Distributing Vertices on Hamiltonian Cycles

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Abstract

Let G be a graph of order n and $3 \leq t \leq \frac{n}{4}$ be an integer. Recently, Kaneko and Yoshimoto [5] provided a sharp $\delta(G)$ condition such that for any set X of t vertices, G contains a hamiltonian cycle H so that the distance along H between any two vertices of X is at least $n/2t$. In this paper, minimum degree and connectivity conditions are determined such that for any graph G of sufficiently large order n and for any set of t vertices $X \subseteq V(G)$, there is a hamiltonian cycle H so that the distance along H between any two consecutive vertices of X is approximately $\frac{n}{t}$. Furthermore, we determine the δ threshold for any t chosen vertices to be appear on a hamiltonian cycle H in a prescribed order, with approximately predetermined distances along H between consecutive chosen vertices.

1 Introduction

In this paper we use the following notation. For a graph G, let $\delta(G)$ be the minimum degree, $\kappa(G)$ be the connectivity of G, $N(v)$ be the set of neighbors of a vertex $v \in V(G)$, $d(v) = |N(v)|$ and $d_A(v)$ be $|N(v) \cap A|$ for any set

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$A \subseteq V(G)$. Also let $G[A]$ denote the subgraph of G induced on the vertices of A.

We denote a path from u to v by (u, v) while a path from u to v, along a path or cycle A, is denoted by $(u, v)_A$. The distance between u and v is denoted by $d(u, v)$ while the distance, along a path or cycle A, is denoted by $d_A(u, v)$. For any subgraph $H \subseteq G$, we define the order of H as the number of vertices on H, that is, $|H|$. All other notation may be found in [1].

Recently Kaneko and Yoshimoto [5] proved the following result.

Theorem 1 *Let G be a graph of order n, $d \leq \frac{n}{4}$ a positive integer and A a set of at most $\frac{n}{2d}$ vertices. If $\delta(G) \geq \frac{n}{2}$ then there exists a hamiltonian cycle in G with the distance, along the cycle, between any pair of vertices of A at least d.*

The key restriction here is that $\delta(G) \geq \frac{n}{2}$ only guarantees that G is 2-connected. Our results show that, with slightly stronger assumptions and for n sufficiently large, there exists a hamiltonian cycle with approximately given distances between the chosen vertices on a hamiltonian cycle.

Along with Theorem 1, our proofs use the following powerful result of Szemerédi [7].

Theorem 2 *For every real number $0 < \delta < 1$ and every positive integer k, there exists a positive integer N such that every subset A of the set $\{1, ..., N\}$ of size at least δN contains an arithmetic progression of length k.*

A graph is said to be *k-linked* if for every choice of $2k$ vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$, there exists a collection of vertex disjoint paths $P_i = (x_i, y_i)$ for all i. We use a recent result of Thomas and Wollan [8].

Theorem 3 *If a graph G is $10k$-connected, then G is k-linked.*

A graph G is said to be *panconnected* if for each pair of vertices $u, v \in V(G)$, there exists a path of length l in G for each l satisfying $d_G(u, v) \leq l \leq n - 1$. Finally, we also make use of the following result of Williamson [9].

Theorem 4 *If $\delta(G) \geq \frac{n+2}{2}$, then the graph G is panconnected.*

Given an integer t, for ease of notation, we consider all indices modulo t. Using the above results, we prove the following theorems.

Theorem 5 *Let $t \geq 3$ be an integer and let $0 < \epsilon \frac{t}{2}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq 2 \left\lceil \frac{t}{2} \right\rceil$. For every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a hamiltonian cycle H such that $d_H(x_i, x_j) \geq (\frac{1}{t} - \epsilon)n$ for all $1 \leq i < j \leq t$.*

Corollary 6 *Let $t \geq 3$ be an integer and let $0 < \epsilon < \frac{t}{2}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq 2 \left\lceil \frac{t}{2} \right\rceil$. For every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a hamiltonian cycle H and an ordering of the elements of X such that $(\frac{1}{t} - \epsilon)n \leq d_H(x_i, x_j) \leq (\frac{1}{t} + \epsilon)n$ for all $1 \leq i \leq t$.*

We also consider the case in which the chosen vertices $\{x_1, \ldots, x_t\}$ appear in a prescribed order along the hamiltonian cycle and at approximately predetermined distances.

Theorem 7 *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+t-1}{2}$ or $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq \frac{3t}{2}$. For every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a hamiltonian cycle H containing the vertices of X in order such that $(\gamma_i - \epsilon)n \leq d_H(x_i, x_{i+1}) \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$.*

The proofs of Theorems 5 and 7 are left to Section 3.

2 Lemmas

We now provide some lemmas which are necessary for the proofs of Theorems 5 and 7. The first lemma tells when and how to absorb vertices into a long cycle.

Lemma 1 *Let $t \geq 3$, $n \geq 5t$ be integers, and let G be a graph of order n having $\delta(G) \geq \frac{n}{2}$. If there exists a cycle C of order at least $\frac{3n}{4}$ containing the vertices of X in order, there exists a hamiltonian cycle H containing the vertices of X in order such that $d_H(x_i, x_{i+1}) \geq d_C(x_i, x_{i+1})$ for all $1 \leq i \leq t$.*

Proof: Proceed by contradiction. Let J be a smallest collection of vertices that cannot be absorbed into C while maintaining $d_C(x_i, x_{i+1})$ for all i. Let J' be a component of smallest order in J. If J' is the single vertex

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v, then since $\delta(G) \geq \frac{n}{2}$ it follows that $d_C(v) \geq \frac{n}{2}$. Since $|C| \leq n-1$, it follows that v is adjacent to two consecutive vertices u and u^+ of C, hence v can be absorbed into C.

