij}$$

for some $i, j = 0, 1, \dots, M-1$ and $i > j$.

It is easy to find an example for which the probability of error is bounded away from zero if the conditions are violated and the binary proof can be adapted and simulated $M(M-1)/2$ times to show that the conditions are also sufficient.

In [HC70] an ingenious scheme was presented (and generalized in [K75]) demonstrating that if the Markov chain is constructed with one more node than the number of the discrete hypotheses, then there exist sets of transition probabilities that will enable the chain to converge to a unique (true) state. This idea could be appropriately adapted to our hypothesis testing framework so that decision rules for the DMs be derived for the infinite tandem team to *always* achieve zero probability of error if the number of the communication messages between DMs is one higher than the number of the hypotheses.

Our results are true if the probability of error is generalized by a cost function which assigns different costs for hypothesis misclassification. They can also be extended to teams which have more general acyclic (tree) architectures, as long as the team does not include a DM who receives an infinite number of messages from other DMs and to teams whose DMs are not identical. The proofs are variants of the proof of Proposition 1.

4. THE SUBOPTIMAL INFINITE TANDEM TEAM

The non-linear coupled equations of (2)-(4) are very hard to solve; their solution does not have to be unique and their computational complexity increases exponentially with N . Moreover, as was discussed in the previous section and will be verified with the computer simulations of section 5, when the conditions of Proposition 1 hold, the convergence of the optimal team probability of error to zero is slow.

REMARK. This is not true for the cases where the conditions of Proposition 1 do not hold. In those cases the convergence to the optimal probability of error occurs geometrically fast at a rate of $\alpha/(\alpha+1)^2$. If $\eta = 1$, the difference in probability of error between a team of $N = 2k + 1$ DMs and the (asymptotically optimal) infinite team is:

$$\frac{1}{(\alpha^2 + \alpha + 1)(\alpha + 1)} \left[\frac{\alpha}{(\alpha + 1)^2} \right]^k \quad \text{for } k = 0, 1, 2, \dots$$

For the above reasons it is worthwhile to suggest the following suboptimal decision scheme which has two important properties: its computational requirements increase *linearly* with N and, under slightly more restrictive conditions than those of Proposition 1, the probability of error goes to zero as $N \rightarrow \infty$.

Under the suboptimal decision scheme each DM tries to minimize the probability of error of its personal decision; that is it acts as if it was the last DM in the team. Therefore, this is a *selfish* decision rule since each DM effectively ignores all of its successors in the team and instead of optimizing the global team decision, each DM tries to "optimize" the performance (i.e., minimize the probability of error) of its own decision. Note that this suboptimal decision scheme can be implemented easily and efficiently even if the DMs of the team are not identical. Formally, the suboptimal decision rules for any DMs are defined as the following likelihood ratio tests with constant thresholds⁵:

$$\Lambda(y_1) \underset{u_1=0}{\overset{u_1=1}{\gtrless}} \eta \equiv \mu_1 \quad (9)$$

$$\Lambda(y_n) \underset{u_n=0}{\overset{u_n=1}{\gtrless}} \frac{P(u_{n-1}=i | H_0)}{P(u_{n-1}=i | H_1)} \eta \equiv \mu_n^i \quad (10)$$

for $n = 2, \dots, N$ and $i = 0, 1$

From now on we refer to the decision rules just introduced as the "suboptimal decision rules" and to the team which employs them the "suboptimal team."

There exists one more attractive attribute of this decision scheme which is in fact the primary reason we analyzed it. One of the more intriguing questions of organizational design deals with the response of the team to a change in the team architecture. How does the team react to the addition or deletion of a DM? If a change occurs in the team when the optimal decision scheme is implemented, the team can not make use of its current status to facilitate the computation of the new optimal decision rules; the problem needs to be solved again from the beginning. This is not true under the suboptimal decision rule, where it is relatively easy for the team to adapt to such a change. The reason for this is that under the suboptimal decision rule, the DMs behave less than

⁵ Comparing the optimal decision rules of (6)-(8) to the suboptimal decision rules of (9) and (10), notice that the suboptimal ones are *decoupled*.

team members and more like individuals; still the analysis will provide some insight and bounds on the optimal team performance. The main result of this section follows.

PROPOSITION 3. *Consider a tandem team which consists of N identical DMs and performs binary hypothesis testing; suppose that the team employs the suboptimal decision scheme. Then, as $N \rightarrow \infty$, the team will achieve zero probability of error w.p.1, for any prior probabilities, if and only if both the initial slope of the ROC curve of the individual DM is infinite and its final slope is zero.*

REMARK. The necessary and sufficient conditions of Proposition 3 are stricter than the ones of Proposition 1, since they require that *both* the initial slope of the ROC curve be infinite and the final slope be zero, instead of *either* the initial slope be infinite or the final slope be zero.

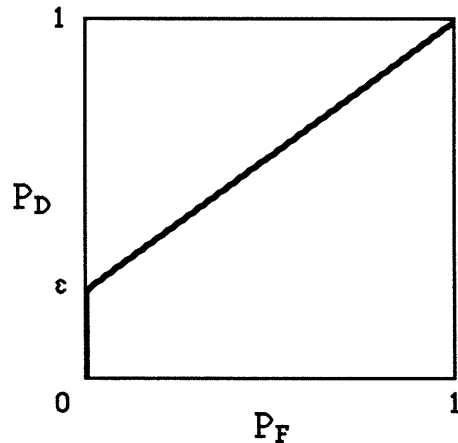


Figure 3. A ROC Curve with Infinite Initial Slope

Proof. To prove the *only if* part recall from Proposition 1 that if the ROC curve of the individual DM has both finite initial slope and non-zero final slope, the asymptotic team probability of error is bounded away from zero even if the optimal decision rules are employed; therefore in that case, it is bounded away from zero when the suboptimal decision rules are employed.

Consider the case where either the initial slope is infinite or the final slope is zero, but not both. We need to show that the asymptotic team probability of error under the suboptimal decision rules is bounded away from zero, even though it goes to zero under the optimal decision rules (as was established in Proposition 1). For this suppose without loss of generality that the ROC curve of the individual DM has infinite initial slope and non-zero final slope; then consider the DM whose ROC curve is presented in Figure 3 along with the underlying probability distributions in Table 2.

