1968

Distributed parameter RC network analysis

Robert C. Peirson
Missouri University of Science and Technology

Department: Electrical and Computer Engineering

DISTRIBUTED PARAMETER $\overline{RC}$ NETWORK ANALYSIS

by

ROBERT CHARLES PEIRSON,

A DISSERTATION

Presented to the Faculty of the Graduate School of the

UNIVERSITY OF MISSOURI - ROLLA

In Partial Fulfilment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

IN

ELECTRICAL ENGINEERING

Rolla, Missouri

1968

Approved by

E C Bullard
(advisor)

J R Boatwright

J J Bongiorno

Robert F Gibson

R E Douglas Jones

ABSTRACT

Methods of analyzing uniform and tapered distribute parameter $\overline{RC}$ networks are reviewed and correlated in this dissertation. Multilayer networks and simple three layer networks are considered.

Analogies between two-wire transmission line theory and multilayer distributed network theory are developed, i.e., standing wave and traveling wave concepts.

An accurate computer-aided steady state analysis procedure is presented which has a solution accuracy that can be determined prior to actual analysis computations.

The impulsive response of the uniform $\overline{RC}$ network is investigated with the goal of measuring the network's parameters. Procedures to determine the parameters R and C of a uniform $\overline{RC}$ microcircuit from transient measurements are included.

ACKNOWLEDGEMENTS

The author wishes to express gratitude to his major professor, Dr. Edward C. Bertnolli, for the many helpful suggestions and encouragement given throughout this work. The author also wishes to express gratitude to Dr. W. W. Happ, Chief, Design Criteria Branch, National Aeronautics and Space Administration, for introducing him to his distributed parameter $\overline{RC}$ analysis and for posing the measurement problem promulgated herein.

The author is indebted to the National Aeronautics and Space Administration, the National Science Foundation and to the University of Missouri - Rolla for financial assistance which together made possible the studies culminating in this dissertation.

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LIST OF SYMBOLS

A line under a quantity denotes that quantity as a matrix.

α	Scalar constant
$A(t)$	"A" matrix
A	Constant matrix evaluated by boundary conditions
B	Constant matrix evaluated by boundary conditions
$c(x)$	Per unit capacitance
C	Total capacitance
d	Total length
e	Naperian base, 2.71898
$F_j(x)$	Coefficient matrix
G_n	Happ and Gupta's [9] "G-function"
i	Electrical current, time varying
I	Electrical current
I_0	Electrical current at x = 0
I_n	Unit matrix of order n (not to be confused with I, electrical current)
j	$\sqrt{-1}$
K	Equals matrix $\begin{bmatrix} 0 & \underline{R} \\ s\underline{C} & 0 \end{bmatrix}$
L	Indicates Laplace transform operation with respect to t
L_x	Indicates Laplace transform operation with respect to x
0, 0_n	Matrices of all zero elements
r	Reciprocal time, i.e., r = 1/t

$r(x)$	Per unit resistance
R	Total resistance
s	Laplace transform variable with respect to t
t	Time parameter
$T(x,s)$	Transmission matrix
ν	Electrical voltage, time varying
V	Electrical voltage
V_0	Electric voltage at x = 0
x	Distance parameter
$x(t)$	State vector
Z_0	Characteristic impedance
Z_R	Receiving end (load) impedance
Z_G	Generating end impedance
$\alpha,\beta,\gamma,\delta$	All constant matrices, evaluated by boundary conditions
ε	Error matrix
Θ_i	i^{th} theta function
ξ	Dummy variable
$\xi(x)$	Vector of vector $\underline{V}(x,s)$ and $\underline{I}(x,s)$, (see Section 3.5)
$\phi(t,t_0)$	State transition matrix
ρ	Laplace transform variable with respect to x
Ω_0^d	The matrizant (see Section 3.3)
-1	Indicates the inverse of a matrix value used as a superscript
$\frac{d}{d(\ )}$	Indicates total deviation
$\frac{\partial}{\partial(\ )}$	Indicates partial deviation

Chapter 1: INTRODUCTION TO DISTRIBUTED $\overline{RC}$ NETWORKS

1.1 Purpose of the Distributed Parameter $\overline{RC}$ Network

With the trend in electronics toward component and system miniaturization, devices which can be constructed using thin film techniques are much in demand. Because of their negligible volume and useful characteristics, layered distributed parameter $\overline{RC}$ networks provide the most promising source of new passive components for future microelectronic systems. These components can act not only as guided signal transmission devices interconnecting active devices but also as frequency selective components.

1.2 Definition of a Distributed Parameter $\overline{RC}$ Network

A distributed parameter $\overline{RC}$ network is an electrical network composed of alternate layers of resistive and dielectric films. Figure 1.1a shows an example of a distributed $\overline{RC}$ network. In this work it is assumed that these layers are deposited on a perfectly conducting film or ground plane (although this does not have to be so), the dielectric layers are non-conductive, and there is no inductance associated with any part of the network. Further, it will be assumed that each layer is electrically homogenous and of uniform thickness. A bar above the letters R and C, specifically $\overline{RC}$, indicates a distributed parameter network.

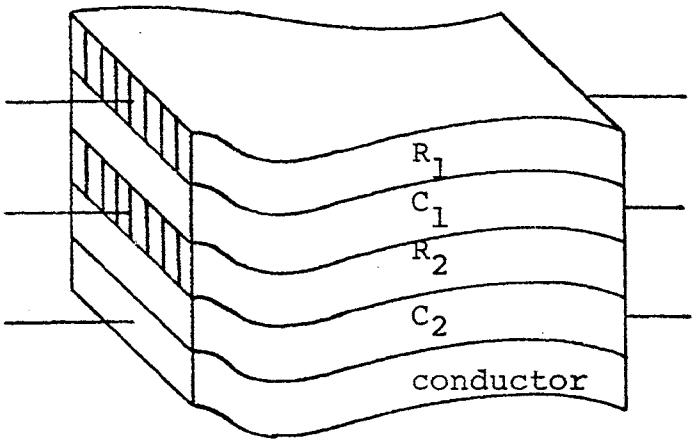


Figure 1.1a The Four Layered Distributed $\overline{RC}$ Network

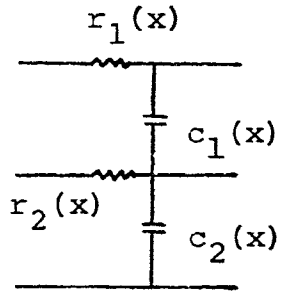


Figure 1.1b The Incremental Model

1.3 Review of Previous Work

Sinusoidal steady state response of distributed parameter networks with certain tapers have been found by Woo and Bartlemay [1], Ghausi and Hershkowitz [2] and Su[3]. Delay and rise time calculations for four terminal $\overline{RC}$ networks were presented by Protonotarios and Wing [4]. These calculations are given in the form of an infinite series of multiple integrals and were obtained from a limit process. Rice [5] found the exact steady-state solution for the multi-wired general transmission line having constant parameter, R, L, C, and G per unit length. Rice discussed two methods for approximating his solution; however, these are not suited to speedy computation.

Vandivort [6] found the steady state transfer matrix solution for an n-port exponentially tapered $\overline{RC}$ network by using a Taylor's series representation. Bertnolli [7] presented an analysis of the n-wire exponential line using a matrix transformation.

Bertnolli and Halijak [8] presented numerical methods for solving the multi-terminal variable parameter $\overline{RC}$ network problem.

Happ and Gupta [9][10] presented a transient analysis of two port uniform $\overline{RC}$ lines in terms of Poisson derived functions in the reciprocal time domain. Peirson and Bertnolli [11] discussed impulse techniques for testing

distributed parameter $\overline{RC}$ microcircuits. This analysis is based on the theta-function and follows Happ and Gupta's paper [10].

1.4 Scope of this Dissertation

The motivation for this work stems from an expressed need by W. W. Happ of the National Aeronautics and Space Administration. Dr. Happ introduced the author to his distributed parameter $\overline{RC}$ network analysis in the reciprocal time domain and posed the measurement problem.

The goal of this thesis is to develop systematic procedures for sinusoidal and transient analysis of both uniform and tapered $\overline{RC}$ networks. The uniform multilayered distributed parameter $\overline{RC}$ transmission line is discussed in Chapter 2. Chapter 3 deals with the methods of obtaining the solution for non-uniform or tapered multilayered distributed parameter $\overline{RC}$ line.

Chapter 4 applies the analysis of the preceeding sections to derive the impulsive response of the uniform distributed parameter $\overline{RC}$ line. This impulsive response is then used in Chapter 5 in the experimental determination of the parameters of $\overline{RC}$ transmission lines by testing them with various terminations.

Chapter 2: THE UNIFORM MULTI-LAYERED LINE

2.1 Introduction

This chapter is concerned with the analysis of the uniform multilayered distributed parameter $\overline{RC}$ networks. This analysis is accomplished by casting the telegrapher's equations into the matrix formulation and solving. These solutions are found using different means.

2.2 Formulation of the Problem

Figure 2.1 depicts a uniform multi-layered distributed parameter $\overline{RC}$ network. The equivalent circuit for a section of this 2n + 2 terminal distributed $\overline{RC}$ network of length Δx is shown in Figure 2.2.

One formulation of the Laplace transformed Kirchoff's voltage and current law equations for the network is the matrix difference equation.

$$\begin{bmatrix} V_1(x+\Delta x,s) \\ V_2(x+\Delta x,s) \\ \cdot \\ \cdot \\ \cdot \\ I_n(x+\Delta x,s) \end{bmatrix} - \begin{bmatrix} V_1(x,s) \\ V_2(x,s) \\ \cdot \\ \cdot \\ \cdot \\ I_n(x,s) \end{bmatrix} = \Delta x \begin{bmatrix} \underline{O}_n & \underline{R} \\ \\ \\ \\ \underline{C} & \underline{O}_n \end{bmatrix} \begin{bmatrix} V_1(x+\Delta x,s) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ I_n(x+\Delta x,s) \end{bmatrix} \qquad (2.1)$$

where

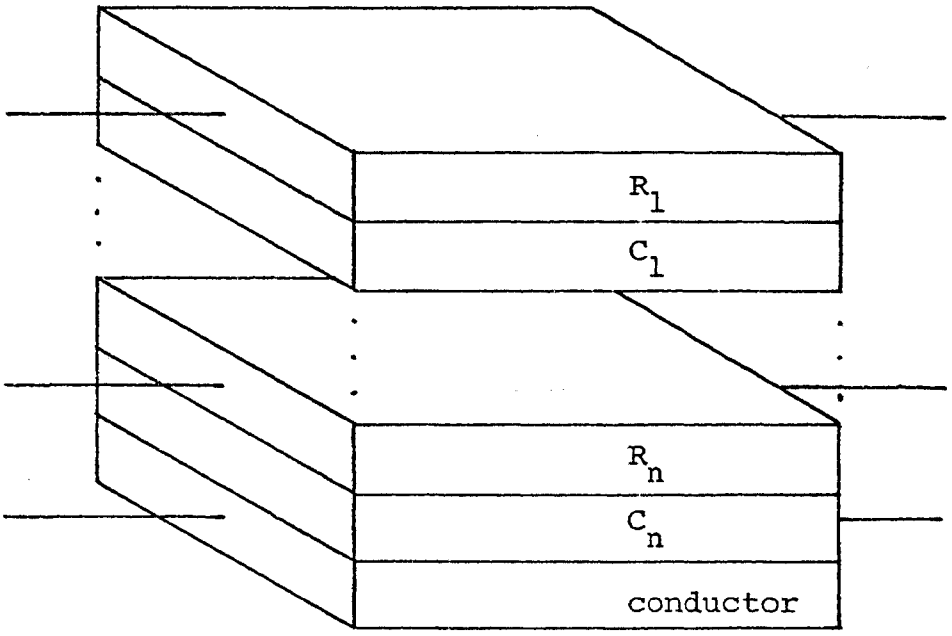


Figure 2.1 The 2n + 2 Terminal Uniform $\overline{RC}$ Microcircuit

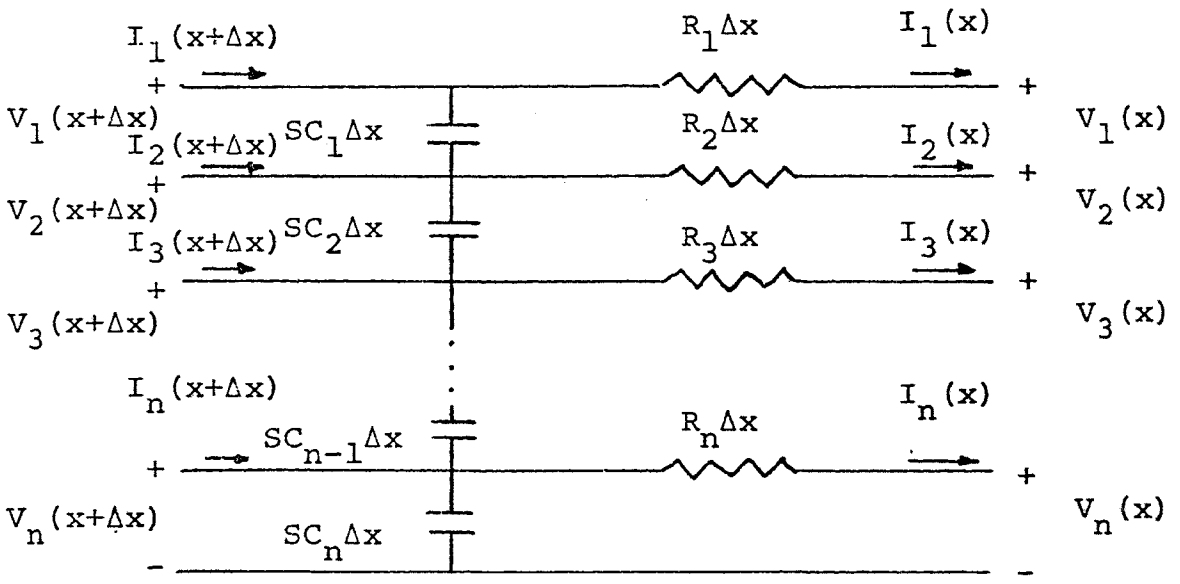


Figure 2.2 Equivalent Circuit of an Incremental Length (Δx) of a 2n-Layered Distributed $\overline{RC}$ Network

$$\underline{C} \equiv \begin{bmatrix} C_1 & -C_1 & 0 & 0 \\ -C_1 & C_1+C_2 & & 0 \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -C_{n-1} \\ \cdot & \cdot & -C_{n-1} & C_{n-1}+C_n \end{bmatrix} \tag{2.2a}$$

and

$$\underline{R} \equiv \text{diag } (R_1, R_2, \cdots, R_n). \tag{2.2b}$$

Dividing Equation (2.1) by the scalar quantity Δx and taking the limit as Δx approaches zero produces the state formulation of the 2n terminal network's equations

$$\frac{\partial}{\partial x} \begin{bmatrix} V_1(x,s) \\ \cdot \\ \cdot \\ \cdot \\ I_n(x,s) \end{bmatrix} = \begin{bmatrix} \underline{O}_n & \underline{R} \\ & \\ s\underline{C} & O_n \end{bmatrix} \begin{bmatrix} V_1(x,s) \\ \cdot \\ \cdot \\ \cdot \\ I_n(x,s) \end{bmatrix} . \tag{2.3}$$

Defining column vectors

$$\begin{aligned} \underline{V}(x,s) &= \text{col } [V_1(x,s), V_2(x,s), \cdots, V_n(x,s)] \\ \underline{I}(x,s) &= \text{col } [I_1(x,s), I_2(x,s), \cdots, I_n(x,s)] \end{aligned} \tag{2.4}$$

Equations (2.3) may be written as [6]

$$\frac{\partial}{\partial x} \begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} = \begin{bmatrix} \underline{O}_n & R \\ s\underline{C} & \underline{O}_n \end{bmatrix} \begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} \tag{2.5}$$

2.3 The Traveling Wave Solution

Differentiating Equation (2.5) with respect to x yields

$$\frac{\partial^2}{\partial x^2}\begin{bmatrix}\underline{V}(x,s)\\ \underline{I}(x,s)\end{bmatrix} = \begin{bmatrix}\underline{0}_n & \underline{R}\\ s\underline{C} & \underline{0}_n\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}\underline{V}(x,s)\\ \underline{I}(x,s)\end{bmatrix}$$

$$= \begin{bmatrix}s\underline{RC} & \underline{0}_n\\ \underline{0}_n & s\underline{CR}\end{bmatrix}\begin{bmatrix}\underline{V}\\ \underline{I}\end{bmatrix} \tag{2.6}$$

or

$$\frac{\partial^2}{\partial x^2}\begin{bmatrix}\underline{V}\\ \underline{I}\end{bmatrix} - \begin{bmatrix}s\underline{RC} & 0_n\\ 0_n & s\underline{CR}\end{bmatrix}\begin{bmatrix}\underline{V}\\ \underline{I}\end{bmatrix} = 0 \tag{2.7}$$

Assuming a traveling wave solution of the form

$$\begin{bmatrix}\underline{V}(x,s)\\ \underline{I}(x,s)\end{bmatrix} = e^{+\begin{bmatrix}\sqrt{s\underline{RC}} & 0\\ 0 & \sqrt{s\underline{CR}}\end{bmatrix}x}\underline{A} + e^{-\begin{bmatrix}\sqrt{s\underline{RC}} & 0\\ 0 & \sqrt{s\underline{CR}}\end{bmatrix}x}\underline{B} \tag{2.8}$$

Then

$$\frac{\partial^2}{\partial x^2}\begin{bmatrix}\underline{V}(x,s)\\ \underline{I}(x,s)\end{bmatrix} = \begin{bmatrix}s\underline{RC} & 0\\ 0 & s\underline{CR}\end{bmatrix}e^{+\begin{bmatrix}\sqrt{s\underline{RC}} & 0\\ 0 & \sqrt{s\underline{CR}}\end{bmatrix}x}\underline{A} +$$

$$\begin{bmatrix}s\underline{RC} & 0\\ 0 & s\underline{CR}\end{bmatrix}e^{-\begin{bmatrix}\sqrt{s\underline{RC}} & 0\\ 0 & \sqrt{s\underline{CR}}\end{bmatrix}x}\underline{B} \tag{2.9}$$

Inserting this and Equation (2.8) into Equation (2.6) yields

$$\begin{bmatrix} sRC & 0 \\ 0 & sCR \end{bmatrix} e + \begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sCR} \end{bmatrix} x \underline{A} + \begin{bmatrix} sRC & 0 \\ 0 & sCR \end{bmatrix} e - \begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sRC} \end{bmatrix} x \underline{B} =$$

$$\begin{bmatrix} sRC & 0 \\ 0 & sCR \end{bmatrix} e + \begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sCR} \end{bmatrix} x \underline{A} + \begin{bmatrix} sRC & 0 \\ 0 & sCR \end{bmatrix} e - \begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sCR} \end{bmatrix} x \underline{B} \qquad (2.10)$$

which verifies that Equation (2.8) is ideed a solution The coefficient matrices $\underline{A}$ and $\underline{B}$ are evaluated by specifying boundary conditions on voltage and current at $x = 0$ and $x = d$.

A better method of expressing the solution is by hyperbolic functions. This method is the standing wave solution and is discussed in the next section.

2.4 The Standing Wave Solution

This method expresses the variables $\underline{V}(x,s)$ and $\underline{I}(x,s)$ as standing waves rather than traveling waves and results in a transfer matrix of hyperbolic functions as a solution. The procedure is to assume a solution of the form

$$\begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} = \begin{bmatrix} A(x) & B(x) \\ C(x) & D(x) \end{bmatrix} \begin{bmatrix} \underline{V}_0(s) \\ \underline{I}_0(s) \end{bmatrix} \qquad (2.11)$$

Substitution of (2.11) into Equation (2.6) yields

$$\frac{\partial^2}{\partial x^2}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} sRC & 0 \\ 0 & sCR \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2.12}$$

Assuming a solution of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \underline{\alpha}\, e^{\begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sCR} \end{bmatrix} x} + \underline{\beta}\, e^{-\begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sCR} \end{bmatrix} x} \tag{2.13}$$

or

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \underline{\gamma} \cosh \begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sCR} \end{bmatrix} x + \underline{\delta} \sinh \begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sCR} \end{bmatrix} x \tag{2.14}$$

Performing the required differentiation and substitution of Equation (2.14) into (2.12) yields

$$\begin{aligned} &\underline{\gamma}\begin{bmatrix} sRC & 0 \\ 0 & sCR \end{bmatrix} \cosh \begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sCR} \end{bmatrix} x + \\ &\quad \underline{\delta}\begin{bmatrix} sRC & 0 \\ 0 & sCR \end{bmatrix} \sinh \begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sCR} \end{bmatrix} x \\ &= \begin{bmatrix} sRC & 0 \\ 0 & sCR \end{bmatrix} \underline{\gamma} \cosh \begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sCR} \end{bmatrix} x + \\ &\quad \begin{bmatrix} sRC & 0 \\ 0 & sCR \end{bmatrix} \underline{\delta} \sinh \begin{bmatrix} \sqrt{sRC} & 0 \\ 0 & \sqrt{sCR} \end{bmatrix} x \end{aligned} \tag{2.15}$$

If $\underline{\gamma}$ and $\underline{\delta}$ both commute with $\begin{bmatrix} s\underline{RC} & 0 \\ 0 & s\underline{CR} \end{bmatrix}$, the solution is verified.