If $|J'| = 2$, $J' = \{u, v\}$, then since J' is connected, $G[J']$ contains the edge uv. If one of u or v is adjacent to consecutive vertices along the cycle, then we can make the same insertion as above. Also if u and v are adjacent to vertices u' and v' respectively with $u'v' \in E(C)$, then we may replace $u'v'$ with $u'uvv'$ to absorb u and v into C. Thus, suppose neither of the above cases occurs.

Since $d_C(u), d_C(v) \geq \frac{n}{2} - 1$ and $|C| \leq n - 2 = 2(\frac{n}{2} - 1)$, we know u and v must both be adjacent to every other vertex along C and $N_C(u) = N_C(v)$. Therefore, since $n \geq 5t$, there must exist some vertex $w \in C - (N(u) \cup N(v))$ with $w \notin X$. Let w^- and w^+ be the vertices adjacent to w along C and without loss of generality select uw^- and vw^+. Then $C' = (..., w^-, u, v, w^+, ...)$ contradicts the maximality of C.

Finally, suppose $|J'| \geq 3$. Then there exists a path (v_1, v_2, v_3) in J'. Clearly $d_C(v_i) > \frac{n}{2} - |J'|$ for all i. Also note that $|C| \leq n - |J'|$. Therefore, since $|J'| \leq |J| \leq \frac{n}{4}$, we get:

$$d_C(v_1) + d_C(v_2) + d_C(v_3) > \frac{3n}{2} - 3|J'| \geq n - |J'| \geq |C|.$$

It follows that there exists v_i and v_j for $1 \leq i < j \leq 3$ which are adjacent to distinct vertices w_1 and $w_2 \in C$ with $dist_C(w_1, w_2) \leq 2$ such that any vertex between w_1 and w_2 along C is not in X. Let w be the vertex between w_1 and w_2 along C (if one exists) and let $v = v_2$ if $i, j \neq 2$. We may now replace the path (w_1, w, w_2) or (w_1, w_2) with the path (w_1, v_i, v, v_j, w_2) or (w_1, v_i, v_j, w_2) to again contradict the choice of J' and finish the proof of Lemma 1. $\square$

In all that follows, let $\gamma_1, \ldots, \gamma_t > 0$, $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$ and let $x_1, \ldots, x_t$ be a set of t prescribed vertices in G. Given a hamiltonian cycle H, let P_i be the path $(x_i, x_{i+1})_H$ and let $\mathscr{P}$ be the collection of paths P_i. Let $f(i) = \lceil \gamma_i n \rceil - |P_i|$. Order the paths P_i such that $f(i) \geq f(i+1)$ (not depending on the order in which they appear in the cycle). Define:

$$\mu(H) = \sum_{i:f(i)>0} t^{f(i)}.$$

We choose a hamiltonian cycle H such that $\mu(H)$ is minimum. Notice if $|P_i| > \lceil (\gamma_i - \frac{\epsilon}{t})n \rceil$ for all i, then $|P_i| < \lceil (\gamma_i + \epsilon)n \rceil$ for all i. Since we will assume the graph does not contain the desired hamiltonian cycle, we may assume $f(1) > \frac{\epsilon}{t}n$ so $\mu(H) > t^{\lceil n\epsilon/t \rceil}$.

Let k be the smallest integer such that $f(k) - f(k+1) > \frac{\epsilon}{t^2}n$. Since $f(1) > \frac{\epsilon}{t}n$, we know k exists and $|P_k| < \gamma_i n$. Let $\mathscr{B}$ be the collection of paths $\{P_i\}_{i=1}^{k}$ and let $\mathscr{A} = \mathscr{P} \setminus \mathscr{B}$.

The next two lemmas are the main components of our proofs. The first explains how to move path segments from one subpath of H to another. The second describes conditions when we may abandon the path structure and build the desired cycle directly.

Lemma 2 *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n}{2}$. For every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, if G contains a hamiltonian cycle H with the vertices of X in order with $dist_H(x_i, x_{i+1}) \geq \epsilon n$ and the number of edges between the sets $\mathscr{A}$ and $\mathscr{B}$ given by $e(\mathscr{A}, \mathscr{B}) \geq h_1 n^2$, then either $\mu(H) \leq t^{\lceil n\epsilon/t \rceil}$ or there exists a hamiltonian cycle H' with $\mu(H') < \mu(H)$.*

Proof: The goal of this lemma is to swap a segment of a path in $\mathscr{A}$ with a segment of a path in $\mathscr{B}$ producing a hamiltonian cycle H' with $\mu(H') < \mu(H)$. This is done by finding a pair of 4-tuples (u_1, v_1, w_1, y_1) and (u_2, v_2, w_2, y_2) with $u_i, v_i \in V(A)$ and $w_i, y_i \in V(B)$ for some $A \in \mathscr{A}$ and $B \in \mathscr{B}$ (as seen in Figure 1) and moving the path (y_1, w_2) from B to A and the path (v_1, u_2) from A to B. Such an ordered 4-tuple will be called a *crossing pair* and the process of moving the segments will be called a *swap*.
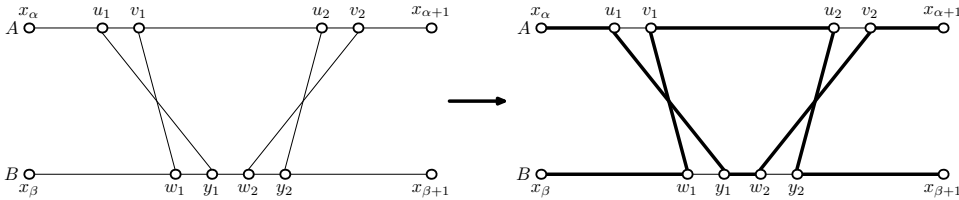

Figure 1: A Crossing Pair.

