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Distributed Economic Dispatch Control via Saddle Point Dynamics and Consensus Algorithms

Lu Bai, Maojiao Ye, Chao Sun, and Guoqiang Hu

***Abstract*—In this paper, a distributed control algorithm is proposed to solve the economic dispatch problem. Without a central control unit, the generators work collaboratively to minimize the generation cost while balancing the supply and demand. The proposed method is based on consensus protocols and the saddle point dynamics. The consensus protocols are employed to estimate the global information in a distributed fashion, and the saddle point dynamics are leveraged to search for the optimal solution of the economic dispatch problem. By utilizing Lyapunov stability analysis, exponential stability of the optimal solution is derived if the capacity limits of the generators are not considered; with the capacity limits, practical stability of the optimal solution is obtained. No global information is needed in the proposed method and the requirement on initial conditions of the state variables is mild. Several case studies on the IEEE 9-bus and IEEE 118-bus systems are presented to demonstrate the effectiveness of the proposed algorithms.**

***Index Terms*—Distributed control, economic dispatch, saddle point dynamics, consensus algorithms.**

I. Introduction

With the development of smart grids, the control and optimization problems (e.g., micro grid operation [1], demand response [2], [3], economic dispatch [4]) have received renewed attention in recent years. In a smart grid, integrated distributed energy resources work collaboratively to serve loads over a common distribution network [5]. Based on the load demand, economic dispatch produces generation references for the distributed energy resources such that the total generation cost is minimized. The basic economic dispatch problem studies how to economically manage the thermal distributed generators such that the total generation cost is minimized under the balance and capacity constraints. It emerges to be an optimization problem of both theoretical significance and practical relevance [5].

Centralized algorithms have been proposed to solve the economic dispatch problem traditionally. The existing methods include analytical algorithms (e.g., Lagrangian relaxation [6], λ-iteration [7]) and heuristic algorithms (e.g., genetic algorithm [8], particle swarm algorithm [9]), etc. However, the centralized approaches are usually expensive to implement in large scale systems, the existence of the central control unit will result in system fragility with respect to central node failure, and a centralized architecture is usually unscalable. To overcome the drawbacks of centralized approaches and meet the requirements of smart grid, distributed algorithms have been proposed to solve the economic dispatch problem.

In the distributed methods, a lot of works utilize consensus algorithms on the incremental cost to solve the economic dispatch problem in a distributed manner. In [10], the incremental costs are synchronized in a distributed manner to solve the economic dispatch problem under different communication topologies. In [11], based on the consensus over the incremental costs, an innovation term with a vanishing learning gain is designed to accommodate the balance constraint. In [4], the generators collectively learn the power mismatch for the computation of the incremental costs. An average consensus based bisection approach is proposed in [12], where a bisection algorithm, based on the average value of the load, is used to calculate the incremental costs. Communication information losses are investigated for the economic dispatch problem in [13], in which a robust distributed system incremental cost estimation algorithm is introduced. In [14], the θ-logarithmic barrier method and a consensus approach are utilized to develop a distributed event-triggered scheme. A projected gradient and finite-time average consensus based strategy is proposed for the economic dispatch problem in [15], where both thermal generators and wind turbines are considered. By incorporating the box constraints into the objective function using non-smooth penalties, an initialization-free distributed coordination algorithm for economic dispatch is proposed in [16]. In this method, a dynamic average consensus algorithm is utilized to estimate the mismatch between the generation and the load in a distributed manner, and based on these estimates, a distributed Laplacian-nonsmooth-gradient algorithm is designed to dynamically allocate the unit generation levels. In [17], a category of proportional-integral type consensus and projection based continuous time algorithm is proposed to solve the distributed optimization problem in an initialization-free manner.

Nevertheless, in the majority of the existing distributed methods, the initial values of the consensus variables need to meet stringent requirements to achieve the balance between supply and load, or global information, such as the entire communication network topology and the total number of the generators, is needed or calculated. These requirements impose an initialization procedure in these algorithms, which is restrictive since it involves global coordination and has to be repeatedly performed if the network configuration changes, load changes, or any distributed generator plugs in or leaves off [18]. Moreover, an error in the initial values will result in power mismatch between the total generation and total load.

In this paper, distributed initialization-free algorithms based

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on saddle point dynamics and consensus algorithms are proposed to solve the economic dispatch problem. The consensus algorithms are employed to estimate the global information in a distributed fashion. With the estimated information, the saddle point dynamics are adapted to search for the optimal solution of the economic dispatch problem. The stability of the closed loop system is proved through Lyapunov stability analysis and the singular perturbation theory, which is often used to analyze the stability of a singular perturbed system with two time scales, such as the stability of algorithms which utilize consensus protocols [19]–[22]. The methods proposed in this paper are distributed as only local information is required for each node to implement the proposed method. Furthermore, no initialization procedure is required to implement the algorithm since no global information, such as the entire communication network topology and the total number of the buses, is needed and the requirement on the initial values of the state variables is mild. Specifically, the initial values of the variables in the algorithm can be arbitrarily set except that the Lagrange multipliers corresponding to the capacity constraints need to be positive initially.

The rest of this paper is organized as follows. In Section II, some preliminaries are provided. The problem is formulated in Section III. Section IV presents the main results, where the algorithms are presented and the stability and optimality of the solutions are proved through singular perturbation theory and Lyapunov stability analysis. Case studies based on the IEEE 9-bus and IEEE 118-bus systems are given in Section V. Finally, conclusion is drawn in Section VI.

II. Preliminaries

Throughout this paper, $\mathbf{1}_n$ and $\mathbf{0}_n$ denote n dimensional column vectors composed of 1 and 0, respectively, $\mathbf{0}_{n\times n}$ denotes an $n\times n$ dimensional matrix composed of 0, and I_n denotes an $n\times n$ dimensional unit matrix. Furthermore, $\text{diag}\{x_i\}$ for $i\in\{1,2,...,n\}$ represents an $n\times n$ dimensional diagonal matrix with ith diagonal element equals x_i, and $[x_i]_{\text{vec}}$ for $i\in\{1,2,...,n\}$ is defined as $[x_i]_{\text{vec}}=[x_1,x_2,...,x_n]^T$.

A graph is defined as $\mathcal{G}\triangleq(\mathcal{V},\mathcal{E})$, where $\mathcal{V}\triangleq\{1,2,...,n\}$ is a set of vertices and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is a set of edges. It is undirected if for every $(i,j)\in\mathcal{E}$, $(j,i)\in\mathcal{E}$. An undirected graph is connected if there exists a path between any two distinct vertices. Denote the set of neighbors of vertex i by

$$\mathcal{N}_i(\mathcal{E})\triangleq\{j\in\mathcal{V}|(i,j)\in\mathcal{E}\}.$$

The adjacency matrix A of a graph represents the connectivity of a graph. For an undirected graph, the elements a_{ij} in A is defined as $a_{ii}=0$, $a_{ij}=a_{ji}>0$ if node j is connected with node i, and $a_{ij}=0$ otherwise. The degree matrix D of a graph is a diagonal matrix with the ith diagonal element being $\sum_{j\in\mathcal{N}_i}a_{ij}$. The Laplacian matrix L of a graph is defined as $L\triangleq D-A$. For an undirected and connected graph, the eigenvalues of L are all positive except for one zero eigenvalue.

