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Distributed dislocation approach for cracks in couple-stress elasticity: shear modes

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Abstract The distributed dislocation technique proved to be in the past an effective approach in studying crack problems within classical elasticity. The present work aims at extending this technique in studying crack problems within couple-stress elasticity, i.e. within a theory accounting for effects of microstructure. As a first step, the technique is introduced to study finite-length cracks under remotely applied shear loadings (mode II and mode III cases). The mode II and mode III cracks are modeled by a continuous distribution of glide and screw dislocations, respectively, that create both standard stresses and couple stresses in the body. In particular, it is shown that the mode II case is governed by a singular integral equation with a more complicated kernel than that in classical elasticity. The numerical solution of this equation shows that a cracked material governed by couple-stress elasticity behaves in a more rigid way (having increased stiffness) as compared to a material governed by classical elasticity. Also, the stress level at the crack-tip region is appreciably higher than the one predicted by classical elasticity. Finally, in the mode III case the corresponding governing integral equation is hypersingular with a cubic singularity. A new mechanical quadrature is introduced here for the numerical solution of this equation. The results in the mode III case for the crack-face displacement and the near-tip stress show significant departure from the predictions of classical fracture mechanics.

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1 Introduction

The present work is concerned with the study of mode II and mode III *finite-length* cracks in a material with microstructure. We assume that the response of the material is governed by couple-stress elasticity. This theory falls into the category of generalized continuum theories and is a particular case of the general approaches of Toupin (1962), Mindlin (1964), and Green and Rivlin (1964). As is well-known, ideas underlying couple-stress elasticity were advanced first by Voigt (1887) and the Cosserat brothers (1909), but the subject was generalized and reached maturity only with the works of Toupin (1962), Mindlin and Tiersten (1962), Mindlin (1964), and Koiter (1964).

Earlier application of the couple-stress elasticity, mainly on stress-concentration problems, met with some success providing solutions physically more adequate than solutions based on classical elasticity (see e.g. Mindlin and Tiersten 1962; Weitsman 1965; Bogy and Sternberg 1967a, b). Work employing couple-stress theories on elasticity and plasticity problems is also continued in recent years (see e.g. Vardoulakis and Sulem 1995; Huang et al. 1997; Chen et al. 1998;

Anthoine 2000; Lubarda and Markenscoff 2000; Bardet and Vardoulakis 2001; Georgiadis and Velgaki 2003; Grentzelou and Georgiadis 2005).

Nevertheless, there is only a limited number of studies concerning the effects of couple-stresses in crack problems. One of the earlier works in this subject is that of Sternberg and Muki (1967) who considered the mode I finite-length crack by employing the method of dual integral equations. They provided only asymptotic results and showed that both the stress and couple-stress fields exhibit a square-root singularity while the rotation field is bounded at the crack-tip. The same method was adopted by Ejike (1969) for a circular (penny-shaped) crack in couple-stress elasticity and by Paul and Sridharan (1980, 1981) for a finite-length crack in micropolar elasticity. Using the Wiener-Hopf technique, Atkinson and Leppington (1977) studied the problem of a semi-infinite crack with exponentially decayed normal tractions on the crack faces. More recently, Huang et al. (1997) provided near-tip asymptotic fields for the mode I and mode II crack problems, in couple-stress elasticity, by using the method of eigenfunction expansions. Also, Zhang et al. (1998) by employing the Wiener-Hopf technique investigated the mode III semi-infinite crack in couple-stress elasticity in the special case where the second couple-stress moduli is set equal to zero. Moreover, using a similar approach, Huang et al. (1999) obtained full-field solutions for semi-infinite cracks under mode I and mode II loadings in elastic-plastic materials with strain-gradient effects.

Here, we aim at providing full-field solutions to the mode II and mode III *finite-length* crack problems within couple-stress elasticity by introducing an approach based on *distributed dislocations*. Since the pioneering work of Bilby et al. (1963), Bilby and Eshelby (1968) the distributed-dislocation technique has been employed to analyze various crack problems in classical elasticity. A thorough exposition of the technique can be found in the treatise by Hills et al. (1996). The strength of this analytical/numerical technique lies in the fact that it gives detailed *full-field* solutions for crack problems at the expense of relatively little analytical demands as compared to the elaborate technique of dual integral equations and, also, of relatively little computational demands as compared to the Finite Element and Boundary Element methods. Although the technique has proven to be very successful in studying crack problems within classical elasticity, it appears that there is no work at all in modeling cracks with distribution of dislocations in materials with microstructure. Therefore, the present work aims at extending the technique in couple-stress elasticity. In another recent work by the present authors (Gourgiotis and Georgiadis 2007) the mode I crack problem was also considered within the same framework. A comparison between the mode II case studied here and the mode I case leads to the conclusion that the opening mode is mathematically more involved than the shear mode. This is in some contrast with situations of classical elasticity where the two plane-strain crack modes involve equivalent mathematical effort.

As in analogous situations of classical elasticity, a superposition scheme will be followed. Thus, the solution to the basic problem (body with a traction-free crack under remote shear field) will be obtained by the superposition of the stress field arising in the un-cracked body (of the same geometry) to the 'corrective' stresses and couple-stresses induced by a continuous distribution of dislocations chosen so that the crack-faces become traction-free. The stress field for a discrete glide and screw dislocation in couple-stress elasticity will serve, respectively, as the Green's function for the mode II and mode III problem. However, we note that deriving the stress field of a discrete dislocation within generalized continua is by no means a straightforward task. Within the framework of couple-stress elasticity a lot of research has been devoted to dislocations. Representative references include work by Kroner (1963), Misicu (1965), Teodosiu (1965), Cohen (1966), Anthony (1970), Knesl and Semela (1972) and Nowacki (1974). Finally, it is shown that due to the nature of the above Green's functions and the boundary conditions that arise in couple-stress elasticity, the aforementioned procedure results for the mode II case in a singular integral equation (SIE), whereas for the mode III case in a hypersingular integral equation (IE) with a cubic singularity. In order to solve this hypersingular IE, a new mechanical quadrature is constructed.

2 Basic concepts and equations of couple-stress elasticity

In this Section, we briefly present the basic ideas and equations of couple-stress elasticity. The theory employed here is a particular case of form III in the general Mindlin's (1964) approach. Nevertheless, we chose to present an alternative approach to Mindlin's variational approach. Indeed, our derivation of basic results relies on the *momentum balance laws*, which—in our opinion—provide more physical insight. It should also be mentioned that versions of the quasi-static couple-stress theory were given by, among others, Aero and Kuvshinskii (1960), Mindlin and Tiersten (1962), Koiter (1964), Palmov (1964), and Muki and Sternberg (1965). The basic equations of dynamical couple-stress theory (including the effects of micro-inertia) were given by Georgiadis and Velgaki (2003).

In the absence of inertia effects, for a control volume CV with bounding surface S, the balance laws for the linear and angular momentum read

$$\int_S T_i^{(n)} dS + \int_{CV} F_i d(CV) = 0, \tag{1}$$

$$\int_S \left(x_j T_k^{(n)} e_{ijk} + M_i^{(n)} \right) dS + \int_{CV} (x_j F_k e_{ijk} + C_i) d(CV) = 0, \tag{2}$$

where $T_i^{(n)}$ is the surface force per unit area (force traction), F_i is the body force per unit volume, $M_i^{(n)}$ is the surface moment per unit area (couple traction), and C_i is the body moment per unit volume.

Next, pertinent *force-stress* and *couple-stress* tensors are introduced by considering the equilibrium of the elementary material tetrahedron and enforcing (1) and (2), respectively. The force-stress tensor σ_{ij} (which is asymmetric) is defined by

$$T_i^{(n)} = \sigma_{ji} n_j, \tag{3}$$

and the couple-stress tensor μ_{ij} (which is also asymmetric) by

$$M_i^{(n)} = \mu_{ji} n_j, \tag{4}$$

where n_j are the direction cosines of the outward unit vector $\mathbf{n}$, which is normal to the surface. In addition just like the third Newton's law $\mathbf{T}^{(\mathbf{n})} = -\mathbf{T}^{(-\mathbf{n})}$ is proved to hold by considering the equilibrium of a material 'slice', it can also be proved that $\mathbf{M}^{(\mathbf{n})} = -\mathbf{M}^{(-\mathbf{n})}$. The couple-stresses μ_{ij} are expressed in dimensions of [force][length]$^{-1}$. Further, σ_{ij} can be decomposed into a symmetric and anti-symmetric part

$$\sigma_{ij} = \tau_{ij} + \alpha_{ij}, \tag{5}$$

with $\tau_{ij} = \tau_{ji}$ and $\alpha_{ij} = -\alpha_{ji}$, whereas it is advantageous to decompose μ_{ij} into its deviatoric $\mu_{ij}^{(D)}$ and spherical $\mu_{ij}^{(S)}$ part in the following manner

$$\mu_{ij} = m_{ij} + \frac{1}{3}\delta_{ij}\mu_{kk}, \tag{6}$$

where $m_{ij} = \mu_{ij}^{(D)}$, $\mu_{ij}^{(S)} = (1/3)\,\delta_{ij}\mu_{kk}$ and δ_{ij} is the Kronecker delta. Now, with the above definitions in hand and with the help of the divergence theorem, one may obtain the equations of equilibrium. Thus, Eq. 2 leads to the following moment equation

$$\partial_i \mu_{ij} + \sigma_{ki} e_{ijk} + C_j = 0, \tag{7}$$

which can also be written as

$$\frac{1}{2}\partial_i \mu_{il} e_{jkl} + \alpha_{jk} + \frac{1}{2} C_l e_{jkl} = 0, \tag{8}$$

since by its definition the anti-symmetric part of stress is written as $\boldsymbol{\alpha} \equiv -(1/2)\,\mathbf{I} \times (\boldsymbol{\sigma} \times \mathbf{I})$, where $\mathbf{I}$ is the idemfactor. Also, Eq. 1 leads to the following force equation

$$\partial_j \sigma_{jk} + F_k = 0, \tag{9}$$

or, by virtue of (5), to the equation

$$\partial_j \tau_{jk} + \partial_j \alpha_{jk} + F_k = 0. \tag{10}$$

Further, combining (8) and (10) yields the *single* equation

$$\partial_j \tau_{jk} - \frac{1}{2}\partial_j \partial_i \mu_{il} e_{jkl} + F_k - \frac{1}{2}\partial_j C_l e_{jkl} = 0. \tag{11}$$

Finally, in view of Eq.6 and by taking into account that $\text{curl}\left(\text{div}\left((1/3)\,\delta_{ij}\mu_{kk}\right)\right) = 0$, we write (11) as

$$\partial_j \tau_{jk} - \frac{1}{2}\partial_j \partial_i m_{il} e_{jkl} + F_k - \frac{1}{2}\partial_j C_l e_{jkl} = 0. \tag{12}$$

Equation 12 is therefore the single equation of equilibrium.