By assumption, $\mathscr{A}$ and $\mathscr{B}$ are nonempty and $e(\mathscr{A}, \mathscr{B}) \geq h_1 n^2$. This implies that there exists a pair of paths $A = (x_\alpha, x_{\alpha+1}) \in \mathscr{A}$ and $B =$

$(x_\beta, x_{\beta+1}) \in \mathscr{B}$ with at least $\frac{4h_1}{t^2}n^2 > 0$ edges between A and B. Let $an = |A| = d_H(x_\alpha, x_{\alpha+1}) + 1$ and $bn = |B| = dist_H(x_\beta, x_{\beta+1}) + 1$. By our assumptions, $0 < a, b < 1$.

Recall $f(\beta) - f(\alpha) > \frac{\epsilon}{t^2}$ by the definition of $\mathscr{B}$. Let $M = \frac{\epsilon}{2t^2}n$. Notice that if the swap uses less than M vertices, then the corresponding new $f(\alpha)$ remains less than the new $f(\beta)$.

For some $h_2 > 0$, there exists a positive fraction of n vertices $v \in A$ with $d_B(v) \geq h_2 n$. Call the collection of such vertices A'. Let $k = \left\lceil \frac{a}{M} \left(\frac{33b^2}{h_2^2} + \frac{18b}{h_2} + \frac{3b^3 - 6b^2 h_2}{h_2^3} + 1 \right) \right\rceil$ be the size of a desired subset of A'. The constant k is chosen in this way to ensure that the distance between consecutive vertices of this subset is small. Since $h_2 \leq b$, we know $a > M$ so $k > \frac{33b^2}{h_2^2} + \frac{18b}{h_2} + \frac{3b^3 - 6b^2 h_2}{h_2^3} + 1 \geq 49$.

Label the vertices of A with increasing consecutive positive integers. For n sufficiently large, by Theorem 2, there exists an arithmetic progression of length k on the labels of the vertices of A'. Call the set of associated vertices E. It follows that $d_H(u, v) \leq \frac{an}{k}$ for all consecutive vertices $u, v \in E$ (in the natural ordering of E).

By the pigeon hole principle and using the fact that $d_B(v) \geq h_2 n$, for every choice of at least $\frac{b}{h_2} + 1$ vertices of E, there exists a crossing pair (u, v, w, y) with $dist_H(w, y) \leq \frac{b}{h_2}$. By this argument, if we choose any set E' of at least $\frac{b}{h_2} + 1$ vertices with $E' \subseteq E$, there exists the desired crossing pair with $u, v \in E'$. Therefore, if we choose $c = 3\frac{b}{h_2} + 1$, we may find a crossing pair (u, v, w, y), remove w and y from B, and we have not decreased the degrees of vertices in A by more than 2. This implies we may still find another crossing pair which is disjoint with respect to vertices of B. Let h be the number of vertices we have removed from B. As long as $h_2 n - h \geq \frac{b}{c-1}n$, there will exist yet another crossing pair. Therefore, we may repeat this process until $h = \frac{b}{c-1}n$, which means we find $\frac{2b}{c-1}n$ crossing pairs which share no vertices in B.

Call a collection of $c = 3\frac{b}{h_2} + 1$ consecutive vertices of E a *block*. By the above argument, each block is involved in at least $\frac{b}{c-1}n$ crossing pairs which are vertex disjoint in B. Consider a collection of c disjoint blocks such that each pair of consecutive blocks has $2c + 1 + 2\frac{b}{h_2} + \frac{b^2 - 2bh_2}{h_2^2} + 1$ vertices of E between them. This collection of blocks uses exactly

$$c^2 + (c-1)\left(2c + 1 + 2\frac{b}{h_2} + \frac{b^2 - 2bh_2}{h_2^2} + 1\right)$$

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vertices of E. Clearly, E is large enough since we chose

$$k \;=\; \tfrac{a}{M}\left(\tfrac{33b^2}{h_2^2} + \tfrac{18b}{h_2} + \tfrac{3b^3-6b^2h_2}{h_2^3} + 1\right)$$

$$\geq\; \tfrac{33b^2}{h_2^2} + \tfrac{18b}{h_2} + \tfrac{3b^3-6b^2h_2}{h_2^3} + 1$$