Lemma 1. *[23] Let $\mathcal{G}$ be an undirected and connected graph, L be the Laplacian matrix of $\mathcal{G}$. Then, for any constant input $u\in\mathbb{R}^n$ and any initial states $x(0),\lambda(0)\in\mathbb{R}^n$, the trajectories of the system*

$$\begin{bmatrix}\dot{x}\\ \dot{\lambda}\end{bmatrix}=\begin{bmatrix}-I-L & -L\\ L & \mathbf{0}_{n\times n}\end{bmatrix}\begin{bmatrix}x\\ \lambda\end{bmatrix}+\begin{bmatrix}u\\ \mathbf{0}_n\end{bmatrix}$$

represented by $x(t)$ and $\lambda(t)$ converge to constant vectors, and more specifically, $x(t)\to\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^Tu$ exponentially as $t\to\infty$.

Lemma 2. *[24] Let $\mathcal{G}$ be an undirected and connected graph. Then, $x_i\to r_0$ as $t\to\infty$ if at least one $a_{i0}>0$ for $i\in\{1,...,n\}$ under the following algorithm,*

$$\dot{x}_i=-\sum_{j=1}^{n}a_{ij}(x_i-x_j)-a_{i0}(x_i-r_0),$$

where r_0 is a time-invariant signal, $a_{i0}>0$ if node i can get the information of r_0 and $a_{i0}=0$, otherwise.

Lemma 3. *[25] Let $\mathcal{A}\in\mathbb{R}^{n\times n}$ and $\mathcal{B}\in\mathbb{R}^{n\times m}$. If $\mathcal{A}$ is Hurwitz and rank$(\mathcal{B})=m$, then the matrix $\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ -\mathcal{B}^T & \mathbf{0}_{m\times m}\end{bmatrix}$ is Hurwitz.*

III. Problem Formulation

Consider a power grid with distributed thermal generators and distributed loads. Suppose there are n buses, without loss of generality, assume that each bus contains one generator (G) and one local load (L). Each bus is equipped with a local agent (A), which can communicate with its neighboring agents and is used to produce the optimal generation reference for the generator. The control architecture is illustrated in Fig. 1.

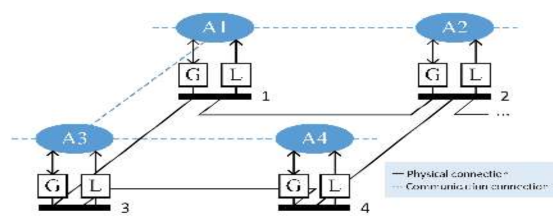


Fig. 1. Illustration of the distributed control architecture.

Let P_{Gi} denote the active power generated by the generator in bus i and P_{Di} denote the active power needed by the load in bus i. The basic economic dispatch problem is formulated as

$$\begin{aligned}&\text{minimize } f(P_G)=\sum_{i=1}^{n}f_i(P_{Gi}),\\ &\text{subject to } \sum_{i=1}^{n}P_{Gi}=\sum_{i=1}^{n}P_{Di},\\ &P_{Gi}^{\min}\le P_{Gi}\le P_{Gi}^{\max},\ i=1,2,...,n,\end{aligned}\tag{1}$$

where $P_G=[P_{Gi}]_{\text{vec}}$, $f_i(P_{Gi})$ is the cost function of the generator in bus i, which is modeled as a quadratic function given by [5]

$$f_i(P_{Gi})=a_iP_{Gi}^2+b_iP_{Gi}+c_i,$$

where $a_i>0$, $b_i\ge0$ and $c_i\ge0$ are the cost parameters. In (1), $P_{Gi}^{\min}$ and $P_{Gi}^{\max}$ are the lower and upper bounds of P_{Gi} with $P_{Gi}^{\min}<P_{Gi}^{\max}$.

Remark 1. *If in bus i, there is no load, then $P_{Di} = 0$; if there are more than one load, they can be aggregated into one single load. If there is no generator in bus i, then $P_{Gi}(t) = 0$, $\forall t$; if there are more than one generator in one bus, they can be modeled as several buses with one generator located in each bus.*

The rest of the paper is based on the following assumptions.

Assumption 1. *The communication graph among the agents is undirected and connected.*

Assumption 2. *The economic dispatch problem in* (1) *is feasible, i.e.,*

$$\sum_{i=1}^{n} P_{Gi}^{\min} \le \sum_{i=1}^{n} P_{Di} \le \sum_{i=1}^{n} P_{Gi}^{\max}.$$

IV. Main Results

In this section, a distributed method is proposed to solve the problem in (1). In Section IV.A, the inequality constraints are ignored, while in Section IV.B, the inequality constraints are considered. In Section IV.C, the line flow constraints are taken into consideration on the basis of the problem in (III).

A. Algorithm Design without Inequality Constraints

1) Centralized Approach Based on the Saddle Point Dynamics: Without the inequality constraints, the Lagrangian function of the problem in (1) is

$$L_1(P_G, \lambda_0) = f(P_G) + \lambda_0 \left(\sum_{i=1}^{n} P_{Di} - \sum_{i=1}^{n} P_{Gi} \right),$$

where $\lambda_0 \in \mathbb{R}$ is the Lagrange multiplier corresponding to the equality constraint. Denote the optimal solution of the problem in (1) (without capacity limits) as P_G^*. Since the problem in (1) is convex, there is a λ_0^* such that (P_G^*, λ_0^*) is the saddle point of $L_1(P_G, \lambda_0)$ [26]. Inspired by the saddle point dynamics [27], the following equations

$$\dot{P}_{Gi} = -\frac{\partial L_1}{\partial P_{Gi}} = -\left(\frac{df_i(P_{Gi})}{dP_{Gi}} - \lambda_0 \right), \ i \in \{1, ..., n\}, \quad (2a)$$

$$\dot{\lambda}_0 = \frac{\partial L_1}{\partial \lambda_0} = \sum_{i=1}^{n} P_{Di} - \sum_{i=1}^{n} P_{Gi}, \quad (2b)$$

can be utilized to search for the saddle point (P_G^*, λ_0^*).

However, λ_0 in (2a), the values of $\sum_{i=1}^{n} P_{Di}$ and $\sum_{i=1}^{n} P_{Gi}$ in (2b) are global information. Hence, a central unit is needed to implement the algorithm in (2). In the following, a distributed method is developed.