As for the kinematical description of the continuum, the following quantities are defined within the geometrically linear theory

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right), \tag{13}$$

$$\omega_{ij} = \frac{1}{2}\left(\partial_j u_i - \partial_i u_j\right), \tag{14}$$

$$\omega_i = \frac{1}{2} e_{ijk} \partial_j u_k, \tag{15}$$

$$\kappa_{ij} = \partial_i \omega_j, \tag{16}$$

where ε_{ij} is the strain tensor, ω_{ij} is the rotation tensor, ω_i is the rotation vector, and κ_{ij} is the curvature tensor (i.e. the gradient of rotation or the curl of the strain) expressed in dimensions of [length]$^{-1}$. Notice also that Eq. 16 can alternatively be written as

$$\kappa_{ij} = \frac{1}{2} e_{jkl} \partial_i \partial_k u_l = e_{jkl} \partial_k \varepsilon_{il}. \tag{17}$$

Equation 17 expresses compatibility for curvature and strain fields. In addition, there is an identity, i.e. $\partial_k \kappa_{ij} = \partial_k \partial_i \omega_j = \partial_i \partial_k \omega_j = \partial_i \kappa_{kj}$, which expresses compatibility for the curvature components. The compatibility equations for the strain components are the usual Saint Venant's compatibility equations. We notice also that $\kappa_{ii} = 0$ because $\kappa_{ii} = \partial_i \omega_i = (1/2)\, e_{ijk} u_{k,ji} = 0$ and, therefore, that κ_{ij} has only eight independent components. The tensor κ_{ij} is obviously an *asymmetric* tensor.

Now, regarding the traction boundary conditions, we note that at first sight, it might seem plausible that the surface tractions (i.e. the force-traction and the couple-traction) can be prescribed arbitrarily on the external surface of the body through relations (3) and (4), which stem from the equilibrium of the material tetrahedron. However, as Koiter (1964) pointed out, the resulting number of six traction boundary conditions (three force-tractions and three couple-tractions) would be in contrast with the *five* geometric boundary conditions that can be imposed. Indeed, since the rotation vector ω_i in couple-stress elasticity is not independent of the displacement vector u_i (cf. (15)), the normal component of the rotation is fully specified by the distribution of tangential displacements over the boundary. Therefore, only the three displacement and the two tangential rotation components can be prescribed independently. As a consequence, only *five* surface tractions (i.e. the work conjugates of the above five independent kinematical quantities) can be specified at a point of the bounding surface of the body. These are three *reduced* force-tractions and two tangential couple-tractions (Mindlin and Tiersten 1962; Koiter 1964)

$$P_i^{(n)} = \sigma_{ji} n_j - \frac{1}{2} e_{ijk} n_j \partial_k m_{(nn)}, \tag{18}$$

$$R_i^{(n)} = m_{ji} n_j - m_{(nn)} n_i, \tag{19}$$

where $m_{(nn)} = n_i n_j m_{ij}$ is the normal component of the deviatoric couple-stress tensor m_{ij}. Finally, it is worth noting that in the micropolar (Cosserat) theory of elasticity (see e.g. Nowacki 1972), the traction boundary conditions are six since the rotation is fully independent of the displacement vector. In this case the tractions can directly be derived from the equilibrium of the material tetrahedron, i.e. the relations between tractions and stresses are given by (3) and (4).

Introducing the constitutive equations of the theory is now in order. We assume a linear and isotropic material response, in which case the potential-energy density takes the form

$$W \equiv W\left(\varepsilon_{ij}, \kappa_{ij}\right) = \frac{1}{2}\lambda \varepsilon_{ii}\varepsilon_{jj} + \mu \varepsilon_{ij}\varepsilon_{ij} + 2\eta \kappa_{ij}\kappa_{ij} + 2\eta' \kappa_{ij}\kappa_{ji}, \tag{20}$$

where $\left(\lambda, \mu, \eta, \eta'\right)$ are material constants. Then, Eq. 20 leads, through the standard variational manner, to the following constitutive equations

$$\tau_{ij} \equiv \sigma_{(ij)} = \frac{\partial W}{\partial \varepsilon_{ij}} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}, \tag{21}$$

$$m_{ij} = \frac{\partial W}{\partial \kappa_{ij}} = 4\eta \kappa_{ij} + 4\eta' \kappa_{ji}. \tag{22}$$

In view of (21) and (22), the moduli (λ, μ) have the same meaning as the Lamé constants of classical elasticity theory, whereas the moduli $\left(\eta, \eta'\right)$ account for couple-stress effects.

Finally, the following points are of notice: (i) The couple-stress moduli $\left(\eta, \eta'\right)$ are expressed in dimensions of [force]. (ii) Since $\kappa_{ii} = 0$, $m_{ii} = 0$ is also valid and therefore the tensor m_{ij} has only eight independent components. (iii) The scalar $(1/3)\,\mu_{kk}$ of the couple-stress tensor does not appear in the final equation of equilibrium, nor in the reduced boundary conditions and the constitutive equations. Consequently, $(1/3)\,\mu_{kk}$ is left indeterminate within the couple-stress theory. (iv) The following restrictions for the material constants should prevail on the basis of a positive definite potential-energy density (Mindlin and Tiersten 1962)

$$3\lambda + 2\mu > 0, \quad \mu > 0, \quad \eta > 0, \quad -1 < \frac{\eta'}{\eta} < 1. \tag{23a,b,c,d}$$

3 Plane problems of couple-stress elasticity

The cases of plane strain and anti-plane strain are examined here and the basic equations are given. In what follows, vanishing body forces and body couples are assumed.

3.1 Plane-strain

For a body that occupies a domain in the (x, y)-plane under conditions of plane strain, the displacement field takes the general form

$$u_x \equiv u_x(x, y) \neq 0, \qquad u_y \equiv u_y(x, y) \neq 0, \qquad u_z \equiv 0. \tag{24a,b,c}$$

By virtue of (13)–(16), the non-vanishing components of strain, rotation and curvature are given as

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right), \tag{25a,b,c}$$

$$\omega_z = \omega_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right), \tag{26}$$

$$\kappa_{xz} = \frac{\partial \omega_z}{\partial x}, \quad \kappa_{yz} = \frac{\partial \omega_z}{\partial y}. \tag{27a,b}$$

Also, from the constitutive Eqs. 21 and 22, the following relations are derived between stress and strain and between couple-stress and curvature

$$\tau_{xx} = (2\mu + \lambda)\,\varepsilon_{xx} + \lambda\varepsilon_{yy}, \quad \tau_{yy} = (2\mu + \lambda)\,\varepsilon_{yy} + \lambda\varepsilon_{xx}, \quad \tau_{xy} = 2\mu\varepsilon_{xy}, \tag{28a,b,c}$$

$$m_{xz} = 4\eta\kappa_{xz}, \quad m_{yz} = 4\eta\kappa_{yz}, \tag{29a,b}$$

whereas, the remaining components are given by

$$\tau_{zz} = -\frac{\lambda}{2(\lambda + \mu)}\left(\tau_{xx} + \tau_{yy}\right), \quad m_{zx} = \frac{\eta'}{\eta} m_{xz}, \quad m_{zy} = \frac{\eta'}{\eta} m_{yz}. \tag{30a,b,c}$$

Next, the non-vanishing components of the anti-symmetric part of the force-stress tensor are obtained from (8) as

$$\alpha_{xy} = -\alpha_{yx} = -\frac{1}{2}\left(\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y}\right) = -2\eta\nabla^2\omega_z. \tag{31}$$

It should be noticed that the independence upon the coordinate z of *all* components of the force-stress and couple-stress tensors, under the assumption (24c), was proved by Muki and Sternberg (1965). Indeed, it is noteworthy that, contrary to the respective plane-strain case in the conventional theory, this independence is not obvious within the couple-stress theory.

Mindlin's stress functions

As Mindlin (1963) indicated, the equations of equilibrium in (7) and (9), in a plane-strain state, are identically satisfied when the stresses are derived from two stress functions $\Phi(x, y)$ and $\Psi(x, y)$ in the following manner

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial y \partial x}, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial x \partial y}, \tag{32a,b}$$

$$\sigma_{xy} = -\frac{\partial^2 \Phi}{\partial y \partial x} - \frac{\partial^2 \Psi}{\partial y^2}, \quad \sigma_{yx} = -\frac{\partial^2 \Phi}{\partial y \partial x} + \frac{\partial^2 \Psi}{\partial x^2}, \tag{33a,b}$$

$$m_{xz} = \frac{\partial \Psi}{\partial x}, \quad m_{yz} = \frac{\partial \Psi}{\partial y}. \tag{34a,b}$$

where the functions Φ and Ψ satisfy the following PDEs

$$\nabla^4\Phi = 0, \quad \nabla^2\left(\ell^2\nabla^2 - 1\right)\Psi = 0. \tag{35a,b}$$

According to the compatibility equations between curvature and strain in (17), the stress functions are related through the following equations

$$\frac{\partial}{\partial x}\left(\Psi - \ell^2\nabla^2\Psi\right) = -2(1 - \nu)\,\ell^2\frac{\partial}{\partial y}\left(\nabla^2\Phi\right), \tag{36}$$

$$\frac{\partial}{\partial y}\left(\Psi - \ell^2\nabla^2\Psi\right) = 2(1 - \nu)\,\ell^2\frac{\partial}{\partial x}\left(\nabla^2\Phi\right), \tag{37}$$

where ν is the Poisson's ratio and $\ell \equiv (\eta/\mu)^{1/2}$ is a *characteristic material length.*

3.2 Anti-plane strain

For a body occupying a domain in the (x, y)-plane under conditions of anti-plane strain, the displacement field takes the general form

$$u_x \equiv 0, \quad u_y \equiv 0, \quad u_z = w(x, y) \neq 0. \tag{38a,b,c}$$

Again, by virtue of (13)–(16), the non-vanishing components of strain, rotation and curvature are given as

$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\frac{\partial w}{\partial x}, \quad \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\frac{\partial w}{\partial y}, \tag{39a,b}$$

$$\omega_x = \omega_{yz} = \frac{1}{2}\frac{\partial w}{\partial y}, \quad \omega_y = \omega_{xz} = -\frac{1}{2}\frac{\partial w}{\partial x}, \tag{40a,b}$$

$$\kappa_{xx} = -\kappa_{yy} = \frac{1}{2}\frac{\partial^2 w}{\partial x \partial y}, \quad \kappa_{xy} = -\frac{1}{2}\frac{\partial^2 w}{\partial x^2},$$
$$\kappa_{yx} = \frac{1}{2}\frac{\partial^2 w}{\partial y^2}. \tag{41a,b,c}$$