$$\geq\; c^2 + (c-1)(2c+1+2\tfrac{b}{h_2} + \tfrac{b^2-2bh_2}{h_2^2} + 1).$$

Consequently, since n was chosen to be sufficiently large, there exists two blocks in this collection containing crossing pairs (u_1, v_1, w_1, y_1) and (u_2, v_2, w_2, y_2) such that $w_2 > y_1$ and $0 < d_B(y_1, w_2) \leq \frac{b^2-2bh_2}{h_2^2} + 1$ (see Figure 2). This number is found by counting the number of crossing pairs from a single block that share vertices in the segment (w_i, y_i) in B. One would then use this to find the largest possible average distance between crossing pairs.
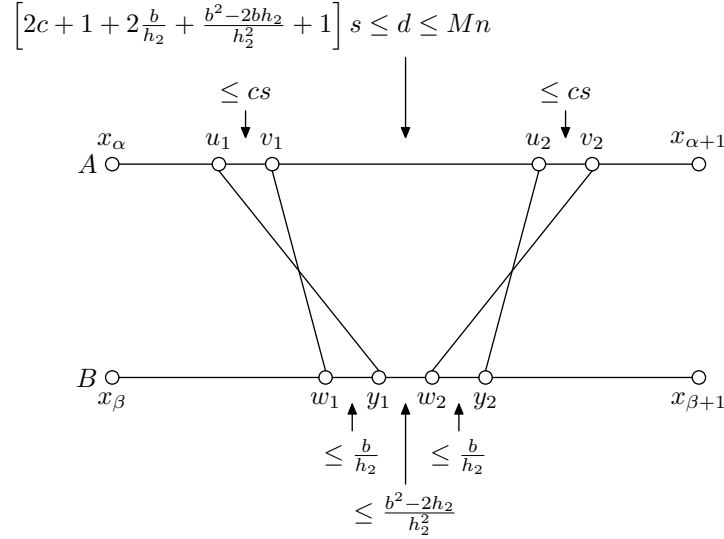

Figure 2: A crossing pair.

At this point we make the swap by redefining $B = (x_\beta, x_{\beta+1})$ to be:

$$B' = (x_\beta, ..., w_1, v_1, ..., u_2, y_2, ..., x_{\beta+1})$$

and redefining $A = (x_\alpha, x_{\alpha+1})$ to be:

$$A' = (x_\alpha, ..., u_1, y_1, ..., w_2, v_2, ..., x_{\alpha+1}).$$

The segments (u_1, v_1), (u_2, v_2), (w_1, y_1) and (w_2, y_2) are no longer included in the paths after we make the swap. The segments (w_i, y_i), for $i = 1, 2$, are of length at most $\frac{b}{h_2}$. If we let s be the number of vertices between consecutive elements of E along A, then the segments (u_i, v_i), for $i = 1, 2$, are of length at most cs. Therefore the new cycle is at most $2(\frac{b}{h_2} + cs)$ vertices shorter.

From the swap, $|A'|$ gained at most $\frac{b^2 - 2bh_2}{h_2^2}$ vertices from B and $f(\alpha)$ is certainly still at most $f(\beta)$. We also know that:

$$
\begin{aligned}
|B'| - |B| &\geq \left(2c + 1 + \tfrac{2b}{h_2} + \tfrac{b^2 - 2bh_2}{h_2^2} + 1\right)s - 2\tfrac{b}{h_2} \\
&> 2cs + \tfrac{b^2 - 2bh_2}{h_2^2} + 1.
\end{aligned}
$$

Conversely, we know:

$$
\begin{aligned}
|B'| - |B| &\leq s[c^2 + (c-1)(2c + 1 + \tfrac{b}{h_2} + \tfrac{b^2 - 2bh_2}{h_2^2} + 1)] \\
&\leq \tfrac{an}{k}[c^2 + (c-1)(2c + 1 + \tfrac{b}{h_2} + \tfrac{b^2 - 2bh_2}{h_2^2} + 1)] \\
&= Mn.
\end{aligned}
$$

Therefore we have constructed a cycle H' which misses at most $2cs + o(n)$ vertices of G with $|B'| > |B|$.

Let J be the missing vertices from the above transfer. Recall $|J| < M = \frac{\epsilon n}{2t^2} < \frac{n}{4}$ by the choice of M. Now by Lemma 1, these vertices can be absorbed into the cycle without decreasing the distances between any of our chosen vertices $\{x_i\}$. This creates a new hamiltonian cycle H'' and, since $f(\beta)$ is now smaller and $f(\alpha)$ is still less than $f(\beta)$, we know that $\mu(H'') < \mu(H)$. This completes the proof of Lemma 2. $\qquad\square$

If there exists a partition of G into two sets with very few edges from one set to the other, then the next lemma constructs the desired hamiltonian cycle directly.

Lemma 3 *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ positive real numbers having $\sum_{i=1}^t \gamma_i = 1$ and $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+t-1}{2}$ or $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq \frac{3t}{2}$. If*

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there exists a partition of $V(G)$ into sets A and B with $e(A,B) < \frac{\epsilon^2 n^2}{1600}$ then for every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a hamiltonian cycle H containing the vertices of X in order such that $(\gamma_i - \epsilon)n \leq d_H(x_i, x_{i+1}) \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$.

Proof: The proof of this lemma is broken into cases based on the connectivity of G.

Case 1 *Suppose $\kappa(G) \geq 5t$.*

Let D_A (or D_B) be the set of vertices in A (respectively B) with more than $\frac{\epsilon}{40}n$ edges into B (respectively A) and let $D = D_A \cup D_B$. From the hypotheses of the lemma, $|D_A|, |D_B| < \frac{\epsilon}{40}n$. Note that $\delta(G[A \setminus D_A]) \geq \frac{n}{2} - \frac{\epsilon}{40}n - \frac{\epsilon}{40}n = \frac{n}{2} - \frac{\epsilon}{20}n$ and, in particular, for any $A' \subseteq A \setminus D_A$ we have $\delta(G[A']) \geq |A'| - \frac{\epsilon}{10}n$. Thus given $A' \subseteq A \setminus D_A$ with $|A'| \geq \frac{\epsilon}{5}n + 2$, we have $\delta(G[A']) \geq \frac{|A'|+2}{2}$. Hence, by Theorem 4, we know $G[A']$ is panconnected. A similar argument holds for $B' \subseteq B \setminus D_B$.