On this assumption, evaluate the arbitrary matrices, $\underline{\gamma}$ and $\underline{\delta}$, using the boundary conditions on $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. These are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{x=0} = I_{2n} \quad \text{and} \quad \frac{\partial}{\partial x}\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{x=0} = \begin{bmatrix} 0 & \underline{R} \\ s\underline{C} & 0 \end{bmatrix} \tag{2.16}$$

Equation (2.14), evaluated at $x = 0$, gives

$$\underline{\gamma} = I_{2n} \tag{2.17}$$

The second boundary condition leads to the equation

$$\frac{\partial}{\partial x}\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{x=0} = \begin{bmatrix} 0 & \underline{R} \\ s\underline{C} & 0 \end{bmatrix} = \underline{\delta}\begin{bmatrix} \sqrt{s\underline{RC}} & 0 \\ 0 & \sqrt{s\underline{CR}} \end{bmatrix} \tag{2.18}$$

Postmultiplying Equation (2.18) by $\begin{bmatrix} \sqrt{s\underline{RC}} & 0 \\ 0 & \sqrt{s\underline{CR}} \end{bmatrix}^{-1}$ yields

$$\underline{\delta} = \begin{bmatrix} 0 & \underline{R} \\ s\underline{C} & 0 \end{bmatrix}\begin{bmatrix} (\sqrt{s\underline{RC}})^{-1} & 0 \\ 0 & (\sqrt{s\underline{CR}})^{-1} \end{bmatrix} = \begin{bmatrix} 0 & \underline{R}(\sqrt{s\underline{CR}})^{-1} \\ s\underline{C}(\sqrt{s\underline{CR}})^{-1} & 0 \end{bmatrix} \tag{2.19}$$

Since $\underline{\gamma}$ and $\underline{\delta}$ have now both been found, the assumption that both $\underline{\gamma}$ and $\underline{\delta}$ commute with $\begin{bmatrix} s\underline{RC} & 0 \\ 0 & s\underline{CR} \end{bmatrix}$ is checked.

$\underline{\gamma}$ and $\begin{bmatrix} s\underline{RC} & 0 \\ 0 & s\underline{CR} \end{bmatrix}$ certainly commute since $\underline{\gamma}$ is a unit matrix. The fact that $\underline{\delta}$ and $\begin{bmatrix} s\underline{RC} & 0 \\ 0 & s\underline{CR} \end{bmatrix}$ commute is demonstrated as follows

$$\begin{bmatrix} 0 & \underline{R}(\sqrt{s\underline{CR}})^{-1} \\ s\underline{C}\sqrt{s\underline{RC}}^{-1} & 0 \end{bmatrix} \begin{bmatrix} s\underline{RC} & 0 \\ 0 & s\underline{CR} \end{bmatrix} =$$

$$\begin{bmatrix} s\underline{RC} & 0 \\ 0 & s\underline{CR} \end{bmatrix} \begin{bmatrix} 0 & \underline{R}\sqrt{s\underline{CR}}^{-1} \\ s\underline{C}\sqrt{s\underline{RC}}^{-1} & 0 \end{bmatrix} \tag{2.20}$$

$$\begin{bmatrix} 0 & \underline{R}\sqrt{s\underline{CR}}^{-1}s\underline{CR} \\ s\underline{C}\sqrt{s\underline{RC}}^{-1}s\underline{RC} & 0 \end{bmatrix} = \begin{bmatrix} 0 & s\underline{RC}\ \underline{R}\sqrt{s\underline{CR}}^{-1} \\ s\underline{CR}s\underline{C}\sqrt{s\underline{RC}} & 0 \end{bmatrix} \tag{2.21}$$

or

$$\begin{bmatrix} 0 & \underline{R}\sqrt{s\underline{CR}} \\ s\underline{C}\sqrt{s\underline{RC}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \underline{R}\sqrt{s\underline{CR}} \\ s\underline{C}\sqrt{s\underline{RC}} & 0 \end{bmatrix} \tag{2.22}$$

Therefore the assumption of commutativity was valid.

2.5 The Matrix Exponential Solution

Another form of the solution to Equation (2.3), repeated here for convenience,

$$\frac{\partial}{\partial x}\begin{bmatrix}\underline{V}(x,s)\\ \underline{I}(x,s)\end{bmatrix} = \begin{bmatrix}0 & \underline{R}\\ s\underline{C} & 0\end{bmatrix}\begin{bmatrix}\underline{V}(x,s)\\ \underline{I}(x,s)\end{bmatrix} = K\begin{bmatrix}\underline{V}(x,s)\\ \underline{I}(x,s)\end{bmatrix} \tag{2.23}$$

is found by returning to the scalar case and forming an analogy between the solution to the scalar differential equation

$$\frac{\partial}{\partial x} V(x,s) = a\, V(x,s) \tag{2.24}$$

and Equation (2.23). The solution to Equation (2.24) is

$$V(x,s) = e^{ax} V_0(s) \tag{2.25}$$

The analagous solution for Equation (2.23) is

$$\begin{bmatrix}\underline{V}(x,s)\\ \underline{I}(x,s)\end{bmatrix} = e^{Kx}\begin{bmatrix}\underline{V}_o(s)\\ \underline{I}_0(s)\end{bmatrix} \tag{2.26}$$

The matrix e^{Kx} is interpreted as and evaluated from the infinite series

$$e^{Kx} = I + Kx + \frac{K^2x^2}{2!} + \frac{K^3x^3}{3!} + \cdots \tag{2.27}$$

This series, e^{Kx}, converges absolutely and uniformly in any interval of the x axis.

Another method of evaluating e^{Kx} is that of using the Laplace transform method. The method presented herein is a generalization of the method presented by Ogata [12]. Ogata presented the scalar form of this method; here the matrix form is developed. This is accomplished by taking the Laplace transform of Equation (2.23) with respect to x

$$\begin{bmatrix} \underline{V}(\rho,s) \\ \underline{I}(\rho,s) \end{bmatrix} - \begin{bmatrix} \underline{V}(0,s) \\ \underline{I}(0,s) \end{bmatrix} = K \begin{bmatrix} \underline{V}(\rho,s) \\ \underline{I}(\rho,s) \end{bmatrix} \tag{2.28}$$

r

$$\rho I - K) \begin{bmatrix} \underline{V}(\rho,s) \\ \underline{I}(\rho,s) \end{bmatrix} = \begin{bmatrix} \underline{V}(0,s) \\ \underline{I}(0,s) \end{bmatrix} \tag{2.29}$$

here ρ is the transform variable. Equation (2.29) is now ritten

$$\begin{bmatrix} \underline{V}(\rho,s) \\ \underline{I}(\rho,s) \end{bmatrix} = (\rho I - K)^{-1} \begin{bmatrix} \underline{V}(0,s) \\ \underline{I}(0,s) \end{bmatrix} \tag{2.30}$$

Since the solution of Equation (2.23) is given by

$$\begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} = \{e^{Kx}\} \begin{bmatrix} \underline{V}(0,s) \\ \underline{I}(0,s) \end{bmatrix} \tag{2.31}$$

ıd the Laplace transform of Equation (2.23) can be written

$$\begin{bmatrix} \underline{V}(\rho,s) \\ \underline{I}(\rho,s) \end{bmatrix} = L_x \{e^{Kx}\} \begin{bmatrix} \underline{V}(0,s) \\ \underline{I}(0,s) \end{bmatrix} \tag{2.32}$$

: is obvious that

$$L_x \; e^{Kx} = (\rho I - K)^{-1} \tag{2.33}$$

ıbstitution of $K = \begin{bmatrix} 0 & \underline{R} \\ s\underline{C} & 0 \end{bmatrix}$ in Equation (2.33) yields

$$L_x e^{Kx} = \begin{bmatrix} \rho I_n & -\underline{R} \\ -s\underline{C} & \rho I_n \end{bmatrix}^{-1} .$$

Since this is a partitioned matrix, the formula found in Reference (13) is helpful in taking the inverse. This formula is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & (D-CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \tag{2.35}$$

Applying this formula results in

$$\begin{bmatrix} \rho I_n & -\underline{R} \\ -s\underline{C} & \rho I_n \end{bmatrix} = \begin{bmatrix} \rho[\rho^2 I_n - s\underline{RC}]^{-1} & [\rho^2 I_n - s\underline{RC}]^{-1}\underline{R} \\ [\rho^2 I_n - s\underline{CR}]^{-1}s\underline{C} & \rho[\rho^2 I_n - s\underline{CR}]^{-1} \end{bmatrix} \tag{2.36}$$

From Appendix B, the inverse Laplace transform of Equation (2.36), with respect to ρ, is

$$e^{Kx} = \begin{bmatrix} \cosh\sqrt{s\underline{RC}}\, x & \sinh\sqrt{s\underline{RC}}\, x\, \sqrt{s\underline{RC}}^{-1}\underline{R} \\ \sinh\sqrt{s\underline{CR}}\, x\, \sqrt{s\underline{CR}}^{-1}s\underline{C} & \cosh\sqrt{s\underline{CR}}\, x \end{bmatrix} \tag{2.37}$$

A method of evaluating the matric e^{Kx}, given by many textbooks on "state variables", [12], is that of transforming K into the Jordan canonical form. This gives

$$e^{Kx} = P\, e^{P^{-1}KP}\, P^{-1} \tag{2.38}$$

This method is not generally applicable to the multi-layer network problem because the eigenvalues are not usually distinct.

The Jordan canonical form can still be used if K has multiple eigenvalues, but $P^{-1}KP$ is not diagonal. One then still has the problem of evaluating $e^{P^{-1}KP}$. Generally it is just as simple to use Sylvester's formula or one of the methods discussed herein to find e^{Kx} and not bother transforming it into the Jordan canonical form.

Chapter 3: THE TAPERED MULTI-LAYERED LINE

3.1 Introduction

A general method of analysis for tapered $\overline{RC}$ networks, resulting in transfer matrix solution, for the n-layer, 2n-port network, shown in Figure 3.1, with its equivalent circuit, Figure 3.2, is as follows.

3.2 Formulation of the Problem

As was shown in the preceeding chapter, the network's differential equation in the s domain is [8]

$$\frac{\partial}{\partial x}\begin{bmatrix}\underline{V}(x,s)\\ \\ \underline{I}(x,s)\end{bmatrix} = \begin{bmatrix}0 & \underline{r}(x)\\ \\ s\underline{c}(x) & 0\end{bmatrix}\begin{bmatrix}\underline{V}(x,s)\\ \\ \underline{I}(x,s)\end{bmatrix} \qquad (3.1)$$

in which $\underline{r}$ and $\underline{c}$ are respectively the resistance and capacitance matrices of the network and are now functions of x as defined in Equation (2.2).

Equation (3.1) has the same form as the "state space" equation, i.e.,

$$\dot{x}(t) = A(t)\, x(t), \qquad x(t_0) = x_0 \qquad (3.2)$$

where x(t) is an n vector and A(t) is an n x n matrix, except the independent variable is x instead of t. Abundant discussion of the solutions of this equation is found in various "Control Systems" textbooks, DeRusso, Roy and Close [14] and Ogata [12] in particular. The solution of Equation (3.2) is

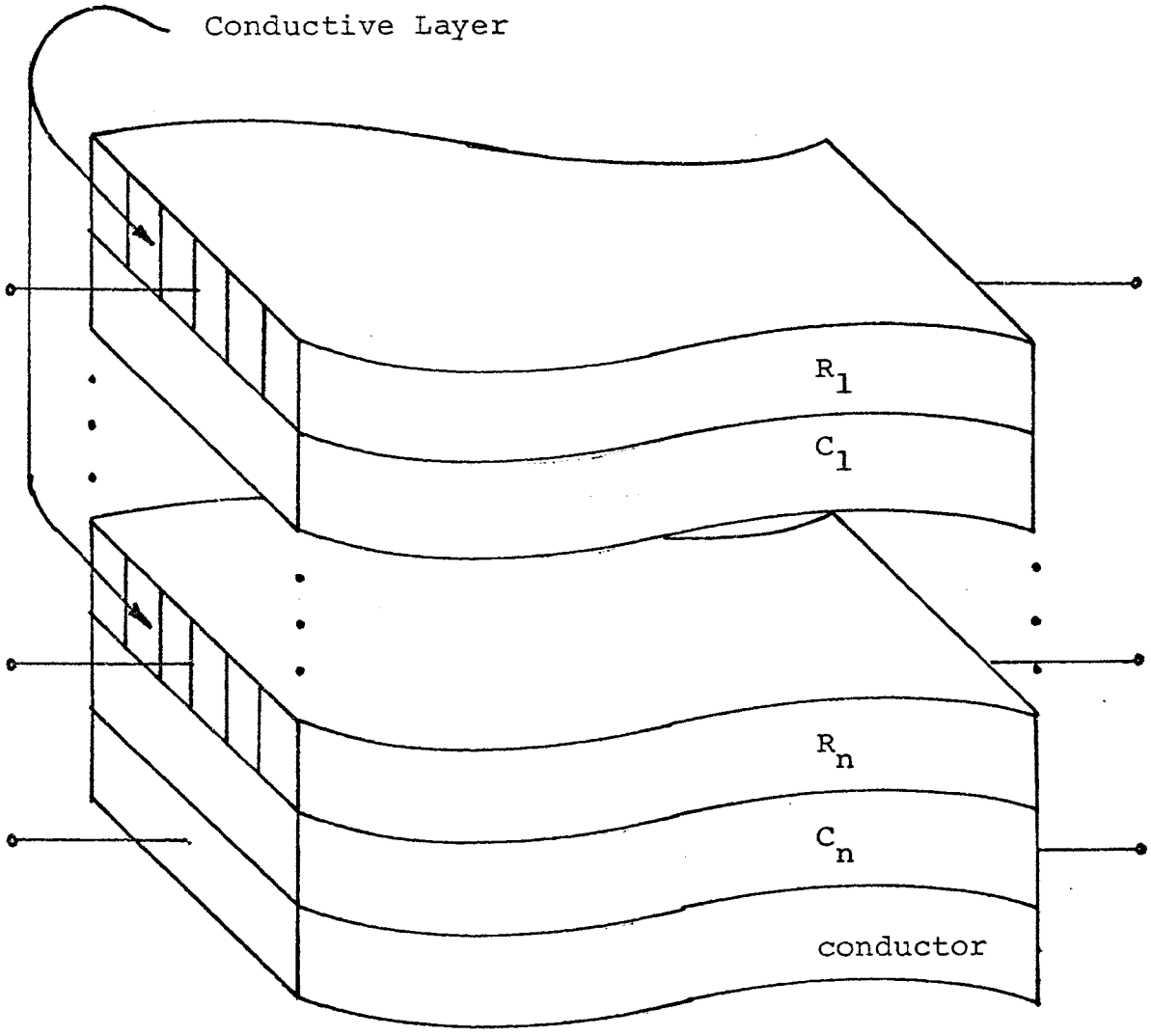


Figure 3.1 The 2n + 2 Terminal Tapered $\overline{RC}$ Microcircuit

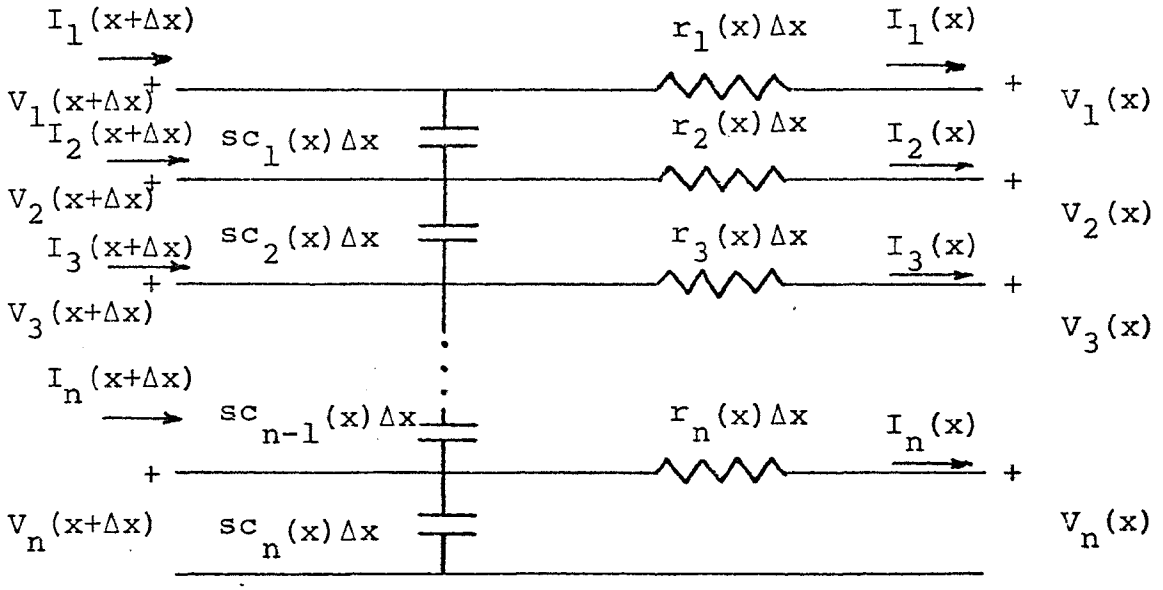


Figure 3.2 Equivalent Circuit of Length Δx for a $2n + 2$ Terminal Tapered $\overline{RC}$ Network

$$x(t) = \emptyset\ (t,\ t_0)\ x(t_0) \tag{3.3}$$

where $\emptyset\ (t,\ t_0)$ an n x n matrix, is called the "state transition matrix". Since Equation (3.1) has independent variable x and is the matrix representation of the transmission line equations, its solution is called the "transfer matrix" and is usually written

$$\begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V(0,s) \\ I(0,s) \end{bmatrix} \tag{3.4}$$

This "transfer matrix" relates the voltages and currents at some position x to the voltages and currents at a reference point, x = 0, as per Figure (3.3).

3.3 The Series Solution

The solution for the general case, i.e., the variable parameter line, where the matrix K is a function of x, is given by

$$\exp \int_0^x K(x)\,dx \tag{3.5}$$

only if K(x) and $\int_0^x K(x)\,dx$ commute. If K is a constant matrix or if K(x) is a diagonal matrix, then clearly K(x) and $\int_0^x K(x)\,dx$ commute. This commutativity does not usually hold.

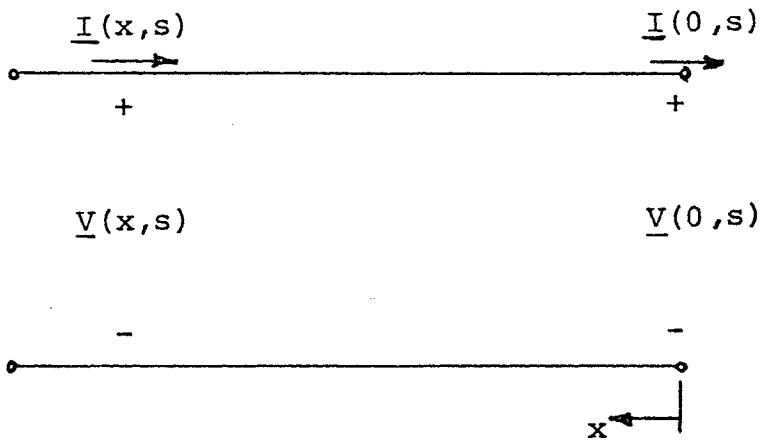


Figure 3.3 The Transmission Line with Reference Current, Voltage and Distance Directions

For the general case, where the commutativity condition is not satisfied, the transfer matrix is not given by Equation (3.5). However, the solution is in terms of an infinite series, which defines a square matrix, called the matrizant [15]. The matrizant is written as follows

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \Omega_0^d[K(x)] = I + \int_0^d K(x_1)\,dx_1 + \int_0^d \int_0^{x_1} K(x_1)K(x_2)\,dx_1 dx_2 +$$

$$+ \cdots + \int_0^d \int_0^{x_1} \cdots \int_0^{x_{2-1}} K(x_1)K(x_2)\cdots K(x_i)\,dx_1 \cdots dx_i + \cdots \tag{3.6}$$

The difficulty in using the matrizant approach is apparent from Equation (3.6). Unless the series (3.6) converges rapidly, the computation becomes quite lengthy.

A method of determining the transmission matrix of a tapered multilayered line is that presented in Reference (16). This method produces a power series in s,

$$T(x,s) = \sum_{j=0}^{\infty} F_j(x)\, s^j \tag{3.7}$$

For the case of a single line for which T(x,s) is (2,2), Equation (3.7) is written

$$\begin{bmatrix} t^{11} & t^{12} \\ t^{21} & t^{22} \end{bmatrix} = \begin{bmatrix} f_0^{11} & f_0^{12} \\ f_0^{21} & f_0^{22} \end{bmatrix} + s \begin{bmatrix} f_1^{11} & f_1^{12} \\ f_1^{21} & f_1^{22} \end{bmatrix} + \cdots \tag{3.8}$$

The coefficients F_j are functions of x alone, because

$$F_j = \frac{1}{j!} \frac{\partial^j}{\partial s^j} \{\Omega_0^d [K(x,s)]\}\Big|_{s=0} \tag{3.9}$$

is independent of s.