It should be obvious that if the prior probabilities are such that $\eta < 1 - \varepsilon$ the optimal decision for every DM is $u = 1$ ⁶; this leads to:

$$P^e(n) = \frac{\eta}{\eta + 1} \quad ; \text{ for } n = 1, 2, 3, \dots$$

Thus also:

$$P^e(\infty) = \frac{\eta}{\eta + 1} > 0$$

The proof of the *only if* part is now complete.

y H	0	1
H ₀	1	0
H ₁	1 - ε	ε

Table 2. Probability Distributions for the DM of Figure 3

The proof of the *if* part consists of proposing decision rules for the DMs of the team which will result in a zero asymptotic probability of error; that is, it will be shown that given any δ such that $1 > \delta > 0$, if the proposed decision rules are employed, the probability of error can be made less than δ as $N \rightarrow \infty$. It will also be shown that the proposed decision rules are indeed the suboptimal decision rules. The detailed proof is presented in the appendix. **Q.E.D.**

⁶It is also easy to see that if $\eta \geq 1 + \varepsilon$ then the suboptimal decision rules coincide with the optimal decision rules and are:

$$u_1(y_1) = \begin{cases} 0 & ; y_1 = 0 \\ 1 & ; y_1 = 1 \end{cases}$$

$$u_n(y_n, u_{n-1}) = \begin{cases} 0 & ; y_n = 0 \text{ and } u_{n-1} = 0 \\ 1 & ; \text{ otherwise} \end{cases}$$

for $n = 2, 3, 4, \dots$

These of course lead to $P^e(\infty) = 0$ (Proposition 1).

Let $\Lambda(\cdot)$ denote the likelihood ratio of y the observation of each individual DM and let $\Lambda(Y)$ denote the range of its possible values. Then, Proposition 3 can also be stated in the following equivalent form:

PROPOSITION 3a. *Consider the infinite tandem team which employs the suboptimal decision rules and performs binary hypothesis testing. The team probability of error is bounded away from zero, for any prior probabilities, if and only if there exists some $B > 0$ such that either:*

$$\log[\Lambda(Y)] < B$$

or:

$$1/B < \log[\Lambda(Y)].$$

Similarly, for the M -ary hypothesis testing case, Proposition 2 can be modified as follows:

PROPOSITION 4. *Consider the infinite tandem team which employs the suboptimal decision rules and performs M -ary hypothesis testing. The team probability of error is bounded away from zero, for any prior probabilities, if and only if there exists some $B_{ij} > 0$ such that either:*

$$\log[\Lambda_{ij}(Y)] < B_{ij}$$

or:

$$1/B_{ij} < \log[\Lambda_{ij}(Y)]$$

for some $i, j = 0, 1, \dots, M-1$ and $i > j$.

It is easy to find an example for which the probability of error is bounded away from zero if the condition is violated and the binary proof may be adapted to show that the conditions are also sufficient. Since under the suboptimal decision scheme each DM tries to minimize his personal probability of error, it does not make sense to assign to each DM a number of messages different from the number of the alternative hypotheses. Our results are true if the probability of error is generalized by a cost function which assigns different costs for hypothesis misclassification. They may be extended to teams with more general acyclic (tree) architectures, as long as the team does not include a DM who receives messages from an infinite number of other DMs. Finally, as was mentioned above, they can also be applied to teams whose DMs are not identical. The proofs are variants of the proof of Proposition 3.

5. NUMERICAL RESULTS

In this section we perform numerical studies of the optimal and the suboptimal decision rules. The probability distributions of the observations are Gaussian with variance $\sigma^2 = 100$; under H_0 the mean is $\mu_0 = 0$ and under H_1 the mean is $\mu_1 = 10$. Note that in this case the initial slope of the ROC curve of the individual DM is $m_0 = \infty$ and that the final slope is $m_1 = 0$; thus the team probability of error should go to zero asymptotically under both the optimal and the suboptimal decision scheme, according to the theoretical results presented above.

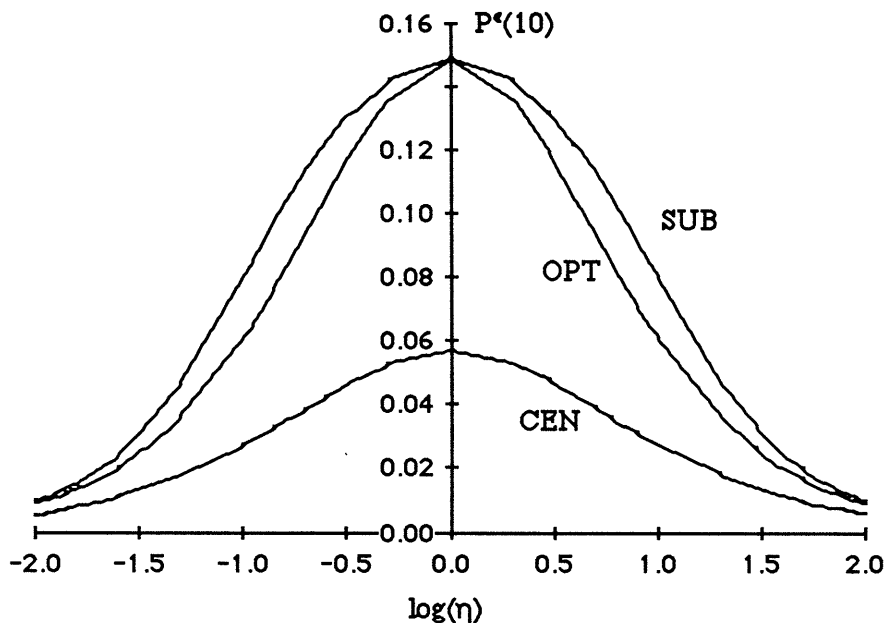


Figure 4. $P^e(10)$: Probability of Error for a Ten DM Team ($N = 10$) vs. η in a log. scale

In Figure 4, we present the probability of error for the centralized case, the optimal case and the suboptimal case for a team consisting of $N = 10$ DMs as a function of the threshold η (namely the prior probabilities). The number $N = 10$ was selected because it was shown in [P88] that it is sufficient for a team to achieve performance very close to the asymptotic one, i.e., as $N \rightarrow \infty$, for a special class of problems. Also, note that for $\eta = 1$ (i.e., equal priors) the optimal and the suboptimal decision rules achieve identical performance; in fact the suboptimal decision rules become optimal in this case. It is interesting to note that as the prior probabilities become very unequal (the minimum prior probability is less than 0.333), the performance of both decision schemes again becomes roughly equal (though never exactly equal).