Choose a system $X' = \{u_1, v_1, u_2, v_2, \ldots, u_t, v_t\}$ of two distinct representatives for each of the vertices of X with $x_i u_i, x_i v_i \in E(G)$ for all i such that $X' \subseteq G \setminus (X \cup D)$. By our degree conditions, there exists such a set X'. Since G is $5t$-connected, we know there exists a set of $2t$ vertex disjoint paths from $A \setminus D$ to $B \setminus D$ in $G \setminus (X \cup X')$. Let M be the collection of shortest such paths (see Figure 3).

Suppose we have paths $P_1, \ldots, P_{i-1}$ for some $1 \leq i < t$ where $P_j = (v_j, u_{j+1})$ for $j < i$. Let $v_i, u_{i+1} \in X'$ and, without loss of generality, suppose $v_i \in A$. Let $Q_i = V(P_1) \cup \cdots \cup V(P_{i-1})$ and let $A' = [A \setminus (D \cup X \cup X' \cup M \cup Q_i)] \cup \{v_i, u\}$ for some $u \in A \cap M \setminus (D \cup Q_i)$. If $\gamma_i n \leq |A'| - (\frac{\epsilon}{3}n + 3t + 2) - (2t - i + 1)$, we use the fact that $G[A']$ is panconnected to construct a path P'_i of order $\gamma_i n - 3$ in A' from v_i to u.

By the argument above, it follows that $G[A' \setminus P'_i]$ is panconnected. Also for $u_{i+1} \in A$, we know that $G[(A' \setminus P'_i) \cup u_{i+1}]$ is panconnected, consequently a path of length 2 from u to u_{i+1} exists and let $P_i = (v_i, \ldots, u, \ldots, u_{i+1})$ be the desired v_i, u_{i+1} path. If $u_{i+1} \in B$ and we let v be the vertex in B such that (u, v) is a path of M, we may use a similar argument to construct the desired path $P_i = (v_i, \ldots, u, \ldots, v, \ldots, u_{i+1})$ using the panconnectivity of $G[B \setminus (X \cup X' \cup D \cup M \cup Q_i) \cup \{u_{i+1}, v\}]$.

If $\gamma_i n > |A'| - (\frac{\epsilon}{3}n + 3t + 2) - (2t - i + 1)$, we again use the fact that $G[A']$ is panconnected to create a path from v_i to u. Let v be the vertex of $M \cap B$
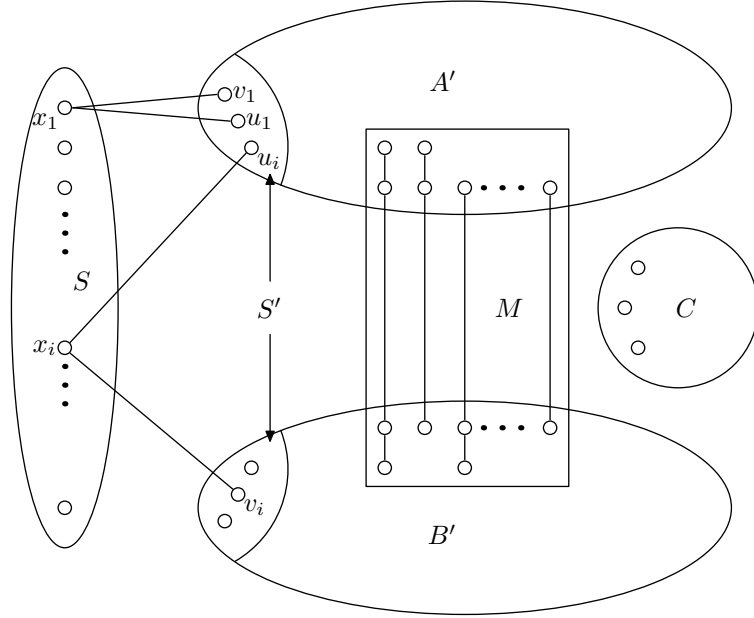
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Figure 3: Graph G

such that (u, v) is a path of M. First suppose $\gamma_i n \le |B'| - (\frac{\epsilon}{3}n + 3t + 2) - (t - i)$ where $B' = [B \setminus (D \cup X \cup X' \cup M \cup Q_i)] \cup v$. We take the path of length 2 from v_i to u, take the path from u to B through M, and then, using the panconnectivity of $G[B']$, create a path of the desired length within B'. Again breaking this into cases as above, based on whether u_{i+1} is in A or B, construct P_i. If $\gamma_i n > |B'| - (\frac{\epsilon}{3}n + 3t + 2) - (t - i)$, we mark v_i as reserved and construct the associated path later. This reservation of vertices happens at most twice.

Suppose, without loss of generality, that v_t is the single remaining vertex in X' (whether it was reserved or not) and $v_t \in A$ and let u be a remaining vertex of $A \cap M \setminus (D \cup Q_t)$. If $u_1 \in B$, then use the panconnectivity of $G[A']$ to connect v_t to u using all of A', take the path in M from u to B and use the panconnectivity of B' to pick up all of B' on the path. This creates a path of order l_t for $\gamma_t n \ge l_t > \gamma_t n - |D| - |M| > (\gamma_t - \frac{\epsilon}{2})n$ as long as n is sufficiently large. If $u_1 \in A$, we take a path (v_t, u) of length 2, use all that remains of B' on a path between two vertices of M, come back to A and pick up all of A' to again construct the desired path.