If the solution, assumed to be of the form,

$$\begin{bmatrix} \underline{V}(x,s) \\ \\ \underline{I}(x,s) \end{bmatrix} = T(x,s) \begin{bmatrix} \underline{V}(0,s) \\ \\ \underline{I}(0,s) \end{bmatrix} \tag{3.10}$$

is substituted in Equation (3.1), the equation

$$\frac{\partial}{\partial x} T(x,s) = K(x,s)\ T(x,s) \tag{3.11}$$

results.

To find the solution, T(x,s), the matrix K(x,s) is decomposed as

$$K(x,s) = K_1(x) + sK_2(x) \tag{3.12}$$

where

$$K_1(x) = \begin{bmatrix} 0 & \underline{r}(x) \\ \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K_2(x) = \begin{bmatrix} 0 & 0 \\ \\ \underline{c}(x) & 0 \end{bmatrix} \tag{3.13}$$

Substituting Equation (3.12) and (3.7) into (3.11), it is found that $F_0(x)$ satisfies the homogeneous differential equation

$$\frac{d}{dx} F_0(x) = K_1(x)\ F_0(x) \tag{3.14}$$

and that for $j = 1, 2, 3, \cdots$,

$$\frac{d}{dx} F_j(x) = K_1(x)\ F_j(x) + K_2(x)\ F_{j-1}(x) \tag{3.15}$$

The initial condition of Equation (3.11) will be satisfied for all s if the coefficients are chosen such that

$$F_0(0) = I_n$$

$$F_j(0) = \underline{0}$$

$$j = 1, 2, 3, \cdots$$

Equations (3.14) and (3.15) can be solved with the aid of a digital computer by integrating Equations (3.14) and (3.15) using the classical Runge-Kutta technique.

Note that this method of solution involves solving a set of non-homogenous, variable coefficient differential equations. Even with the aid of the digital computer and the Runge-Kutta algorithm, this method seems to be quite devious. A more straightforward procedure for obtaining the power series form of the solution matrix is to expand the matrizant in its Maclaurin series in s.

The Maclaurin series in the complex variable s is

$$f(x) = f(0) + sf'(0) + \frac{s^2}{2!} f''(0) + \frac{s^3}{3!} f''(0) + \cdots \tag{3.16}$$

Expanding the matrizant solution in the Maclaurin series yields

$$\Omega_0^d \begin{bmatrix} 0 & \underline{r}(x) \\ s\underline{c}(x) & 0 \end{bmatrix} = \begin{bmatrix} I_n & \underline{R}_T \\ 0 & I_n \end{bmatrix} +$$

$$s \begin{bmatrix} \int_0^d \underline{r}(x_1) \int_0^{x_1} \underline{c}(x_2)\,dx_2 dx_1 & \int_0^d \underline{r}(x_1) \int_0^{x_1} \underline{c}(x_2) \int_0^{x_2} \underline{r}(x_3)\,dx_3 dx_2 dx_1 \\ \underline{C}_T & \int_0^d \underline{c}(x_1) \int_0^{x_1} \underline{r}(x_2)\,dx_2 dx_1 \end{bmatrix} +$$

$$s^2 \begin{bmatrix} \int_0^d \underline{r}(x_1) \int_0^{x_1} \underline{c}(x_2) \int_0^{x_2} \underline{r}(x_3) \int_0^{x_3} \underline{c}(x_4)\,dx_4 dx_3 dx_2 dx_1 & \int_0^d \underline{r}(x_1) \int_0^{x_1} \underline{c}(x_2) \int_0^{x_2} \underline{r}(x_3) \int_0^{x_3} \underline{c}(x_4) \int_0^{x_4} \underline{r}(x_5)\,dx_5 dx_4 dx_3 dx_2 dx_1 \\ \int_0^d \underline{c}(x_1) \int_0^{x_1} \underline{r}(x_2) \int_0^{x_2} \underline{c}(x_3)\,dx_3 dx_2 dx_1 & \int_0^d \underline{c}(x_1) \int_0^{x_1} \underline{r}(x_2) \int_0^{x_2} \underline{c}(x_3) \int_0^{x_3} \underline{r}(x_4)\,dx_4 dx_3 dx_2 dx_1 \end{bmatrix}$$

$$+ \cdots \qquad (3.17)$$

where

$$\underline{R}_T = \int_0^d \underline{r}(x)\ dx$$

$$\underline{C}_T = \int_0^d \underline{c}(x)\ dx$$

Expansion of the matrizant in its Maclaurin series is a relatively easy procedure. Another method of obtaining the power series in s is that of evaluating the matrizant itself, Equation (3.6), and collecting powers of s, thus forming the same series as Equation (3.8).

Another method of forming the Maclaurin series using the matrizant is as follows. The matrix K(x,s) is decomposed, the same as Ghausi's method, Equation (3.12),

$$K(x,s) = K_1(x) + s\ K_2(x)$$

Equation (3.1) then becomes

$$\frac{\partial}{\partial x}\begin{bmatrix}\underline{V}(x,s)\\ \underline{I}(x,s)\end{bmatrix} = \left[\begin{bmatrix}0 & \underline{r}(x)\\ 0 & 0\end{bmatrix} + s\begin{bmatrix}0 & 0\\ \underline{c}(x) & 0\end{bmatrix}\right]\begin{bmatrix}\underline{V}(x,s)\\ \underline{I}(x,s)\end{bmatrix} \tag{3.18}$$

The solution to Equation (3.18) is

$$\Omega_0^x\ [K_1(x) + s\ K_2(x)] \tag{3.19}$$

from Equation (3.6).

Reference (15), page 219, displays a method of expanding this solution. This is by the formula

$$\Omega_0^x [u + v] = \Omega_0^x [u] \; \Omega_0^x [V] \tag{3.20}$$

where

$$V = \Omega_0^x [u]^{-1} \; v \; \Omega_0^x [u] \tag{3.21}$$

and $u = K_1(x)$

$v = s\, K_2(x)$

Forming $\Omega_0^x [u] = \Omega_0^x [K_1]$ gives

$$\Omega_0^x \begin{bmatrix} 0 & \underline{r}(x) \\ 0 & 0 \end{bmatrix} = I + \int_0^x \begin{bmatrix} 0 & \underline{r}(x) \\ 0 & 0 \end{bmatrix} dx + \int_0^x \int_0^{x_1} \begin{bmatrix} 0 & \underline{r}(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \underline{r}(x_1) \\ 0 & 0 \end{bmatrix} dx_1 dx + \cdots \tag{3.22}$$

$$\Omega_0^x \begin{bmatrix} 0 & \underline{r}(x) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} + \begin{bmatrix} 0 & \underline{R}_x \\ 0 & 0 \end{bmatrix} \tag{3.23}$$

where $\underline{R}_x = \int_0^x \underline{r}(x)dx$. Note that the third and all higher terms are zero.

The next step is to find V,,

$$V = \Omega_0^x [u]^{-1} \; v \; \Omega_0^x [u] \tag{3.24}$$

$$\Omega_0^x [K_1]^{-1} = \begin{bmatrix} I_n & -\underline{R}_x \\ 0 & I_n \end{bmatrix} \tag{3.25}$$

Therefore,

$$V = \begin{bmatrix} I_n & -\underline{R}_x \\ 0 & I_n \end{bmatrix} \; s \begin{bmatrix} 0 & 0 \\ \underline{c}(x) & 0 \end{bmatrix} \begin{bmatrix} I_n & \underline{R}_x \\ 0 & I_n \end{bmatrix} =$$

$$s\underline{c}(x) \begin{bmatrix} -\underline{R}_x & -\underline{R}_x^2 \\ I_n & \underline{R}_x \end{bmatrix} \qquad (3.26)$$

Having found V, determine $\Omega_0^d[V]$.

$$\Omega_0^x[V] = \Omega_0^x \left[s\underline{R}_x c(x) \begin{bmatrix} -I_n & -\underline{R}_x \\ R_x^{-1} & I_n \end{bmatrix} \right] =$$

$$I + s\int_0^x \underline{c}(x) \begin{bmatrix} -\underline{R}_{x1} & \underline{R}_{x1}^2 \\ I_n & \underline{R}_{x1} \end{bmatrix} dx_1 +$$

$$s^3 \int_0^x \int_0^{x_1} \int_0^{x_2} \underline{c}(x_1)\underline{c}(x_2)\underline{c}(x_3) \begin{bmatrix} -\underline{R}_{x1} & \underline{R}_{x1}^2 \\ I_n & \underline{R}_{x1} \end{bmatrix} \begin{bmatrix} -\underline{R}_{x2} & \underline{R}_{x2}^2 \\ I_n & \underline{R}_{x2} \end{bmatrix} \begin{bmatrix} -\underline{R}_{x3} & \underline{R}_{x3}^2 \\ I_n & \underline{R}_{x3} \end{bmatrix} dx_3 dx_2 dx_1 + \cdots$$

(3.27)

The above matrix $\Omega_0^x[V]$ is then pre-multiplied by

$\Omega_0^x[u]$ as shown in Equation (3.20). This finally generates the Maclaurin series in s.

Note in this analysis the matrix K(x,s) was decomposed the same way Ghausi [16] did it. Then using the formula

$$\overset{x}{\underset{0}{\Omega}}[u + r] = \overset{x}{\underset{0}{\Omega}}[u] \; \overset{x}{\underset{0}{\Omega}}[V] \tag{3.28}$$

The final solution was found to be

$$\overset{x}{\underset{0}{\Omega}}[K_1 + sK_2] =$$

$$\overset{x}{\underset{0}{\Omega}}\begin{bmatrix} 0 & \underline{r}(x) \\ 0 & 0 \end{bmatrix} \cdot \overset{x}{\underset{0}{\Omega}}\left[\Omega_0^x \begin{bmatrix} 0 & \underline{r}(x) \\ 0 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & 0 \\ s\underline{c}(x) & 0 \end{bmatrix} \cdot \Omega_0^x \begin{bmatrix} 0 & \underline{r}(x) \\ 0 & 0 \end{bmatrix}\right] \tag{3.29}$$

The matrix, $\Omega_0^x \begin{bmatrix} 0 & \underline{r}(x) \\ 0 & 0 \end{bmatrix}$ is recognized to be the dc solution. The matrix

$$\overset{x}{\underset{0}{\Omega}}\left[\Omega_0^x \begin{bmatrix} 0 & \underline{r}(x) \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ s\underline{c}(x) & 0 \end{bmatrix} \Omega_0^x \begin{bmatrix} 0 & \underline{r}(x) \\ 0 & 0 \end{bmatrix}\right]$$

, since it is a power series in s, see Equation (3.27), contains all higher frequency terms. This Equation, (3.29), is a different way of writing Ghausi's matrix power series in s, Equation (3.7). This definitely displays that Ghausi's power series is most accurate at low frequency.

An error criteria for this method is found as follows. Rewriting Equation (3.27),

$$\overset{x}{\underset{0}{\Omega}}[K(x,s)] =$$

$$\overset{x}{\underset{0}{\Omega}}[K_1] \cdot \overset{x}{\underset{0}{\Omega}}\{\Omega_0^x[K_1]^{-1} K_2 \; \Omega_0^x[K_1]\} \tag{3.30}$$

Now, since $\Omega_0^x[\Omega_0^x[K_1]^{-1} K_2 \Omega_0^x[K_1]]$ is expressed as a series, Equation (3.27), the error caused by truncation is found by writing Equation (3.30) as

$$\Omega_0^x[K_1] = [A][B + \varepsilon] = AB + A\varepsilon \tag{3.31}$$

where ε is the matrix of truncated terms.

The Euclidian norm of the error caused by truncation is

$$|\Omega_0^d[K_1] - AB| = |A\varepsilon| \tag{3.32}$$

$|AB|$ is the norm of the approximate solution. The error term $|A\varepsilon|$ can be written

$$|A\varepsilon| \leq |A||\varepsilon| \tag{3.33}$$

The bound on ε, using the Maclaurin series remainder formula, is [17]

$$|\varepsilon| \leq \frac{s_i}{n!} \frac{\partial^n}{\partial s^n} |\Omega_0^x[V]| \tag{3.34}$$

where s_i is a value of s in the interval of interest.

3.4 The Perturbation Method

An interesting alternative solution has been proposed by Kinariwala [18] and modified by Pipes [19]. The approach is to decompose K(x) into two matrices, $K_0(x)$ and $K_1(x)$, where $K_0(x)$ satisfies the commutativity conditions stated in Section 3.3. Thus

$$K(x) = K_0(x) + K_1(x) \tag{3.35}$$

or

$$\frac{\partial}{\partial x}\begin{bmatrix}\underline{V}(x,s)\\ \\ \underline{I}(x,s)\end{bmatrix} = [K_0(x) + K_1(x)]\begin{bmatrix}\underline{V}(x,s)\\ \\ \underline{I}(x,s)\end{bmatrix} \tag{3.36}$$

where $K_1(x)$ is interpreted as a perturbation upon $K_0(x)$.

An effective procedure for selecting $K_0(x)$, resulting in a constant K_0, is to average the elements of matrix $K(x)$ in the range $0 \leq x \leq d$, so that

$$K_0 = \frac{1}{d}\int_0^d K(x)\ dx \tag{3.37}$$

The unperturbed equation

$$\frac{\partial}{\partial x}\begin{bmatrix}\underline{V}(x,s)\\ \\ \underline{I}(x,s)\end{bmatrix} = K_0\begin{bmatrix}\underline{V}(x,s)\\ \\ \underline{I}(x,s)\end{bmatrix} \tag{3.38}$$

has the solution

$$\begin{bmatrix}\underline{V}(x,s)\\ \\ \underline{I}(x,s)\end{bmatrix} = \begin{bmatrix}A & B\\ \\ C & D\end{bmatrix}_0\begin{bmatrix}\underline{V}(0,s)\\ \\ \underline{I}(0,s)\end{bmatrix} \tag{3.39}$$

where

$$\begin{bmatrix}A & B\\ \\ C & D\end{bmatrix}_0 = \exp\int_0^x K_0\ dx \tag{3.40}$$

or, since K_0 is a constant,

$$\begin{bmatrix}A & B\\ \\ C & D\end{bmatrix}_0 = e^{K_0 x} \tag{3.41}$$

The solution to Equation (3.36) is assumed to be of the same form as Equation (3.40), but with successive corrections added to take the perturbations into account.

The perturbations $K_1(x) \begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix}$ are equivalent to a forcing function term, i.e.,

$$\frac{\partial}{\partial x} \begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} = K_0(x) \begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} + K_1(x) \begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} \tag{3.42}$$

Using the superposition theorem, the solution for $\begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix}$ is then given by

$$\begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}_0 \begin{bmatrix} \underline{V}(0) \\ \underline{I}(0) \end{bmatrix} + \int_0^x \begin{bmatrix} A & B \\ C & D \end{bmatrix}_0 K_1(x) \begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} dx \tag{3.43}$$

This will be recognized as a Volterra equation. The series solution is found to be

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = I_n + \int_0^x \begin{bmatrix} A & B \\ C & D \end{bmatrix}_0 K_1(x_1) \begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} dx_1 + \cdots +$$

$$\int_0^x \int_0^{x_1} \cdots \int_0^{x_2} \begin{bmatrix} A & B \\ C & D \end{bmatrix}_0 K_1(x_1) \cdots \tag{3.44}$$

If the elements of $K_1(x)$ are all relatively small, then only the first few terms of Equation (3.44) are necessary for an adequate approximation.

Another form of deriving Equation (3.44) is that found by using the formula displayed in Equation (3.20),

$$\overset{x}{\underset{0}{\Omega}}[u + v] = \overset{x}{\underset{0}{\Omega}}[u] \quad \overset{x}{\underset{0}{\Omega}}[V] \tag{3.45}$$

where

$$V = \overset{x}{\underset{0}{\Omega}}[u]^{-1} \; v \; \overset{x}{\underset{0}{\Omega}}[u] \tag{3.46}$$

Let

$$u = K_0(s) \tag{3.47}$$

the constant matrix of averaged elements, and

$$v = K_1(x,s) \tag{3.48}$$

such that

$$K(x,s) = K_0(s) + K_1(x,s) \tag{3.49}$$

In line with previous notation, define

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_0 = \overset{x}{\underset{0}{\Omega}}[u] = \overset{x}{\underset{0}{\Omega}}[K_0] \tag{3.50}$$

Forming V, the following equation results

$$V = \overset{x}{\underset{0}{\Omega}}[K_0]^{-1} \; K_1(x) \; \overset{x}{\underset{0}{\Omega}}[K_0] \tag{3.51}$$

Finally, the solution is

$$\overset{x}{\underset{0}{\Omega}}[K_0] \; \overset{x}{\underset{0}{\Omega}}[V] = \overset{x}{\underset{0}{\Omega}}[K_0] \; \overset{x}{\underset{0}{\Omega}}[\overset{x}{\underset{0}{\Omega}}[K_0]^{-1} \; K_1(x) \; \overset{x}{\underset{0}{\Omega}}[K_0]] \tag{3.52}$$

or

$$= \begin{bmatrix} A & B \\ C & D \end{bmatrix}_0 \Omega_0^x \left[\begin{bmatrix} A & B \\ C & D \end{bmatrix}_0^{-1} K_1(x) \begin{bmatrix} A & B \\ C & D \end{bmatrix}_0 \right]. \qquad (3.53)$$

which is another way of expressing Equation (3.44).

A comment should be made about choosing the constant matrix, K_0. As stated earlier, Pipes [19] used the method of averaging the elements of the matrix K(s) over the range $0 \leq x \leq d$. This is probably a good method of determining K_0, but bear in mind that it is not the only method of choosing K_0. The maximum element values of K(x) or the minimum element values of K(x) may also be used. The main objective is to produce a matrix $K_1(x)$ in which all elements are relatively small.

3.5 An Accurate Analysis of the Tapered Line

3.5a Introduction

An excellent numerical method for solving the matrix differential equation

$$\frac{\partial}{\partial x} \begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} = \begin{bmatrix} 0 & R(x) \\ sC(x) & 0 \end{bmatrix} \begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} \qquad (3.54)$$

or

$$\dot{\xi}(x) = K(x)\ \xi(x) \qquad (3.55)$$

is now described. This method, which includes an expression for solution error, enables selection of the solution accuracy prior to the calculation of the actual solution.

This method consists of subdividing the transmission line into n equal subdivisions of length $\Delta x = d/n$. The transfer matrix of each of these subdivisions is then approximated by a Taylor's series.

3.5b The Transfer Matrix Solution

The transfer matrix solution to Equation (3.55) at $x = d$ has been shown in Section (3.1) to be the matrizant of $K(x)$ [15]. Consider the subdivision of (0, d) into n equal subintervals of length $\Delta x = d/n$. Corresponding to this subdivision, $\xi(x)$ can be written [15],

$$\xi(x) = \{\Omega_{(n-1)\Delta x}^{d} [K(x)] \cdots \Omega_{i\Delta x}^{(i+1)\Delta x} [K(x)] \cdots \Omega_{0}^{\Delta x} [K(x)]\}\xi(0)$$

$$\text{or} \qquad [T_n \cdots T_1]\, \xi(0) \tag{3.56}$$

Consider the incremental transfer matrix, $[T_i]$, $\Omega_{i\Delta x}^{(i+1)\Delta x} [K(x)]$, which relates the network variable vector at distance $(i + 1)\Delta x$ to the network variable vector at distance $i\Delta x$,

$$\xi[(i+1)\Delta x] = \Omega_{i\Delta x}^{(i+1)\Delta x} [K(x)] \cdot \xi(i\Delta x) = [T_i]\xi(i\Delta x) \tag{3.57}$$

The transfer matrix relating $\xi[(i+1)\Delta x]$ and $\xi(i\Delta x)$ may also be expanded in a Taylor's series in x [8].

$$T_i = I + T_i^{(1)}(i\Delta t)\Delta x + T_i^{(2)}(i\Delta x)\frac{\Delta x^2}{2!} + \cdots \tag{3.58}$$

$$\Omega_{i\Delta t} \quad \text{where} \quad T_i = \Omega_{i\Delta x}^{(i+1)\Delta x} [K(x)] \tag{3.59}$$

and $T_i^{(r)}(i\Delta x)$ indicates $\frac{\partial^r}{\partial x^r}[T_i(x)]$ evaluated at $x = i\Delta x$.