2) Distributed Approach: The distributed updating laws for P_{Gi} and λ_0 are designed as

$$\dot{P}_{Gi} = -\bar{k}_i \left(\frac{df_i(P_{Gi})}{dP_{Gi}} - \lambda_i \right), i = \{1, ..., n\}, \quad (3a)$$

$$\dot{\lambda}_0 = \bar{m}_0 z_q, \quad (3b)$$

where $\bar{k}_i = \delta k_i$, $\bar{m}_0 = \delta m_0$, δ is a small positive parameter to be determined, k_i and m_0 are fixed positive parameters, λ_i is agent i's estimation of λ_0, and z_q is agent q's estimation of $\frac{1}{n}\sum_{i=1}^{n}(P_{Di} - P_{Gi})$. Any z_q, $q \in \{1, ..., n\}$ can be selected to calculate λ_0. The estimation algorithms are designed based on the dynamic average consensus algorithm in Lemma 1 and the leader-following consensus algorithm in Lemma 2 as follows:

$$\dot{z}_i = -z_i - \sum_{j=1}^{n} a_{ij}(z_i - z_j) - \sum_{j=1}^{n} a_{ij}(\theta_i - \theta_j) + P_{Di} - P_{Gi}, \quad (4a)$$

$$\dot{\theta}_i = \sum_{j=1}^{n} a_{ij}(z_i - z_j), \quad (4b)$$

$$\dot{\lambda}_i = -\sum_{j=1}^{n} a_{ij}(\lambda_i - \lambda_j) - a_{i0}(\lambda_i - \lambda_0), \quad (4c)$$

where θ_i for $i \in \{1, ..., n\}$ are intermediate variables.

In algorithms (3) and (4), each agent runs (3a) and (4), and one agent q, which can be arbitrarily selected, runs (3b). Specifically, each agent runs the dynamic average consensus algorithm (4a) and (4b) to collaboratively find the average value of the mismatch between total load and total generation and propagate this information among the network. Then, one agent q calculates λ_0 by (3b) using z_q, and each agent calculates λ_i by (4c) using λ_0. Finally, each agent updates P_{Gi} by (3a) using λ_i and other local information.

Theorem 1. *Suppose that the total load $\sum_{i=1}^{n} P_{Di}$ is finite and Assumption 1 hold. Then, there exists a positive constant δ^* such that for every $0 < \delta < \delta^*$, P_G converges to P_G^* exponentially under algorithms* (3) *and* (4).

Proof. See Appendix A for the proof. □

Remark 2. *In this algorithm, agent q serves as a leader which generates the signal λ_0. The algorithm is distributed since only a subset of the agents need to be connected with agent q. Furthermore, failure of agent q can be easily recovered by appointing any other agent, e.g., agent r, to calculate λ_0 using its own z_r. However, in centralized approaches, the failure of the central node is disastrous.*

B. Algorithm Design with Inequality Constraints

Assumption 3. *There is at least one generator whose optimal generation is not equal to its upper or lower bound. That is, $\exists$ i, such that $P_{Gi}^{\min} < P_{Gi}^* < P_{Gi}^{\max}$.*

With the inequality constraints considered, the Lagrangian function of the problem in (1) is

$$L_2(P_G, \gamma) = f(P_G) + \lambda_0 \left(\sum_{i=1}^{n} P_{Di} - \sum_{i=1}^{n} P_{Gi} \right) + \sum_{i=1}^{n} \alpha_i (P_{Gi} - P_{Gi}^{\max}) + \sum_{i=1}^{n} \beta_i (P_{Gi}^{\min} - P_{Gi}),$$

where $\gamma = [\lambda_0, \alpha_1, \beta_1, ..., \alpha_n, \beta_n]^T$, α_i and β_i, $i \in \{1, ..., n\}$ are the Lagrange multipliers corresponding to the inequality constraints. Since the problem in (1) is convex and the Slater's condition is satisfied indicated by Assumption 3, there exists $\gamma^* = [\lambda_0^*, \alpha_1^*, \beta_1^*, ..., \alpha_n^*, \beta_n^*]^T$, such that (P_G^*, γ^*) is the saddle point of $L_2(P_G, \gamma)$ with P_G^* being the optimal solution of the problem in (1), and (P_G^*, γ^*) being the solution of the following Karush-Kuhn-Tucker (KKT) conditions,

$$\frac{df_i}{dP_{Gi}}(P_{Gi}^*) - \lambda_0^* + \alpha_i^* - \beta_i^* = 0,\ \sum_{i=1}^n P_{Di} - \sum_{i=1}^n P_{Gi}^* = 0,$$
$$P_{Gi}^* - P_{Gi}^{\max} \le 0,\ P_{Gi}^{\min} - P_{Gi}^* \le 0,\ \alpha_i^*(P_{Gi}^* - P_{Gi}^{\max}) = 0,$$
$$\beta_i^*(P_{Gi}^{\min} - P_{Gi}^*) = 0,\ \alpha_i^* \ge 0,\ \beta_i^* \ge 0. \tag{5}$$

Motivated by [25], the distributed updating law for agent i is designed as

$$\dot{P}_{Gi} = -\bar{k}_i\left(\frac{df_i(P_{Gi})}{dP_{Gi}} - \lambda_i + \alpha_i - \beta_i\right), \tag{6a}$$
$$\dot{\alpha}_i = \bar{m}_{i1}\alpha_i(P_{Gi} - P_{Gi}^{\max}),\ \dot{\beta}_i = \bar{m}_{i2}\beta_i(P_{Gi}^{\min} - P_{Gi}), \tag{6b}$$

where $\bar{m}_{ij} = \delta m_{ij}$, m_{ij} for $i \in \{1, ..., n\}, j \in \{1, 2\}$ are fixed positive parameters, and $\alpha_i(0) > 0, \beta_i(0) > 0$ for $i \in \{1, ..., n\}$. For one agent $q \in \{1, ..., n\}$, except (6), it needs to calculate λ_0 the same as that in Section IV.A as

$$\dot{\lambda}_0 = \bar{m}_0 z_q. \tag{7}$$

The estimation algorithms are the same as (4).

Define $o = [P_G^T\ \alpha^T\ \beta^T\ \lambda_0\ \lambda^T\ z^T\ \theta_1^{T1}]^T$ and $o^* = [P_G^{*T}\ \alpha^{*T}\ \beta^{*T}\ \lambda_0^*\ \lambda_0^*\mathbf{1}_n^T\ \frac{1}{n}\mathbf{1}_n^T(P_G^* - P_D)\mathbf{1}_n^T\ \theta_1^{eT}(P_G^*)^1]^T$ with $P_D = [P_{Di}]_{\text{vec}}$. Then, the following result can be derived.