Then, the constitutive equations in 21 and 22 provide

$$\tau_{xz} = 2\mu\varepsilon_{xz} = \mu\frac{\partial w}{\partial x}, \quad \tau_{yz} = 2\mu\varepsilon_{yz} = \mu\frac{\partial w}{\partial y}, \tag{42a,b}$$

$$m_{xx} = 4\left(\eta + \eta'\right)\kappa_{xx} = 2\left(\eta + \eta'\right)\frac{\partial^2 w}{\partial x \partial y}, \tag{43a}$$

$$m_{yy} = 4\left(\eta + \eta'\right)\kappa_{yy} = -2\left(\eta + \eta'\right)\frac{\partial^2 w}{\partial x \partial y} = -m_{xx}, \tag{43b}$$

$$m_{xy} = 4\eta\kappa_{xy} + 4\eta'\kappa_{yx} = -2\eta\frac{\partial^2 w}{\partial x^2} + 2\eta'\frac{\partial^2 w}{\partial y^2}, \tag{43c}$$

$$m_{yx} = 4\eta\kappa_{yx} + 4\eta'\kappa_{xy} = 2\eta\frac{\partial^2 w}{\partial y^2} - 2\eta'\frac{\partial^2 w}{\partial x^2}. \tag{43d}$$

Further, the non-vanishing components of the anti-symmetric part of the force-stress tensor are obtained from (8)

$$\alpha_{zx} = -\alpha_{xz} = \frac{1}{2}\left(\frac{\partial m_{xy}}{\partial x} + \frac{\partial m_{yy}}{\partial y}\right) = \eta\frac{\partial}{\partial x}\left(\nabla^2 w\right), \tag{44a}$$

$$\alpha_{zy} = -\alpha_{yz} = -\frac{1}{2}\left(\frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y}\right) = \eta\frac{\partial}{\partial y}\left(\nabla^2 w\right). \tag{44b}$$

Finally, by taking into account (5) and (44), the components of the force-stress tensor can be written as

$$\sigma_{xz} = \mu\frac{\partial}{\partial x}\left(w - \ell^2\nabla^2 w\right),$$
$$\sigma_{zx} = \mu\frac{\partial}{\partial x}\left(w + \ell^2\nabla^2 w\right), \tag{45a,b}$$

$$\sigma_{yz} = \mu\frac{\partial}{\partial y}\left(w - \ell^2\nabla^2 w\right),$$
$$\sigma_{zy} = \mu\frac{\partial}{\partial y}\left(w + \ell^2\nabla^2 w\right). \tag{46a,b}$$

In view of the above and by enforcing equilibrium, a single PDE of the fourth order for the displacement component is obtained

$$\nabla^2 w - \ell^2\nabla^4 w = 0. \tag{47}$$

4 Discrete dislocations in couple-stress elasticity

4.1 Glide dislocation

Consider a glide dislocation with Burgers vector $\mathbf{b} = (b, 0, 0)$ imposed in an infinite medium along the plane $x > 0$, $y = 0$. The appropriate Mindlin's stress functions for this problem were given by Cohen (1966), Knesl and Semela (1972), and Nowacki (1974)

$$\Phi = -\frac{\mu b r}{4\pi(1-\nu)}(2\ln r + 1)\sin\theta, \tag{48}$$

$$\Psi = \frac{2\mu b\ell}{\pi}\left[K_1(r/\ell) - \ell/r\right]\cos\theta, \tag{49}$$

where $r = \left(x^2+y^2\right)^{1/2}, \theta = \tan^{-1}(y/x)$ and $K_i(r/\ell)$ is the i^{th}-order modified Bessel function of the second kind. Further, the stresses induced at a point (x, y) may be found from the above stress functions by using Eqs. 32–34

$$\begin{aligned}\sigma_{xx} = &-\frac{\mu b}{4\pi(1-\nu)r}(3\sin\theta + \sin 3\theta)\\ &+\frac{2\mu b}{\pi r}\left[\frac{2\ell^2}{r^2} - K_2(r/\ell)\right]\sin 3\theta\\ &-\frac{\mu b}{4\pi\ell^2}r\left[K_2(r/\ell) - K_0(r/\ell)\right]\\ &\times(\sin\theta + \sin 3\theta),\end{aligned} \tag{50}$$

$$\begin{aligned}\sigma_{yy} = &\frac{\mu b}{4\pi(1-\nu)r}(\sin 3\theta - \sin\theta)\\ &-\frac{2\mu b}{\pi r}\left[\frac{2\ell^2}{r^2} - K_2(r/\ell)\right]\sin 3\theta\\ &+\frac{\mu b}{4\pi\ell^2}r\left[K_2(r/\ell) - K_0(r/\ell)\right]\\ &\times(\sin\theta + \sin 3\theta),\end{aligned} \tag{51}$$

$$\begin{aligned}\sigma_{xy} = &\frac{\mu b}{4\pi(1-\nu)r}(\cos\theta + \cos 3\theta)\\ &-\frac{2\mu b}{\pi r}\left[\frac{2\ell^2}{r^2} - K_2(r/\ell)\right]\cos 3\theta\\ &-\frac{\mu b}{4\pi\ell^2}r\left[K_2(r/\ell) - K_0(r/\ell)\right]\\ &\times(\cos\theta - \cos 3\theta),\end{aligned} \tag{52}$$

$$\begin{aligned}\sigma_{yx} = &\frac{\mu b}{4\pi(1-\nu)r}(\cos\theta + \cos 3\theta)\\ &-\frac{2\mu b}{\pi r}\left[\frac{2\ell^2}{r^2} - K_2(r/\ell)\right]\cos 3\theta\\ &+\frac{\mu b}{4\pi\ell^2}r\left[K_2(r/\ell) - K_0(r/\ell)\right]\\ &\times(3\cos\theta + \cos 3\theta),\end{aligned} \tag{53}$$

$$m_{xz} = \frac{\mu b}{\pi}\left[\frac{2\ell^2}{r^2} - K_2(r/\ell)\right]\cos 2\theta - \frac{\mu b}{\pi}K_0(r/\ell), \quad (54)$$

$$m_{yz} = \frac{\mu b}{\pi}\left[\frac{2\ell^2}{r^2} - K_2(r/\ell)\right]\sin 2\theta. \quad (55)$$

Examining now the asymptotic behavior of the above stress field (to determine the possibility of singularities), we note that as $r \to 0$ the following asymptotic relations hold (see e.g. Abramowitz and Stegun 1964)

$$\frac{1}{r}\left(\frac{2\ell^2}{r^2} - K_2(r/\ell)\right) = O\left(r^{-1}\right),$$
$$r\left[K_2(r/\ell) - K_0(r/\ell)\right] = O\left(r^{-1}\right),$$
$$K_0(r/\ell) = O(\ln r). \quad (56a,b,c)$$

In view of (56), as the dislocation core ($r \to 0$) is approached, the components of the force-stress tensor $\left(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{yx}\right)$ exhibit a Cauchy singularity (just as in classical elasticity), the couple-stress m_{xz} becomes logarithmically unbounded, while m_{yz} remains bounded. Finally, when $\ell \to 0$ the stress field of classical elasticity for a discrete glide dislocation is recovered.

4.2 Screw dislocation

For a screw dislocation with strength b the displacement field in couple-stress elasticity is given as (see our derivation in Appendix A)

$$w = \frac{b}{2\pi}\theta - \frac{b}{4\pi}(1+\beta)\left[\frac{2\ell^2}{r^2} - K_2(r/\ell)\right]\sin 2\theta, \quad (57)$$

where the ratio $\beta \equiv \eta'/\eta$ should satisfy the following inequality $-1 < \beta < 1$. The stress and couple-stress fields corresponding to (57) are obtained from Eqs. 42–46 as

$$\tau_{xz} = -\frac{\mu b}{2\pi r}\sin\theta, \quad \tau_{yz} = \frac{\mu b}{2\pi r}\cos\theta, \quad (58a,b)$$

$$\sigma_{xz} = -\frac{\mu b}{2\pi r}\sin\theta + \frac{\mu b \ell^2(1+\beta)}{\pi r^3}\sin 3\theta,$$
$$\sigma_{yz} = \frac{\mu b}{2\pi r}\cos\theta - \frac{\mu b \ell^2(1+\beta)}{\pi r^3}\cos 3\theta, \quad (59a,b)$$

$$m_{yy} = -m_{xx} = \frac{\mu \ell^2(1+\beta)b}{\pi r^2}\cos 4\theta - \frac{3\mu b \ell^2(1+\beta)^2}{\pi r^2}\left(\frac{2\ell^2}{r^2} - K_2(r/\ell)\right)\cos 4\theta + \frac{\mu b(1+\beta)^2}{2\pi}K_2(r/\ell)\cos 4\theta - \frac{\mu b(1+\beta)^2}{8\pi}K_0(r/\ell)(3\cos 4\theta + 1), \quad (60a,b)$$

$$m_{yx} = \frac{3\mu b \ell^2(1+\beta)^2}{\pi r^2}\left(\frac{2\ell^2}{r^2} - K_2(r/\ell)\right)\sin 4\theta - \frac{\mu b(1+\beta)^2}{4\pi}K_2(r/\ell)(2\sin 4\theta + \sin 2\theta) + \frac{3\mu b(1+\beta)^2}{8\pi}K_0(r/\ell)\sin 4\theta - \frac{\mu b(1+\beta)}{\pi}\left(\frac{2\ell^2}{r^2} - K_2(r/\ell)\right)\sin 2\theta, \quad (61)$$

$$m_{xy} = m_{yx} - 2\mu\ell^2(1-\beta)\nabla^2 w, \quad (62)$$

The following points are of notice now: (i) Using the well known asymptotic properties of the modified Bessel functions, we conclude that as $r \to 0$ the asymmetric and the symmetric shear stresses behave as $\sim r^{-3}$ and $\sim r^{-1}$, respectively, whereas the couple-stresses behave as $\sim r^{-2}$. (ii) When $\beta = -1$ (i.e. when $\eta = -\eta'$), the above stress field degenerates into the respective one in classical elasticity for a screw dislocation.

5 Formulation of crack problems by a distribution of dislocations

5.1 Mode II crack

Consider a straight crack of length $2a$ embedded in the xy-plane of infinite extend in a field of pure shear (Fig. 1). The crack faces are traction free and the body is considered to be in plane-strain conditions. The crack faces are defined by $\mathbf{n} = (0, \pm 1)$. Then, according to (18) and (19), the boundary conditions along the crack faces are written as

$$\sigma_{yx} = 0, \quad \sigma_{yy} = 0, \quad m_{yz} = 0 \quad \text{for } |x| < a, \quad (63a,b,c)$$

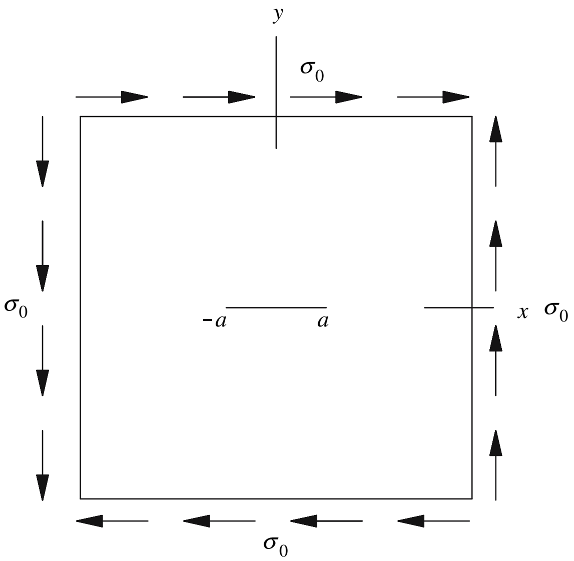

Fig. 1 Cracked body under remote shear in plane strain

whereas the regularity conditions at infinity are

$$\sigma_{yx}^{\infty}=\sigma_{xy}^{\infty}=\tau_{xy}^{\infty}\rightarrow\sigma_0,\quad \sigma_{yy}^{\infty},\sigma_{xx}^{\infty}\rightarrow 0,$$
$$m_{xz}^{\infty},m_{yz}^{\infty}\rightarrow 0\quad as\quad r\rightarrow\infty, \tag{64a,b,c}$$

where $r=\left(x^2+y^2\right)^{1/2}$ now is the distance from the origin and the constant σ_0 denotes the remotely applied shear loading.