The terms of the series may be generated by repeated differentiation of Equation (3.59). When this differentiation is performed, Equation (3.58) becomes

$$T_i = I + \Delta x\, K(i\Delta x) + [\dot{K}(i\Delta x) + K^2(i\Delta x)]\frac{\Delta x^2}{2!} + [\ddot{K}(i\Delta x) + K(i\Delta x)\dot{K}(i\Delta x) + 2\dot{K}(i\Delta x)K(i\Delta x) + K^3(i\Delta x)]\frac{\Delta x^3}{3!} + \cdots \tag{3.60}$$

3.5c Approximation of the Transfer Matrix

The desired transfer matrix, from Equation (3.6), is

$$[T] = \Omega_0^d [K(x)] \tag{3.61}$$

This transfer matrix is given by the ordered product of the incremental transfer matrices $[T_i]$

$$[T] = \prod_{i=1}^{n} [T_i] = \Omega_{(n-1)\Delta x}^{d} \cdots \Omega_{i\Delta x}^{(i+1)\Delta x} \cdots \Omega_0^{\Delta x} \tag{3.62}$$

where the function notation for $K(x)$ has been suppressed. The error in the transfer matrix $[T]$ due to any errors in the incremental transfer matrices $[T_i]$ is found by taking the differential of $[T]$,

$$\delta[T] = \delta T_n [\Omega_0^{(n-1)\Delta x}] + \Omega_{(n-1)\Delta x}^{d} [\delta T_{n-1}] \Omega_0^{(n-2)\Delta x} + \cdots$$

$$+ \Omega_{(i+1)\Delta x}^{d} \delta T_i \, \Omega_0^{(i+1)\Delta x} + \cdots + \Omega_{\Delta t}^{d} \delta T_1 \qquad (3.63)$$

Using $|A + B| \leq |A| + |B|$ & $|AB| \leq |A||B|$ from Reference (20), the Euclidian norm of $\delta[T]$ is

$$|\delta[T]| \leq |\delta T_n| |\Omega_0^{(n-1)\Delta x}| + |\Omega_{(n-1)\Delta x}^{d}| |\delta T_{n-1}| |\Omega_0^{(n-2)\Delta x}| + \cdots$$

$$+ |\Omega_{(i+1)\Delta x}^{d}| |\delta T_i| |\Omega_0^{(i+1)\Delta x}| + \cdots + |\Omega_{\Delta x}^{d}| |\delta T_i| \qquad (3.64)$$

The matrizant for a uniform line is

$$\Omega_\alpha^\beta [K_0] = e^{K_0(\beta-\alpha)} \qquad (3.65)$$

since K_0 is independent of x.

For a tapered line a constant matrix K_0 can be chosen such that

$$|\Omega_\alpha^\beta [K(x)]| \leq | e^{K_0(\beta-\alpha)} | \qquad (3.66)$$

by selecting K_0 such that $|K_0| > |K(x)|_{max}$ over (0,d)

Now, the error in the transfer matrix is bounded by the inequality

$$|\delta[T]| \leq |\delta T_n| |e^{K_0(n-1)\Delta x}| + |e^{K_0 \Delta x}| |\delta T_{n-1}| |e^{K_0(n-2)\Delta x}| + \cdots$$

$$+ |e^{K_0(n-i+1)\Delta x}| |\delta T_i| |e^{K_0(i+1)\Delta t}| + \cdots + |e^{K_0(n-1)\Delta x}| |\delta T_1| \qquad (3.67)$$

Expanding $e^{K_0 x}$ in its series yields

$$e^{K_0 x} = I_n + K_0 x + \frac{K_0^2 x^2}{2!} + \frac{K_0^3 x^3}{3!} + \cdots + \frac{K_0^n x^n}{n!} \tag{3.68}$$

Taking the norm of both sides yields

$$|e^{K_0 x}| = \left| I_n + K_0 x + \frac{K_0^2 x^2}{2!} + \cdots + \frac{K_0^n x^n}{n!} \right| \tag{3.69}$$

$$= I_n + K_0 x + \frac{K_0 x^2 2}{2!} + \cdots + \frac{K_0^n x^n}{n!} = e^{|K_0| x}$$

The above results can now be used to place a stronger bound on $|\delta[T]|$

$$|\delta[T]| \leq |\delta T_n| e^{|K_0|(n-1)\Delta x} + e^{|K_0|\Delta x} |\delta T_n| e^{|K_0|(n-2)\Delta x} + \cdots$$

$$+ e^{|K_0|(n-i-1)\Delta x} |\delta T_i| e^{|K_0|(i-1)\Delta x} + \cdots$$

$$+ e^{|K_0|(n-1)\Delta x} |\delta T_1| \tag{3.70}$$

now

$$e^{|K_0|(n-1)\Delta x} \leq e^{|K_0| d} \quad \text{and} \quad 1 \leq e^{|K_0| d} \tag{3.71}$$

Therefore

$$|\delta[T]| \leq e^{|K_0| d} |\delta T_n| e^{|K_0| d} + e^{|K_0| d} |\delta T_{n-1}| e^{|K_0| d} + \cdots$$

$$+ e^{|K_0| d} |\delta T_i| e^{|K_0| d} + \cdots + e^{|K_0| d} |\delta T_1| e^{|K_0| d} \tag{3.72}$$

$$|\delta T| \leq |\delta T_i|_{mx} e^{2|K_0| d} \tag{3.73}$$

$|\delta T_0|_{mx}$ is the largest of the normed errors encountered when calculating the incremental transfer matrices T_i.

If the incremental transfer matrix series (3.58) is truncated after (r terms, the resulting error is given by[21]

$$R_{r+1} = \frac{T_i^{(r+1)}\ (\tau)\ (\Delta x)^{r+1}}{(r+1)!} \tag{3.74}$$

and taking the norm gives

$$|R_{r+1}| = \left|\frac{T_i^{(r+1)}\ (\tau)\ (\Delta x)^{r+1}}{(r+1)!}\right| \leq \frac{\left|T_i\ (\tau)^{(r+1)}\right| (\Delta x)^{r+1}}{(r+1)!} \tag{3.75}$$

where τ is between $(i+1)\Delta x$ and $i\Delta x$.

Since $\Delta x = d/n$, Equation (3.75) can be written

$$|R_{r+1}| \leq \frac{\left|T_i(\tau)^{(r+1)}\right| d^{r+1}}{(r+1)!\ n^{r+1}} \leq \frac{\left|T_{i\,max}^{(r+1)}\right| d^{r+1}}{(r+1)!\ n^{r+1}} \tag{3.76}$$

The total error from Equation (3.73) is

$$|\delta[T]| \leq \frac{e^{2|K_0|d}\left|T_{i\ max}^{(r+1)}\right| d^{r+1}}{(r+1)!\ n^r} \tag{3.77}$$

From the above inequality, it can be seen that the calculation error for this method is a function of n and r, the number of time increments and the order of the Taylor's series respectively. This demonstrates the convergence of the method and that the solution error decreases approximately in proportion to $(r+1)!\ n^r$. Computer time depends directly on n, the number of increments, while the amount of work needed to program this

method depends on r, the order of the Taylor's series. Therefore, one must trade off programming time versus computer run time. Expression (3.77) gives the relationship between these two calculation parameters and the calculation accuracy to aid in selecting n and r.

3.5d Example of the Method

As an example, the transfer matrix for an exponentially tapered three wire transmission line, as shown in Figure 3.4 , will be determined. The $\underline{r}(x)$ and $\underline{c}(x)$ per unit length parameter matrices are

$$\underline{r}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^{2x}, \quad \underline{c}(x) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} e^{-2x} \tag{3.78}$$

For s = 0.0 + j1.0 and a line one meter long, the exact transfer matrix for this line is

$$[T] = \begin{bmatrix} 0.79955610 & 0.30072160 & 3.14600100 & 0.07278376 \\ 0.30072210 & 0.49873280 & 0.07278329 & 3.07321600 \\ -0.13433320 & 0.20138410 & 0.95972330 & 0.06039867 \\ 0.20138400 & -0.33571670 & 0.06039860 & 0.89932530 \end{bmatrix} \tag{3.79}$$

$$+ j \begin{bmatrix} 1.07987300 & -1.06943700 & 0.49715330 & -0.49544380 \\ -1.06943800 & 2.14931200 & -0.49544310 & 0.99259530 \\ 0.41591490 & -0.40606480 & 0.28070960 & -0.27883480 \\ -0.40606480 & 0.82197910 & -0.27883470 & 0.55954410 \end{bmatrix}$$

Using the numerical procedure described here, the transfer matrix is

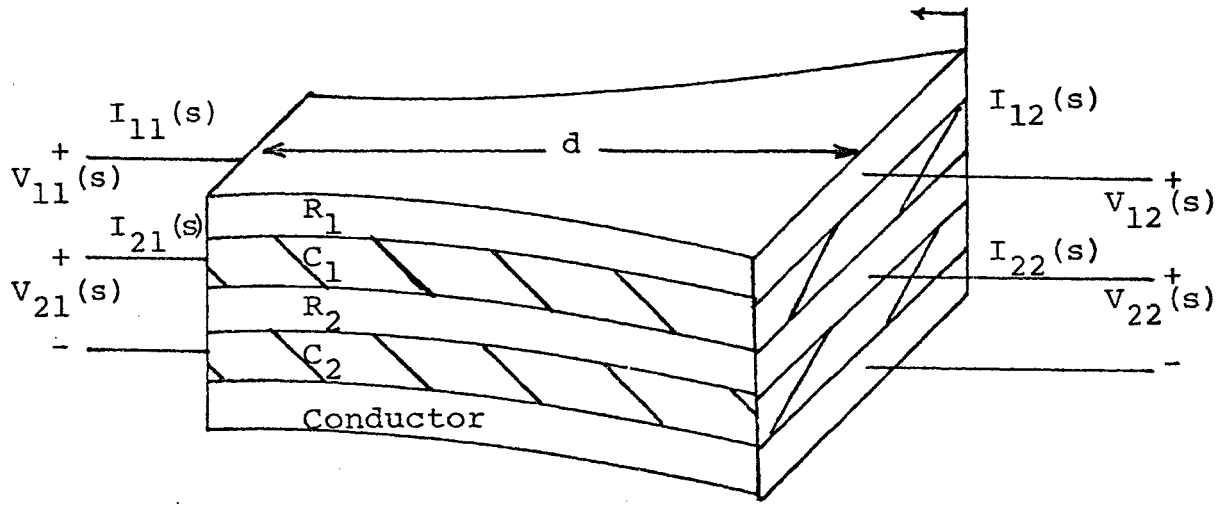


Figure 3.4 The 6 Terminal Exponentially Tapered $\overline{RC}$ Microcircuit

$$[T] \doteq \begin{bmatrix} 0.79945715 & 0.30071925 & 3.14598903 & 0.07277925 \\ 0.30071925 & 0.49873790 & 0.07277925 & 3.07320977 \\ -0.13433415 & 0.20138563 & 0.95972442 & 0.06039733 \\ 0.20138563 & -0.33571977 & 0.06039733 & 0.89932709 \end{bmatrix}$$

(3.80)

$$\begin{bmatrix} 1.07987778 & -1.06944552 & 0.49714625 & -0.49543814 \\ -1.06944552 & 2.14932330 & -0.49543814 & 0.99258440 \\ 0.41591685 & -0.40606686 & 0.28070897 & -0.27883477 \\ -0.40606686 & 0.82198271 & -0.27883477 & 0.55954375 \end{bmatrix}$$

This solution was obtained using n = 60 and r = 3. Using an IBM 360/50 computer system, 123 seconds were required to compile and execute evaluation of the transfer matrix (3.80).

To illustrate convergence, computer runs were made for r of 1, 2, and 3 and for n varying from 10 to 60 in steps of 10. Both the log of the bound on the normed error predicted by Equation (3.73) and the log of the actual error (the left hand side of (3.43)) are plotted versus n in Figure 3.5. By inspection, the information in Figure 3.5 satisfies the inequality (3.73).

A recent publication by Liou [22] describes the transition matrix solution for a linear time invarient system. His system's state form relation has a constant A matrix. For n = 1, r variable, the calculation procedure promulgated here coincides with Liou's evaluation of e^{At}, except Liou used a different norm than the Euclidian norm used here.

Legend:

	r=1	r=2	r=3
$\log \left\{ \frac{e^{2\lvert K_0 \rvert d} \lvert \xi(\tau) \rvert^{r+1} d^{r+1}}{(r+1)!\, n^r} \right\}$	o	Δ	◇
$\log \lvert \delta T \rvert$	x	+	↑

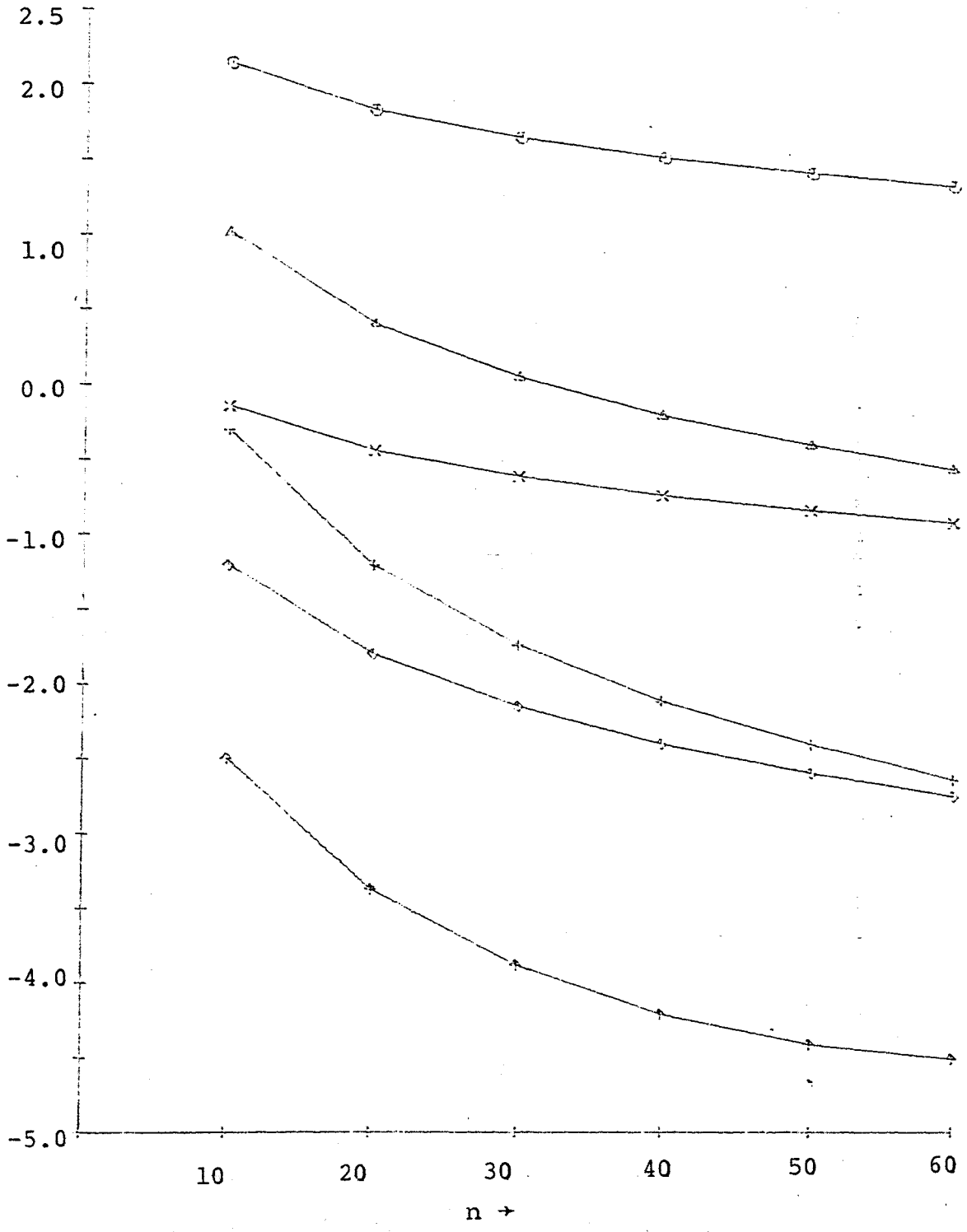


Figure 3.5 Convergence Data for the Exponential Transmission Line

Chapter 4: IMPULSIVE RESPONSE OF $\overline{RC}$ MICROCIRCUITS

4.1 Introduction

The goal of this chapter is to apply the analysis of previous sections to the single layer uniform distributed parameter $\overline{RC}$ network, shown in Figure (4.1). Convergence of the series solutions obtained will be investigated to derive simpler expressions which one may then compare with experimental results to determine the parameters of the $\overline{RC}$ network.

4.2 The Open-Circuited $\overline{RC}$ Microcircuit

The impulsive response of a distributed parameter $\overline{RC}$ network is found using the network's transfer function shown in Table 4-1. For an open-circuited uniform $\overline{RC}$ network, driven by an ideal voltage generator, the voltage transfer function is

$$\frac{V(d,s)}{V(0,s)} = \text{sech } \sqrt{RCs} \qquad (4.1)$$

From this, the impulsive response is found to be

$$V(d,s) = \text{sech } \sqrt{RCs} \qquad (4.2)$$

where R is the total distributed resistance and C is the total distributed capacitance, i.e., R = rd and C = cd, where r and c are the per unit length parameters. The inverse Laplace transform of Equation (4.1) in the reciprocal time domain, given in terms of the G functions, presented by Happ and Gupta [9], is

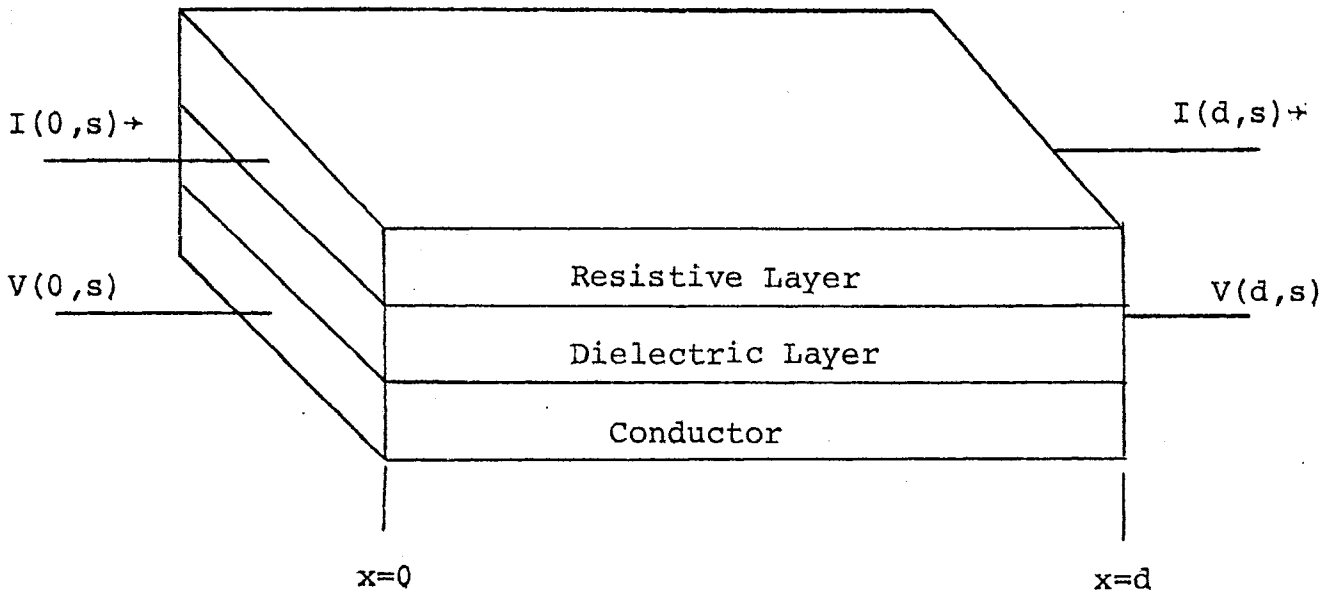


Figure 4.1 The Single Layer Uniform Distributed Parameter $\overline{RC}$ Network

Table 4-1

Table of Transfer Functions

Z_R \ Z_G	0	Z_0
0	$I_d = \frac{\cosh \gamma}{Z_0} V_s$	$I_d = \frac{1}{Z_0} e^{-\gamma} V_s$
∞	$V_d = \text{sech } \sqrt{sRC}\, V_s$	$V_d = e^{-\sqrt{sRC}} V_s$
Z_0	$V_d = e^{-\sqrt{sRC}} V_s$	$V_d = \frac{1}{2} e^{-\sqrt{sRC}} V_s$
KZ_0	$V_d = \frac{V_s}{\cosh \gamma + \frac{\sinh \gamma}{K}}$	$\frac{K}{K-1} e^{-\gamma} V_s$

where $\gamma = \sqrt{sRC}$

$$v(d,r) = \sum_{o}^{\infty} \frac{(-1)^n(2n+1)}{RC} G_n\left(RCr; \frac{1}{2}; 2\right) \tag{4.3}$$

where

$$G_n(r;m,k) = \frac{e^{-(n+m)^2 r}\, r^{K-\frac{1}{2}}}{\sqrt{\pi}} \tag{4.4}$$

Using Equation (4.4), Equation (4.3) may also be written

$$v(d,r) = \sum_{o}^{\infty} (-1)^n \frac{(2n+1)r^{\frac{3}{2}}\sqrt{RC}}{\sqrt{\pi}} e^{-(n+\frac{1}{2})^2 RCr} \tag{4.5}$$

A more straightforward method of calculating the inverse Laplace transform of Equation (4.2) is by using the formula found in Reference (21), i.e.,

$$L^{-1}\{\operatorname{sech}(\sqrt{s}\,a)\} = -\frac{1}{a^2}\left[\frac{\partial}{\partial \nu}\,\theta_1\left(\frac{\nu}{2}, t/a^2\right)\right]_{\nu=0} \tag{4.6}$$

where

$$\theta_1\left(\frac{\nu}{2}, t/a^2\right) = \frac{a}{\sqrt{\pi t}} \sum_{-\infty}^{\infty} (-1)^n e^{\frac{-a^2}{t}\left(\frac{\nu}{2}-\frac{1}{2}+n\right)^2} \tag{4.7}$$

Performing the differentiation indicated in (4.6) yields

$$v(d,t) = L^{-1}\{\operatorname{sech}\sqrt{sRC}\} = \frac{-\sqrt{RC}}{\sqrt{\pi}\, t^{3/2}} \sum_{-\infty}^{\infty} (-1)^n\left(n-\frac{1}{2}\right) e^{\frac{-RC}{t}\left(n-\frac{1}{2}\right)^2} \tag{4.8}$$

for the inverse Laplace transform of (4.2). Replacing t with 1/r, casts Equation (4.8) into the reciprocal time domain, giving

$$v(d,r) = \frac{r^{\frac{3}{2}}\sqrt{RC}}{\sqrt{\pi}} \sum_{-\infty}^{\infty} (-1)^{n+1}\left(n-\frac{1}{2}\right) e^{-RC\left(n-\frac{1}{2}\right)^2 r} \tag{4.9}$$

This series (4.8) can also be expressed as

$$v(d,r) = \frac{r^{\frac{3}{2}}\sqrt{RC}}{\sqrt{\pi}} \sum_{0}^{\infty} 2(-1)^n (n + \tfrac{1}{2}) e^{-RC(n+\frac{1}{2})^2 r} \tag{4.10}$$

Equation (4.10) is the same as Equation (4.5).