Theorem 2. *Suppose that Assumptions 1-3 hold. Then, for each pair of positive numbers (Δ, ζ), there exists a positive constant $\delta^*(\Delta, \zeta)$, such that for every $0 < \delta < \delta^*$, under algorithms* (4), (6), *and* (7) *with $\alpha(0) > 0, \beta(0) > 0$, there exists a $T > 0$ such that $\|o - o^*\| \le \zeta,\ \forall t > T$, for $\|o(0) - o^*\| \le \Delta$.*

Proof. See Appendix B for the proof. □

Remark 3. *The method can be directly extended to solve a more general problem as follows:*

$$\text{minimize } F(P_G) = \sum_{i=1}^n F_i(P_{Gi}),$$
$$\text{subject to } \sum_{i=1}^n P_{Gi} = \sum_{i=1}^n P_{Di},$$
$$g_{ij}(P_{Gi}) \le 0,\ i \in \{1, 2, ..., n\}, j \in \{1, ..., m_i\}, \tag{8}$$

where $P_{Gi} \in \mathbb{R}^m$, $F_i(P_{Gi})$ is twice continuously differentiable and strictly convex, $g_{ij}(P_{Gi})$ is twice continuously differentiable and convex, and m_i is the number of local inequality constraints for P_{Gi}. With slight revision, the control algorithm (6) *becomes*

$$\dot{P}_{Gi} = -\bar{k}_i\left(\frac{dF_i(P_{Gi})}{dP_{Gi}} - \lambda_i + \sum_{j=1}^{m_i}\alpha_{ij}\frac{dg_{ij}(P_{Gi})}{dP_{Gi}}\right), \tag{9a}$$
$$\dot{\alpha}_{ij} = \bar{m}_{ij}\alpha_{ij}g_{ij}(P_{Gi}). \tag{9b}$$

Under the following assumptions: 1. The solution of (8) *exists and is finite; 2. The slater's condition is satisfied; 3. $\forall i$, there exists at most one $j \in \{1, ..., m_i\}$ such that $g_{ij}(P_{Gi}^*) = 0$; 4. $\exists$ i, such that $g_{ij}(P_{Gi}^*) < 0$, $j \in \{1, ..., m_i\}$. Then, under algorithms* (4), (7), *and* (9), *the result in Theorem 2 holds.*

[1] θ_1 is a function of θ. $\theta_1^{eT}(P_G^*)$ is a function of P_G^*. The details can be found in the proof of Theorem 1 in Appendix A.

Remark 4. *In this algorithm,* (6b) *are utilized to make α and β satisfy the KKT conditions, such that P_G satisfies the inequality constraints. To handle the inequality constraints, [16] uses non-smooth penalty functions to incorporate the box constraints into the objective function, and [17] uses projection method on the basis that the local feasible sets can be explicitly obtained. Different from these intialization-free methods, our work utilizes the function of the inequality constraints as shown in Remark 3.*

C. Economic Dispatch with Line Flow Constraints

In this section, the line flow constraints are considered upon the basic economic dispatch problem. Suppose the transmission network is represented by graph $\mathcal{G}_t = (\mathcal{N}, \mathcal{E}_t)$ with $\mathcal{E}_t = \{1, ..., m\}$ being the transmission lines set. $(i, j) \in \mathcal{E}_t$ means bus i and bus j are connected by a transmission line $l \in \mathcal{E}_t$. Arbitrarily assign direction to each line l as the reference power flow direction, define the incidence matrix $B \in \mathbb{R}^{n\times m}$ with $B_{il} = 1$ if line l origins from bus i, $B_{il} = -1$ if line l goes to bus i, and $B_{il} = 0$, otherwise. The problem is formulated as [17]

$$\text{minimize } f(P_G) = \sum_{i=1}^n f_i(P_{Gi})$$
$$\text{subject to } P_{Gi} - \sum_{l=1}^m B_{il}P_{Ll} - P_{Di} = 0,\ i = 1, ..., n,$$
$$P_{Gi}^{\min} \le P_{Gi} \le P_{Gi}^{\max},\ i = 1, ..., n,$$
$$|P_{Ll}| \le P_{Ll}^{\max},\ l = 1, ..., m, \tag{10}$$

where P_{Ll} is the power flow on transmission line l, $P_{Ll}^{\max} > 0$ is the line flow limit of transmission line l. The solution of the problem in (10) is denoted by (P_G^*, P_L^*) with $P_L^* = [P_{Ll}^*]_{\text{vec}}$. Note that P_G^* is unique.

The augmented Lagrangian function of the problem in (10) is

$$L_3 = \sum_{i=1}^n (f_i(P_{Gi}) + \lambda_i(P_{Di} + \sum_{l=1}^m B_{il}P_{Ll} - P_{Gi})$$
$$+ \alpha_i(P_{Gi} - P_{Gi}^{\max}) + \beta_i(P_{Gi}^{\min} - P_{Gi}))$$
$$+ \sum_{l=1}^m (u_l(P_{Ll} - P_{Ll}^{\max}) + v_l(-P_{Ll} - P_{Ll}^{\max}))$$
$$+ \frac{1}{2}\sum_{i=1}^n (P_{Di} + \sum_{l=1}^m B_{il}P_{Ll} - P_{Gi})^2,$$

where $\lambda_i, \alpha_i, \beta_i, u_l, v_l \in \mathbb{R}$ are the Lagrange multipliers. By setting the communication graph $\mathcal{G}$ the same as the transmission graph $\mathcal{G}_t$, the distributed algorithm is designed as follows:

$$\dot{P}_{Gi} = -\kappa_{Gi}(\frac{df_i(P_{Gi})}{dP_{Gi}} - \lambda_i + \alpha_i - \beta_i$$
$$- (P_{Di} + \sum_{l=1}^m B_{il}P_{Ll} - P_{Gi})),\ i = 1, ..., n, \tag{11a}$$

$$\dot{P}_{Ll} = -\kappa_{Ll}(\sum_{i=1}^{n} B_{li}\lambda_i + u_l - v_l + \sum_{i=1}^{n} B_{li}(P_{Di} + \sum_{l=1}^{m} B_{il}P_{Ll} - P_{Gi})),\ l = 1, ..., m, \tag{11b}$$

$$\dot{\lambda}_i = \kappa_{\lambda i}(P_{Di} + \sum_{l=1}^{m} B_{il}P_{Ll} - P_{Gi}),\ i = 1, ..., n, \tag{11c}$$

$$\dot{\alpha}_i = \kappa_{\alpha i}\alpha_i(P_{Gi} - P_{Gi}^{\max}),\ i = 1, ..., n, \tag{11d}$$

$$\dot{\beta}_i = \kappa_{\beta i}\beta_i(P_{Gi}^{\min} - P_{Gi}),\ i = 1, ..., n, \tag{11e}$$

$$\dot{u}_l = \kappa_{ul}u_l(P_{Ll} - P_{Ll}^{\max}),\ l = 1, ..., m, \tag{11f}$$

$$\dot{v}_l = \kappa_{vl}v_l(-P_{Ll} - P_{Ll}^{\max}),\ l = 1, ..., m, \tag{11g}$$

where $\kappa_{Gi}, \kappa_{Ll}, \kappa_{\lambda i}, \kappa_{\alpha i}, \kappa_{\beta i}, \kappa_{ul}, \kappa_{vl}$ are positive constants, $\alpha_i(0) > 0$, $\beta_i(0) > 0$, $u_l(0) > 0$, and $v_l(0) > 0$.