Finally suppose (v_{t-1}) and (v_t) are the two reserved vertices of X'. One may show that $|A'|, |B'|, \lceil \gamma_{t-1} n \rceil$ and $\lceil \gamma_t n \rceil$ are all within $\frac{\epsilon}{2} n$ of each other. As above, we create short paths from v_{t-1} and u_t to vertices $v'_{t-1}, u'_t \in A$ and from v_t and u_1 to vertices $v'_t, u'_1 \in B$. We now use the panconnectivity of $G[A']$ and $G[B']$ to construct a path $P'_{t-1} = (v'_{t-1}, u'_t)$ of length $|A'|$ and a path $P'_t = (v'_t, u'_1)$ of length $|B'|$. We then let $P_i = (v_i, \ldots, v'_i, \ldots, u'_{i+1}, \ldots, u_{i+1})$ for $i = t-1, t$. One may easily check that these paths are of length l_i with:

$$(\gamma_i - \frac{\epsilon}{3})n + 3t + 2 - |D_A| < l_i < (\gamma_i + \frac{\epsilon}{3})n + 3t + 2$$

so since n was chosen to be large enough, we get:

$$(\gamma_i - \frac{\epsilon}{2})n < l_i < (\gamma_i + \frac{\epsilon}{2})n.$$

Notice, in this process, we can miss at most $|D_A| + |D_B| + \frac{\epsilon}{5}n + 2 + 6t < \frac{\epsilon}{2}n$ for n sufficiently large. Applying Lemma 1, the desired hamiltonian cycle results.

Case 2 *Suppose $\frac{3t}{2} \leq \kappa(G) < 5t$.*

Let K be a minimum cutset of G with $\frac{3t}{2} \leq |K| < 5t$. Since $\delta(G) \geq \frac{n}{2}$, there cannot be more than two components of $G \setminus K$. Call these components A and B.

We call a vertex $v \in K$ *blocked to A* (or B) if for every edge e from v into A (respectively B), $e = vx_i$ for some $x_i \in X$. For each vertex $v \in K \setminus X$ which is blocked to A, we choose a distinct vertex of $x_i \in N(v) \cap X \cap A$. Call this the *blocking vertex*. We call the vertices of $K \cap X$ with only one edge to either $A \setminus X$ (or $B \setminus X$) *half-blocked* to A (or B).

For $v \in K \setminus X$ which is blocked by a vertex $x_i \in A \cap X$, remove all edges to $A \cap X \setminus x_i$ and move v to B and move x_i to K. By the choice of these removed edges, they will not affect the connectivity. Also we have only removed a small number of edges so this will have no effect on our other properties of the graph. We have now eliminated all the blocked vertices of $K \setminus X$ and possibly created more half-blocked vertices.

We next remove all edges between vertices of X to create a new graph G'. Let K' be a minimum cutset in G' containing the maximum number of vertices of X and observe that we have the following facts about G'. The following are true:

- There are no blocked vertices in K'.

- $\kappa(G') \geq \kappa(G) - \frac{t}{2} \geq t$.

- No half-blocked vertices could also have been blocked or blocking.

For the sake of notation, we distinguish four different types of paths that we would like to connect. A path P_i from x_i to x_{i+1} is of Type I if $x_i \in K'$ and $x_{i+1} \notin K'$ or $x_i \notin K'$ and $x_{i+1} \in K'$. A path P_i is of Type II if $x_i, x_{i+1} \in A$ or both are in B. A path P_i is of Type III if $x_i, x_{i+1} \in K'$. Finally a path P_i is of Type IV if $x_i \in A$ and $x_{i+1} \in B$ or $x_i \in B$ and $x_{i+1} \in A$. See Figure 4.
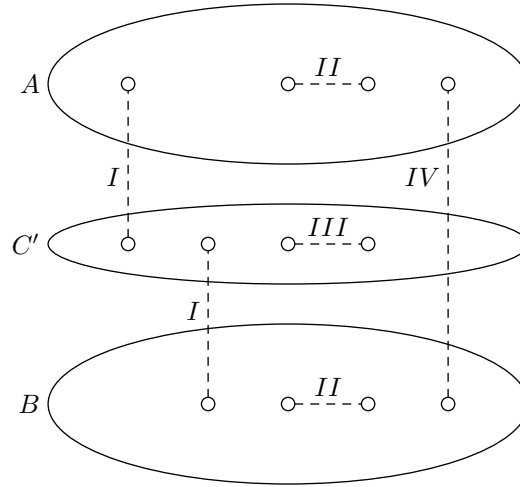

Figure 4: Types of paths.

Since $\delta(G) \geq \frac{n}{2}$ and $|K'| < 5t$, we know $\frac{n}{2} - 5t \leq |A|, |B| \leq \frac{n}{2} + 5t$ and $G[A]$ and $G[B]$ are panconnected by Theorem 4. Using the same argument as in the previous case, as long as there are enough paths from A to B, we may construct all paths as desired. If $\kappa(G') > t$, the reader may verify that there are enough paths from A to B to complete the above argument. If $\kappa(G') = t$, we know every vertex of X was either blocked or blocking. This implies that all the paths are of Types II or IV. The reader may again verify that the paths may be constructed as above to get the desired hamiltonian cycle.

Case 3 *Suppose* $\kappa(G) < \frac{3t}{2}$.

By assumption, $\kappa(G) < \frac{3t}{2}$ implies $\delta(G) \geq \frac{n+t-1}{2}$. Let $k = k_a + k_b$ where k_a is the number of blockings or half-blockings into A and likewise for B. From the above arguments, we know that if $\kappa(G) \geq t + 1 + k$ then we may connect the paths to get the desired hamiltonian cycle. Consider a vertex $v \in A$ and a vertex $w \in B$ which are not involved in any half-blocking. The vertex v is adjacent to at most $\kappa(G) - k_a$ vertices and w is adjacent to at most $\kappa(G) - k_b$ vertices of K. Therefore $|A| \geq d(v) + 1 - (\kappa(G) - k_a) \geq \frac{n+t+1}{2} - \kappa(G) + k_a$ and similarly $|B| \geq \frac{n+t+1}{2} - \kappa(G) + k_b$. Hence $n = |A| + |B| + \kappa(G) \geq n + t + 1 - \kappa(G) + k_a + k_b$ or $\kappa(G) \geq t + k_a + k_b + 1$ and we have our result.