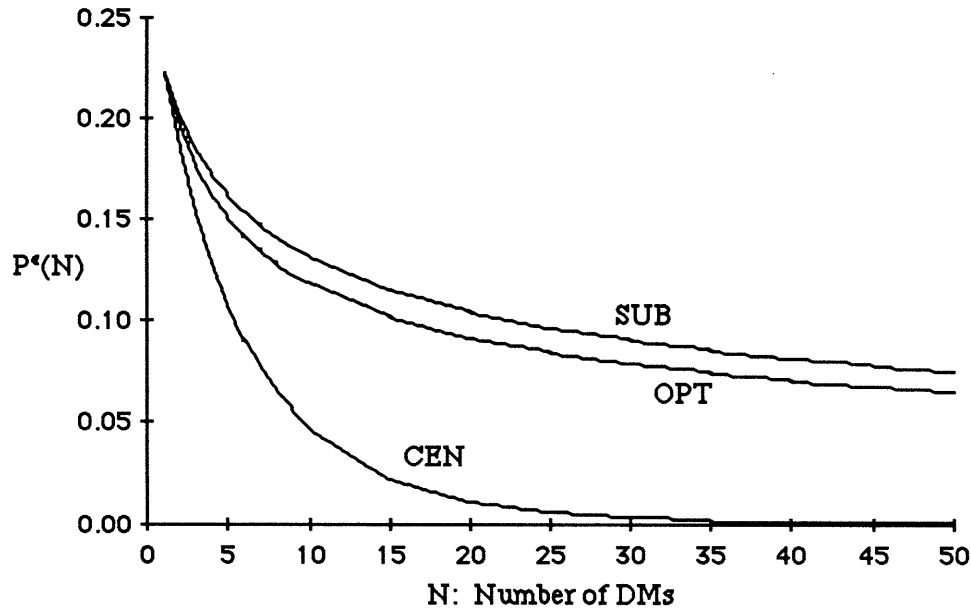


Figure 5. $P^e(N)$: Probability of Error for N DM Team ($\eta = 3$)

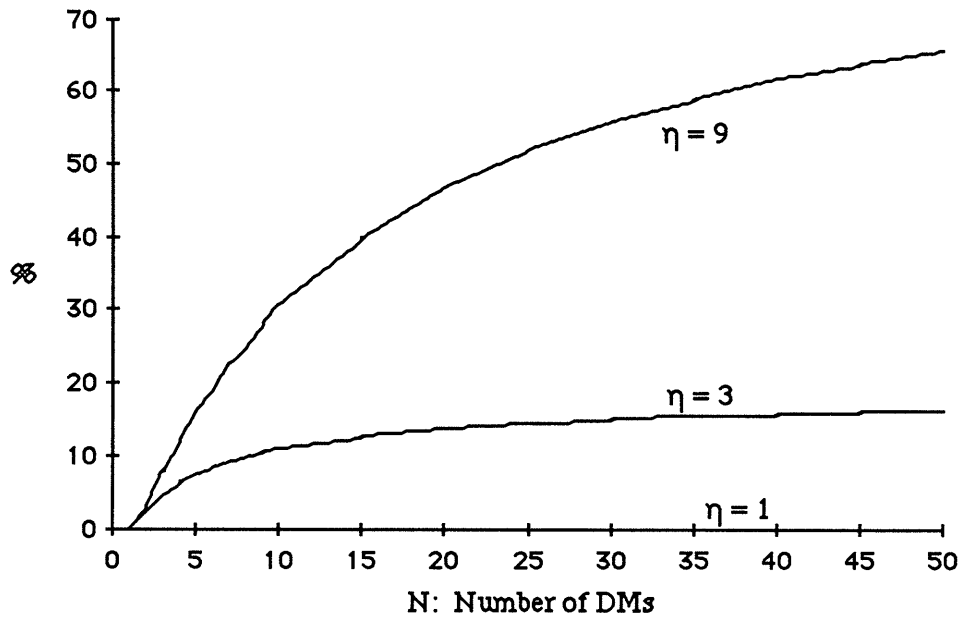


Figure 6. Percentage Deterioration in Performance

In Figure 5, we compare the performance of the centralized case, the optimal case and the suboptimal case for $\eta = 3$ as a function of N , the number of DMs in the team. The probability of error of the centralized case decreases exponentially to zero. On the other hand, under both the

other two decision schemes it decreases slowly to zero. Finally, observe that the percentage deterioration in the team performance when the team changes from the optimal to the suboptimal decision rules reaches a steady-state level as the number of DMs increases. In Figure 6 the percentage deterioration in the team performance is presented for the cases of $\eta = 1$ (where there is no deterioration because the decision rules coincide, as was mentioned above), $\eta = 3$ and $\eta = 9$; as η increases the deterioration in performance between the two decision schemes also increases.

We would like to repeat that the implementation of the suboptimal decision rules is much easier than the one of the optimal decision rules. Each DM of the team needs to know only very little about the rest of the team in order to implement the suboptimal decision rules; for example, it does not need to know anything about its successors. The suboptimal decision rules are much less rigid and can be easily adapted to accommodate for changes in the team; also, for the optimal decision rules the computational requirements grow exponentially with time, while for the suboptimal decision rule they just grow linearly with N . Thus, depending on the particular application of the team and on the available resources, the team designer has to evaluate the trade offs and decide which decision rules should be implemented.

ACKNOWLEDGEMENT

The authors would like to thank Prof. John N. Tsitsiklis for his valuable suggestions.