Assuming that the problem in (10) is feasible, under algorithm (11), P_G converges to P_G^* asymptotically.

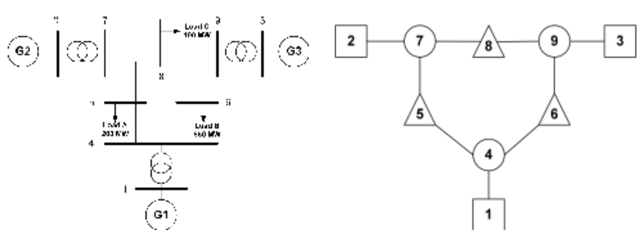
The conclusion can be obtained by following the proof of the stability of the auxiliary system in Theorem 2 noticing that (x^*, y^*), where $x^* = (P_G^*, P_L^*)$ and $y^* = (\lambda^*, \alpha^*, \beta^*, [u_l^*]_{\text{vec}}, [v_l^*]_{\text{vec}})$ being the optimal values of the Lagrange multipliers, is a saddle point of L_3, and $L_3(x, y^*) = L_3(x^*, y^*)$ implies (x, y^*) is a saddle point of L_3 [27].

V. Case Studies

A. Simulation Setup

First, simulation is conducted on the IEEE 9-bus system. Then, the algorithm is tested on the larger scale IEEE 118-bus system.

In the IEEE 9-bus system shown in Fig. 2 (a), there are 9 buses, 3 generators (located in bus 1, 2, 3) and 3 loads (located in bus 5, 6, 8). The parameters of the cost



(a) Simplified illustration (b) Communication topology

Fig. 2. Simplified illustration and the communication topology of the IEEE 9-bus system.

functions are given in Table I [13]. The power demand of

TABLE I
Generator Parameters

P_G	a_i	b_i	c_i	$P_{Gi}^{\min}$(MW)	$P_{Gi}^{\max}$(MW)
P_{G1}	0.001562	7.92	561	150	600
P_{G2}	0.00194	7.85	310	100	400
P_{G3}	0.00482	7.97	78	50	200

loads located in bus 5, 6, and 8 are 200 MW, 550 MW, and 100 MW, respectively. By direct calculation, the optimal generations are $P_{G1}^* = 393.1698$ MW, $P_{G2}^* = 334.6038$ MW, $P_{G3}^* = 122.2264$ MW, and $\lambda_0^* = 9.1483$. The communication topology is set the same as the physical connection of the buses shown in Fig. 2 (b). The squares represent buses with generators in them and the triangles represent buses with loads in them. Node 4 is selected to calculate λ_0. The initial values of the variables are randomly selected except that $\alpha(0) > \mathbf{0}_n$ and $\beta(0) > \mathbf{0}_n$.

B. Simulation Results

1) Without Capacity Limits: In this simulation, the capacity limits are not considered. The parameters are set as $\bar{k} = 1000 \times \mathbf{1}_9$ and $\bar{m}_0 = 0.003$. The simulation results produced by (3) and (4) are shown in Fig. 3. From the simulation results, it can

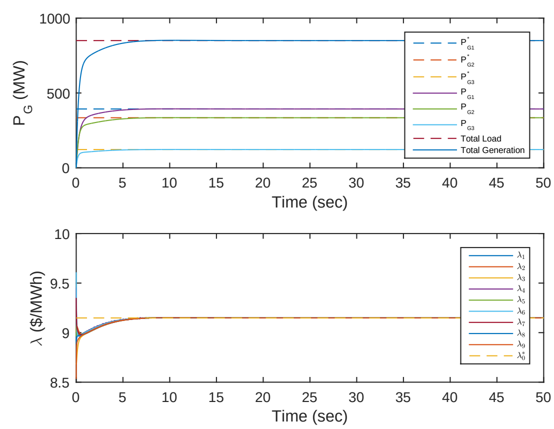


Fig. 3. Power generations of the generators and estimated incremental costs produced by (3) and (4) when capacity limits are not considered. The dashed lines denote the optimal values.

be seen that P_G converges to the optimal generation, the total generation converges to the total load, and all λ_is converge to λ_0^*.

2) With Capacity Limits: In this simulation and thereafter, the generator capacity limits are considered. The updating laws used are (4), (6), and (7). The parameters are set as $\bar{k} = 1000 \times \mathbf{1}_9$, $\bar{m}_0 = 0.003$, and $\bar{m}_1 = \bar{m}_2 = \mathbf{1}_9$. To check the effectiveness of the proposed algorithm, the upper capacity limit of generator 3 is revised to 80 MW, which is lower than $P_{G3}^* = 122.2264$ MW. In this situation, the new optimal solutions are P_{G1}^*=416.5620 MW, P_{G2}^*=353.4380 MW, P_{G3}^*=80.0000 MW, and $\lambda_0^* = 9.2213$ by direct calculation. The simulation results are shown in Fig. 4. It can be seen that generator 3's final generation decreases and equals 80 MW, which is its upper bound. The new incremental cost increases due to lower power generation of generator in bus 3.

3) With line flow constraints: In this section, the line flow constraints are imposed on the IEEE 9-bus system. From Section V.A, the optimal generations of the three generators located in buses 1, 2, and 3 are $P_{G1}^* = 393.1698$ MW, $P_{G2}^* = 334.6038$ MW, and $P_{G3}^* = 122.2264$ MW, respectively. To see the influence of the line flow constraints, we impose a line flow limit of 250 MW (smaller than P_{G2}^*) on the transmission line which connects bus 2 and bus 7, and 500 MW on other transmission lines. By direct calculation,

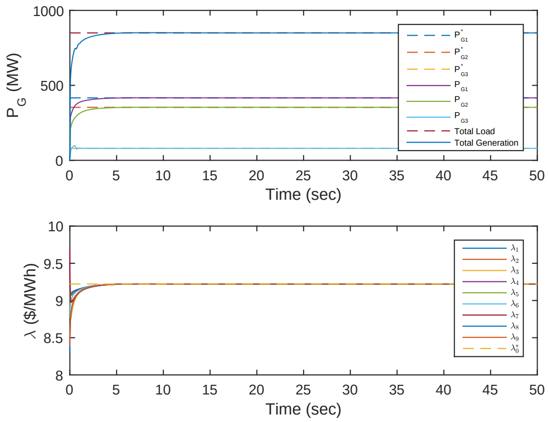


Fig. 4. Power generations and estimated incremental costs produced by (4), (6), and (7), and when capacity limits are contradicted. The dashed lines denote the new optimal generations.