This completes the proof of Lemma 3. □

Our final lemma provides some structure similar to that in Theorem 1 but with the chosen vertices in a given order on the hamiltonian cycle.

Lemma 4 *Let $t \geq 3$ be an integer and for sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+t-1}{2}$ or $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq \frac{3t}{2}$. For every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a hamiltonian cycle H containing the vertices of X in order such that $d_H(x_i, x_{i+1}) \geq (\frac{1}{6400t^3(1-\frac{1}{2t})})n$ for all $1 \leq i \leq t$.*

Proof: Let n be sufficiently large and G be as stated. Let $\{x_1, \ldots, x_t\} \in V(G)$ and let $\epsilon = \frac{1}{2t}$. If there exists a partition of $V(G)$ into two sets A and B, having $|A|, |B| \geq \epsilon n$ such that $e(A, B) < \frac{\epsilon^2}{1600}n^2$ then we may apply Lemma 3 to get the desired hamiltonian cycle. Subsequently, we need only show how to proceed if such a partition does not exist.

Claim 1 *Suppose we are given a graph G of sufficiently large order n with $\delta(G) \geq \frac{n}{2}$ and a real number $\epsilon > 0$. If, for every partition of $V(G)$ into two sets A and B with $|A|, |B| \geq \epsilon n$, we have $e(A, B) \geq \frac{\epsilon^2}{1600}n^2$, then $\kappa(G) \geq \frac{\epsilon^2}{1600(1-\epsilon)}n$.*

Proof of Claim 1: Let K be a cutset of order less than $\frac{\epsilon^2}{1600(1-\epsilon)}n$. Since $\delta(G) \geq \frac{n}{2}$, we know there are only two components (call them A and B) of $G \setminus K$ and since $|K| < \frac{\epsilon^2}{1600(1-\epsilon)}n$ and $\delta(G) \geq \frac{n}{2}$, we know $|A|, |B| > \epsilon n$. Let $A' = A \cup K$. By assumption, $e(A', B) \geq \frac{\epsilon^2}{1600}n^2$ and all these edges must

be incident to vertices in K. Therefore, there exists a vertex $v \in K$ such that,

$$d_B(v) \geq \frac{\frac{\epsilon^2 n^2}{1600}}{\frac{\epsilon^2 n}{1600(1-\epsilon)}} = (1-\epsilon)n.$$

However, since $|A| > \epsilon n$, this is a contradiction, completing the proof of Claim 1. $\square$

Since $\delta(G) \geq \frac{n}{2}$, we may choose a system X' of two distinct representatives from the neighborhood of each vertex of X. Also since $\delta(G) \geq \frac{n}{2}$ we may create a collection $\mathscr{P}$ of $2t$ vertex disjoint paths in $G \backslash X$ of length $\frac{\epsilon^2 n}{1600(1-\epsilon)2t} - 6$ two of which start at each vertex of X. Let $P = \cup V(P_i$ for $P_i \in \mathscr{P}$. Notice $|P| \leq \kappa(G) - 10t$ so $G \setminus (P \cup X)$ is at least $10t$-connected. By Theorem 3, we know $G \setminus (P \cup X)$ is t-linked. This implies that we may link, using only vertices of $G \setminus (P \cup X)$, the ends of the paths of P to create a cycle of length at least $\frac{\epsilon^2}{1600(1-\epsilon)}n - 11t$ containing the vertices of X in order.

Choose a longest cycle H having $d_H(x_i, x_j) \geq \frac{\epsilon^2 n}{1600(1-\epsilon)t} - 5$ for $i \neq j \leq t$. We may assume $|H| < \frac{3n}{4}$; otherwise applying Lemma 1, the desired hamiltonian cycle results. Now suppose $|H| \leq \frac{n+t-1}{2}$. This implies $d_{G \backslash H}(v) \geq 1$ for all $v \in H$. Also any vertex of $G \backslash H$ may not be adjacent to consecutive vertices of H so $\delta(G[G \setminus H]) \geq \frac{n+t-1-|H|}{2} > \frac{|G \backslash H|}{2}$. By Dirac's Theorem [3], this implies $G[G \setminus H]$ is hamiltonian connected. At this point we simply choose two consecutive vertices $v, v^+ \in V(H)$ and neighbors of these vertices $u, u' \in G \backslash H$. Now create H' from H by removing the edge vv^+ and inserting the hamiltonian path $(v, u, \ldots, u', v^+)$. Notice $|H'| > |H|$, which contradicts the choice of H.

Now suppose $\frac{n+t-1}{2} < |H| < \frac{3n}{4}$. If $J = G \setminus H$, then $\frac{n}{4} < |J| < \frac{n-t+1}{2}$. By assumption, $e(H, J) \geq \frac{\epsilon^2 n^2}{1600}$, hence it follows that there exists a path $P_i = (x_i, x_{i+1})_H$ such that $e(P_i, J) \geq \frac{\epsilon^2 n^2}{1600t}$. Consequently, there are at least $\frac{\epsilon^2 n^2}{1600t|J|} - 1 \geq \frac{\epsilon^2 n}{800t} - 1$ vertices $v \in P_i$ with $d_J(v) \geq 2$. Since $|P_i| < \frac{3n}{4}$, the average distance between such vertices is at most:

$$\frac{|P_i|}{\frac{\epsilon^2 n}{800t} - 1} < \frac{300t}{\epsilon^2} < \frac{n}{4} < |J|$$

if n is sufficiently large. Therefore, there exist two vertices $u, v \in P_i$ with $d_{P_i}(u, v) < |J|$ such that $d_J(u), d_J(v) \geq 2$.