Then, the crack problem is decomposed into the following two auxiliary problems.

The un-cracked body

It can readily be verified that the appropriate Mindlin's stress functions for the un-cracked body of infinite extent subjected to boundary conditions (64a,b,c) are as follows

$$\Phi=-\sigma_0 xy,\quad \Psi=0. \tag{65a,b}$$

The stress field that corresponds to the above stress functions can be found from (32)–(34) as

$$\sigma_{yx}(x,y)=\sigma_{xy}(x,y)=\sigma_0,$$
$$\sigma_{xx}=\sigma_{yy}=0,\quad m_{xz}=m_{yz}=0. \tag{66a,b,c}$$

Notice, that there are no couple-stresses induced in the un-cracked body, the body being in a state of pure shear.

The corrective solution

Consider a body geometrically identical to the initial cracked body (Fig. 1) but with no remote loading now. The only loading applied is along the crack faces. This consists of equal and opposite tractions to those generated in the un-cracked body. The boundary conditions along the faces of the crack are written as

$$\sigma_{yx}=-\sigma_0,\quad \sigma_{yy}=0,\quad m_{yz}=0$$
$$\text{for } |x|<a. \tag{67a,b,c}$$

The corrective stresses (67a,b,c) may be generated by a continuous distribution of discrete glide dislocations along the crack faces. The stresses and couple-stresses induced by the continuous distribution of dislocations can be derived by integrating the effect of a discrete glide dislocation (i.e. by the use of Eqs. 50–55). We note that (67b,c) are automatically satisfied since a discrete glide dislocation does not produce normal stresses σ_{yy} or couple-stresses m_{yz} along the crack-line. Then, satisfaction of the boundary condition (67a) leads to a *single* IE. Separating the singular part from the regular part of the kernels, we obtain the governing SIE of the mode II problem in couple-stress elasticity as

$$-\sigma_0=\frac{\mu(3-2\nu)}{2\pi(1-\nu)}\fint_{-a}^{a}\frac{B(\xi)}{x-\xi}d\xi+\frac{\mu}{\pi}\int_{-a}^{a}$$
$$B(\xi)\,k(x,\xi)\,d\xi,\quad |x|<a, \tag{68}$$

where $\fint$ signifies Cauchy principal value integration and $B(\xi)=db/d\xi$ is the dislocation density at a point ξ ($|\xi|<a$), this density being defined in the same way as in classical elasticity (see e.g. Hills et al. 1996).

The kernel $k(x,\xi)$ is defined as

$$k(x,\xi)=-\frac{2}{x-\xi}\left[\frac{2\ell^2}{(x-\xi)^2}-K_2(|x-\xi|/\ell)-\frac{1}{2}\right]$$
$$-\frac{(x-\xi)}{\ell^2}\left[\frac{2\ell^2}{(x-\xi)^2}-K_2(|x-\xi|/\ell)\right.$$
$$\left.+K_0(|x-\xi|/\ell)\right]. \tag{69}$$

To show that $k(x,\xi)$ is regular, we expand the latter in series as $x\rightarrow\xi$ (see e.g. Abramowitz and Stegun 1964) and obtain

$$k(x,\xi)=(a_1+a_2\ln|x-\xi|)(x-\xi)$$
$$+O\left((x-\xi)^3\ln|x-\xi|\right), \tag{70}$$

where a_i are constants depending on the characteristic material length ℓ. Since $\lim_{x\rightarrow\xi}(x-\xi)^n\ln|x-\xi|=0$ for $n>0$, we conclude that $k(x,\xi)$ is regular in the closed domain $-a\leq(x,\xi)\leq a$.

The solution $B(\xi)$ in (68) is determined in the class of Hoelder continuous functions and may be written as a product of a regular bounded function and a fundamental solution. Asymptotic analysis, within the framework of the couple-stress elasticity, showed that the displacement u_x behaves as $\sim r^{1/2}$ in the crack tip region, where r denotes now the polar distance from the crack tip (Huang et al. 1997). Consequently, the dislocation density is expressed in the form

$$B(\xi) = f(\xi)\left(\alpha^2 - \xi^2\right)^{-1/2}, \tag{71}$$

where $f(\xi)$ is bounded and continuous in the interval $|\xi| \leq \alpha$. Further, in order to render the problem determinate, the dislocation density should also satisfy an *auxiliary* condition expressing the requirement that there be no net relative tangential displacement between one end of the crack and the other, i.e.

$$\int_{-a}^{a} B(\xi) d\xi = 0. \tag{72}$$

Before proceeding with the solution of the governing integral equation, it is interesting to consider two limit cases concerning the behavior of (68) w.r.t. ℓ. By letting $\ell \to 0$ and noting that $\lim_{\ell \to 0} k(x, \xi) = -1/(x - \xi)$, Eq. 68 degenerates into the counterpart equation governing the mode II problem in classical elasticity, i.e.

$$-\sigma_0 = \frac{\mu}{2\pi(1-\nu)} \fint_{-a}^{a} \frac{B(\xi)}{x - \xi} d\xi, \qquad |x| < a. \tag{73}$$

On the other hand, by letting $\ell \to \infty$ and noting that $\lim_{\ell \to \infty} k(x, \xi) = 0$, (68) takes the form

$$-\sigma_0 = \frac{\mu(3 - 2\nu)}{2\pi(1-\nu)} \fint_{-a}^{a} \frac{B(\xi)}{x - \xi} d\xi, \qquad |x| < a. \tag{74}$$

It can readily be shown, that the ratio of the crack-face displacements obtained by the solution of (74) and (73), respectively, is $1/(3 - 2\nu)$. Equation 74 shows mathematically that there is a lower bound for the crack-face displacement u_x when $\ell \to \infty$. The same ratio of displacements was also obtained by Sternberg and Muki (1967) for a mode I crack in couple-stress elasticity.

For the numerical solution of the SIE in (68), the Gauss-Chebyshev quadrature developed by Erdogan and Gupta (1972) is used. After the appropriate normalization over the interval $[-1, 1]$, the integral equation takes the discretized form

$$-\sigma_0 = \frac{\mu}{n} \sum_{i=1}^{n} \left[\frac{(3 - 2\nu)}{2(1-\nu)(t_k - s_i)} + k(t_k, s_i)\right] f(s_i), \tag{75}$$

where

$$k(t_k, s_i) = -\frac{2}{t_k - s_i} \left[\frac{2}{p^2 (t_k - s_i)^2} - K_2(p|t_k - s_i|) - \frac{1}{2}\right] - p^2 (t_k - s_i)\left(\frac{2}{p^2 (t_k - s_i)^2} - K_2(p|t_k - s_i|) + K_0(p|t_k - s_i|)\right), \tag{76}$$

with $p = a/\ell$, $t = x/a$, and $s = \xi/a$. The integration and collocation points are given, respectively, as

$$T_n(s_i) = 0, \ \ s_i = \cos[(2i - 1)\pi/2n], \ \ i = 1, \ldots, n, \tag{77a}$$

$$U_{n-1}(t_k) = 0, \ \ t_k = \cos[k\pi/n], \ \ k = 1, \ldots, n - 1, \tag{77b}$$

where $T_n(x)$ and $U_n(x)$ are the Chebyshev polynomials of the first and second kind, respectively. Formula (75) is a standard Gauss–Chebyshev quadrature with the requirement that the collocation points t_k must satisfy (77b), i.e. that t_k be the roots of U_{n-1}. The auxiliary condition in (72) can be written in discretized form as

$$\frac{\pi}{n} \sum_{i=1}^{n} f(s_i) = 0. \tag{78}$$

Equations 75 and 78 provide an algebraic system of n equations in the n unknown functions $f(s_i)$. A computer program was written that solves the above system of equations.

Some numerical results are presented now. In Fig. 2 the dependence of the tangential crack-face displacement on the ratio a/ℓ in couple-stress elasticity is depicted. It is noteworthy that as the crack length becomes comparable to the characteristic length ℓ, the material exhibits a more *stiff* behavior, i.e. the tangential crack-face displacements become smaller and smaller in magnitude. Finally, we note that the displacements obtained within the classical theory of elasticity serve as an upper *bound* of couple-stress elasticity.

Next, the near-tip behavior of the shear stress σ_{yx} given as the expression in the RHS of (68) plus σ_0, is determined. Due to the symmetry of the problem (in geometry and loading) with respect to y-axis we confine attention only to the right crack tip. Now, as

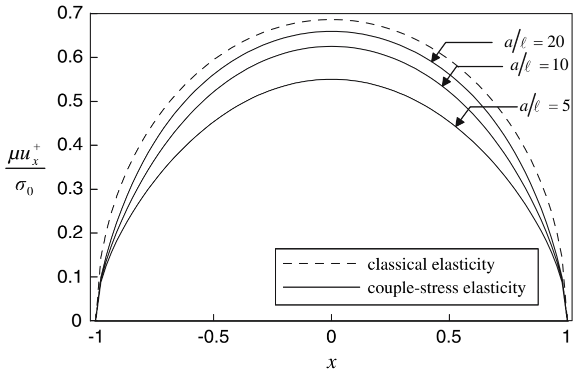

Fig. 2 Normalized upper-half tangential crack displacement profile ($\nu = 0.3$)

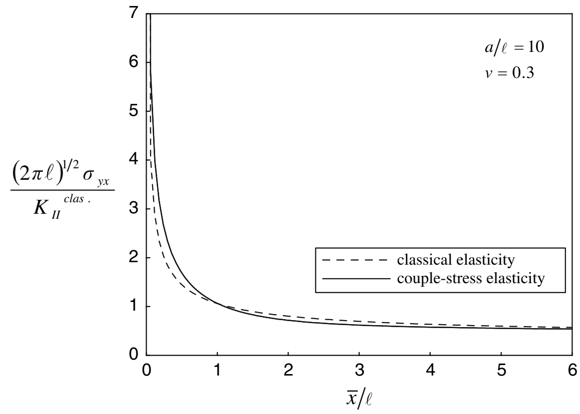

Fig. 4 Distribution of the shear stress ahead of the cracktip

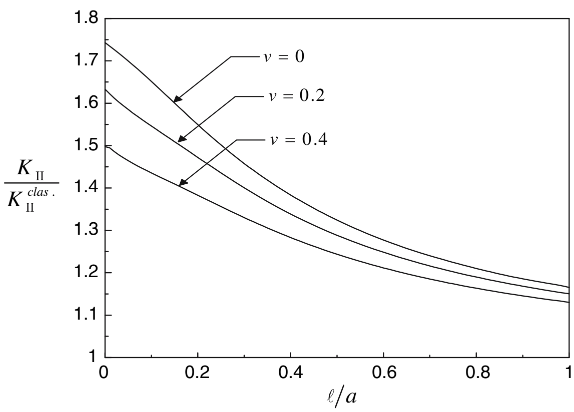

Fig. 3 Variation of the ratio of stress intensity factors in couple-stress elasticity and in classical elasticity

$x \to a^+$ the following asymptotic relations hold

$$\int_{-a}^{a} \frac{B(\xi)}{x-\xi} d\xi = \mathrm{O}\left((x-a)^{-1/2}\right),$$
$$\int_{-\alpha}^{\alpha} B(\xi)\, k(x,\xi)\, d\xi = \mathrm{O}(1), \qquad x > a, \qquad \text{(79a,b)}$$

where the dislocation density is defined in (71). Thus, we conclude that σ_{yx} exhibits a square root singularity at the crack tip. In light of the above, we define the stress intensity factor in couple-stress elasticity as $K_{II} = \lim_{x\to a^+} [2\pi(x-a)]^{1/2}\sigma_{yx}(x, y=0)$ for the right crack tip $(x > a)$ The dependence of the ratio of the stress intensity factor in couple-stress elasticity K_{II} to the one in classical elasticity upon ℓ/a is given in Fig. 3.