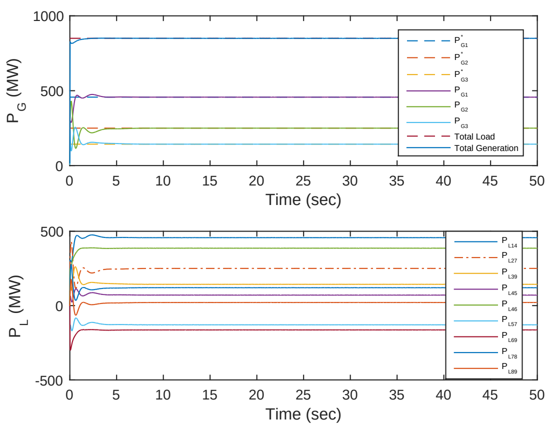


Fig. 5. Power generations and power flows produced by (11) with line flow constraints.

the optimal generations of the three generators with line flow constraints are $P^*_{G1} = 457.0673$ MW, $P^*_{G2} = 250.0$ MW, and $P^*_{G3} = 142.9327$ MW, respectively. By using algorithm (11) and setting the control gains as $\kappa_{Gi} = \kappa_{Li} = 1000$, $\kappa_{\lambda i} = 1$, and $\kappa_{\alpha i} = \kappa_{\beta i} = \kappa_{ul} = \kappa_{vl} = 0.02$ for $i = 1, ..., n$ and $l = 1, ..., m$, we get the simulation results shown in Fig. 5. It can be seen that the generations converge to the new optimal generations and the power flow on all transmission lines are within their line flow limits.

4) Implementation on IEEE 118-bus System: In this section, simulation on the IEEE 118-bus system is conducted to test the performance of the proposed algorithm. There are 118 buses and 54 generators. The data used is adopted from [28] and the communication graph is set the same as the physical connection of the buses. Agent 49 is selected to calculate λ_0. In the simulation, the parameters are set as $\bar{k} = 2300 \times \mathbf{1}_{118}$, and $\bar{m}_0 = \bar{m}_1 = \bar{m}_2 = 0.15 \times \mathbf{1}_{118}$. The simulation results are shown in Fig. 6. It can be seen that the algorithm is still valid for this large scale power system, and the convergence time is comparable with the case for the IEEE 9-bus system.

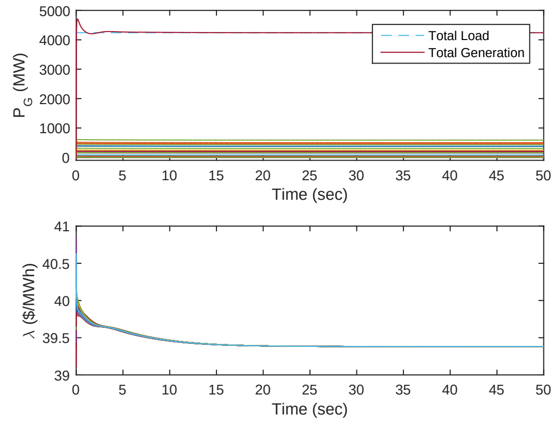


Fig. 6. Power generations and estimated incremental costs produced by (4), (6), and (7) tested on the IEEE 118-bus system.

VI. Conclusion

In this paper, distributed algorithms based on dynamic average consensus, leader-following consensus and the saddle point dynamics are proposed to solve the economic dispatch problem. The proposed algorithm does not require any initialization procedure to design the initial values of the variables and find the global information such as the total load and the entire communication topology. Global exponential stability of the optimal solution is derived if the capacity limits are not considered, and practical stability of the optimal solution is proved when the capacity limits are considered. Several case studies are conducted to show the effectiveness of the proposed algorithms.

Appendix A
Proof of Theorem 1

Proof. Let $U = [U_1 \; U_2] \in \mathbb{R}^{n\times n}$ with $U_1 \in \mathbb{R}^{n\times(n-1)}$ and $U_2 \in \mathbb{R}^n$ be an orthogonal matrix such that $U_2^T L = 0$. Define $[\theta_1^T \; \theta_2]^T \triangleq U^T\theta$ where $\theta_1 \in \mathbb{R}^{n-1}$ and $\theta_2 \in \mathbb{R}$. From (4a) and (4b), it can be derived that

$$\begin{aligned}\begin{bmatrix} \dot{z} \\ \dot{\theta}_1 \end{bmatrix} =& \begin{bmatrix} -(I+L) & -LU_1 \\ U_1^T L & \mathbf{0}_{(n-1)\times(n-1)} \end{bmatrix} \begin{bmatrix} z \\ \theta_1 \end{bmatrix} \\ &+ \begin{bmatrix} P_D - P_G \\ \mathbf{0}_{n-1} \end{bmatrix}, \quad (12a)\\ \dot{\theta}_2 =& 0. \quad (12b)\end{aligned}$$

Hence, $\theta_2(t) = \theta_2(0)$, which is a stable state. Let $\tau = \delta t$. Equations (3), (4c) and (12a) in the τ-time scale can be written as

$$\frac{dP_G}{d\tau} = -K\left(\frac{\partial f(P_G)}{\partial P_G} - \lambda\right), \quad (13a)$$

$$\frac{d\lambda_0}{d\tau} = m_0 z_q, \quad (13b)$$

$$\delta\frac{d\lambda}{d\tau} = -(L + A_0)\lambda + A_0\mathbf{1}_n\lambda_0, \quad (13c)$$

$$\begin{bmatrix} \delta\frac{dz}{d\tau} \\ \delta\frac{d\theta_1}{d\tau} \end{bmatrix} = \begin{bmatrix} -(I+L) & -LU_1 \\ U_1^T L & \mathbf{0}_{(n-1)\times(n-1)} \end{bmatrix} \begin{bmatrix} z \\ \theta_1 \end{bmatrix}$$

$$+\left[\begin{array}{c} P_D-P_G \\ \mathbf{0}_{n-1} \end{array}\right], \tag{13d}$$

where $K = \text{diag}\{k_i\}$. The system given by (13) attains a singular perturbed form [29].