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APPENDIX

Proof of Lemma 1. By induction on N . It is trivial to confirm that it is true for the case of $N = 2$. To prove the general case first observe that since the problem is symmetric with respect to the two hypotheses, the team ROC curve will be symmetric with respect to the line $P_D = -P_F + 1$. Thus, only (5a), (5b) and (5c) need to be proved. For this consider whether N is odd or even; then (5) can be written accordingly:

CASE A. $N = 2k$ for $k = 1, 2, 3, \dots$

$$(P_F^0, P_D^0) = (0,0) \tag{11a}$$

$$(P_F^{2j-1}, P_D^{2j-1}) = \left[\left(\frac{1}{\alpha+1} \right)^{2k-2(j-1)} \sum_{i=0}^{j-1} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i, \left(\frac{\alpha}{\alpha+1} \right)^{2k-2(j-1)} \sum_{i=0}^{j-1} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right]$$

for $j = 1, 2, \dots, k$ (11b)

$$(P_F^{2j}, P_D^{2j}) = \left[\left(\frac{1}{\alpha+1} \right)^{2k-2j-1} \left[1 - \frac{\alpha}{\alpha+1} \sum_{i=0}^j \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right], \left(\frac{\alpha}{\alpha+1} \right)^{2k-2j-1} \left[1 - \frac{1}{\alpha+1} \sum_{i=0}^j \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right] \right]$$

for $j = 1, 2, \dots, k-1$ (11c)

$$(P_F^{2k-1+j}, P_D^{2k-1+j}) = (1 - P_D^{2k-j}, 1 - P_F^{2k-j})$$

for $j = 1, 2, \dots, 2k$ (11d)

CASE B. $N = 2k + 1$ for $k = 1, 2, 3, \dots$

$$(P_F^0, P_D^0) = (0, 0)$$

(12a)

$$(P_F^{2j-1}, P_D^{2j-1}) = \left[\left(\frac{1}{\alpha+1} \right)^{2k-2j+3} \sum_{i=0}^{j-1} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i, \left(\frac{\alpha}{\alpha+1} \right)^{2k-2j+3} \sum_{i=0}^{j-1} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right]$$

for $j = 1, 2, \dots, k$ (12b)

$$(P_F^{2j}, P_D^{2j}) = \left[\left(\frac{1}{\alpha+1} \right)^{2k-2j} \left[1 - \frac{\alpha}{\alpha+1} \sum_{i=0}^j \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right], \left(\frac{\alpha}{\alpha+1} \right)^{2k-2j} \left[1 - \frac{1}{\alpha+1} \sum_{i=0}^j \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right] \right]$$

for $j = 1, 2, \dots, k$ (12c)

$$(P_F^{2k+j}, P_D^{2k+j}) = (1 - P_D^{2k+1-j}, 1 - P_F^{2k+1-j})$$

for $j = 1, 2, \dots, 2k + 1$ (12d)

Suppose that a team of N identical DMs has a ROC curve defined by (11) ((12)). Then it is a matter of tedious but straightforward algebra to show that adding a new DM to the team will yield a new team whose ROC curve will be given by (12) ((11) respectively).

Consider only the case where $N = 2k$ since the proof for the case where $N = 2k + 1$ is analogous; the ROC curve of the team consists of the (P_F^n, P_D^n) points defined by (11) for $\eta = 0, 1, \dots, 2N - 1$. The new DM $N+1$ has two decision rules to choose from:

$$(i). \quad u_{N+1} = \begin{cases} 1 & ; \quad u_N = 1 \text{ and } y_N = 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$(ii). \quad u_{N+1} = \begin{cases} 0 & ; \quad u_N = 0 \text{ and } y_N = 0 \\ 1 & ; \quad \text{otherwise} \end{cases}$$

Then, if decision rule (i). is employed:

$$(P_F^n, P_D^n) = \left(\frac{1}{\alpha + 1} P_F^n, \frac{\alpha}{\alpha + 1} P_D^n \right) \quad (13)$$

and if decision rule (ii). is employed:

$$(P_F^n, P_D^n) = \left(\frac{\alpha}{\alpha + 1} P_F^n + \frac{1}{\alpha + 1} 1, \frac{1}{\alpha + 1} P_D^n + \frac{\alpha}{\alpha + 1} 1 \right) \quad (14)$$

When decision rule (i). is employed the point (0,0) yields again (0,0), the points of (11b) yield the points of (12b), the points of (11c) yield the first $(k-1)$ points of (12c) and the first point of (11c) (i.e., the point corresponding to $j = 1$) yields the last point of (12c) (i.e., the point corresponding to $j = k$). Similarly, the points of (12d) are obtained by symmetry. Thus all the points of (12) can be obtained by the team of $N+1$ DMs.

It is not hard to verify analytically that all the other points that can be obtained from the points of (11) and the two decision rules will never lie above the ROC curve defined by (12). **Q.E.D.**

Proof of Lemma 2. Again distinguish between N odd and even.

CASE A. $N = 2k$ for $k = 1, 2, 3, \dots$

Note two facts; first that (P_F^{N-1}, P_D^{N-1}) is given by (5b) for $j = k$ and second that instead of taking the limit as N goes to infinity, we can equivalently take the limit as $k = N/2$ goes to infinity. Thus:

$$(P_F^{2k-1}, P_D^{2k-1}) = \left[\left(\frac{1}{\alpha+1} \right)^2 \sum_{i=0}^{k-1} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i, \left(\frac{\alpha}{\alpha+1} \right)^2 \sum_{i=0}^{k-1} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right] \quad (15)$$

and taking the limit as k goes to infinity:

$$\begin{aligned} \lim_{k \rightarrow \infty} (P_F^{2k-1}, P_D^{2k-1}) &= \left[\left(\frac{1}{\alpha+1} \right)^2 \frac{(\alpha+1)^2}{\alpha^2 + \alpha + 1}, \left(\frac{\alpha}{\alpha+1} \right)^2 \frac{(\alpha+1)^2}{\alpha^2 + \alpha + 1} \right] = \\ &= \left(\frac{1}{\alpha^2 + \alpha + 1}, \frac{\alpha^2}{\alpha^2 + \alpha + 1} \right) \end{aligned} \quad (16)$$

CASE B. $N = 2k + 1$ for $k = 1, 2, 3, \dots$

This time (P_F^{N-1}, P_D^{N-1}) is given by (5c) for $j = k$:

$$(P_F^{2k}, P_D^{2k}) = \left[1 - \frac{\alpha}{\alpha+1} \sum_{i=0}^k \left(\frac{\alpha}{(\alpha+1)^2} \right)^i, 1 - \frac{1}{\alpha+1} \sum_{i=0}^k \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right] \quad (17)$$

Taking the limit as k goes to infinity:

$$\begin{aligned} \lim_{k \rightarrow \infty} (P_F^{2k}, P_D^{2k}) &= \left[1 - \frac{\alpha}{\alpha+1} \frac{(\alpha+1)^2}{\alpha^2 + \alpha + 1}, 1 - \frac{1}{\alpha+1} \frac{(\alpha+1)^2}{\alpha^2 + \alpha + 1} \right] = \\ &= \left(\frac{1}{\alpha^2 + \alpha + 1}, \frac{\alpha^2}{\alpha^2 + \alpha + 1} \right) \end{aligned} \quad (18)$$