It is observed that for $\ell/a \to 0$ and Poisson's ratio $\nu = 0.4$, there is a 50% increase in K_{II} when couple-stress effects are taken into account, while for $\nu = 0.2$ and $\nu = 0$ the increase is 62% and 73%, respectively. It should be noted that when $\ell/a = 0$ (no couple-stress effects) the above ratio becomes evidently $K_{II}/K_{II}^{clas.} = 1$. Therefore, the ratio plotted in Fig. 3 exhibits a finite jump discontinuity at $\ell/a = 0$; the ratio at the tip of the crack rises abruptly as ℓ/a departures from zero. The same discontinuity was observed by Sternberg and Muki (1967), who attributed that kind of behavior to the severe *boundary-layer* effects predicted by the couple-stress elasticity in stress-concentration problems. Finally, it can be shown that the ratio decreases monotonically with increasing values of ℓ/a and tends to unity as $\ell/a \to \infty$. The case $\ell/a \to \infty$ is rather impractical since generally the relation between lengths in a usual crack problem will be $\ell << a$, i.e. the crack length will be much greater than the material length. However, in an attempt to explain the latter finding, we note that the case $\ell \to \infty$, with $a \neq 0$, resembles a situation where, in a sense, there is *no* microstructure in the body, since the 'building blocks' of the material are of infinite size. Of course, this case has an obscure physical meaning, but, as far as stresses are concerned, the solution shows that the material exhibits a behavior similar to the one for a material governed by the classical theory.

Further, the distribution of the shear stress σ_{yx} ahead of the crack tip (see Fig. 4) shows that the couple-stress effects are dominant for $x < \ell$, whereas outside this zone σ_{yx} gradually approaches the distribution of the classical solution. For convenience, a new variable $\bar{x} = x - a$ is introduced measuring now distance from the crack tip in the RHS of Fig. 1.

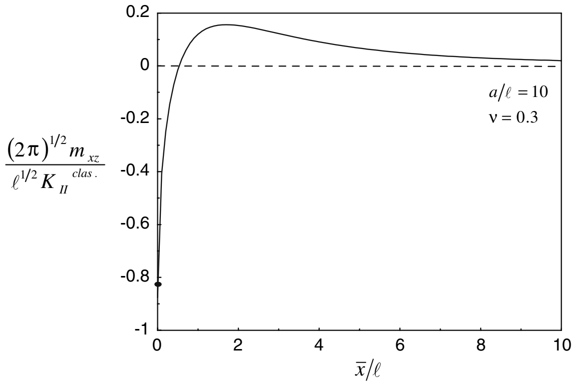

Fig. 5 Distribution of the couple-stress ahead of the crack tip

Finally, taking into account that m_{xz} exhibits a logarithmic singularity in the case of a glide dislocation and that

$$\int_{-\alpha}^{\alpha} \ln |x - \xi| \, B(\xi) \, d\xi = O(1) \quad \text{as} \quad x \to a^{+}, \quad (80)$$

we conclude that m_{xz} given as the integral of (54) is bounded at the crack tip. This observation is in agreement with the asymptotic results of Huang et al. (1997) for a mode II crack. Figure 5 depicts the distribution of the couple-stress m_{xz} ahead of the crack tip. In particular, we observe that m_{xz} takes finite *negative* values immediately ahead of the crack tip in the RHS. Then, as the position (observation point) moves away from the crack tip, m_{xz} changes sign and gradually reaches zero for $x > 10\ell$. It should be noted, though, that m_{xz} exhibits the property of anti-symmetry w.r.t. the y- axis (see Fig. 1): Therefore, m_{xz} is *positive* immediately ahead of the LHS crack tip. An anti-symmetric distribution of the couple-stress is required for the moment equation in (7) to be satisfied.

Finally, as we show in Appendix C, the orders of singularities of the above stress and couple-stress fields lead to an integrable strain-energy density in the vicinity of crack tips and also lead to a bounded value of the J-integral.

5.2 Mode III crack

Consider a straight crack of length $2a$ embedded in the (x, y)-plane of infinite extent under a remotely applied anti-plane shear loading (see Fig. 6). The crack faces are assumed to be traction free. The boundary conditions along the crack faces are written as (cf. (18) and (19))

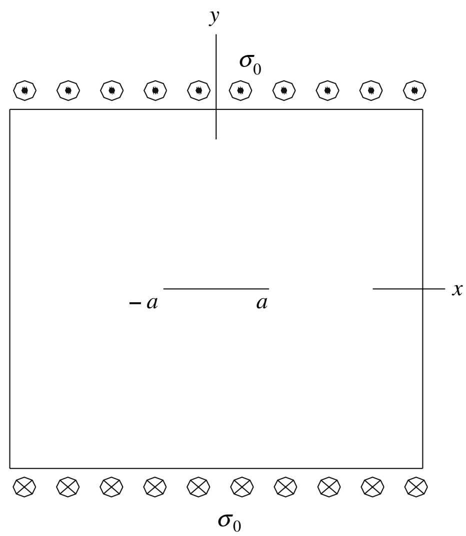

Fig. 6 Cracked body under remote shear in anti-plane strain

$$\sigma_{yz} + \frac{1}{2}\partial_x m_{yy} = 0, \qquad m_{yx} = 0 \quad \text{for} \quad |x| < a, \qquad (81a,b)$$

whereas the regularity conditions at infinity are given as

$$\sigma_{yz}^{\infty} = \tau_{yz}^{\infty} \to \sigma_0, \quad \sigma_{xz}^{\infty} \to 0, \quad m_{xx}^{\infty}, m_{yy}^{\infty}, m_{yx}^{\infty}, m_{xy}^{\infty} \to 0 \quad \text{as} \quad r \to \infty, \qquad (82a,b,c)$$

The ‘reduced’ boundary condition in (81a) is also justified physically from the fact that the displacement w and the rotation $\omega_y = -(1/2)(\partial w/\partial x)$ cannot be prescribed independently on the crack faces. This situation is analogous to the one in Kirchhoff’s plate theory regarding the effective shear force.

Again, the crack problem is decomposed into the following two auxiliary problems.

The *un-cracked* body

It can be readily shown that the un-cracked body subjected to the boundary conditions (82a,b,c) is in a state of pure anti-plane shear. The only non-zero stresses are

$$\sigma_{yz}(x, y) = \tau_{yz}(x, y) = \sigma_0. \qquad (83)$$

Note that there are no couple-stresses induced in the un-cracked body.

The corrective solution

Consider a body geometrically identical to the initial cracked body in Fig. 6 but with no remote loading now. The applied loading along the crack faces consists of equal and opposite tractions to those generated in the un-cracked body, i.e.

$$\sigma_{yz} + \frac{1}{2}\partial_x m_{yy} = -\sigma_0, \quad m_{yx} = 0 \quad \text{for } |x| < a \text{ and } \quad y = 0. \tag{84a,b}$$

The corrective stresses in (84a,b) may be generated by a continuous distribution of discrete screw dislocations along the crack faces. The stresses induced by the continuous distribution of dislocations are obtained as integrals of Eqs. 59–62. Note that (84b) is automatically satisfied since a discrete screw dislocation does not give rise to couple-stresses m_{yx} along the crack line. Then, satisfaction of the boundary condition (84a) leads, after lengthy calculations, to the governing hypersingular IE of the mode III problem in couple-stress elasticity ($|x| < a$)

$$-\sigma_0 = \neq\!\!\!\!\int_{-a}^{a}\left[\frac{c_1}{x-\xi} + \frac{c_2\ell^2}{(x-\xi)^3} + c_3 k\,(x,\xi)\right] B\,(\xi)\,d\xi \tag{85}$$

where $\neq\!\!\!\!\int$ signifies Hadamard's finite-part integration (see e.g. Kutt 1975; Paget 1981), $B\,(\xi)$ is the dislocation density function at the point ξ ($|\xi| < a$), and

$$c_1 = \frac{\mu\left(\beta^2 + 2\beta + 9\right)}{16\pi}, \quad c_2 = \frac{\mu\,(1+\beta)\,(\beta-3)}{2\pi}, \quad c_3 = \frac{\mu\,(1+\beta)^2}{\pi}. \tag{86}$$

Further, the kernel $k\,(x,\xi)$ is defined as

$$k(x,\xi) = -\frac{\ell^2}{(x-\xi)^3}\left[6\left(K_2(|x-\xi|/\ell) - \frac{2\ell^2}{(x-\xi)^2}\right) + \frac{1}{2}\right] + \frac{1}{4(x-\xi)}\left[3K_0(|x-\xi|/\ell) - 5K_2(|x-\xi|/\ell) - \frac{1}{4}\right]. \tag{87}$$

Expanding $k\,(x,\xi)$ in series as $x \to \xi$ and using the asymptotic properties of the modified Bessel functions, it can be readily shown that $k\,(x,\xi)$ is regular in the closed domain $-a \le (x,\xi) \le a$. We also note that when $\beta = -1$ (i.e. when $\eta = -\eta'$), Eq. 85 degenerates into the SIE that governs the counterpart problem in classical elasticity.