Let λ^e, z^e and θ_1^e be the solutions of the following equations,

$$\begin{aligned} &-(L+A_0)\lambda^e + A_0\mathbf{1}_n\lambda_0 = \mathbf{0}_n, \\ &\left[\begin{array}{cc} -(I+L) & -LU_1 \\ U_1^T L & \mathbf{0}_{(n-1)\times(n-1)} \end{array}\right]\left[\begin{array}{c} z^e \\ \theta_1^e \end{array}\right] + \left[\begin{array}{c} P_D-P_G \\ \mathbf{0}_{n-1} \end{array}\right] \\ &= \mathbf{0}_{2n-1}. \end{aligned}$$

Since $L+A_0$ is positive definite, $\lambda^e = \mathbf{1}_n\lambda_0$. By Lemma 1 and Lemma 2.1 in [30], $z^e = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T(P_D-P_G)$, and θ_1^e is a linear function of P_G.

The reduced system of (13) is

$$\left[\begin{array}{c} \frac{dP_G}{d\tau} \\ \frac{d\lambda_0}{d\tau} \end{array}\right] = H_2\left[\begin{array}{c} P_G \\ \lambda_0 \end{array}\right] + \mathbf{b}_1,$$

where

$$H_2 = \left[\begin{array}{cc} -\text{diag}\{2k_i a_i\} & [k_i]_{\text{vec}} \\ -\frac{m_0}{n}\mathbf{1}_n^T & 0 \end{array}\right], \quad \mathbf{b}_1 = \left[\begin{array}{c} -[k_i b_i]_{vec} \\ \frac{m_0}{n}\mathbf{1}_n P_D \end{array}\right].$$

Define

$$P \triangleq \left[\begin{array}{cc} \text{diag}\{\sqrt{k_i}\} & \mathbf{0}_n \\ \mathbf{0}_n^T & \sqrt{\frac{m_0}{n}} \end{array}\right].$$

It can be derived that

$$P^{-1}H_2P = \left[\begin{array}{cc} -\text{diag}\{2k_i a_i\} & \sqrt{\frac{m_0}{n}}[\sqrt{k_i}]_{vec} \\ -\sqrt{\frac{m_0}{n}}[\sqrt{k_i}]_{vec}^T & 0 \end{array}\right].$$

$P^{-1}H_2P$ is Hurwitz by Lemma 3. As a result, H_2 is Hurwitz due to similarity transformation. Define $\tilde{\phi} \triangleq \left[\begin{array}{c} P_G-P_G^* \\ \lambda_0-\lambda_0^* \end{array}\right]$, by noticing that $\left[\begin{array}{c} P_G^* \\ \lambda_0^* \end{array}\right] = -H_2^{-1}\mathbf{b}_1$, it can be derived that

$$\frac{d\tilde{\phi}}{d\tau} = H_2\tilde{\phi}. \tag{15}$$

Thus, the reduced system is exponentially stable.

Define $\psi \triangleq [\lambda^T,\ z^T,\ \theta_1^T]^T$, $\psi^e \triangleq [\lambda^{eT},\ z^{eT},\ \theta_1^{eT}]^T$ and $\tilde{\psi} \triangleq \psi - \psi^e$. Then we have the boundary layer model

$$\delta\frac{d\tilde{\psi}}{d\tau} = H_1\tilde{\psi}, \tag{16}$$

where

$$H_1 = \left[\begin{array}{ccc} -(L+A_0) & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times(n-1)} \\ \mathbf{0}_{n\times n} & -(I+L) & -LU_1 \\ \mathbf{0}_{(n-1)\times n} & U_1^T L & \mathbf{0}_{(n-1)\times(n-1)} \end{array}\right].$$

Since $\left[\begin{array}{cc} -(I+L) & -LU_1 \\ U_1^T L & \mathbf{0}_{(n-1)\times(n-1)} \end{array}\right]$ is Hurwitz due to Lemma 3 and $-(L+A_0)$ is Hurwitz, H_1 is Hurwitz. Thus, the boundary-layer system is exponentially stable.

Therefore, from Theorem 11.4 in [29] and noticing that the closed loop system is linear, it can be concluded that there exists a positive constant δ^* such that for every $0<\delta<\delta^*$, P_G converges to P_G^* exponentially. □

Appendix B
Proof of Theorem 2

Proof. First, we check the stability of the following auxiliary system,

$$\begin{aligned} \dot{P_{Gi}} &= -k_i\left(\frac{df_i(P_{Gi})}{dP_{Gi}} - \lambda_0 + \alpha_i - \beta_i\right), && (17a) \\ \dot{\alpha_i} &= m_{i1}\alpha_i(P_{Gi} - P_{Gi}^{\max}), && (17b) \\ \dot{\beta_i} &= m_{i2}\beta_i(P_{Gi}^{\min} - P_{Gi}), && (17c) \\ \dot{\lambda}_0 &= \frac{m_0}{n}(\sum_{i=1}^n P_{Di} - \sum_{i=1}^n P_{Gi}). && (17d) \end{aligned}$$

Consider the Lyapunov function candidate inspired by [31],

$$\begin{aligned} V_r =& \sum_{i=1}^n \frac{1}{2k_i}(P_{Gi}-P_{Gi}^*)^2 + \frac{n}{2m_0}(\lambda_0-\lambda_0^*)^2 \\ &+ \sum_{i=1}^n\left(\frac{1}{m_{i1}}(\alpha_i - \alpha_i^* - \alpha_i^*\log(\alpha_i) + \alpha_i^*\log(\alpha_i^*))\right. \\ &\left.+ \frac{1}{m_{i2}}(\beta_i - \beta_i^* - \beta_i^*\log(\beta_i) + \beta_i^*\log(\beta_i^*))\right), \end{aligned}$$

where (P_G^*, γ^*) is a saddle point of $L_2(P_G,\gamma)$. Denoting the derivative of V_r along (17) as $\dot{V}_{rr}$. Then we have

$$\dot{V}_{rr} \le L_2(P_G^*,\gamma) - L_2(P_G,\gamma^*) \le 0.$$

Because (P_G^*,γ^*) is the saddle point of $L_2(P_G,\gamma)$, $\dot{V}_{rr}=0$ if and only if $L_2(P_G^*,\gamma) - L_2(P_G^*,\gamma^*) = 0$ and $L_2(P_G^*,\gamma^*) - L_2(P_G,\gamma^*) = 0$. Since $L_2(P_G,\gamma)$ is strictly convex in P_G, $L_2(P_G^*,\gamma^*) - L_2(P_G,\gamma^*) = 0$ if and only if $P_G = P_G^*$. When $P_G = P_G^*$, $P_{Gi}^* - P_{Gi}^{\max} \le 0$, $P_{Gi}^{\min} - P_{Gi}^* \le 0$ and $\sum_{i=1}^n P_{Di} - \sum_{i=1}^n P_{Gi}^* = 0$. $L_2(P_G^*,\gamma) - L_2(P_G^*,\gamma^*) = 0$ implies

$$\begin{aligned} &\sum_{i=1}^n \alpha_i(P_{Gi}^* - P_{Gi}^{\max}) + \sum_{i=1}^n \beta_i(P_{Gi}^{\min} - P_{Gi}^*) \\ =& \sum_{i=1}^n \alpha_i^*(P_{Gi}^* - P_{Gi}^{\max}) + \sum_{i=1}^n \beta_i^*(P_{Gi}^{\min} - P_{Gi}^*) = 0. \end{aligned}$$