From (16) and (18), we conclude that the Lemma be true. **Q.E.D.**

Proof of Lemma 3. Since the points of (7b) are symmetric to the points of (7a), it suffices to prove (7a). Denote: $m = -\lfloor \frac{N}{2} \rfloor - 1 + j$. Then the points of (7b) can be written as:

$$\begin{aligned} &\left[\left(\frac{1}{\alpha+1} \right)^{N-2\lfloor \frac{N}{2} \rfloor - 2m} \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor + m} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i, \left(\frac{\alpha}{\alpha+1} \right)^{N-2\lfloor \frac{N}{2} \rfloor - 2m} \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor + m} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right] \\ & \qquad \qquad \qquad m = -1, -2, \dots, -\lfloor \frac{N}{2} \rfloor \end{aligned} \quad (19)$$

Similarly, denote: $m = -\lfloor \frac{N}{2} \rfloor + j$. Then the points of (5c) can be written as:

$$\left[\left(\frac{1}{\alpha+1} \right)^{N-2\lfloor \frac{N}{2} \rfloor - 2m-1} \left[1 - \frac{\alpha}{\alpha+1} \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor + m} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right] \right. \\ \left. \left(\frac{\alpha}{\alpha+1} \right)^{N-2\lfloor \frac{N}{2} \rfloor - 2m-1} \left[1 - \frac{1}{\alpha+1} \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor + m} \left(\frac{\alpha}{(\alpha+1)^2} \right)^i \right] \right] \\ m = -1, -2, \dots, -\lfloor \frac{N}{2} \rfloor + 1 \quad (20)$$

When the limit is taken as $N \rightarrow \infty$ the summations of the above equations are constant for all finite m ; in fact they were calculated in the proof of Lemma 2. Then by examining separately the cases of N odd and N even, it can be shown that (19) and (20) can be written in the form of (7a). **Q.E.D.**

Proof of Proposition 1.

STEP 1: Choose some large number η^* . Then consider the point $(P_F(\eta^*), P_D(\eta^*))$ of the ROC curve of the individual DM which has slope η^* and denote by $\varepsilon_0(\eta^*) > 0$ the probability of detection at that point (Figure 7). Then from the concavity of the ROC curve:

$$\eta^* P_F(\eta^*) < \varepsilon_0(\eta^*) \Rightarrow P_F(\eta^*) < \frac{\varepsilon_0(\eta^*)}{\eta^*} \quad (21)$$

Note that such a point can always be obtained regardless of how large η^* is because of the assumption that the initial slope of the ROC curve is: $m_0 = \infty$.

STEP 2: For some integer N^* , define the following decision rules for the DMs of the team:

For DM 1:

$$u_1 = \gamma_1(y_1) = \begin{cases} 0 & ; \Lambda(y_1) \leq \eta^* \\ 1 & ; \Lambda(y_1) > \eta^* \end{cases} \quad (22a)$$

For DM n ; $n = 2, 3, \dots, N^*$:

$$u_n = \gamma_n(y_n, u_{n-1}) = \begin{cases} 0 & ; \Lambda(y_n) \leq \eta^* \text{ and } u_{n-1} = 0 \\ 1 & ; \text{otherwise} \end{cases} \quad (22b)$$

For DM n ; $n = N^*+1, N^*+2, N^*+3, \dots$

$$u_n = \gamma_n(y_n, u_{n-1}) = u_{n-1} \quad (22c)$$

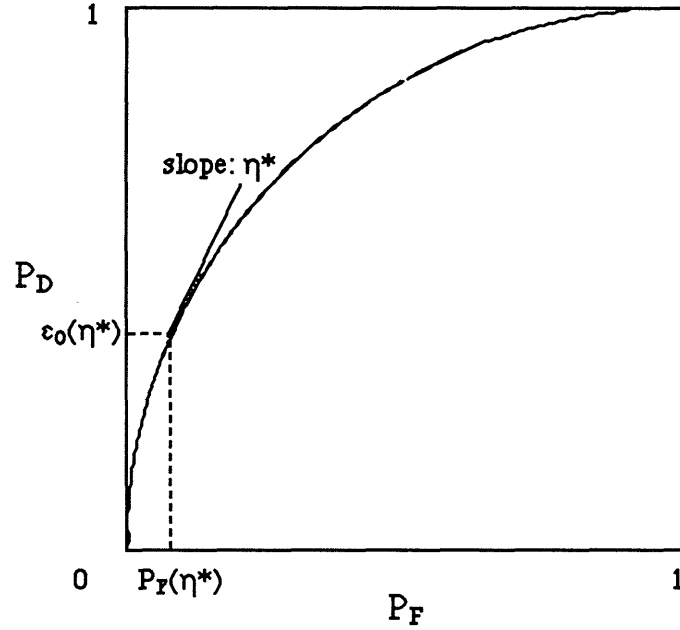


Figure 7. A ROC Curve with Infinite Initial Slope ($m_0 = \infty$)

Let us elaborate on the decision rules of (22). According to these only the first N^* DMs influence the decision of the team. A decision $u_i = 0$ is propagated through the first N^* DMs of the team until some DM n^* receives an observation which makes it decide $u_{n^*} = 1$. In that case the team decision will be $u_N = 1$; otherwise the team decision will be $u_N = 0$. Note that since η^* was chosen to be large, there is a very small probability for a DM n^* who receives $u_{n^*-1} = 0$ to decide $u_{n^*} = 1$ under both hypotheses, but if it does its observation y_{n^*} suggests that there is an overwhelming probability that H_1 be the correct hypothesis.