Plotting the first three terms of series (4.10) with $\sqrt{RC} = 1$ for n = 0, 1, and 2, versus r generates Figure 4.2. It is obvious from this figure that rapid convergence is obtained for large r or small t. An increasing number of terms is needed for r small or t large. This has been shown by Happ and Gupta [9].

At this point, series (4.9) will be transformed into a form which will yield rapid convergence for r small or t large. This new form along with Equation (4.10) will describe the response of an $\overline{RC}$ microcircuit for both t small and t large.

This series transformation can be derived by the use of Poisson's sum formula [24]

$$\sum_{-\infty}^{\infty} f(x + n) = \sum_{-\infty}^{\infty} e^{2\pi i k x} \int_{-\infty}^{\infty} f(\xi) e^{-2\pi i k \xi} \, d\xi \tag{4.11}$$

Choosing

$$f(x) = \frac{\sqrt{RC}}{\sqrt{\pi}} (-1)^{x+1} (x - \tfrac{1}{2}) r^{2/3} e^{-RC(x-\frac{1}{2})^2 r} \tag{4.12}$$

and substituting this into Equation (4.11) yields

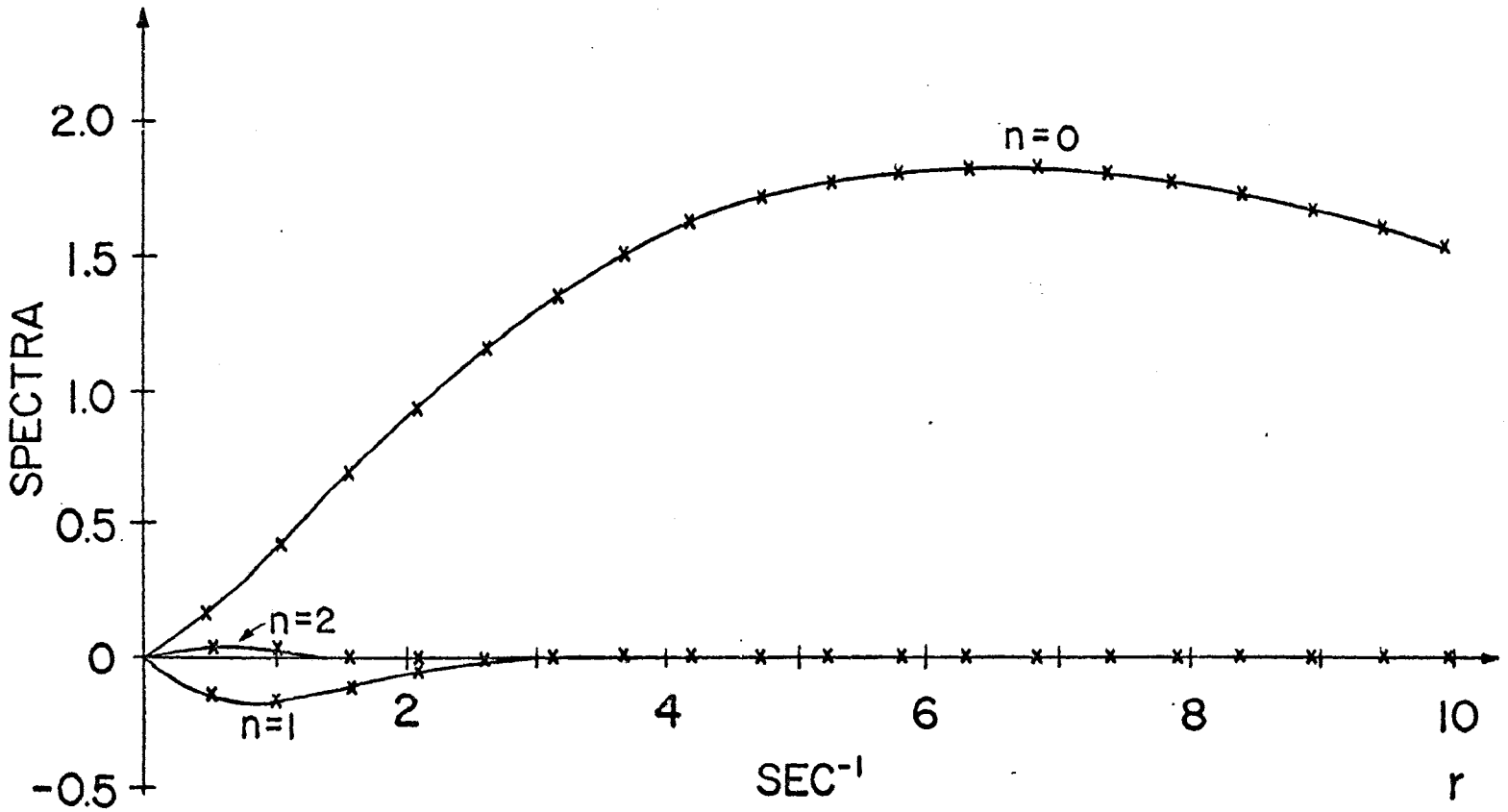


Figure 4.2 Impulse Response Spectra in the Reciprocal Time Domain

$$\sum_{-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{RC}}{\sqrt{\pi}} (-1)^{x+n+1} (x+n-\tfrac{1}{2}) r^{\frac{3}{2}} e^{-RC(x+n-\frac{1}{2})^2 r}$$

(4.13)

$$= \sum_{n=-\infty}^{\infty} e^{2\pi i k x} \int_{-\infty}^{\infty} \frac{\sqrt{RC}}{\sqrt{\pi}} (-1)^{\xi+1} (\xi-\tfrac{1}{2}) r^{\frac{3}{2}} e^{-RC(\xi-\frac{1}{2})^2 r - 2\pi k \xi} d\xi$$

Setting x = 0 in Equation (4.13) gives

$$\sum_{-\infty}^{\infty} \frac{\sqrt{RC}}{\sqrt{\pi}} (-1)^{n+1} (n - \tfrac{1}{2}) r^{3/2} e^{-(n-\frac{1}{2})^2 rRC}$$

(4.14)

$$= \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sqrt{RC}}{\sqrt{\pi}} (-1)^{\xi+1} (\xi-\tfrac{1}{2}) r^{\frac{3}{2}} e^{-RC(\xi-\frac{1}{2})^2 r - 2\pi i k \xi} d\xi$$

The left-hand side of (4.14) may be recognized as series (4.9), v(d,r). The integration indicated in the right-hand side of (4.14), found in Appendix A, gives

$$\sum_{-\infty}^{\infty} \frac{\sqrt{RC}}{\sqrt{\pi}} (-1)^{n+1} (n - \tfrac{1}{2}) r^{\frac{3}{2}} e^{-RC(n-\frac{1}{2})^2 r}$$

(4.15)

$$= \sum_{-\infty}^{\infty} (-1)^{n+1} \frac{(n - \frac{1}{2})\pi}{RC} e^{\frac{-(n-\frac{1}{2})^2 \pi^2}{rRC}}$$

for v(d,r).

Another form of this series is

$$v(d,r) = \sum_{\circ}^{\infty} \frac{2\pi}{RC} (-1)^n (n+\tfrac{1}{2}) e^{-(n+\frac{1}{2})^2 \pi^2 / rRC} \qquad (4.16)$$

or with

$$t = 1/r$$

$$v(d,t) = \sum_{\circ}^{\infty} \frac{2\pi}{RC} (-1)^n (n + \tfrac{1}{2})\ G_n\ (\frac{\pi^2 t}{RC};\ \tfrac{1}{2};\ \tfrac{1}{2})$$

Plotting the first three terms of series (4.16), with $\sqrt{RC} = 1$ for n = 0, 1, and 2, versus r generates Figure (5.2).

As can be seen from Figure 4.3, this series converges rapidly for r small, t large.

Equations (4.10) and (4.16) are expressions for the impulsive response of an $\overline{RC}$ microcircuit that converge rapidly for t small and t large respectively. Using first term approximations gives

$$v(d,t) \doteq \frac{\sqrt{RC}}{\sqrt{\pi}\ t^{3/2}}\ e^{\frac{-RC}{4t}} \qquad \text{for small t} \qquad (4.17a)$$

$$v(d,t) \doteq \frac{\pi}{RC}\ e^{\frac{-\pi^2 t}{4RC}} \qquad \text{for large t} \qquad (4.17b)$$

Equations (4.17) will be referred to later in the section on experimental parameter determination and are summarized in Table 5-1.

4.3 The Short Circuited $\overline{RC}$ Microcircuit

Consider an $\overline{RC}$ microcircuit driven by an ideal voltage generator and terminated with a short circuit. The input voltage, output current transfer function is

$$\frac{I(d,s)}{V(0,x)} = \sqrt{\frac{sC}{R}}\ \cosh\ \sqrt{RCs} \qquad (4.18)$$

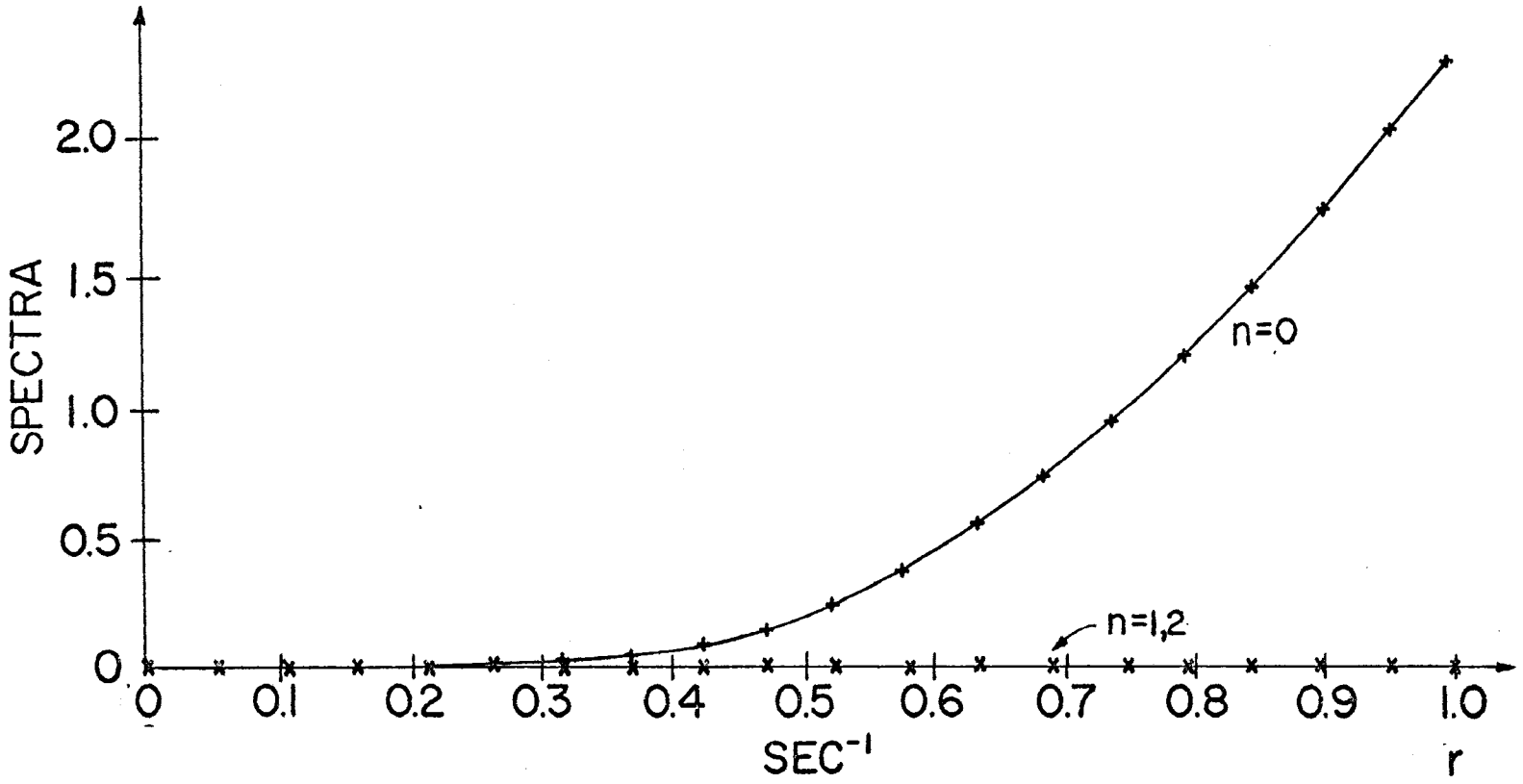


Figure 4.3 Impulse Response Spectra in the Reciprocal Time Domain

Again, since the impulse response is being considered, let $V(0,s) = 1$. Equation (4.18) is now written

$$I(d,s) = \sqrt{\frac{sC}{R}} \cosh \sqrt{RCs} \tag{4.19}$$

The H transform of Equation (4.19), presented by Happ and Gupta [9]

$$i(d,r) = \frac{1}{R^2C}\{2 \sum_{0}^{\infty} (n+\tfrac{1}{2})^2 G_n(RCr;\tfrac{1}{2};e) - \sum_{0}^{\infty} G_n(RCr;\tfrac{1}{2};2)\} \tag{4.20}$$

expressing this in its series form, using

$$G_n(r;\ m,\ k) = \frac{e^{-(n-m)^{2n}}\ r^{k-\frac{1}{2}}}{\sqrt{\pi}} \tag{4.21}$$

yields

$$i(d,r) = \frac{C}{R\pi}\{2r^{\frac{5}{2}} \sum_{0}^{\infty} RC(n+\tfrac{1}{2})^2 e^{-RC(n+\frac{1}{2})^2 r} - r^{\frac{3}{2}} \sum_{0}^{\infty} e^{-RC(n+\frac{1}{2})^2 r}\} \tag{4.22}$$

This series (4.22) is identical to the one derived using the following inverse Laplace transform formula, then replacing t with 1/r.

$$L^{-1}\{\sqrt{s} \cosh \sqrt{RCs}\} = \frac{d}{dt} L^{-1}\{\frac{\cosh\sqrt{sRC}}{\sqrt{s}}\} + L^{-1}\{\frac{\cosh\sqrt{sRC}}{\sqrt{s}}\}\Big|_{t=0}$$

$$= \frac{d}{dt}\{\frac{1}{\sqrt{RC}} \theta_4(0,\ t/RC)\} + \frac{1}{\sqrt{RC}} \theta_4(0,\ t/RC)\Big|_{t=0} \tag{4.23}$$

where

$$\theta_4(\nu,x) = \frac{1}{\sqrt{\pi x}} \sum_{-\infty}^{\infty} e^{-\frac{1}{x}(\nu+\frac{1}{2}+n)^2} \tag{4.24}$$

is a theta function.

Series (4.22) is an expression that will be useful (have good convergence) for r large or t small.

A rapidly converging expression for r small, t large can be found using Equation (4.22) and a different representation of the theta function. This form of the theta function, derived using Poisson's sum formula as in the first part of this chapter, is [24]

$$\theta_4(\nu,\xi) = \sum_{-\infty}^{\infty} e^{-\pi n^2 \xi + 2\pi i n(\nu+\frac{1}{2})} \tag{4.25}$$

Let $\xi = \pi t/a^2$

and $\nu = 0$

$$\theta_4(0,\ t/a^2) = \sum_{-\infty}^{\infty} (-1)^n\ e^{\frac{-\pi^2 n^2 t}{a^2}} \tag{4.26}$$

Therefore, from

$$i(d,t) = \frac{d}{dt}\left\{\frac{1}{\sqrt{RC}}\,\theta_4(0,\ t/RC)\right\} + \frac{1}{\sqrt{RC}}\,\theta_4(0,\ t/RC)\Big|_{t=0} \tag{4.27}$$

$$i(d,t) = \frac{C}{R}\,\frac{\pi^2}{RC\sqrt{RC}} \sum_{-\infty}^{\infty} n^2(-1)^{n+1}\ e^{-n^2\pi^2 t/RC} \tag{4.28}$$

Casting this into the reciprocal time domain with $t = 1/r$ yields

$$i(d,r) = \frac{\pi^2}{R^2C} \sum_{-\infty}^{\infty} n^2(-1)^{n+1}\ e^{-n^2\pi^2/RCr} \tag{4.29}$$

This expression can also be written

$$i(d,r) = \frac{2\pi^2}{R^2C} \sum_{-\infty}^{\infty} n^2(-1)^{n+1} \; e^{\frac{-n^2\pi^2}{RCr}} \qquad (4.30)$$

Convergence is rapid for r small, t large. Using one term (for n = 1)

$$i(d,r) \doteq \frac{2\pi^2}{R^2C} \; e^{-\pi^2/RCr} \qquad (4.31)$$

The preceeding analysis provides rapidly converging expressions for both t large and t small for the short circuit distributed parameter $\overline{RC}$ network.

These expressions are:

$$i(d,t) \doteq \frac{C}{R\pi}\{- \frac{1}{2t^{3/2}} e^{\frac{-RC}{4t}} + \frac{RC}{4t^{5/2}} e^{\frac{-RC}{4t}}\} \text{ for t small} \qquad (4.32a)$$

and

$$i(d,t) \doteq \frac{2\pi^2}{R^2C} \; e^{\frac{-\pi^2 t}{RC}} \qquad \text{for t large} \qquad (4.32b)$$

Equations (4.32) will be used in the following section on experimental determination of parameters and are summarized in Table 5-1.

4.4 The Open-Circuited $\overline{RC}$ Microcircuit with Current Excitation

The s domain impulsive response for the open circuited $\overline{RC}$ microcircuit with current excitation is

$$V(d,s) = \sqrt{\frac{R}{sC}} \; \text{csch}\sqrt{RCs} \qquad (4.33)$$

The inverse Laplace transform is

$$L^{-1}\{\frac{b}{\sqrt{s}} \text{ csch } a\sqrt{s}\} = \frac{b}{a}\,\theta_4(0,\ t/a^2) \tag{4.34}$$

from Equation (4.25)

$$\theta_4(\nu,\xi) = \sum_{-\infty}^{\infty} e^{-\pi n^2\xi + 2\pi i n(\nu+\frac{1}{2})} \tag{4.35}$$

Let $\xi = \pi t/a^2$

and $\nu = 0$

$$\theta_4(0,\ t/a^2) = \sum_{-\infty}^{\infty} (-1)^n\ e^{\frac{-\pi^2 n^2 t}{a^2}} \tag{4.36}$$

Therefore,

$$v(d,t) = \frac{1}{C} \sum_{-\infty}^{\infty} (-1)^n\ e^{-\frac{\pi^2 n^2 t}{RC}} \tag{4.37}$$

Series (4.37) has good convergence for t large or r small.

To find a form good for small t, another form of the theta function is used,

$$\theta_4(\nu,\xi) = \frac{1}{\sqrt{\pi\xi}} \sum_{-\infty}^{\infty} e^{-\frac{1}{\xi}(\nu+\frac{1}{2}+n)^2} \tag{4.38}$$

$$\theta_4(0,t/RC) = \frac{\sqrt{RC}}{\sqrt{\pi t}} \sum_{-\infty}^{\infty} e^{-(u+\frac{1}{2})^2 RC/t} \tag{4.39}$$

From

$$v(d,t) = L^{-1}\{\sqrt{\frac{R}{sC}} \text{ csch } \sqrt{RCs}\} = \frac{R}{C}\,\frac{1}{\sqrt{\pi t}} \sum_{-\infty}^{\infty} e^{-(n+\frac{1}{2})^2 RC/t} \tag{4.40}$$

or

$$v(d,t) = \frac{R}{C} \frac{2}{\sqrt{\pi t}} \sum_{0}^{\infty} e^{-(n+\frac{1}{2})^2 RC/t} \qquad (4.41)$$

Equation (4.41) has good convergence for small t.