Since $P_{Gi}^* - P_{Gi}^{\max} \le 0$, $P_{Gi}^{\min} - P_{Gi}^* \le 0$ and $\alpha_i \ge 0$, $\beta_i \ge 0$, it can be derived that

$$\alpha_i(P_{Gi}^* - P_{Gi}^{\max}) = 0, \quad \beta_i(P_{Gi}^{\min} - P_{Gi}^*) = 0. \tag{18}$$

When $P_G = P_G^*$,

$$\dot{P_{Gi}} = -k_i\left(2a_iP_{Gi}^* + b_i - \lambda_0 + \alpha_i - \beta_i\right) = 0. \tag{19}$$

Above all, γ satisfies the KKT conditions (5). As a result, $\dot{V}_{rr}=0$ only when $P_G = P_G^*$ and (P_G^*,γ) is a saddle point of $L_2(P_G,\gamma)$.

When $P_{Gi}^{\min} < P_{Gi}^* < P_{Gi}^{\max}$, $\alpha_i = \beta_i = 0$ at $\dot{V}_{rr}=0$ from (18). For $i \in \{1,...,n\}$, define $N_{\text{in}} = \{i|P_{Gi}^{\min} < P_{Gi}^* < P_{Gi}^{\max}\}$, $N_{\max} = \{i|P_{Gi}^* - P_{Gi}^{\max} = 0\}$ and $N_{\min} = \{i|P_{Gi}^{\min} - P_{Gi}^* = 0\}$. Since any two of $P_{Gi}^{\min} < P_{Gi}^* < P_{Gi}^{\max}$, $P_{Gi}^* - P_{Gi}^{\max} = 0$ and $P_{Gi}^{\min} - P_{Gi}^* = 0$ cannot be satisfied simultaneously, $N_{\text{in}} \bigcap N_{\max} = \emptyset$, $N_{\text{in}} \bigcap N_{\min} = \emptyset$ and $N_{\max} \bigcap N_{\min} = \emptyset$. Let $e_i \in \mathbb{R}^n$ denote an n dimensional column vector whose ith element is equal to 1 and other elements are equal to zero. Define a matrix $A_f \in \mathbb{R}^{n\times n}$ whose ith column is

$$\text{col}[A_f]_i = \begin{cases} -e_i, & i \in N_{\max} \\ e_i, & i \in N_{\min} \\ \mathbf{0}_n, & i \in N_{\text{in}} \end{cases}.$$

Remove the zero columns of A_f and define the remaining matrix as $A_a \in \mathbb{R}^{n\times(n-|N_{\text{in}}|)}$, where $|N_{\text{in}}|$ is the number of elements of N_{in}. For γ, if $i \in N_{\text{in}}$, remove α_i and β_i; if $i \in N_{\max}$, remove β_i; if $i \in N_{\min}$, remove α_i. Define the remaining vector of γ as γ_a. From (19) it can be derived that

$$[\mathbf{1}_n \ A_a]\gamma_a = \text{diag}\left\{2a_i\right\} P_G^* + [b_i]_{\text{vec}}.$$

Since $[\mathbf{1}_n \ A_a]$ has full column rank by Assumption 3, γ_a is unique. As a result, γ^* is unique. Define $\bar{\phi} = [(P_G - P_G^*)^T \ (\gamma - \gamma^*)^T]^T$. Thus, $\dot{V}_{rr} < 0$ for all $\bar{\phi} \neq 0$. From Lemma 4.3 in [29], there exists a $\mathcal{K}$ function $W(\|\bar{\phi}\|)$ such that $\dot{V}_{rr} \leq -W(\|\bar{\phi}\|)$.

Following the variable transformation in Theorem 1, the consensus algorithm (4) can be expressed as

$$\dot{\tilde{\psi}} = H_1\tilde{\psi} - \dot{\psi}^e. \tag{20}$$

Since H_1 is Hurwitz, there exists symmetric positive definite matrix P_1 such that $H_1P_1 + P_1H_1^T = -Q_1$ for any given symmetric positive definite matrix Q_1.

Let $V = \tilde{\psi}^T P_1 \tilde{\psi} + V_r$. For every (P_G, λ_0) that belongs to a compact set, calculating the derivative of V along (6), (7) and (20) gives

$$\begin{aligned}\dot{V} =& -\tilde{\psi}^T Q_1 \tilde{\psi} - 2\tilde{\psi}^T P_1 \dot{\psi}^e + \delta \dot{V}_{rr} + \delta(P_G - P_G^*)^T \tilde{\lambda} \\ & + \delta n(\lambda_0 - \lambda^*)\tilde{z}_q \\ \leq & -\lambda_{\min}(Q_1)\|\tilde{\psi}\|^2 + \delta d_1\|\tilde{\psi}\|^2 + \delta d_2\|\tilde{\psi}\|\|\bar{\phi}\| - \delta W(\|\bar{\phi}\|) \\ \leq & -(\lambda_{\min}(Q_1) - \delta d_1 - \delta d_2\epsilon_1)\|\tilde{\psi}\|^2 - \delta W(\|\bar{\phi}\|) + \frac{\delta d_2}{4\epsilon_1}\|\bar{\phi}\|^2,\end{aligned}$$

where d_1, d_2 are some positive constants, and ϵ_1 is a positive constant that can be arbitrarily chosen. For a positive constant ϵ_2, by selecting $\delta < \frac{\lambda_{\min}(Q_1)}{d_1 + d_2\epsilon_1 + \epsilon_2} \triangleq \delta^*$, then

$$\begin{aligned}\dot{V} \leq & -\delta\epsilon_2\|\tilde{\psi}\|^2 - \delta W(\|\bar{\phi}\|) + \frac{\delta d_2}{4\epsilon_1}\|\bar{\phi}\|^2 \\ \leq & -\delta W'(\|(\tilde{\psi}, \bar{\phi})\|) + o(\delta\epsilon_1^{-1}) \\ \leq & -\frac{\delta}{2}W'(\|(\tilde{\psi}, \bar{\phi})\|), \quad \forall \|(\tilde{\psi}, \bar{\phi})\| > W'^{-1}(2o(\epsilon_1^{-1})),\end{aligned}$$

where $W'(\|(\tilde{\psi}, \bar{\phi})\|) = \epsilon_2\|\tilde{\psi}\|^2 + W(\|\bar{\phi}\|)$. Thus, from Theorem 4.18 in [29], the conclusion is obtained. □

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