In addition, Zhang et al. (1998) showed, by using the Williams eigenfunction asymptotic analysis, that the crack face displacement behaves as $\sim r^{3/2}$ in the crack tip region, where r denotes the polar distance from the crack tip. Thus, the dislocation density $B\,(\xi)$ can be expressed as

$$B\,(\xi) = f\,(\xi)\,\left(a^2 - \xi^2\right)^{1/2}, \tag{88}$$

where $f\,(\xi)$ is a continuous bounded function in $\xi \le |a|$. Finally, to ensure uniqueness the dislocation density must satisfy the following auxiliary condition stemming from the requirement of single-valuedness of the displacement along a closed loop around the crack

$$\int_{-a}^{a} B\,(x)dx = 0. \tag{89}$$

Now, the *near-tip* behavior of the stress and couple-stress field for the mode III problem can be determined from the singular nature of the respective stress and couple-stress field of a discrete screw dislocation. Again, confining our attention to the RHS crack tip and taking into account the following result (Chan et al. 2003)

$$\frac{\partial^n}{\partial x^n}\int_{-a}^{a}\frac{B\,(\xi)}{x-\xi}d\xi = O\left((x-a)^{1/2-n}\right), \text{ for } n \ge 0 \text{ as } x \to a^+, x > a \tag{90}$$

with the dislocation density being given by (88), we conclude that $\left(\tau_{yz}, \sigma_{yz}\right)$ given as the integrals of (58b) and (59b) behave as $\sim \bar{x}^{-3/2}$ and $\sim \bar{x}^{1/2}$, respectively, whereas the couple-stresses $\left(m_{xx}, m_{yy}\right)$ given by the integration of (60a,b) exhibit a square root singularity at the crack tip. Again, $\bar{x} = x - a$ is the distance from the RHS crack tip along the crack line. Finally, in light of the above, the *total* shear stress defined as $t_{yz} = \sigma_{yz} + (1/2)\,\partial_x m_{yy}$ has the following asymptotic behavior $t_{yz} \sim \bar{x}^{-3/2}$ near the crack tip. Such a behavior was detected before in the mode III crack problem of gradient elasticity (Georgiadis 2003). The two problems present similarities in their mathematical analysis. Finally, as we show in Appendix C, despite the hypersingular nature of the above stress field, the strain-energy density is integrable in the vicinity of crack tips and, also, the J-integral takes a bounded value.

For the numerical solution of the hypersingular integral equation in 85, the appropriate quadrature is constructed here by taking into account the cubic singularity of the integral equation and the endpoint behavior of the dislocation density (details are given in Appendix B). Equation 85 after the appropriate normalization over the interval $[-1, 1]$ takes the discretized form

$$-\sigma_0 = -\frac{\pi c_2}{p^2}\sum_{i=1}^{n}(-1)^{i+k}\left[\frac{t_k}{2\left(1-t_k^2\right)} - \frac{1}{t_k - s_i}\right] \times \frac{f(s_i)\left(1-s_i^2\right)}{(t_k - s_i)\left(1-t_k^2\right)^{1/2}} + \sum_{i=1}^{n}\frac{\pi}{(1+n)} \times \left[\frac{c_1}{(t_k - s_i)} + \frac{c_2}{p^2(t_k - s_i)^3} + c_3 k(t_k, s_i)\right] \times f(s_i)\left(1-s_i^2\right), \quad (91)$$

where

$$k(t_k, s_i) = -\frac{1}{p^2(t_k - s_i)^3}\left[6\left(K_2(p|t_k - s_i|) - \frac{2}{p^2(t_k - s_i)^2} + \frac{1}{2}\right)\right] + \frac{1}{4(t_k - s_i)}\left[3K_0(p|t_k - s_i|) - 5K_2(p|t_k - s_i|) - \frac{1}{4}\right], \quad (92)$$

with $p = a/\ell$, and the set of the n discrete integration points are given by

$$s_i : U_n(s_i) = 0, \quad s_i = \cos(i\pi/(n+1)), \quad i = 1, \ldots, n, \quad (93a)$$

while the $n+1$ collocation points are given by

$$t_k : T_{n+1}(t_k) = 0, \quad t_k = \cos((2k-1)\pi/2(n+1)), \quad k = 1, \ldots, n+1. \quad (93b)$$

The auxiliary condition in (89) can be written in discretized form as

$$\sum_{i=1}^{n}\frac{\left(1-s_i^2\right) f(s_i)}{1+n} = 0, \quad (94)$$

Then, Eqs. 91 and 94 provide a system of $n+2$ algebraic equations. The system is solved in the least-squares sense.

In Fig. 7, the crack-face displacements are shown for the special case $\beta = 0$ (i.e. $\eta' = 0$). It is observed that in the crack-tip vicinity, the crack closes more smoothly as compared to the classical result. Further, it is also noted that when the characteristic material length ℓ becomes comparable to the crack length the material behaves in a more rigid way (having increased stiffness).

Both couple-stress and classical elasticity ($K_{III}^{clas.}$ field) distributions ahead of the right crack tip are shown in Fig. 8. The total shear stress t_{yz} is employed to depict the couple-stress elasticity solution. As in the analogous gradient elasticity solution (Georgiadis 2003), we observe that for a very small zone in the crack-tip region ($x < 0.5\ell$) the total stress t_{yz} takes on negative values exhibiting therefore a *cohesive-traction* character along the prospective fracture zone. Also, t_{yz} exhibits a bounded maximum. As $\beta \to -1$, the *cohesive* zone becomes significantly smaller whereas the maximum value of the total shear stress increases. The behavior of t_{yz} reminds typical *boundary-layer* behavior as, e.g., that found for the surface pressure near the leading edge of Joukowski airfoil (Van Dyke 1964). Finally, we note that at points lying outside the domain where the effects of

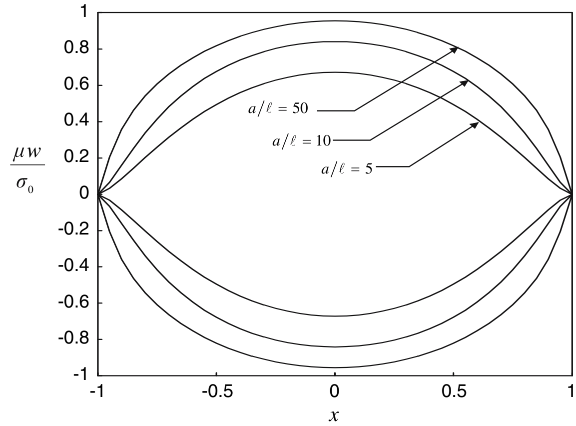

Fig. 7 Normalized upper and lower crack displacement profiles under remote mode III loading ($\beta = 0$)

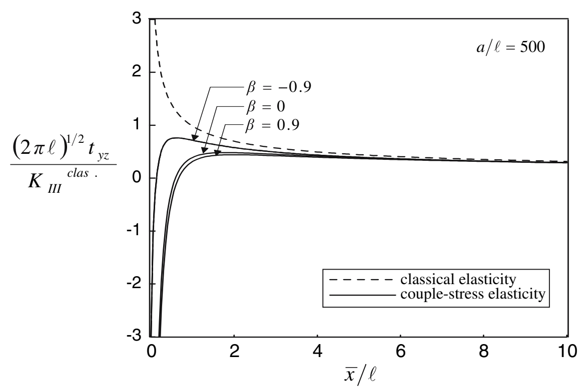

Fig. 8 Distribution of the total shear stress ahead of the crack tip

microstructure are pronounced (i.e. for $x > \ell$) the total shear stress tends to the classical $K_{III}^{clas.}$ shear stress.

6 Concluding remarks

In this paper, the technique of the distributed dislocations was used in order to solve finite-length shear crack problems in couple-stress elasticity. The technique provides an alternative approach to the elaborate analytical method of dual integral equations used before to attack asymptotically the mode I crack problem. Moreover, the present approach is capable to provide a full-field solution. In fact, we have obtained here the stress distribution ahead of the crack tips and the crack-face displacements (i.e. our results are not restricted to the crack-tip region). Also, our solution to the finite-length crack in mode III is quite novel in the literature.

The governing integral equations are derived using the discrete-dislocation stress fields in couple-stress elasticity, as the Green's functions of crack problems. In particular, it is shown that the mode II problem is governed by a single singular integral equation. In the mode III case, the governing integral equation is found to be hypersingular with a cubic singularity. For the solution of the latter equation, a new efficient quadrature is constructed.

The results of our analysis indicate that when the microstructure of the material is taken into account the material behaves in a more rigid way. In particular, in the mode II problem, the crack face displacements become significantly smaller than their counterparts in classical elasticity, when the length of the crack is comparable to the characteristic length ℓ of the material. Further, stresses retain the same order of singularity as in the classical theory, while the couple-stress field is found to be bounded in the crack-tip region. In the mode III problem, the results for the near-tip field show significant departure from the predictions of classical fracture mechanics. It is shown that cohesive stresses develop in the immediate vicinity of the crack-tip and that, ahead of the small cohesive zone, the stress distribution exhibits a local maximum that is bounded. This maximum value may serve, therefore, as a measure of the critical stress level at which further advancement of the crack may occur. In addition, in the vicinity of the crack-tip, the crack-face displacement closes more smoothly as compared to the classical result.

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Appendix A: The screw dislocation in couple-stress elasticity

Let the direct Fourier transform and its inverse be defined as

$$w^*(\xi, y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} w(x, y)\, e^{ix\xi} dx, \tag{A1a}$$

$$w(x, y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} w^*(\xi, y)\, e^{-ix\xi} d\xi, \tag{A1b}$$

where $i \equiv (-1)^{1/2}$. Transforming the field equation (47) with (A1a) gives the following ODE

$$\ell^2 \frac{d^4 w^*}{dy^4} - \left(2\ell^2\xi^2 + 1\right) \frac{d^2 w^*}{dy^2} + \left(\ell^2\xi^4 + \xi^2\right) w^* = 0, \tag{A2}$$

and, further, the general transformed solution for $y \geq 0$

$$w^*(\xi, y) = A_1(\xi)\, e^{-|\xi| y} + A_2(\xi)\, e^{-y \frac{\left(1+\ell^2\xi^2\right)^{1/2}}{\ell}} \tag{A3}$$

Now, we impose at the origin of the infinite (x, y)-plane a single screw dislocation with Burger's vector $b = (0, 0, b)$. In the upper half-plane, the screw dislocation gives rise to the following boundary value problem

$$w\left(x, 0^+\right) = -\frac{b}{2} H(x), \tag{A4a}$$

$$m_{yx}\left(x, 0^+\right) = 0, \tag{A4b}$$

where $H(x)$ is the Heaviside step function and the minus sign in (A4a) is justified from the sign convention that is adopted in dislocation theory. In view now of the constitutive equation (43d) and the properties of the Fourier transform, the boundary conditions (A4a,b) furnish in the transform domain

$$w^*\left(\xi, 0^+\right) = -b(\pi/2)^{1/2}\, \delta_+(\xi), \tag{A5a}$$

$$m^*_{yx}\left(\xi, 0^+\right) = 2\eta \frac{d^2 w^*}{dy^2} + 2\eta' \xi^2 w^* = 0, \tag{A5b}$$

where $\delta_+(\xi) = 1/2\,(\delta(\xi) + i/\pi\xi)$ is the Heisenberg delta function (Roos 1969) and $\delta(\xi)$ is the Dirac distribution. The constants $A_1(\xi)$ and $A_2(\xi)$ are now computed using the transformed boundary conditions i.e.