Rapidly converging expressions for both t large and t small have been found. Using a one term approximation the following equations are derived. These equations will be used in the following chapter on experimental determination of parameters and are summarized in Table 5-1.

$$v(d,t) \doteq \frac{1}{C} e^{-\frac{\pi^2 t}{RC}} \qquad \text{for large t} \qquad (4.42a)$$

$$v(d,t) \doteq \frac{R}{C} \frac{2}{\sqrt{\pi t}} e^{-\frac{RC}{4t}} \qquad \text{for small t} \qquad (4.42b)$$

4.5 The Short Circuited $\overline{RC}$ Microcircuit with Current Excitation

The s domain impulse response for the short circuited $\overline{RC}$ microcircuit with current excitation is the same as for open circuited voltage excitation, i.e.,

$$I(d,s) = \text{sech } \sqrt{RCs}$$

The inverse Laplace transformation of this is

$$i(d,t) = \frac{\sqrt{RC}}{t^{\frac{3}{2}}\sqrt{\pi}} \sum_{0}^{\infty} 2(-1)^n (n+\tfrac{1}{2}) e^{\frac{-RC(n+\frac{1}{2})^2}{t}} \qquad (4.43)$$

analogous to Equation (4.10) except (4.43) is cast in the t domain.

The one term approximation for both t small and t large are

$$i(d,t) \doteq \frac{\sqrt{RC}}{t^{\frac{3}{2}}\sqrt{\pi}} \; e^{-\frac{RC}{4t}} \qquad \text{t small} \qquad (4.44a)$$

$$i(d,t) \doteq \frac{\pi}{RC} \; e^{-\frac{\pi^2 t}{4RC}} \qquad \text{t large} \qquad (4.44b)$$

4.6 The Characteristic Impedance Termination

At this point, an example would be useful in gaining insight into the behavior of these lines. Assume a two wire uniform $\overline{RC}$ transmission line is terminated in its characteristic impedance, i.e., $Z_R(s) = Z_0$, as shown in Figure 4.4. The voltage at $x = d$ is related to the current at $x = d$ by the following relationship.

$$V(d,s) = Z_R(s)\; I(d,s) = Z_0\; I(d,s) \qquad (4.45)$$

Inverting Equation (2.25) and evaluating at $x = d$, gives

$$\begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \begin{bmatrix} \cosh \sqrt{sRC}\, d & -Z_0 \sinh \sqrt{sRC}\, d \\ -\frac{\sinh \sqrt{sRC}\, d}{Z_0} & \cosh \sqrt{sRC}\, d \end{bmatrix} \begin{bmatrix} V(d,s) \\ I(d,s) \end{bmatrix} \qquad (4.46)$$

where R, C are scalars and $\sqrt{\frac{R}{sC}} = Z_0$, since this is only a two wire line. Inserting Equation (4.45) into (4.46) yields

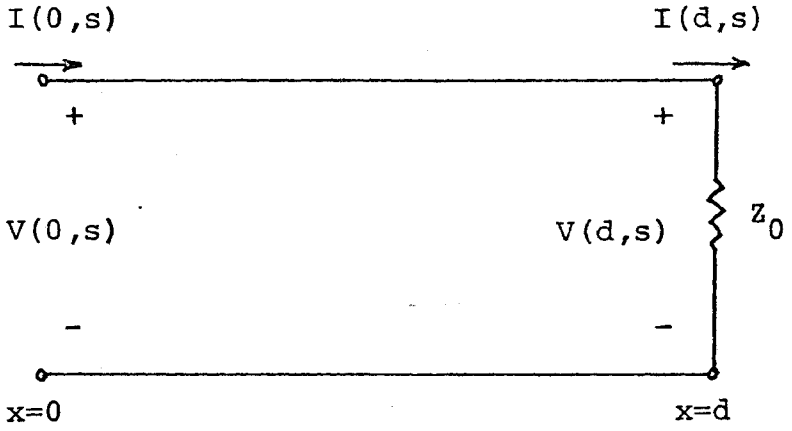


Figure 4.4 The Transmission Line with Matched Load (Z_0)

$$\begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \begin{bmatrix} \cosh \sqrt{sRC}\, d & -Z_0 \sinh \sqrt{sRC}\, d \\ -\frac{\sinh \sqrt{sRC}\, d}{Z_0} & \cosh \sqrt{sRC}\, d \end{bmatrix} \begin{bmatrix} V(d,s) \\ V(d,s)/Z_0 \end{bmatrix} \tag{4.47}$$

Writing only the top equation, gives

$$\begin{aligned} V_0 &= (\cosh \sqrt{sRC}\, d - \sinh \sqrt{sRC}\, d)\, V(d,s) \\ &= e^{-\sqrt{sRC}\, d}\, V(d,s) \end{aligned} \tag{4.48}$$

Let $V(d,s) = 1$, for impulse excitation.

The inverse Laplace transformation of Equation (4.48) yields

$$v(x,t) = \frac{\sqrt{RC}\, x}{2\sqrt{\pi}\, t^{\frac{3}{2}}}\, e^{-RCx^2/4t} \tag{4.49}$$

This "wave" is plotted in Figure 4.5 as a family of curves for various values of time and distance, and in Figure 4.6 as a three dimensional plot.

As in lossless transmission line theory, when a line is terminated in its characteristic impedance, there is no reflected pulse.

Figures 4.5 and 4.6 illustrate the propagation of the pulse along the line. Note that the energy does not propagate in the form of impulses as in the LC or lossless case. The equation governing the behavior of the $\overline{RC}$ line is the partial differential equation known as the heat or diffusion equation. It is well known that heat does not travel in the form of impulses through a material. Since

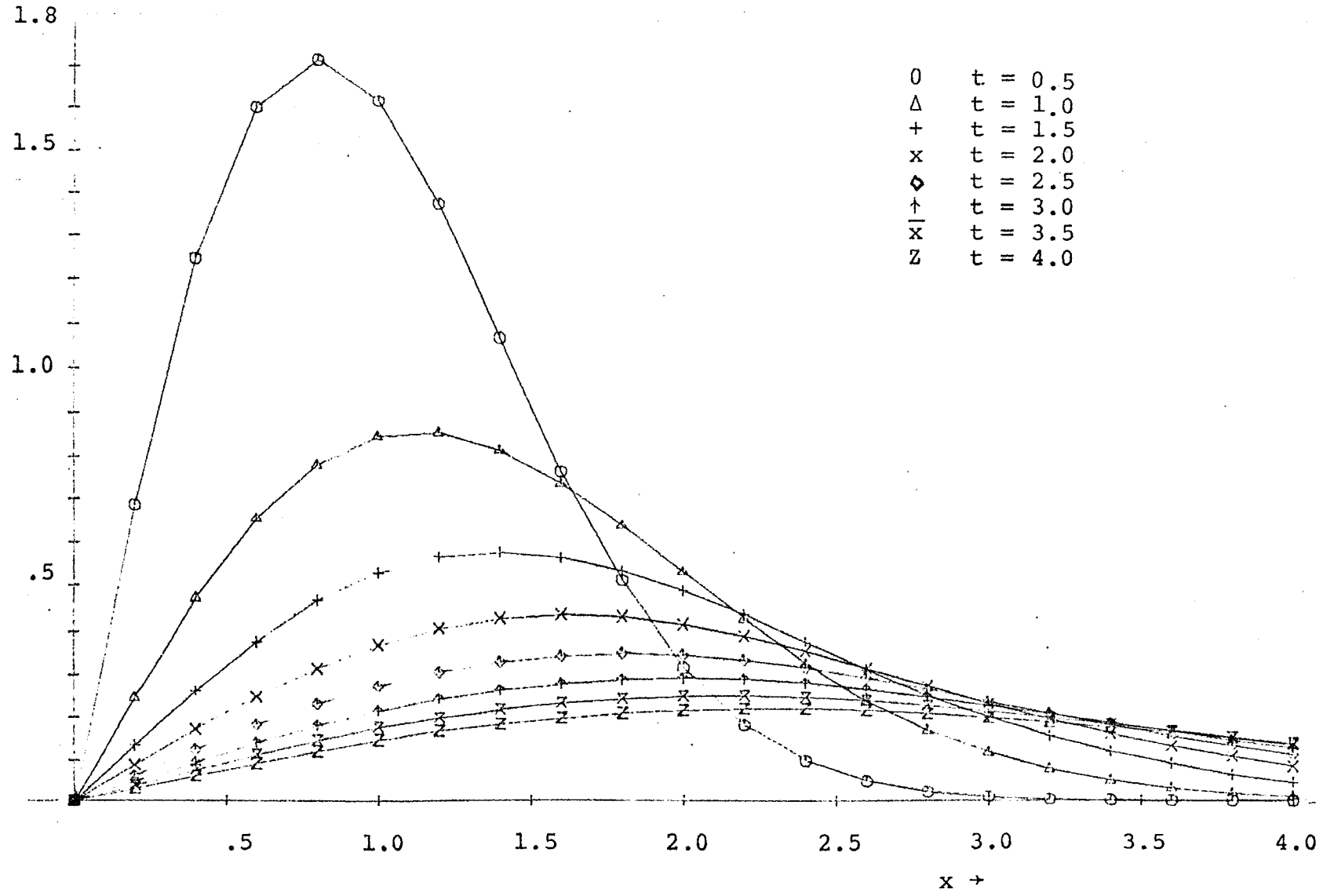


Figure 4.5 The Impulsive Response of a Matched $\overline{RC}$ Microcircuit (two dimensional view)

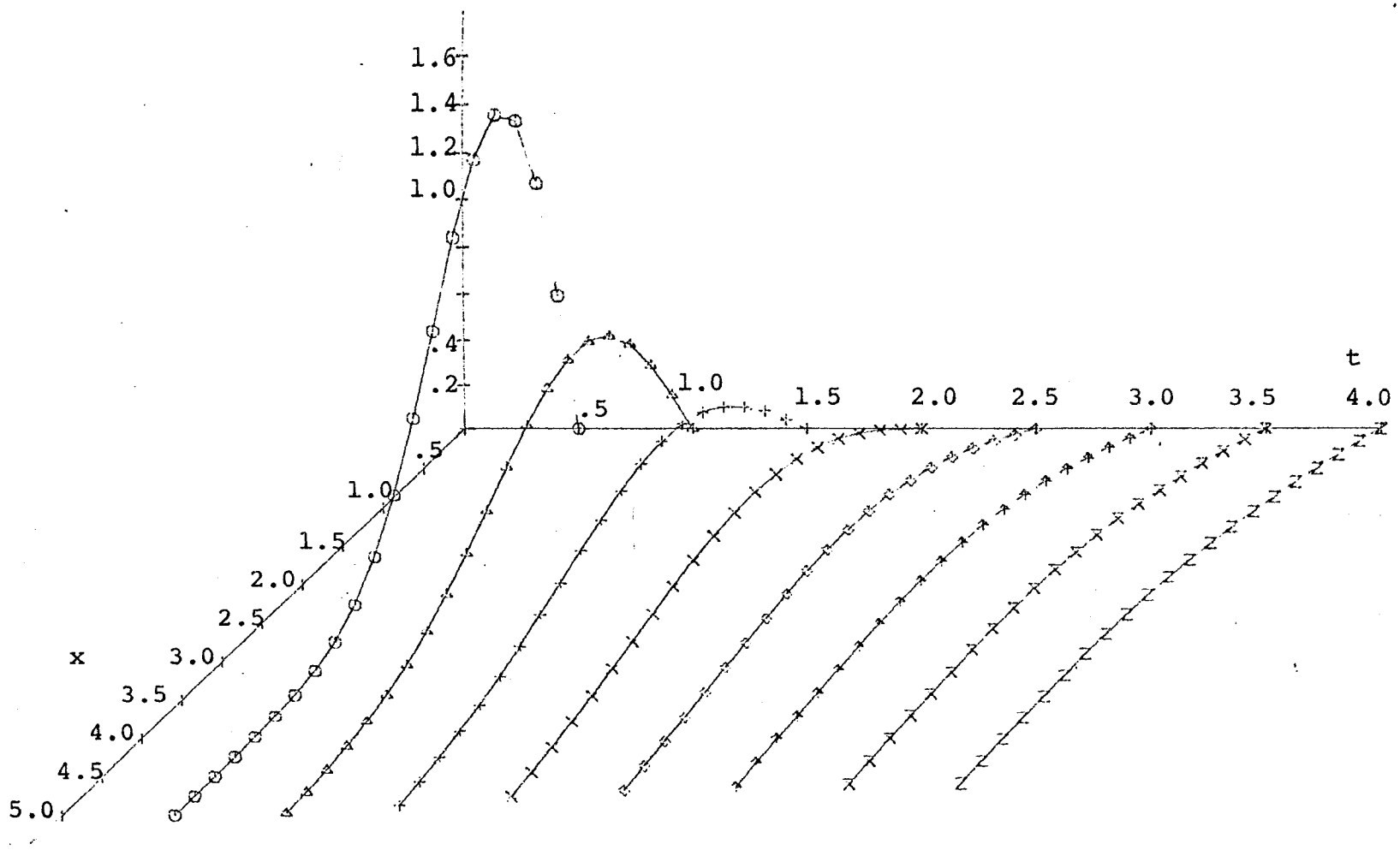


Figure 4.6 The Impulsive Response of a Matched $\overline{RC}$ Microcircuit (three dimensional view)

the same equation governs the action of the $\overline{RC}$ microcircuit, it is natural that it should exhibit this method of propagation.

Figures 4.5 and 4.6 also demonstrate the attenuation of the $\overline{RC}$ microcircuit. This is also natural since it is not a lossless system. Figures 4.5 and 4.6 were plotted using values of R = 1 and C = 1. Choosing a smaller value of R would decrease the amount of attenuation in the network.

Chapter 5: EXPERIMENTAL DETERMINATION OF $\overline{RC}$ MICROCIRCUIT PARAMETERS

The transient analysis reviewed in the preceeding chapters will now be used to develop a procedure for experimentally determining the distributed network parameters R and C. This knowledge of R and C provides a complete description of the $\overline{RC}$ microcircuit.

Table 5-1 lists responses for both open circuited and short circuited $\overline{RC}$ networks with both impulse voltage and impulse current excitation. The impulsive current response is included here because it may be easier to obtain an ideal current generator in actual practice than an "ideal" voltage generator.

The procedure is to record open circuit $\overline{RC}$ network voltage responses and short circuit $\overline{RC}$ network current responses to either impulsive current or voltage excitations. These recorded impulsive responses are then compared with the predicted response (see Table 5-1) in a least squares sense to obtain the parameters R and C.

The general form of the responses listed in Table 5-1 is

$$f(t) = \alpha e^{-\beta t} \tag{5.1}$$

A curve may be fit to Equation (5.1) by first linearizing it,

$$\ln f(t) = \ln \alpha - \beta t \tag{5.2}$$

TABLE 5-1 IMPULSIVE EXCITATION RESPONSES

Impulsive Voltage Excitation

$\delta(t)$  Open Circuit

For Small t

$$v(d,t) \doteq \frac{\sqrt{RC}}{\sqrt{\pi}\, t^{\frac{3}{2}}} e^{-\frac{RC}{4t}}$$

For Large t

$$v(d,t) \doteq \frac{\pi}{RC} e^{-\frac{\pi^2 t}{4RC}}$$

$\delta(t)$  Short Circuit

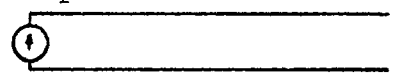
For Small t

$$i(d,t) \doteq \frac{C}{R\pi}\left\{\frac{1}{2t^{\frac{3}{2}}} + \frac{RC}{4t^{\frac{5}{2}}}\right\} e^{-\frac{RC}{4t}}$$

For Large t

$$i(d,t) \doteq \frac{2\pi^2}{R^2 C} e^{-\frac{\pi^2 t}{RC}}$$

Impulsive Current Excitation

$\delta(t)$  Open Circuit

For Small t

$$v(d,t) \doteq \frac{R}{C}\frac{2}{\sqrt{\pi t}} e^{-\frac{RC}{4t}}$$

For Large t

$$v(d,t) \doteq \frac{R}{C}\left\{1-2\, e^{-\frac{\pi^2 t}{RC}}\right\}$$

$\delta(t)$  Short Circuit

For Small t

$$i(d,t) \doteq \frac{\sqrt{RC}}{\sqrt{\pi}\, t^{\frac{3}{2}}} e^{-\frac{RC}{4t}}$$

For Large t

$$i(d,t) \doteq \frac{\pi}{RC} e^{-\frac{\pi^2 t}{4RC}}$$

Now, form the deviation between the observed data, (y_i, t_i), and the theoretical values

$$\ln f(t_i) = y_i = (\ln \alpha - \beta t_i) \tag{5.3}$$

The sum of the squared deviations is

$$s(\alpha,\beta) = \sum_{i=1}^{n} (y_i - \ln \alpha + \beta t_i)^2 \tag{5.4}$$

Then

$$\frac{\partial s}{\partial \alpha} = 2 \sum_{i=1}^{n} (y_i - \ln \alpha + \beta t_i)(-\frac{1}{\alpha}) \tag{5.5}$$

$$\frac{\partial s}{\partial \beta} = 2 \sum_{i=1}^{n} (y_i - \ln \alpha + \beta t_i) t_i \tag{5.6}$$

Equating these partial derivatives to zero determines the solutions, a and b (the least squares estimates of α and β), of the equations

$$\sum_{i=1}^{n} y_i = n \ln a - b \sum_{i=1}^{n} t_i \tag{5.7}$$

$$\sum_{i=1}^{n} t_i y_i = n \ln a \sum_{i=1}^{n} t_i - b \sum_{i=1}^{n} t_i^2 \tag{5.8}$$

The observed data (y_i, t_i) in Equation (5.3) is taken from an oscillograph of the impulsive response of the $\overline{RC}$ microcircuit. This data is then compared with the appropriate curve from Table 5-1 (depending on the conditions of the test, i.e., open circuit or short circuit and voltage or

current excitation). A detailed explanation of this least-square approximation is found in Reference (25).

It should be noted that to completely determine R and C, two sets of measurements must be made. For example, if the line is excited with a voltage source, the approximate open circuit response, for t small, is

$$v(d,t) \doteq \frac{\sqrt{RC}}{\pi t^{\frac{3}{2}}} \; e^{-\frac{RC}{4t}} \tag{5.9}$$

From the least-squares approximation, the term RC can be determined.

Now a short circuit test with the same source should be performed. The approximate response here, from Table 5-1 is

$$i(d,t) \doteq \frac{2\pi^2}{R^2C} \; e^{-\frac{\pi^2 t}{RC}} \tag{5.10}$$

for t large.

The least-squares approximation will give an estimate of R^2C. From these two measurements, R and C may now be determined.

Chapter 6: SUGGESTIONS FOR FURTHER RESEARCH

The research leading to the preparation of this report has lead to several suggestions for further research. Obviously, the next step after the analysis of the distributed parameter $\overline{RC}$ network, is the design and synthesis of these networks. One problem is that of synthesizing a taper given certain input and output requirements. Another, that of determining the parameters of an $\overline{RC}$ network given the taper and input/output requirements. A design scheme for a simple multisection stepped line (a line built of cascaded uniform sections, each of different widths) would be of interest.

A formalized design of multilayered matching networks for interconnecting lumped active devices would be welcome, since this type of problem is inherent in the manufacture of integrated circuits, especially those designed to operate at microwave frequencies and above.

The concept of computer-aided design is becoming more prominent as the third generation computers come into use. The use of computer-aided design techniques can shorten the lapse time between the concept and realization of a system. To use computer-aided design techniques, an accurate model of the system must be found. If the overall system contains one or more distributed parameter devices, it is essential that they also be accurately modeled.

However, for computer-aided design to be an effective tool, methods must be found to relate the models used in the design procedure to the actual devices which are used to produce the hardware. It is necessary to integrate the processes of modeling and measurement of large classes of these devices with eventual application to computer-aided design being a primary consideration.

Better approximation techniques for the analysis of distributed parameter systems are needed. One method of approximation that should be investigated is that of approximating the tapered line with a line of cascaded uniform or even tapered sections. Error criteria must be found before these approximations are useful.

The impulsive response of the uniform $\overline{RC}$ distributed parameter network has been investigated in this work. The impulsive response of certain tapered networks and a method for determining the impulsive response of a general tapered network is needed.

Having analyzed the passive distributed parameter system, the question of analysis of the active-passive distributed parameter system or even the distributed active system comes to mind. Another problem herein is that of building the distributed active system. This may take the form of a "distributed transistor".

The distributed parameter device can be constructed using monolithic techniques with the resistive layers being composed of either n-type or p-type materials and the

dielectric formed by the depletion layer between adjacent n-type and p-type layers. If a dc reverse bias is applied to the p-n junction, the width of the deplection layer changes, changing the capacitance. Can this change in capacitance with dc bias be used effectively to build a parametric amplifier?

Another topic of research is that of investigating the various combinations of distributed-lumped systems, both active and passive, to obtain various types of filters and matching devices. The responses of various configurations could be cataloged to provide a basis for knowledgeable design and synthesis.

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VITA

Robert Charles Peirson was born on September 25, 1943, in Grand Island, Nebraska. He received his primary and secondary education in Sedalia, Missouri. After graduation from high school he attended the University of Missouri - Rolla, Rolla, Missouri. He received a Bachelor of Science in Electrical Engineering in May, 1965, and Master of Science in Electrical Engineering in May, 1966.

The author has held the position of half time instructor in the Electrical Engineering Department from September, 1965, to August, 1968. The author has industrial experience with Collins Radio Company, Cedar Rapids, Iowa, and Braddock, Dunn and McDonald, Inc., El Paso, Texas.

The author is a member of IEEE and Eta Kappa Nu.