STEP 3: If the decision rules of the previous step are implemented the probability of detection of the team is:

$$P_D^t = 1 - (1 - \epsilon_0(\eta^*))^{N^*} \quad (23)$$

In order to make the team probability of detection greater than $1 - \delta$:

$$P_D^t = 1 - (1 - \epsilon_0(\eta^*))^{N^*} > 1 - \delta \Rightarrow N^* > \frac{\log(\delta)}{\log(1 - \epsilon_0(\eta^*))} \quad (24)$$

STEP 4: Similarly, the team probability of false alarm is:

$$P_F^t = 1 - (1 - P_F(\eta^*))^{N^*} \quad (25)$$

Then from (25) and (21):

$$P_F^t < 1 - \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)^{N^*} \quad (26)$$

In order to make the team probability of false alarm smaller than δ :

$$P_F^t < 1 - \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)^{N^*} < \delta \Rightarrow \frac{\log(1 - \delta)}{\log\left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)} > N^* \quad (27)$$

STEP 5: To show that an N^* satisfying both (24) and (27) indeed exists, recall that for $x \approx 0$:

$$\log(1+x) \approx x \quad (28)$$

If η^* is selected sufficiently large, $\varepsilon_0(\eta^*)$ is very close to zero (Figure 7); then from (24), (27) and (28), the *integer* N^* has to satisfy:

$$\begin{aligned} \left\lceil \frac{\log(1 - \delta)}{\log\left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)} \right\rceil &\approx \left\lceil \frac{\log(1 - \delta)}{-\frac{\varepsilon_0(\eta^*)}{\eta^*}} \right\rceil \geq N^* \geq \left\lceil \frac{\log(\delta)}{-\varepsilon_0(\eta^*)} \right\rceil \approx \left\lceil \frac{\log(\delta)}{\log(1 - \varepsilon_0(\eta^*))} \right\rceil \Rightarrow \\ &\left\lceil -\eta^* \frac{\log(1 - \delta)}{\varepsilon_0(\eta^*)} \right\rceil \geq N^* \geq \left\lceil -\frac{\log(\delta)}{\varepsilon_0(\eta^*)} \right\rceil \end{aligned} \quad (29)$$

Thus by choosing η^* sufficiently large, which can be done because of the assumption $m_0 = \infty$, an integer N^* satisfying both (24) and (27) can always be obtained.

STEP 6: If the decision rules of *Step 2* are employed by the team, for any given prior probabilities and consequently any threshold $\eta = P(H_0)/P(H_1)$ (which is not to be confused with our personal choice of large number η^*), the team probability of error is:

$$P^{e(\infty)} = \frac{\eta}{\eta + 1} P_F^t + \frac{1}{\eta + 1} (1 - P_D^t) < \frac{\eta}{\eta + 1} \delta + \frac{1}{\eta + 1} \delta = \delta \quad (30)$$

Thus for any given $\delta > 0$, the infinite team probability of error can be made less than δ . **Q.E.D.**

Proof of Proposition 3.

Denote by P_F^n the probability of false alarm and by P_D^n the probability of detection of the decision of the n th DM of the team. Then (10) can be written in the following form:

$$\Lambda(y_n) \begin{cases} u_n = 1 \\ \geq \\ u_n = 0 \end{cases} \begin{cases} \frac{1 - P_F^{n-1}}{1 - P_D^{n-1}} \eta = \mu_n^0 & ; u_{n-1} = 0 \\ \frac{P_F^{n-1}}{P_D^{n-1}} \eta = \mu_n^1 & ; u_{n-1} = 1 \end{cases} \quad (31)$$

for $n = 2, 3, 4, \dots$

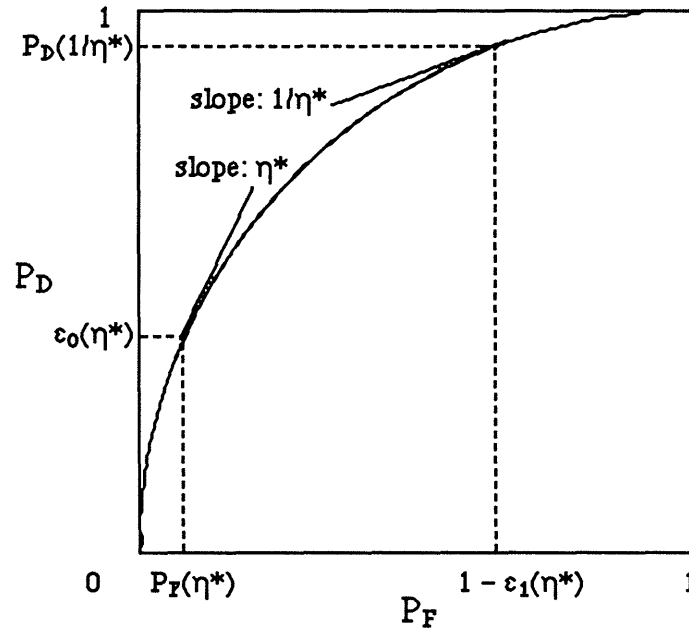


Figure 8. A ROC Curve with Infinite Initial Slope and Zero Final Slope ($m_0 = \infty, m_1 = 0$)

STEP 1: Pick some large number η^* . Then consider the point $(P_F(\eta^*), P_D(\eta^*))$ of the ROC curve of the individual DM which has slope η^* and denote by $\epsilon_0(\eta^*) > 0$ the probability of detection at that point (Figure 8). Then from the concavity of the ROC curve:

$$\eta^* P_F(\eta^*) < \epsilon_0(\eta^*) \Rightarrow P_F(\eta^*) < \frac{\epsilon_0(\eta^*)}{\eta^*} \quad (32)$$

Note that such a point can always be obtained regardless of how large η^* is because of the assumption that the initial slope of the ROC curve is: $m_0 = \infty$.

STEP 2: Similarly, for the same large number η^* , consider the point $(P_F(1/\eta^*), P_D(1/\eta^*))$ of the ROC curve of the individual DM which has slope $1/\eta^*$ and denote by $1 - \varepsilon_1(\eta^*)$ the probability of false alarm at that point (Figure 8). Then from the concavity of the ROC curve:

$$\frac{1}{\eta^*} [(1 - \varepsilon_1(\eta^*)) - 1] + 1 < P_D(1/\eta^*) \Rightarrow P_D(1/\eta^*) > 1 - \frac{\varepsilon_1(\eta^*)}{\eta^*} \quad (33)$$

Note that such a point can always be obtained regardless of how large η^* is because of the assumption that the initial slope of the ROC curve: $m_1 = 0$.