$$A_1(\xi) = -(\pi/2)^{1/2}\, b\left(1 + \ell^2(1+\beta)\,\xi^2\right)\delta_+(\xi),$$
$$A_2(\xi) = (\pi/2)^{1/2}\, b\ell^2(1+\beta)\,\xi^2\delta_+(\xi), \qquad \text{(A6a,b)}$$

where $\beta = \eta'/\eta$. With the aid of the inversion formula in (A1b), we obtain the integral representation for the displacement field due to a screw dislocation

$$w(x,y) = -\frac{b}{2}\int_{-\infty}^{\infty}\left[e^{-y|\xi|} - (\xi\ell)^2(1+\beta)\left(e^{-y|\xi|} - e^{-\frac{y\left(1+\ell^2\xi^2\right)^{1/2}}{\ell}}\right)\right]\delta_+(\xi)\,e^{-ix\xi}d\xi. \qquad \text{(A7)}$$

Using the properties of the Heisenberg delta function and the Dirac distribution, we finally obtain

$$w(x,y) = -\frac{b}{4} - \frac{b}{2\pi}\int_0^{\infty}\frac{e^{-y\xi}}{\xi}\sin(\xi x)d\xi - \frac{b\ell^2(1+\beta)}{2\pi}\int_0^{\infty} \times\,\xi\left(e^{-y\xi} - e^{-\frac{y\left(1+\ell^2\xi^2\right)^{1/2}}{\ell}}\right)\sin(\xi x)\,d\xi. \qquad \text{(A8)}$$

The above integrals can be determined in closed form. In particular, we have

$$\int_0^{\infty}\frac{e^{-y\xi}}{\xi}\sin(\xi x)\,d\xi = \tan^{-1}\frac{x}{y},$$
$$\int_0^{\infty}\xi e^{-y\xi}\sin(\xi x)d\xi = 2\frac{xy}{r^4},$$
$$\int_0^{\infty}\xi e^{-\frac{y\left(1+\ell^2\xi^2\right)^{1/2}}{\ell}}\sin(\xi x)\,d\xi = \frac{xy}{(r\ell)^2}K_2\left(\frac{r}{\ell}\right). \qquad \text{(A9a,b,c)}$$

In light of the above results, the displacement can be written as

$$w = \frac{b}{2\pi}\theta - \frac{b}{4\pi}(1+\beta)\left[\frac{2\ell^2}{r^2} - K_2(r/\ell)\right]\sin 2\theta. \qquad \text{(A10)}$$

Appendix B: Construction of numerical quadrature

The problem of finding a numerical quadrature for integrals with order of singularity greater than two ($a > 2$) arises naturally in generalized continuum theories where the field equations and the boundary conditions are of higher order than the respective ones in classical elasticity. Although a lot of work has been done in the literature for Hadamard type integrals ($a = 2$) (see e.g. Kutt 1975; Paget 1981; Ioakimidis 1983, 1995; Kaya and Erdogan 1987; Monegato 1987, 1994; Tsamasphyros and Dimou 1990; Korsunsky 1998; Kabir et al. 1998; Hui and Shia 1999), only a few papers have been published concerning integrals with $a > 2$. In a recent work by Chan et al. (2003), a systematic treatment of hypersingular integrals was presented based on the Kaya/Erdogan approach. This approach leads to very good results, with the only caveat that when the kernel cannot be explicitly given in terms of a sum of the hypersingular part and a remainder, the extraction of a strong singularity may lead to a loss of accuracy. Our intention here is to derive a numerical quadrature for the hypersingular integral

$$S(t) = \fint_{-1}^{1} f(s)\,w(s)\,\frac{ds}{(s-t)^3} \qquad \text{for } |t| < 1, \quad \text{(B.1)}$$

where $f(s)$ is a bounded and continuous function in the interval $[-1, 1]$, and $w(s) = \left(1-s^2\right)^{1/2}$ is the weight function corresponding to the second-kind Chebyshev polynomials U_j. The integral in (B.1) is to be understood in the Hadamard finite-part sense (Kutt 1975; Paget 1981). The basic steps in the development of the quadrature follow the strategy introduced by Korsunsky (1998).

The unknown function can be approximated with a sufficient degree of accuracy by a truncated series of second-kind Chebyshev polynomials

$$f(s) \cong \sum_{j=0}^{p} B_j U_j(s). \qquad \text{(B.2)}$$

Making use of the relation for the Cauchy principal-value integral (Abramowitz and Stegun 1964)

$$\fint_{-1}^{1} U_j(s)\left(1-s^2\right)^{1/2}\frac{ds}{(s-t)} = -\pi T_{j+1}(t), \quad \text{(B.3)}$$

(B.1) can be rewritten as

$$S(t) = \int_{-1}^{1} f(s)\, w(s) \frac{ds}{(s-t)^3} = \frac{1}{2}\frac{d^2}{dt^2} \times \int_{-1}^{1} f(s)\left(1-s^2\right)^{1/2} \frac{ds}{(s-t)} = \frac{1}{2}\frac{d^2}{dt^2}\sum_{j=0}^{p} B_j \int_{-1}^{1} U_j(s)\left(1-s^2\right)^{1/2} \times \frac{ds}{(s-t)} = -\frac{\pi}{2}\sum_{j=0}^{p} B_j T''_{j+1}(t), \quad \text{(B.4)}$$

where the prime denotes differentiation with respect to t. We note that the interchange of the order of differentiation and integration in (B.4) is valid in view of results by Monegato (1994).

Next, we establish the following identity

$$-\frac{T_{j+1}(t)\, U_n(t) - T_{n+1}(t)\, U_j(t)}{U_n(t)} = \sum_{i=1}^{n} \frac{a_i}{s_i - t} \quad \text{for } j < n,\ s_i : U_n(s_i) = 0, \quad \text{(B.5)}$$

where the partial-fraction expansion above is possible because the degree of the numerator in the left hand side of (B.5) is less than that of the denominator. It can easily be found (Korsunsky 1998) that the coefficients a_i in (B.5) are given by the relation

$$a_i = \frac{\left(1-s_i^2\right)}{1+n} U_j(s_i). \quad \text{(B.6)}$$

Equation (B.5) takes now the form

$$-T_{j+1}(t) + \frac{T_{n+1}(t)\, U_j(t)}{U_n(t)} = \sum_{i=1}^{n} \frac{1-s_i^2}{(1+n)(s_i-t)} U_j(s_i). \quad \text{(B.7)}$$

Differentiating (B.7) twice with respect to t and selecting a discrete set of points t_k, $k = 1, \ldots, n+1$ such that $T_{n+1}(t) = 0$, we obtain

$$-T''_{j+1}(t_k) + T''_{n+1}(t_k)\frac{U_j(t_k)}{U_n(t_k)} + 2T'_{n+1}(t_k)\frac{U'_j(t_k)\, U_n(t_k) - U_j(t_k)\, U'_n(t_k)}{U_n^2(t_k)} = 2\sum_{i=1}^{n} \frac{1-s_i^2}{(1+n)(s_i-t)^3} U_j(s_i). \quad \text{(B.8)}$$

Further, employing the well known identities about the derivatives of Chebyshev polynomials

$$T'_{n+1}(t) = (n+1)\, U_n(t), \quad n \ge 0, \quad \text{(B.9)}$$

$$U'_n(t) = \left(1-t^2\right)^{-1}\left[-(n+1)\, T_{n+1}(t) + t U_n(t)\right], \quad n \ge 0, \quad \text{(B.10)}$$

we write (B.8), after some lengthy algebra, under the form

$$-T''_{j+1}(t_k) + 2(1+n)\, U'_j(t_k) - (1+n)\frac{t_k}{1-t_k^2} \times U_j(t_k) = 2\sum_{i=1}^{n} \frac{1-s_i^2}{(1+n)(s_i-t_k)^3} U_j(s_i). \quad \text{(B.11)}$$

Using (B.2), multiplying (B.9) by $\frac{\pi}{2}B_j$ and summing over j from 0 to p, we then get

$$S(t_k) \cong -\pi(1+n)\, f'(t_k) + \frac{\pi}{2}(1+n)\frac{t_k}{1-t_k^2} f(t_k) + \pi\sum_{i=1}^{n} \frac{1-s_i^2}{(1+n)(s_i-t_k)^3} f(s_i). \quad \text{(B.12)}$$

One further step is needed now that would lead to the evaluation of the right hand side of (B.12) only at n points $s_i : U_n(s_i) = 0$. This can be done with the aid of the Lagrange interpolation formula, which will be exact within the class of polynomials chosen to represent $f(t)$

$$f(t) = \sum_{i=1}^{n} \frac{U_n(t)}{U'_n(s_i)(t-s_i)} f(s_i). \quad \text{(B.13)}$$

Differentiating (B.13) with respect to t and then substituting t with $t = t_k : T_{n+1}(t) = 0$, we get

$$f'(t_k) = \sum_{i=1}^{n} U_n(t_k)\left[\frac{t_k}{1-t_k^2} - \frac{1}{(t_k-s_i)}\right] \times \frac{f(s_i)}{U'_n(s_i)(t_k-s_i)}. \quad \text{(B.14)}$$

In light of the above analysis, (B.12) can be written as

$$S(t_k) \cong \pi\sum_{i=1}^{n}\left\{-U_n(t_k)\left[\frac{1}{(s_i-t_k)} + \frac{t_k}{2\left(1-t_k^2\right)}\right] \times \frac{1}{(s_i-t_k)\, T_{n+1}(s_i)} + \frac{1}{(1+n)(s_i-t_k)^3}\right\} f(s_i)\left(1-s_i^2\right), \quad \text{(B.15)}$$

where

$$t_k : T_{n+1}(t_k) = 0,\ t_k = \cos\left(\frac{(2k-1)\pi}{2(1+n)}\right), \quad k = 1, \ldots, n+1,$$

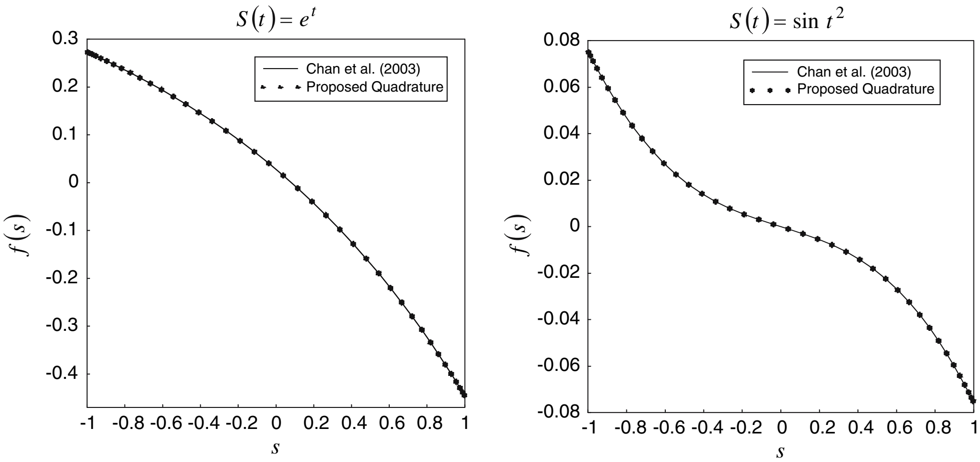

Fig. B1 Solution of the hypersingular integral equation (B.1) using the proposed quadrature and comparison with the semi-analytical method of Chan et al. (2003)

$$s_i : U_n(s_i) = 0, \quad s_i = \cos\left(\frac{i\pi}{1+n}\right), i = 1, \ldots, n.$$