APPENDIX A: DERIVATION OF EQUATION (4.15)

This appendix is concerned with the integration indicated in Equation (4.14). The integral in the right hand side of this equation is

$$\frac{RC}{\pi} \int_{-\infty}^{\infty} (-1)^{\xi+1} \left(\xi - \frac{1}{2}\right) r^{\frac{3}{2}} e^{-RC\left(\xi-\frac{1}{2}\right)^2 r - 2\pi i n \xi} \, d\xi \qquad (A.2)$$

Expressing $(-1)^{\xi+1}$ as $-(e^{i\pi})^{\xi}$ or $-e^{i\pi\xi}$ the integral can be written

$$-\frac{RC}{\pi} \int_{-\infty}^{\infty} \left(\xi - \frac{1}{2}\right) r^{\frac{3}{2}} e^{-RC\left(\xi-\frac{1}{2}\right)^2 r - 2\pi i \left(n-\frac{1}{2}\right)\xi} \, d\xi \qquad (A.2)$$

Let

$$\xi - \frac{1}{2} = y,$$

therefore,

$$\xi = y + \frac{1}{2} \qquad (A.3)$$

and

$$d\xi = dy.$$

The limits on the integral say the same.

Equation (A.2) becomes

$$-\frac{RC}{\pi} \int_{-\infty}^{\infty} y \, r^{\frac{3}{2}} e^{-RCy^2 r - \left(n-\frac{1}{2}\right) 2\pi i \left(y+\frac{1}{2}\right)} \, dy \qquad (A.4)$$

Rearranging yields

$$-\frac{RCr^3}{\pi} e^{-\left(n-\frac{1}{2}\right)\pi i} \int_{-\infty}^{\infty} y \, e^{-rRC\left[y^2 + \frac{\left(n-\frac{1}{2}\right) 2\pi i y}{RC\, r}\right]} \, dy \qquad (A.5)$$

Completing the square in the exponent gives,

$$- \frac{RCr^3}{\pi} e^{-(n-\frac{1}{2})\pi i} \quad e^{-\frac{(n-\frac{1}{2})^2\pi^2}{RCr}} \int_{-\infty}^{\infty} y\, e^{-RCr[y+\frac{(n-\frac{1}{2})\pi i}{RCr}]^2} dy \tag{A.6}$$

Let

$$y + \frac{(n - \frac{1}{2})\pi i}{RCr} = z, \tag{A.7}$$

then

$$dy = dz \tag{A.8}$$

and the limits on the integral say the same. Substitution of this change into Equation (A.6)

$$- \frac{RCr^3}{\pi} e^{-(n-\frac{1}{2})\pi i - \frac{(n-\frac{1}{2})^2\pi^2}{RCr}} \int_{-\infty}^{\infty} [z - \frac{(n - \frac{1}{2})\pi i}{RCr}] e^{-RCrz^2} dz \tag{A.9}$$

Making a final change of variables,

$$q^2 = RCrz^2 \tag{A.10}$$

with

$$dq = \sqrt{RCr}\, dz, \tag{A.11}$$

Equation (A.9) becomes

$$\frac{RCr^3}{\pi} e^{-(n-\frac{1}{2})\pi i - \frac{(n-\frac{1}{2})^2\pi^2}{RCr}} \int_{-\infty}^{\infty} \frac{q}{RCr} e^{-q^2} - \frac{(n-\frac{1}{2})\pi i}{(RCr)^{\frac{3}{2}}} e^{-q^2} dq \tag{A.12}$$

Integrating yields

$$- \frac{RCr^3}{\pi} e^{-(n-\frac{1}{2})\pi i} - \frac{(n-\frac{1}{2})^2\pi^2}{RCr} \qquad 0 - \frac{(n-\frac{1}{2})\pi^{\frac{3}{2}}i}{(RCr)^{\frac{3}{2}}} \tag{A.13}$$

Rewriting gives

$$(-1)^n \frac{(n-\frac{1}{2})\pi}{RC} e^{-\frac{(n-\frac{1}{2})^2\pi^2}{RCr}} \tag{A.14}$$

which is the right hand side of Equation (4.15) in the text.

APPENDIX B: LAPLACE TRANSFORM OF CERTAIN FUNCTIONS OF A MATRIX

The Laplace transform of a function of a matrix is found by drawing an analogy to the scalar operation

$$F(s) = L[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt \qquad (B.1)$$

By analogy, the operation for matrices consists of

$$\underline{F}(s) = L[\underline{f}(t)] = \int_0^\infty \underline{f}(t)\, e^{-sI_n t}\, dt \qquad (B.2)$$

where I_n is the unit matrix of order n.

Now, find the Laplace transform of e^{At}, where A is a square matrix.

$$L[e^{At}] = \int_0^\infty e^{At}\, e^{-sI_n t}\, dt = \int_0^\infty e^{-[sI_n - A]t}\, dt$$

$$= -[sI_n - A]^{-1}\, e^{-[sI_n - A]t} \Big|_0^\infty = [sI_n - A]^{-1} \qquad (B.3)$$

if $sI_n > A$ (element by element)

From this, it is easy to compute the Laplace transform of cosh (At). Since

$$\cosh At = \frac{1}{2}\,[e^{At} + e^{-At}] \qquad (B.4)$$

The Laplace transform is therefore

$$L[\cosh At] = \frac{1}{2}\{L[e^{At}] + L[e^{-At}]\}$$

$$= \frac{1}{2}[[sI_n - A]^{-1} + [sI_n + A]^{-1}]$$

$$= sI_n\,[s^2I_n - A^2]^{-1} \qquad \text{(B.5)}$$

Similarly, for the Laplace transform of the sinh (At),

$$L[\sinh At] = [sI_n - A^2]^{-1}\,A \qquad \text{(B.6)}$$

APPENDIX C: COMPUTER PROGRAMS

```
C        R C PEIRSON    PHD DISSERTATION
C        TAYLORS MATRIX SERIES APPROXIMATION OF THE
C        EXPONENTIAL LINE       IBM 360/50
C     R=3 NEW TAYLOR COMPLEX EXP(AK*(1.0-X))  FIRST
      COMPLEX Z(2,2),Y(2,2),T(4,4),T1(4,4),A(4,4)
     2,ANORM    ,DIFF(4,4),EXACT(4,4)
      WRITE(3,150)
  150 FORMAT(5X11H INPUT DATA)
      READ(1,400) ((Z(I,J),J=1,2),I=1,2)
      READ(1,400) ((Y(I,J),J=1,2),I=1,2)
      READ(1,100) ((EXACT(I,J),J=1,4),I=1,4)
      WRITE(3,200) ((Z(I,J),J=1,2),I=1,2)
      WRITE(3,200) ((Y(I,J),J=1,2),I=1,2)
      WRITE(3,300) ((EXACT(I,J),J=1,4),I=1,4)
      WRITE(3,151)
  151 FORMAT(5X8H ANSWERS)
      W=1.0
      AK=2.0
      DO 95 K1=1,3
      DO 95 N=10,60,10
      WRITE(3,401)K1,N
      X1=N
  401 FORMAT(5X4HR = ,I2,5X4HN = ,I3)
      DX=1.0/X1
      X=1.0-DX
      Q=EXP(AK*X)
      F1=AK*Q
      F2=AK*F1
      F3=AK*F2
      CALL TAYLOR(Q,F1,F2,F3,DX,T,K1,N,X)
    7 X=X-DX
      Q=EXP(AK*X)
      F1=AK*Q
      F2=AK*F1
      F3=AK*F2
      CALL TAYLOR(Q,F1,F2,F3,DX,T1,K1,N,X)
      CALL CGMPRD(T,T1,A,4,4,4)
      DO 11 I=1,4
      DO 11 J=1,4
   11 T(I,J)=A(I,J)
      IF(X)13,13,7
   13 CONTINUE
      WRITE(3,102)
  102 FORMAT(5X18HFINAL ANSWER FOR T)
      WRITE(3,300) ((T(I,J),J=1,4),I=1,4)
      DO 50 I=1,4
      DO 50 J=1,4
   50 DIFF(I,J)=EXACT(I,J)-T(I,J)
      CALL EUCNOR(DIFF,ANORM,4)
      WRITE(3,400) ANORM
      WRITE(3,200) ANORM
      CALL EUCNOR(EXACT,ANORM,4)
      WRITE(3,400) ANORM
      WRITE(3,200) ANORM
      CALL EUCNOR(T,ANORM,4)
```

```
      WRITE(3,400) ANORM
      WRITE(3,200) ANORM
   95 CONTINUE
      CALL EXIT
   00 FORMAT(4E18.8)
   00 FORMAT(4F18.8)
   00 FORMAT(8F16.8)
   00 FORMAT(4E18.8)
      END
      SUBROUTINE TAYLOR(Q,F1,F2,F3,DX,T,K1,N,X)
      COMPLEX Z(2,2),Y(2,2),TK(4,4),C(4,4),B(4,4),D(4,4),
     1E(4,4),UNIT(4,4),A(4,4),T(4,4),T1(4,4),T2(4,4),T3(4,4)
     2,F,CMPLX
     4,H(4,4),G(4,4),T4(4,4)
      COMPLEX    T2A(4,4),T3A(4,4),T4A(4,4),ANORM
      F=Q
      Z(1,1)=(1.0,0.0)
      Z(1,2)=(0.0,0.0)
      Z(2,1)=(0.0,0.0)
      Z(2,2)=(1.0,0.0)
      Y(1,1)=(0.0,1.0)
      Y(1,2)=(0.0,-1.0)
      Y(2,1)=(0.0,-1.0)
      Y(2,2)=(0.0,2.0)
      UNIT(1,1)=(1.0,0.0)
      UNIT(2,2)=(1.0,0.0)
      UNIT(3,3)=(1.0,0.0)
      UNIT(4,4)=(1.0,0.0)
      UNIT(1,2)=(0.0,0.0)
      UNIT(2,1)=(0.0,0.0)
      UNIT(3,4)=(0.0,0.0)
      UNIT(4,3)=(0.0,0.0)
      DO 10 I=1,2
      DO 10 J=3,4
   10 UNIT(I,J)=(0.0,0.0)
      DO 11 I=3,4
      DO 11 J=1,2
   11 UNIT(I,J)=(0.0,0.0)
      DO 21 I=1,2
      DO 21 J=1,2
   21 TK(I,J)=(0.0,0.0)
      DO 22 I=3,4
      DO 22 J=3,4
   22 TK(I,J)=(0.0,0.0)
      DO 23 I=1,2
      DO 23 J=3,4
   23 TK(I,J)=F*Z(I,J-2)
      DO 24 I=3,4
      DO 24 J=1,2
   24 TK(I,J)=Y(I-2,J)/F
      DO 25 I=3,4
      DO 25 J=1,4
   25 A(I,J)=UNIT(I,J)
      DO 26 I=1,2
      DO 26 J=1,4
```

```
26 A(I,J)=-UNIT(I,J)
   CALL CGMPRD(TK,A,C,4,4,4)
   CALL CGMPRD(TK,TK,B,4,4,4)
   CALL CGMPRD(B,TK,D,4,4,4)
   CALL CGMPRD(C,TK,E,4,4,4)
   CALL CGMPRD(E,TK,H,4,4,4)
   CALL CGMPRD(D,TK,G,4,4,4)
   GO TO (41,42,43,44),K1
41 DO 31 I=1,4
   DO 31 J=1,4
   T1(I,J)=UNIT(I,J)+TK(I,J)*DX
31 T(I,J)=T1(I,J)
   RETURN
42 DO 32 I=1,4
   DO 32 J=1,4
   T1(I,J)=UNIT(I,J)+TK(I,J)*DX
   T2A(I,J)=C(I,J)*F1/F+B(I,J)
   T2(I,J)=T2A(I,J)*DX*DX/2.0
32 T(I,J)=T2(I,J)+T1(I,J)
   RETURN
43 DO 33 I=1,4
   DO 33 J=1,4
   T1(I,J)=UNIT(I,J)+TK(I,J)*DX
   T2(I,J)=          (C(I,J)*F1/F+B(I,J))*DX*DX/2.0
   T2A(I,J)=(F1*F1/F*F)*TK(I,J)
   T4A(I,J)=(F1/F)*E(I,J)
   T3A(I,J)=C(I,J)*((F*F2-F1*F1)/(F*P))+((F1*F1)/(F*F))
  1*TK(I,J)+(F1/F)*E(I,J)+D(I,J)
   T3(I,J)=T3A(I,J)*DX*DX*DX/6.0
33 T(I,J)=T3(I,J)+T2(I,J)+T1(I,J)
   RETURN
44 DO 34 I=1,4
   DO 34 J=1,4
   T1(I,J)=UNIT(I,J)+TK(I,J)*DX
   T2(I,J)=          (C(I,J)*F1/F+B(I,J))*DX*DX/2.0
   T3A(I,J)=C(I,J)*((F*F2-F1*F1)/(F*P))+((F1*F1)/(F*F))
  1*TK(I,J)+(F1/F)*E(I,J)+D(I,J)
   T3(I,J)=T3A(I,J)*DX*DX*DX/6.0
   T4A(I,J)=(F3/F-(F2*F1)/(F*F))*C(I,J)+((F1*F1)/(F*F))
  1*TK(I,J)-((2.*F1*F2)/(F*F)-(2.*F1*F1*F1)/(F*F*F))*
  2C(I,J)-((F1*F1*F1)/(F*F*F))*TK(I,J)+((2.*F1*F2)/(F*F)
  3-(2.*F1*F1*F1)/(F*F*F))*TK(I,J)+((F1*F1*F1)/(F*F*F))
  4*C(I,J)+(F2/F-(F1*F1)/(F*F))*E(I,J)+(F1/F)*H(I,J)+
  5(F2/F-(F1*F1)/(F*F))*E(I,J)+((F1*F1)/(F*F))*B(I,J)+
  6(F1/F)*H(I,J)+G(I,J)
   T4(I,J)=T4A(I,J)*DX*DX*DX*DX/24.0
34 T(I,J)=T4(I,J)+T3(I,J)+T2(I,J)+T1(I,J)
   RETURN
   END
   SUBROUTINE CGMPRD(A,B,C,N,M,L)
      CGMPRD     COMPLEX MATRIX MULTIPLICATION
   COMPLEX A(4,4),B(4,4),C(4,4),SUM
   DO 9 I=1,N
   DO 9 K=1,L
   SUM=(0.0,0.0)
```

```
      DO 8 J=1,M
    8 SUM=SUM+A(I,J)*B(J,K)
    9 C(I,K)=SUM
      RETURN
      END
      SUBROUTINE EUCNOR(A,ANORM,N)
C        EUCNOR CALCULATES THE EUCLIDEAN NORM OF A
C        COMPLEX MATRIX        IBM 360/50
      COMPLEX A(4,4),AC(4,4),B(4,4),TR,ANORM,CONJG,CSQRT
      DO 11 I=1,N
      DO 11 J=1,N
   11 AC(I,J)= CONJG(A(J,I))
      CALL CGMPRD(A,AC,B,N,N,N)
      TR=(0.0,0.0)
      DO 12 I=1,N
   12 TR=TR+B(I,I)
      ANORM=CSQRT(TR)
      RETURN
      END
```

```
C         R C PEIRSON    PHD DISSERTATION
C         COMPUTATION OF VANDIVORT'S SOLUTION MATRIX FOR THE
C         EXPONENTIAL LINE       IBM   360/50
C      SOLN OF EXP LINE W/ COMPLEX
       COMPLEX R(2,2),C(2,2),V(2,2),Z1(2,2),Z2(2,2),GAM(2,2),
      1U(2,2),EIG1,EIG2,COSHG(4,4),SINHG(4,4),DET,GAMINV(4,4)
      2,GAMT(4,4),GAMTIN(4,4),COSHGT(4,4),SINHGT(4,4),T(4,4),
      3TRANS(4,4),WJ,EXPM(4,4),A,UNIT(4,4),CSQRT,CEXP,EXPOS,
      4EXNEG,B,CCOSH,CSINH,CCOS,CSIN
       CCOSH(A)=CCOS((0.0,1.0)*A)
       CSINH(A)=(0.0,-1.0)*CSIN((0.0,1.0)*A)
       READ(1,100) ((R(I,J),J=1,2),I=1,2)
       READ(1,100) ((C(I,J),J=1,2),I=1,2)
       READ(1,100) AK
       WRITE(3,150)
  150  FORMAT(5X11H INPUT DATA)
       WRITE(3,300) ((R(I,J),J=1,2),I=1,2)
       WRITE(3,300) ((C(I,J),J=1,2),I=1,2)
       WRITE(3,300) AK
       WRITE(3,151)
  151  FORMAT(5X8H ANSWERS)
       WJ=(0.0,1.0)
       UNIT(1,1)=(1.0,0.0)
       UNIT(1,2)=(0.0,0.0)
       UNIT(2,1)=(0.0,0.0)
       UNIT(2,2)=(1.0,0.0)
       CALL CGMPRD(R,C,U,2,2,2)
       DO 10 I=1,2
       DO 10 J=1,2
   10  V(I,J)=AK*AK*UNIT(I,J)+WJ*U(I,J)
       CALL CMFCN(V,Z1,Z2,EIG1,EIG2)
       DO 13 I=1,2
       DO 13 J=1,2
   13  GAM(I,J)=CSQRT(EIG1)*Z1(I,J)+CSQRT(EIG2)*Z2(I,J)
       WRITE(3,100) ((GAM(I,J),J=1,2),I=1,2)
       CALL CMFCN(GAM,Z1,Z2,EIG1,EIG2)
       DO 16 I=1,2
       DO 16 J=1,2
   16  COSHG(I,J)=CCOSH(EIG1)*Z1(I,J)+CCOSH(EIG2)*Z2(I,J)
       DO 17 I=1,2
       DO 17 J=1,2
   17  SINHG(I,J)=CSINH(EIG1)*Z1(I,J)+CSINH(EIG2)*Z2(I,J)
       CALL CDET(GAM,DET)
       CALL CINV(GAM,DET,GAMINV)
       DO 18 I=1,2
       DO 18 J=1,2
   18  GAMT(I,J)=GAM(J,I)
       CALL CINV(GAMT,DET,GAMTIN)
       CALL CMFCN(GAMT,Z1,Z2,EIG1,EIG2)
       A=CCOSH(EIG1)
       B=CCOSH(EIG2)
       WRITE(3,300) A,B
       DO 19 I=1,2
       DO 19 J=1,2
   19  COSHGT(I,J)=CCOSH(EIG1)*Z1(I,J)+CCOSH(EIG2)*Z2(I,J)
```

```
      DO 20 I=1,2
      DO 20 J=1,2
   20 SINHGT(I,J)=CSINH(EIG1)*Z1(I,J)+CSINH(EIG2)*Z2(I,J)
C     T=1/2(TRANS)
      CALL CGMPRD(GAMINV,SINHG,V,2,2,2)
      DO 23 I=1,2
      DO 23 J=1,2
   23 T(I,J)=COSHG(I,J)-V(I,J)
      CALL CGMPRD(C,GAMINV,V,2,2,2)
      CALL CGMPRD(V,SINHG,U,2,2,2)
      DO 24 I=3,4
      DO 24 J=1,2
   24 T(I,J)=WJ*U(I-2,J)
      CALL CGMPRD(R,GAMTIN,V,2,2,2)
      CALL CGMPRD(V,SINHGT,U,2,2,2)
      DO 25 I=1,2
      DO 25 J=3,4
   25 T(I,J)=U(I,J-2)
      CALL CGMPRD(GAMTIN,SINHGT,V,2,2,2)
      DO 26 I=3,4
      DO 26 J=3,4
   26 T(I,J)=COSHGT(I-2,J-2)+V(I-2,J-2)
      A=(1.0,0.0)
      EXPOS=CEXP(A)
      EXNEG=1.0/EXPOS
      EXPM(1,1)=EXPOS
      EXPM(2,2)=EXPOS
      EXPM(3,3)=EXNEG
      EXPM(4,4)=EXNEG
      DO 27 I=1,2
      DO 27 J=3,4
   27 EXPM(I,J)=(0.0,0.0)
      DO 28 I=3,4
      DO 28 J=1,2
   28 EXPM(I,J)=(0.0,0.0)
      EXPM(1,2)=(0.0,0.0)
      EXPM(2,1)=(0.0,0.0)
      EXPM(3,4)=(0.0,0.0)
      EXPM(4,3)=(0.0,0.0)
      CALL CGMPRD(EXPM,T,TRANS,4,4,4)
      WRITE(3,200) ((TRANS(I,J),J=1,4),I=1,4)
      WRITE(2,100) ((TRANS(I,J),J=1,4),I=1,4)
  100 FORMAT(4E18.8)
  200 FORMAT(8F16.8)
  300 FORMAT(4F18.8)
      CALL EXIT
      END
      SUBROUTINE CMFCN(V,Z1,Z2,EIG1,EIG2)
C        CMFCN    COMPUTES THE CONSTITUTENT IDEMPOTENTS FOR
C        COMPUTATION OF COMPLEX MATRIX FUNCTIONS
      COMPLEX V(2,2),EIG1,EIG2,UNIT(2,2),W1(2,2),W2(2,2),
     181,B2,Z1(2,2),Z2(2,2)
      UNIT(1,1)=(1.0,0.0)
      UNIT(1,2)=(0.0,0.0)
      UNIT(2,1)=(0.0,0.0)
```

```
      UNIT(2,2)=(1.0,0.0)
      CALL CEIG22(V,EIG1,EIG2)
      DO 11 I=1,2
      DO 11 J=1,2
      W1(I,J)=EIG2*UNIT(I,J)
   11 W2(I,J)=EIG1*UNIT(I,J)
      B1=EIG1-EIG2
      B2= -B1
      DO 12 I=1,2
      DO 12 J=1,2
      Z1(I,J)=(V(I,J)-W1(I,J))/B1
   12 Z2(I,J)=(V(I,J)-W2(I,J))/B2
      RETURN
      END
      SUBROUTINE CDET(A,DET)
C        CDET   COMPUTES THE COMPLEX DETERMINANT OF A
C        (2,2) COMPLEX MATRIX
      COMPLEX A(4,4),DET
      DET=A(1,1)*A(2,2)-A(2,1)*A(1,2)
      RETURN
      END
      SUBROUTINE CINV(A,DET,AI)
      COMPLEX A(2,2),DET,AI(2,2)
      AI(1,1)=A(2,2)/DET
      AI(1,2)=-A(1,2)/DET
      AI(2,1)=-A(2,1)/DET
      AI(2,2)= A(1,1)/DET
      RETURN
      END
      SUBROUTINE CGMPRD(A,B,C,N,M,L)
C        CGMPRD    COMPLEX MATRIX MULTIPLICATION
      COMPLEX A(4,4),B(4,4),C(4,4),SUM
      DO 9 I=1,N
      DO 9 K=1,L
      SUM=(0.0,0.0)
      DO 8 J=1,M
    8 SUM=SUM+A(I,J)*B(J,K)
    9 C(I,K)=SUM
      RETURN
      END
      SUBROUTINE CEIG22(A,EIG1,EIG2)
C        CEIG22 COMPUTES THE EIGENVALUES OF A (2,2)
C        COMPLEX MATRIX
      COMPLEX A(2,2),EIG1,EIG2,B
     1,CSQRT
      B=CSQRT((A(2,2)+A(1,1))*(A(2,2)+A(1,1))-4.0*(A(1,1)*
     1A(2,2)-A(2,1)*A(1,2)))
      EIG1=(A(2,2)+A(1,1)+B)/2.0
      EIG2=(A(2,2)+A(1,1)-B)/2.0
      RETURN
      END
```

```
C         R C PEIRSON    PHD DISSERTATION
C       PLOT ERROR CRITERIA
        DIMENSION Y(18),YL(18),DIFF(18),DIFFL(18),X(6),YP(18)
        READ(1,100)Y
        READ(1,100)DIFF
        DO 10 I=1,6
   10   X(I)=I*10
        DO 12 K=1,18
        YL(K)=ALOG10(Y(K))
   12   DIFFL(K)=ALOG10(DIFF(K))
        WRITE(3,100)YL,DIFFL,Y,DIFF
        CALL PENPOS('R.C.PEIRSON',11,1)
        CALL NEWPLT(1.0,1.0,6.5)
        CALL ORIGIN(0.0,-5.0)
        CALL XSCALE(0.0,60.0,5.5)
        CALL YSCALE(-5.0,2.5,7.0)
        CALL XAXIS(10.0)
        CALL YAXIS(0.5)
        DO 20 I=1,6
   20   YP(I)=YL(I)
        DO 11 J=1,3
        CALL XYPLT(X,YP,6,1,J)
        DO 30 I=1,6
   30   YP(I)=YL(I+6*J)
   11   CALL PEN(2)
        DO 21 I=1,6
   21   YP(I)=DIFFL(I)
        DO 13 M=1,3
        ISYM=M+3
        CALL XYPLT(X,YP,6,1,ISYM)
        DO 31 I=1,6
   31   YP(I)=DIFFL(I+6*M)
   13   CALL PEN(2)
        CALL ENDPLT
        CALL LSTPLT
        CALL EXIT
  100   FORMAT(4E18.8)
        END
```

```
C         *** IMPULSE ***
C         TWO DIMENSIONAL PLOT
C         R C PEIRSON    PHD DISSERTATION
      DIMENSION E(99)   ,X(100)
      CALL PENPOS('R.C.PEIRSON',11,1)
      READ(1,10 ) XPOSIN,YPOSIN,TZERO,YZERO,TMIN,TMAX,
     1YMIN,YMAX,DT,DY
      WRITE(3,10 ) XPOSIN,YPOSIN,TZERO,YZERO,TMIN,TMAX,
     1YMIN,YMAX,DT,DY
      CALL NEWPLT(XPOSIN,YPOSIN,10.0)
      CALL ORIGIN(TZERO,YZERO)
      CALL TSCALE(TMIN,TMAX,8.0)
      CALL YSCALE(YMIN,YMAX,5.0)
      CALL TAXIS(DT)
      CALL YAXIS(DY)
      W = 1.0
      DO 100 I=1,8
      T = W/2.
      A = 0.0
      DO 101 J = 1,21
      X(J) = A / 4.
      X(J) = (J-1)/4.
      A = A + 1.
  101 E(J)= (X(J)/(T**1.5))*EXP(-X(J)*X(J)/(4.0*T))
      WRITE(3,30)I,T
      DO 102 K = 1,16,5
      KK = K - 1
  102 WRITE(3,31)((X(J+KK),J = 1,5),(E(J+KK),J=1,5))
      WRITE (3,32)   X(21),E(21)
      CALL TPLT(E,21,1,I)
      W = W + 1.
  100 CONTINUE
      CALL ENDPLT
      CALL LSTPLT
      CALL EXIT
   10 FORMAT(10F6.1)
   30 FORMAT(//' I= 'I2,5X,'T= ',F4.1//)
   31 FORMAT(5X,5F18.4/5E18.8)
   32 FORMAT(5X, F18.4/ E18.8)
      END
```

```
C         R C PEIRSON    PHD DISSERTATION
C         *** IMPULSE ***
C         THREE DIMENSIONAL PLOT PROGRAM
      EXTERNAL AFUN
      DIMENSION X(30),Y(30),G(30),A(30,30),B(30,30),
     1C(30),D(30),F(30)
      READ(1,10) XPOSIN,YPOSIN,XPGIN,XMAX, YMAX,YSZIN1,
     1DX,DY,ZMAX,XSZIN2
     1,DZ,XEVAL,ZEVAL,IXINI,IXFIN,IZINI,IZFIN,IYZPLT,IXYPLT
   10 FORMAT(10F7.3/3F7.3,6I7)
      CALL PENPOS('R. C. PEIRSON',13,1)
      CALL PLOT3D(XPOSIN,YPOSIN,XPGIN,XMAX,YMAX, YSZIN1,
     1DX ,DY,ZMAX,DZ,
     MXEVAL,ZEVAL,IXINI,IXFIN,IZINI, IZFIN,IYZPLT,IXYPLT,
     3XSZIN2,AFUN)
      CALL LSTPLT
      CALL EXIT
      END
      FUNCTION AFUN(X,Z)
      AFUN= (Z/X**1.5)*EXP(-Z*Z/(4*X))
      RETURN
      END
      SUBROUTINE PLOT3D(XPOSIN,YPOSIN,XPGIN,XMAX,YMAX,
     1YSZIN1,DX,DY,
     2ZMAX,DZ,XEVAL,ZEVAL,IXINI,IXFIN,IZINI,IZFIN,IYZPLT,
     3IXYPLT,XSZIN2,AFUN)
      DIMENSION X(30),Y(30),G(30),A(30,30),B(30,30),
     1C(30),D(30),F(30)
      WRITE(3,10)XPOSIN,YPOSIN,XPGIN,XMAX, YMAX,YSZIN1,
     1DX,DY,ZMAX,XSZIN2
     1,DZ,XEVAL,ZEVAL,IXINI,IXFIN,IZINI,IZFIN,IYZPLT,IXYPLT
   10 FORMAT(10F7.3/3F7.3,6I7)
      CALL NEWPLT (XPOSIN,YPOSIN, XPGIN)
      CALL ORIGIN (0.0,0.0)
      XSZIN1 = XSZIN2-YSZIN1
      CALL XSCALE (0.0,XMAX,XSZIN1)
      CALL YSCALE (0.0,YMAX,YSZIN1)
      CALL XAXIS (DX)
      CALL YAXIS (DY)
      XMIN2 = (-YSZIN1) *(XMAX/XSZIN1)
      CALL XSCALE (XMIN2,XMAX,XSZIN2)
      YSZIN2 = 2 * YSZIN1
      YMIN2 = -YMAX
      CALL YSCALE (YMIN2,YMAX,YSZIN2)
      IZTIC =(ZMAX/DZ) +1
      WRITE(3,30) XSZIN1, XMIN2, YSZIN2, IZTIC
   30 FORMAT ( 3E18.8,I4)
      DO 100 I =1,IZTIC
      X(I) = ((I-1)*DZ)*(XMIN2/ZMAX)
  100 Y(I) =  ((I-1)*DZ)* ((YMIN2)/ZMAX)
      CALL XYPLT (X,Y,IZTIC,1,13)
      IFI= IXFIN + 1
      IZFIN = IZFIN + 1
      LL = IXINI-1
      NN = IZINI-1
```

```
      DO 101 I = IXINI,IFI
      X(I-LL) =(I-1)* XEVAL
      DO 101 J= IZINI , IZFIN
      Z = (J-1) * ZEVAL
      A(J-NN,I-LL) = AFUN(X(I-LL),Z) + Z * YMIN2/ZMAX
      B(J-NN,I-LL) = X(I-LL) + Z *(XMIN2/ZMAX)
      IF(A(J-NN,I-LL) -YMAX)101,101,2
    2 YMAX = A(J-NN,I-LL)
      WRITE (3,35) YMAX
   35 FORMAT(' ASSIGNED YMAX NOT LARGE ENOUGH,REPLACED BY'
     1,E18.8)
      CALL XSCALE (XMIN2,XMAX,XSZIN2)
      CALL YSCALE (YMIN2,YMAX,YSZIN2)
  101 CONTINUE
      ISX = IFI-LL
      ISZ = IZFIN -NN
      DO 102 I= 1,ISX
      K = I
      DO 103  J= 1,ISZ
      F(J) =A(J,I)
  103 G(J) = B(J,I)
      WRITE(3,40)(F(J),J=1,ISZ)
      WRITE(3,40)(G(J),J=1,ISZ )
   40 FORMAT(7E18.8)
      IF((I-13).GT.0) K = I - 13
  102 CALL XYPLT(G,F,ISZ,IYZPLT,K)
      IF(IXYPLT.EQ.0) GO TO 50
      DO 104 J= 1,ISZ
      DO 105 I=1,ISX
      C(I) = A(J,I)
  105 D(I) = B(J,I)
  104 CALL XYPLT(D,C,ISX,IXYPLT,-1)
   50 CALL ENDPLT
      RETURN
      END
```