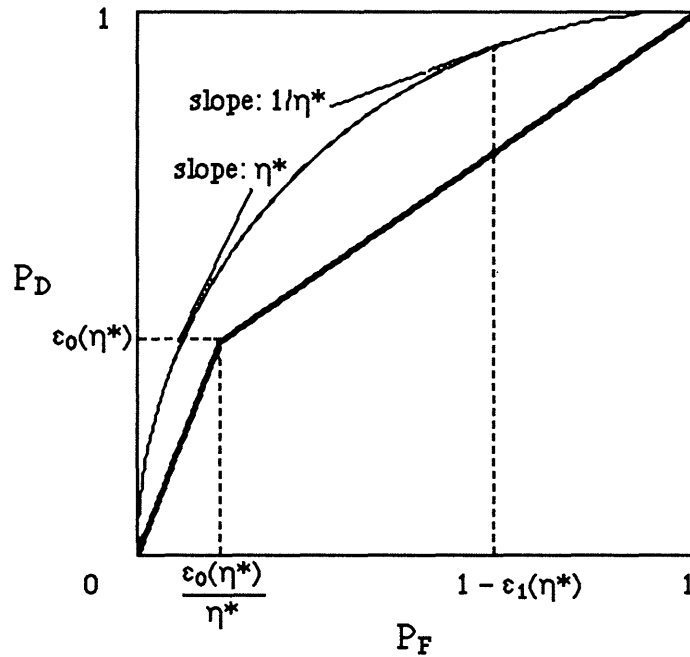


Figure 9. The ROC Curves of the *Worse* DM of *Step 3* and of the Original DM

STEP 3: Assume without loss of generality (since the names of the hypotheses can be interchanged) that the decision threshold η satisfies:

$$\eta > \frac{1 - \varepsilon_0(\eta^*) - \frac{\varepsilon_1(\eta^*)}{\eta^*}}{1 - \frac{\varepsilon_0(\eta^*)}{\eta^*} - \varepsilon_1(\eta^*)} \quad (34)$$

Then replace every DM of the team by a *worse* DM whose ROC curve is presented in Figure 9 and underlying probability distributions are presented in Table 3. *Steps 2–6* of Proposition 1 can now be repeated for the infinite tandem team consisting of the newly defined DMs to show that $P^{e(\infty)} = 0$, but for the sake of completeness they are included here as well.

y H	0	1
H ₀	$1 - \frac{\epsilon_0(\eta^*)}{\eta^*}$	$\frac{\epsilon_0(\eta^*)}{\eta^*}$
H ₁	$1 - \epsilon_0(\eta^*)$	$\epsilon_0(\eta^*)$

Table 3. Probability Distributions for DM of Figure 9

STEP 4: For some integer N^* , define the following decision rules for the DMs of the team:

For DM 1:

$$u_1 = \gamma_1(y_1) = \begin{cases} 0 & ; y_1 = 0 \\ 1 & ; y_1 = 1 \end{cases} \quad (35a)$$

For DM n ; $n = 2, 3, \dots, N^*$:

$$u_n = \gamma_n(y_n, u_{n-1}) = \begin{cases} 0 & ; y_n = 0 \text{ and } u_{n-1} = 0 \\ 1 & ; \text{ otherwise} \end{cases} \quad (35b)$$

For DM n ; $n = N^*+1, N^*+2, N^*+3, \dots$

$$u_n = \gamma_n(y_n, u_{n-1}) = u_{n-1} \quad (35c)$$

These may be equivalently written in terms of likelihood ratios as follows:

For DM 1:

$$u_1 = \gamma_1(y_1) = \begin{cases} 0 & ; \Lambda(y_1) \leq \eta^* \\ 1 & ; \Lambda(y_1) > \eta^* \end{cases} \quad (36a)$$

For DM n ; $n = 2, 3, \dots, N^*$:

$$u_n = \gamma_n(y_n, u_{n-1}) = \begin{cases} 0 & ; \Lambda(y_n) \leq \eta^* \text{ and } u_{n-1} = 0 \\ 1 & ; \text{ otherwise} \end{cases} \quad (36b)$$

For DM n ; $n = N^*+1, N^*+2, N^*+3, \dots$

$$u_n = \gamma_n(y_n, u_{n-1}) = u_{n-1} \quad (36c)$$

STEP 5: It is easy to see that if the decision rules of the previous step are implemented the probability of detection of the team is:

$$P_D^t = 1 - (1 - \varepsilon_0(\eta^*))^{N^*} \quad (37)$$

In order to make the team probability of detection greater than $1 - \delta$:

$$P_D^t = 1 - (1 - \varepsilon_0(\eta^*))^{N^*} > 1 - \delta \Rightarrow N^* > \frac{\log(\delta)}{\log(1 - \varepsilon_0(\eta^*))} \quad (38)$$

STEP 6: Similarly, the team probability of false alarm is:

$$P_F^t = 1 - (1 - P_F(\eta^*))^{N^*} \quad (39)$$

Then from (39) and (32):

$$P_F^t < 1 - \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)^{N^*} \quad (40)$$

In order to make the team probability of false alarm smaller than δ :

$$P_F^t < 1 - \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)^{N^*} < \delta \Rightarrow \frac{\log(1 - \delta)}{\log\left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)} > N^* \quad (41)$$

STEP 7: To show that an N^* satisfying both (38) and (41) indeed exists, recall that if η^* is selected sufficiently large, $\varepsilon_0(\eta^*)$ is very close to zero (Figure 8); then from (28), (38) and (40), the *integer* N^* has to satisfy:

$$\left\lceil \frac{\log(1 - \delta)}{\log\left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)} \right\rceil \approx \left\lceil \frac{\log(1 - \delta)}{-\frac{\varepsilon_0(\eta^*)}{\eta^*}} \right\rceil \geq N^* \geq \left\lceil \frac{\log(\delta)}{-\varepsilon_0(\eta^*)} \right\rceil \approx \left\lceil \frac{\log(\delta)}{\log(1 - \varepsilon_0(\eta^*))} \right\rceil \Rightarrow$$

$$\left\lceil -\eta^* \frac{\log(1 - \delta)}{\varepsilon_0(\eta^*)} \right\rceil \geq N^* \geq \left\lceil -\frac{\log(\delta)}{\varepsilon_0(\eta^*)} \right\rceil \quad (42)$$

Thus by choosing η^* sufficiently large, which can be done because of the assumption $m_0 = \infty$, an integer N^* satisfying both (38) and (41) can always be obtained.

STEP 8: If the decision rules of *Step 4* are employed by the team, for any given decision threshold η which satisfies (34) (and which is not to be confused with our personal choice of large number η^*), the team probability of error is:

$$P^{e(\infty)} = \frac{\eta}{\eta+1} P_F^i + \frac{1}{\eta+1} (1 - P_D^i) < \frac{\eta}{\eta+1} \delta + \frac{1}{\eta+1} \delta = \delta \quad (43)$$

Thus for any given $\delta > 0$, the probability of error of the infinite team which consists of the *worse* DMs (of Figure 9) can be made less than δ ; therefore, the probability of error of the infinite team which consists of the original (*better*) DMs can be made less than δ .