Finally, taking into account that

$$U_n(t_k) = (-1)^{k+1} \Big/ \left(1 - t_k^2\right)^{1/2}, \quad T_{n+1}(s_i) = (-1)^i, \tag{B.16}$$

we write the resulting formula under the form

$$\begin{aligned} &\neq\!\!\!\!\!\!\int_{-1}^{1} f(s)\left(1-s^2\right)^{1/2} \frac{ds}{(s-t)^3} \cong \\ &\pi \sum_{i=1}^{n} \left\{ (-1)^{i+k} \left[\frac{1}{(s_i - t_k)} + \frac{t_k}{2\left(1-t_k^2\right)} \right] \right. \\ &\times \left. \frac{1}{(s_i - t_k)\left(1-t_k^2\right)^{1/2}} + \frac{1}{(1+n)(s_i - t_k)^3} \right\} \\ &\times f(s_i)\left(1 - s_i^2\right). \end{aligned} \tag{B.17}$$

It is noteworthy, that formula (B.17) also holds in precisely the same form for the more general case when the integral kernel is split up into a hypersingular part of order $a = 3$ and a remainder

$$K(s,t) = \frac{1}{(s-t)^3} + k(s,t), \tag{B.18}$$

where the remainder may consist of Cauchy type and regular kernels. In that case, (B.17) takes the form

$$\begin{aligned} &\neq\!\!\!\!\!\!\int_{-1}^{1} K(s,t) f(s)\left(1-s^2\right)^{1/2} ds \\ &\cong \pi \sum_{i=1}^{n} \left\{ (-1)^{i+k} \left[\frac{1}{(s_i - t_k)} + \frac{t_k}{2\left(1-t_k^2\right)} \right] \right. \\ &\times \left. \frac{1}{(s_i - t_k)\left(1-t_k^2\right)^{1/2}} + \frac{K(s_i, t_k)}{(1+n)} \right\} f(s_i)\left(1 - s_i^2\right). \end{aligned} \tag{B.19}$$

To check the validity of the proposed quadrature, we solve the hypersingular integral equation in (B.1) for two cases, i.e. for the loading function $S(t)$ being defined as: (i) $S(t) = e^t$, and (ii) $S(t) = \sin t^2$. For single-valuedness, the following auxiliary condition should also be taken into account

$$\int_{-1}^{1} f(s)\left(1-s^2\right)^{1/2} ds = 0. \tag{B.20}$$

Then, (B.17) and (B.20) form a system of $n+2$ equations in n unknowns which is solved in the least-squares sense. It is shown (see Fig. B1) that our results are in excellent agreement with the ones obtained by using the semi-analytical method of Chan et al. (2003).

Appendix C: Evaluation of the strain-energy density at crack tips and the J-integral

Our aim here is to show the orders of singularities of the stress and couple-stress fields obtained in the main body of the paper lead to an integrable strain-energy density in the vicinity of crack tips and also lead to a bounded value of the J-integral. The procedure followed is analogous in many respects with the one adopted in the work by Georgiadis (2003).

The strain-energy density function in (20) reads, in terms of stresses

$$W = \frac{1}{2\mu}\left[\tau_{ij}\tau_{ij} - \frac{\nu}{1+\nu}\tau_{ii}\tau_{jj} + \frac{1}{4\ell^2\left(1-\beta^2\right)}\left(m_{ij}m_{ij} - \beta m_{ij}m_{ji}\right)\right], \tag{C1}$$

where β is the ratio of the couple-stress moduli defined as $\beta = \eta'/\eta$.

Further, the path-independent J-integral within the couple-stress theory is given by (Atkinson and Leppington 1974; Lubarda and Markenskoff 2000)

$$J = \int_\Gamma \left[W n_x - P_q \frac{\partial u_q}{\partial x} - R_q \frac{\partial \omega_q}{\partial x}\right] d\Gamma = \int_\Gamma \left(W dy - \left[P_q \frac{\partial u_q}{\partial x} + R_q \frac{\partial \omega_q}{\partial x}\right] d\Gamma\right), \tag{C2}$$

where a Cartesian rectangular coordinate system is attached to the RHS crack tip with the distance x measured now from the tip, Γ is a piece-wise smooth simple two-dimensional contour surrounding the crack-tip, W is the strain-energy density, u_q is the displacement, ω_q is the rotation, P_q is the force-traction defined in (18), and R_q is the couple-traction defined in (19).

For the evaluation of the J-integral, we consider the rectangular-shaped contour Γ in Fig. C1 with vanishing "height" along the y- direction and with $\varepsilon \to +0$. This type of contour permits using solely the asymptotic near-tip stress and displacement fields. It is noted that upon this choice of contour, the integral $\int_\Gamma W dy$ in (C2) becomes zero if we allow the 'height' of the rectangle to vanish. In this way, the expression for the J-integral becomes

$$J = \lim_{\varepsilon \to +0}\left\{2\int_{-\varepsilon}^{\varepsilon}\left(P_q \frac{\partial u_q}{\partial x} + R_q \frac{\partial \omega_q}{\partial x}\right) dx\right\}. \tag{C3}$$

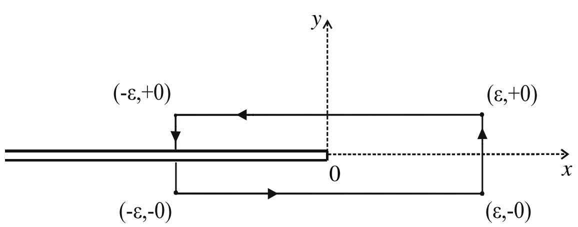

Fig. C1 Rectangular-shaped contour surrounding the cracktip

The cases of mode II and mode III cracks are examined in what follows.

Mode II

In the case of plane-strain, the strain-energy density reads

$$W = \frac{1}{4\mu}\left[(1-\nu)\left(\tau_{xx}^2 + \tau_{yy}^2\right) + 2\tau_{xy}^2 - 2\nu\tau_{xx}\tau_{yy}\right] + \frac{1}{8\mu\ell^2}\left(m_{xz}^2 + m_{yz}^2\right). \tag{C4}$$

As shown before, the couple-stresses $\left(m_{xz}, m_{yz}\right)$ are bounded (non-singular) in the crack-tip vicinity in the mode II case, whereas both the asymmetric and symmetric stresses exhibit a square root singularity (see also Huang et al. 1997). Now, the term in square brackets in (C4) is the same as in classical elasticity and behaves in exactly the same way, while the second term (the one involving couple-stresses) is bounded in the crack-tip vicinity. Therefore, by following the standard procedure to check upon the integrability of the strain-energy density around a singularity (see e.g. Barber 1992), we conclude that the strain-energy density is *integrable* indeed in the crack-tip vicinity.

Further, taking into account that in the mode II case both the normal stress σ_{yy} and the couple-stress m_{yz} are zero along the crack line $\left(y = 0^{\pm}\right)$ and that the crack-faces are defined by $\mathbf{n} = (0, \pm 1)$, the J-integral in (C3) finally takes the form

$$J = \lim_{\varepsilon \to +0}\left\{2\int_{-\varepsilon}^{\varepsilon}\left(\sigma_{yx}(x, y=0) \cdot \frac{\partial u_x(x, y=0)}{\partial x}\right) \times dx\right\}. \tag{C5}$$

Now, in view of the asymptotic behavior of the fields entering (C5), we obtain

$$J = \lim_{\varepsilon \to +0}\left\{\frac{A_{II}^2}{\mu}\int_{-\varepsilon}^{\varepsilon}(x_+)^{-1/2}(x_-)^{-1/2} dx\right\} = \frac{\pi A_{II}^2}{2\mu}, \tag{C6}$$

where the product of distributions inside the integral was obtained by the use of Fisher's theorem (see e.g. Georgiadis 2003), i.e. the operational relation $(x_-)^{\lambda}(x_+)^{-1-\lambda} = -\pi\delta(x)\left[2\sin(\pi\lambda)\right]^{-1}$ with $\lambda \neq -1, -2, -3, \ldots$ and $\delta(x)$ being the Dirac delta distribution. Finally, we note that the amplitude factor A_{II} is connected with the asymptotic results of Huang et al. (1997), in the mode II case, through the relation $A_{II} = 2^{1/2}\left[(3-2\nu)(1-\nu)\right]^{1/2} B_{II}$.

Mode III

In this case, the strain-energy density is given by

$$W = \frac{1}{2\mu}\left(\tau_{xz}^2 + \tau_{yz}^2\right) + \frac{1}{8\mu\ell^2\left(1-\beta^2\right)}\left[(1-\beta) \times \left(m_{xx}^2 + m_{yy}^2\right) + \left(m_{xy}^2 + m_{yx}^2\right) - 2\beta m_{xy} m_{yx}\right]. \quad \text{(C7)}$$

Based on the results of our analysis for the mode III case, we notice that the couple-stresses behave as $\sim r^{-1/2}$ around the crack tip, while the symmetric stresses behave as $\sim r^{1/2}$. Thus, by invoking again the standard procedure involving the evaluation of a volume integral around the singularity (see e.g. Barber 1992), we conclude that the strain-energy density in (C7) is integrable in the crack-tip vicinity and the strain energy itself is bounded.

Next, taking into account that the couple-stress m_{yx} is identically zero along the crack line in the mode III problem, the J-integral takes the following form

$$J = \lim_{\varepsilon \to +0}\left\{2\int_{-\varepsilon}^{\varepsilon}\left(t_{yz}\,(x, y=0) \cdot \frac{\partial w\,(x, y=0)}{\partial x}\right)dx\right\}, \quad \text{(C8)}$$

where $t_{yz} = \sigma_{yz} + (1/2)\,\partial_x m_{yy}$ is the total shear stress which, as shown before, exhibits a near-tip behavior as $-(r^{-3/2})$. In light of the above, we obtain

$$J = \lim_{\varepsilon \to +0}\left\{3\mu A_{III}^2 \int_{-\varepsilon}^{\varepsilon} -\,(x_+)^{-3/2}\,(x_-)^{1/2}\,dx\right\} = \frac{3\pi\mu A_{III}^2}{2}, \quad \text{(C9)}$$

where A_{III} is an amplitude factor (constant) dependent upon both couple-stress moduli and the remote loading. The above result shows that the J-integral is also bounded in the mode III case (despite the hypersingular nature of the near-tip total shear stress). Finally, we note that in the special case where the second couple-stress modulus is set equal to zero (i.e. $\beta = 0$), A_{III} above is connected with the amplitude factor B in the work by Zhang et al. (1998) through the relation $A_{III} = 2B\ell$.

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