DESCRIPTION OF THREE DIMENSIONAL PLOT SUBPROGRAM

Introduction

This is a subroutine for arranging a family of curves, obtained from an equation of two positive independent variables, in a three dimensional isometric plot.

The dependent variable may become negative, but it cannot be so large that, when plotted, it goes below the lowest part of the Z axis. If such a point is generated it will be ignored and a message printed on the listing.

Likewise, a positive point cannot be plotted any higher than the top of the Y axis. However, this problem is eliminated because of a check made in the program.

In all cases, the names of the variables used in the subroutine correspond to the names used in the plotter notes available from the computer center. Also, names for new fixed point variables follow the standard rules.

The main program is used to read in the plot parameters (page size, plot size, maximum and minimum values, etc.), start a plot page (CALL PENPOS), and for calling the subroutine PLOT3D.

The function to be plotted is defined in a function subprogram. Any number of functions may be plotted by calling PLOT3D and substituting the function's name in for AFUN. Each function must be defined in its own function subprogram.

Restrictions

In order to simplify the subroutine and to make it easier to use, some restrictions were imposed which limit the freedom of the user.

The plot will extend equal distances above and below the X axis. The plot, then, can be centered allowing the maximum of 5.5 inches for YSZIN1. If desired, however, two or more plots can be stacked on the same page. In any case, the maximum value of YSZIN1 is YPOSIN as long as YPOSIN is 5.5 or less.

The Z axis is drawn at a 45 degree slope and its length is 1.414*YSZIN1. The programmer need not worry about scaling the Z axis because this is done automatically and is determined by YMAX, XMAX and YSZIN1. Because the Z axis is at an angle of 45 degrees, the number of inches allowed to plot negative X values equals the number of inches allowed to plot negative Y values, which, in turn, is equal to YSZIN1.

Subroutine PLOT3D Parameters

As mentioned in the Introduction, all parameter names used in the subroutine have the same names as those used in the computer center Plotter Notes. Therefore XPOSIN, YPOSIN, XPGIN, XMAX, DX, and DY will not be explained again here.

YSZIN1 = is the same parameter as YSZIN used in the Computer Science notes but in the subroutine YSZIN1 refers to the length in inches of the Y axis. The Y axis is drawn above the origin only. YSZIN2 refers to

the total distance over which the Y variable may be plotted. It is determined by the user when he selects YSZIN1.

YMAX = any positive floating point number, and is needed for drawing the Y axis only. However, to save time, a number equal to or large than the largest number to be plotted should be entered for YMAX. If YMAX is not large enough a message will be printed along with some new value of YMAX. This new value is used by the computer.

XSZIN2 = the number of inches over which the user allows the computer to plot the X variable, and it is also equal to the outside dimension of the plot. Although only positive values of X are allowed in the evaluation of AFUN, negative values of X are generated by the computer in converting to a three dimensional plot. XSZIN2 will always be larger than XSZIN1 in order to accommodate these negative values. XSZIN1 is the length of the X axis in inches. It is calculated in the subroutine, and is determined by the choice of YSZIN1 and XSZIN2. The X axis is drawn to the right of the origin only.

ZMAX = is the maximum value of Z that can be plotted.

DZ = is equivalent to DY and DX but of course refers to the Z axis.

XEVAL = the increment by which X is increased after each evaluation of AFUN.

ZEVAL = is equivalent to XEVAL but refers to the Z variable.

IXINI = determines the first point of evaluation of AFUN along the X axis. For example, if IXINI = 1, AFUN is evaluated at X = 0, for IXINI = 2 AFUN is evaluated at X = 0 + XEVAL or XEVAL.

IZINI = corresponds to IXINI but refers to the Z variable.

IXFIN = determines the final point of evaluation of AFUN along the X axis. It can be determined by the user by dividing the maximum value of the X variable for which AFUN is to be evaluated by XEVAL. The result must be a fixed point number. The maximum value of the X variable must be less than or equal to XMAX.

IZFIN = corresponds to IXFIN but refers to the Z variable.

IYZPLT = if IYZPLT = 1, lines contained in planes parallel to the YZ plane are drawn through the points. If it is zero no lines will be drawn but the points will be marked with a characteristic symbol. The symbol is repeated after every thirteenth curve of a family.

IXYPLT = if IXYPLT = 1, the points previously plotted will be connected by lines contained in planed parallel to the XY plane. If the points were previously connected, a surface is formed. If IXYPLT = 0, the plot routine is finished.

Subroutine Statement Explanation

Statement 5 calls for a new page everytime PLOT3D is called; however, the name of the programmer is not repeated. If it is not desired to call a new page NEWPLT may be put in the main program after PENPOS. YPOSIN must be equal to or larger than YSZIN1. XPGIN must be equal to or larger than XSZIN2.

Statement 7 calculates the length of the X axis. It can be seen that XSZIN2 must be somewhat larger than YSZIN1.

Statements 8-11 scale the plot page to the right of the origin and then draw the positive X and Y axis.

Statements 12-13 calculate the smallest value of the X variable (XMIN2) that the computer will need to convert the plot to 3-D. XMIN2 is determined by previous plotting parameters and is completely independent of AFUN.

Statements 14-16 define new YSCALE parameters and the calls YSCALE.

Statements 17-23 define a DO loop and its parameters. This DO loop creates values which are plotted as the Z axis

by statement 23. It can be seen in statement 21 that, if (I-1)*DZ equals ZMAX, the point XMIN2 will be plotted. Also, from 22, under the same condition, YMIN2 will be plotted. Because the maximum value of (I-1) equals ZMAX/DZ there is no chance for error on the part of the user.

Statements 24-27 define subscript variables and DO loop parameters. The reason for the complicated subscript notation is that it allows first point evaluation at points other than (0,0) with no point deletion or addition. Because of the nature of XYPLT, even though a point at (0,0) was not calculated, it would still be called. Therefore, subscripts were used which convert R(m) to R(1) where R(m) is the first calculated or defined value of an array. DO loop parameters were defined so that there is no chance for 0 or negative values.

Statements 28-33 define the DO loops necessary to evaluate AFUN at the desired values of X and Z. It is seen that the value of AFUN is not explicitly available to the user, but is instead added to another number which creates the 3-D array, A, which is plotted on the Y axis.

From Figure C.1 it can be seen that the Y value to be plotted is the value of AFUN minus some part of the value of Z. This part of Z was found to be equal to Z * YMIN/ZMAX This can be proven by letting Z = ZMAX and AFUN = 0. The result is YMIN2 which is correct. Because of the linearity of the plot all other points must be correct also.

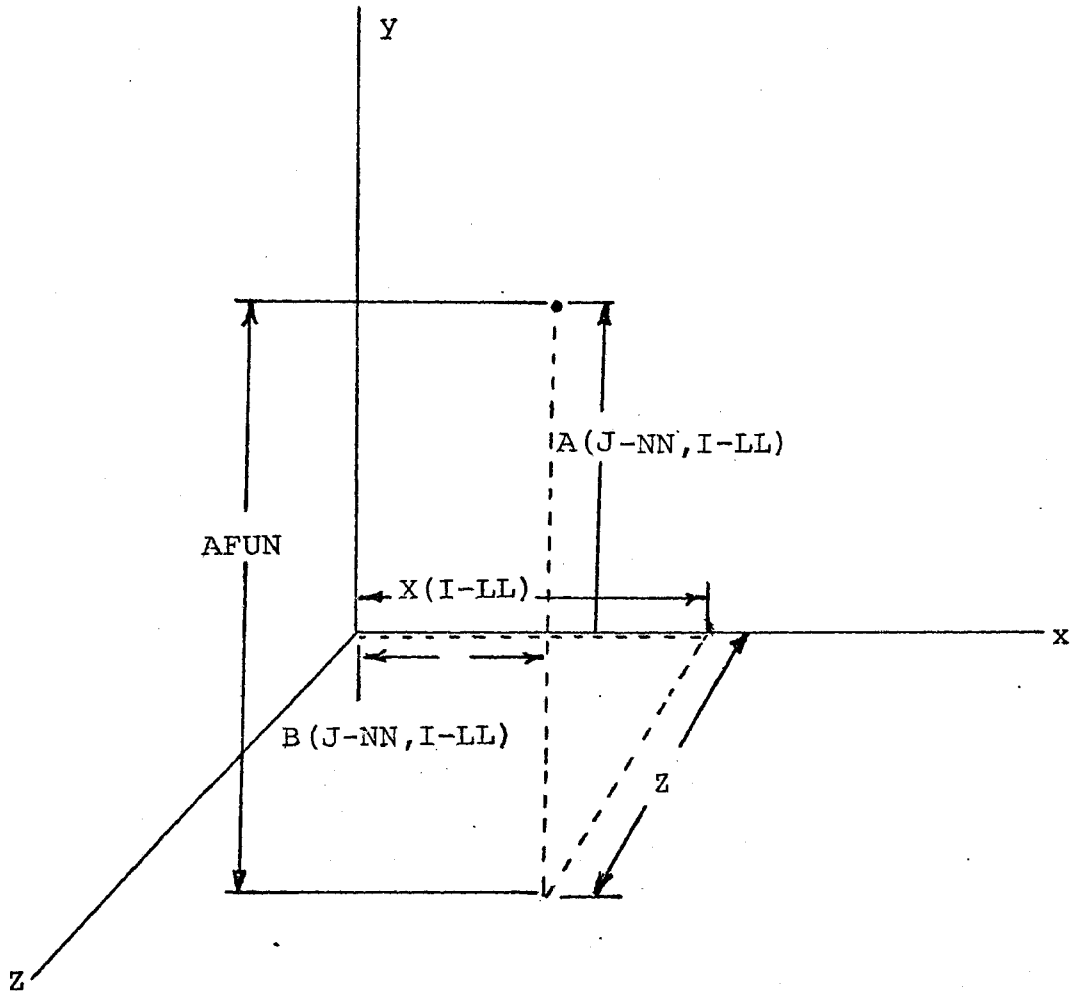

Figure C.1 Three Dimensional Plot Coordinates

The same type of reasoning applies to B and X.

Statements 34-39 test each value of A with YMAX. If A is larger, the value of YMAX is changed and the page is rescaled.

Statements 41-52 define F and G as single subscripted variables which may be called by XYPLT. When F and G are plotted the points are marked by a characteristic symbol. The symbol is different for each curve if the number of curves is less than 13. If greater than 13, the symbols are repeated. The points may or may not be connected with lines depending on the value of IYZPLT.

Statement 53 determines if the points are to be connected by a perpendicular family of curves.

Statements 54-58 define C and D. The subscripts I and J are varied this time so that when plotted the family of curves form a perpendicular set. If the points were connected previously, a surface is formed.

Special acknowledgement must be made to Joseph L. Mueller for assisting the author by writing this three dimensional plot subprogram under his direction.