STEP 9: What remains to be verified is that the decision rules of *Step 4* for the *worse* DMs (of Figure 9) are indeed the suboptimal decision rules of (9) and (10) for those DMs.

It is not hard to verify that, for DM 1, the suboptimal decision rule of (9) and the proposed decision rule of (35a) are identical. Recall that η was assumed to satisfy (34) and that η^* can be selected sufficiently large. Then:

$$\eta^* > \eta > \frac{1 - \varepsilon_0 - \frac{\varepsilon_1}{\eta^*}}{1 - \frac{\varepsilon_0}{\eta^*} - \varepsilon_1} > \frac{1 - \varepsilon_0}{1 - \frac{\varepsilon_0}{\eta^*}} \quad (44)$$

Then it should be clear from Figure 7 that the decision thresholds for both decision rules coincide.

To show this for the other DMs, recall that under the decision rules of *Step 4*:

$$(P_F^n, P_D^n) = \begin{cases} \left(\left(1 - \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*} \right)^n, 1 - (1 - \varepsilon_0(\eta^*))^n \right) & ; n = 1, 2, \dots, N^* \\ \left(\left(1 - \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*} \right)^{N^*}, 1 - (1 - \varepsilon_0(\eta^*))^{N^*} \right) & ; n = N^*+1, N^*+2, N^*+3, \dots \end{cases} \quad (45)$$

Substituting into the thresholds of (31):

$$\mu_n^0 = \begin{cases} \frac{\eta \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*} \right)^{n-1}}{(1 - \varepsilon_0(\eta^*))^{n-1}} & ; n = 2, 3, \dots, N^* \\ \frac{\eta \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*} \right)^{N^*}}{(1 - \varepsilon_0(\eta^*))^{N^*}} & ; n = N^*+1, N^*+2, N^*+3, \dots \end{cases} \quad (46)$$

$$\mu_n^1 = \begin{cases} \eta \frac{1 - \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)^{n-1}}{1 - (1 - \varepsilon_0(\eta^*))^{n-1}} & ; n = 2, 3, \dots, N^* \\ \eta \frac{1 - \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)^{N^*}}{1 - (1 - \varepsilon_0(\eta^*))^{N^*}} & ; n = N^*+1, N^*+2, N^*+3, \dots \end{cases} \quad (47)$$

To show the above thresholds are indeed the thresholds of the suboptimal decision rules, we need to show that:

$$\mu_n^0 \begin{cases} \leq \eta^* & ; n = 2, 3, \dots, N^* \\ > \eta^* & ; n = N^*+1, N^*+2, \dots \end{cases} \quad (48)$$

and:

$$\mu_n^1 \leq \frac{1 - \varepsilon_0(\eta^*)}{1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}} & ; n = 2, 3, 4, \dots \quad (49)$$

For this, we now need to show that there exists some integer N^* satisfying (48). Since we want the team probability of error to go to zero asymptotically, we also need to verify that N^* satisfies (42). Recall that η^* is large; then:

$$\mu_n^0 = \mu_{n-1}^0 \frac{\left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)}{(1 - \varepsilon_0(\eta^*))} > \mu_{n-1}^0 & ; n = 2, 3, \dots, N^* \quad (50)$$

Therefore, it suffices to show that:

$$\mu_{N^*+1}^0 = \eta \frac{\left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)^{N^*}}{(1 - \varepsilon_0(\eta^*))^{N^*}} \geq \eta^* > \eta \frac{\left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)^{N^*-1}}{(1 - \varepsilon_0(\eta^*))^{N^*-1}} = \mu_{N^*}^0 \quad (51)$$

Taking logarithms:

$$\begin{aligned} N^* \left[\log \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right) - \log(1 - \varepsilon_0(\eta^*)) \right] &\geq \log \left(\frac{\eta^*}{\eta}\right) > \\ &> (N^* - 1) \left[\log \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right) - \log(1 - \varepsilon_0(\eta^*)) \right] \end{aligned} \quad (52)$$

Using (28) to approximate:

$$N^* \left[-\frac{\varepsilon_0(\eta^*)}{\eta^*} + \varepsilon_0(\eta^*) \right] \geq \log \left(\frac{\eta^*}{\eta}\right) > (N^* - 1) \left[-\frac{\varepsilon_0(\eta^*)}{\eta^*} + \varepsilon_0(\eta^*) \right] \quad (53)$$

This can be rewritten as:

$$\frac{\log\left(\frac{\eta^*}{\eta}\right)}{\varepsilon_0(\eta^*)\left(1-\frac{1}{\eta^*}\right)} + 1 > N^* \geq \frac{\log\left(\frac{\eta^*}{\eta}\right)}{\varepsilon_0(\eta^*)\left(1-\frac{1}{\eta^*}\right)} \quad (54)$$

Therefore, an integer N^* satisfying (48) indeed exists. Moreover, for η^* sufficiently large:

$$\eta^* \frac{\log(1-\delta)}{-\varepsilon_0(\eta^*)} > \frac{\log\left(\frac{\eta^*}{\eta}\right)}{\varepsilon_0(\eta^*)\left(1-\frac{1}{\eta^*}\right)} + 1 > N^* \geq \frac{\log\left(\frac{\eta^*}{\eta}\right)}{\varepsilon_0(\eta^*)\left(1-\frac{1}{\eta^*}\right)} > \frac{\log(\delta)}{-\varepsilon_0(\eta^*)} \quad (55)$$

Thus, this integer N^* also satisfies (42).

Finally to show that (49) holds, observe that for η^* sufficiently large (which implies that $\varepsilon_0(\eta^*)$ is very small), apply Taylor's theorem to obtain that, for all $n = 2, 3, 4, \dots$:

$$\mu_n^1 = \eta \frac{1 - \left(1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}\right)^{n-1}}{1 - (1 - \varepsilon_0(\eta^*))^{n-1}} \approx \eta \frac{(n-1) \frac{\varepsilon_0(\eta^*)}{\eta^*}}{(n-1) \varepsilon_0(\eta^*)} = \frac{\eta}{\eta^*} < \frac{1 - \varepsilon_0(\eta^*)}{1 - \frac{\varepsilon_0(\eta^*)}{\eta^*}} \quad (56)$$

Thus, the decision rules of *Step 4* correspond indeed the suboptimal decision rules, for the integer N^* defined by (54). **Q.E.D.**