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Recall that no vertex of J may be adjacent to consecutive vertices of H so $\delta(J) \geq \frac{n+t-1}{2} - \frac{|H|}{2} \geq \frac{|J|+2}{2}$ so, by Theorem 4, J is panconnected. Let $u' \in J \cap N(u)$ and let $v' \in J \cap N(v) \setminus \{u'\}$. There exists a hamiltonian path P of J from u' to v'. We now replace $P_i = (x_i, \ldots, u, \ldots, v, \ldots, x_{i+1})$ with the path $P_i' = (x_i, \ldots, u, u', P, v', v, \ldots, x_{i+1})$. Because $|J| > d_H(u, v)$, it follows that $|P_i'| > |P_i|$ contradicting the choice of H, completing the proof of Lemma 4. $\qquad\square$

3   Main Results

Theorem 5: *Let $t \geq 3$ be an integer and let $0 < \epsilon \frac{t}{2}$. for sufficiently large n, let g be a graph of order n having $\delta(g) \geq \frac{n}{2}$ and $\kappa(g) \geq 2\left\lceil \frac{t}{2} \right\rceil$. for every $x = \{x_1, x_2, \ldots, x_t\} \subseteq v(g)$, there exists a hamiltonian cycle h such that $d_h(x_i, x_j) \geq (\frac{1}{t} - \epsilon)n$ for all $1 \leq i < j \leq t$.*

Proof: By Theorem 1, we know there exists a hamiltonian cycle H in G such that, for a given set of t vertices $\{x_i\}$, $d_H(x_i, x_j) \geq \frac{1}{2t}$ for all $i \neq j$. Let $\mathscr{H}$ be the set of hamiltonian cycles which satisfy Theorem 1. For each H in $\mathscr{H}$, define $\mathscr{A}$ and $\mathscr{B}$ as above. We choose $H \in \mathscr{H}$ with $\mu(H)$ minimized. Reorder the vertices in whatever order they fall on the cycle. Then if the number of edges between $\mathscr{A}$ and $\mathscr{B}$ is at least $\frac{\epsilon^2}{1600}n^2$, apply Lemma 2 with $\gamma_i = \frac{1}{t}$ for all i to get the desired result.

Suppose the number of edges between $\mathscr{A}$ and $\mathscr{B}$ is less than $\frac{\epsilon^2}{1600}n^2$. Let K be a minimum cutset of G. By the minimum degree condition, there can only be two components A and B of $G \setminus K$, furthermore, $|K| \geq 2\left\lceil \frac{t}{2} \right\rceil$. For every vertex $x_i \notin K$ make a short path to a vertex v_i of K and contract the path to a new vertex $x_i \in K$. Notice we have only removed at most $2t$ vertices and we have not changed the connectivity of G. If t is even, we may connect x_1 to x_2 through A, x_2 to x_3 through B, and so on to get a hamiltonian cycle with all vertices equally spaced.

If t is odd, there exists at least one vertex $v \in K \setminus X$ and we may again connect most of the paths as above. There may be one path $P_t = (x_t, x_1)$ remaining that cannot fit into only one of A or B. For this path we must use the vertex v to cross between A and B to get the desired hamiltonian cycle.

For sharpness of the minimum degree condition, the graph G consisting of two cliques of order $\frac{n+1}{2}$ sharing a common vertex has $\delta(G) \geq \frac{n}{2} - 1$ with

G not hamiltonian.

For sharpness of the connectivity condition, consider the graph in Figure 5. This graph consists of two sets $A = B = K_{\frac{n-t}{2}}$ and an A, B separating set K with $|K| = 2 \left\lceil \frac{t}{2} \right\rceil - 1$. Notice $|K|$ is always odd and $|K| \leq t$. If all of the vertices X are in A (or B), we would need at least $\left\lceil \frac{t}{2} \right\rceil$ paths to cross into B to construct the desired set of paths. This uses $2 \left\lceil \frac{t}{2} \right\rceil$ vertices of K but $|K| = 2 \left\lceil \frac{t}{2} \right\rceil - 1$ so it is impossible to construct the desired hamiltonian cycle.
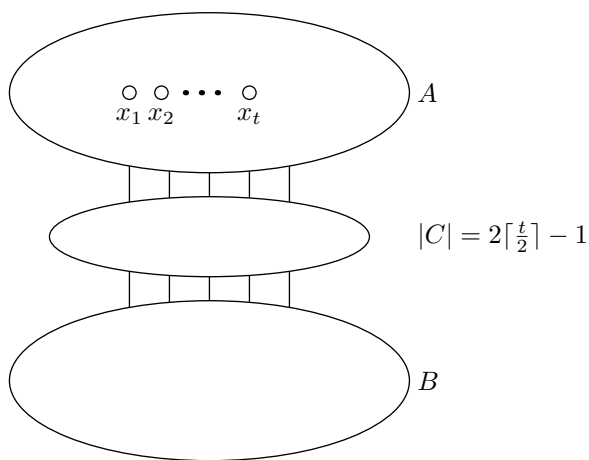

Figure 5: The graph G.

This completes the proof of Theorem 5. □

Theorem 7: *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+t-1}{2}$ or $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq \frac{3t}{2}$. For every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a hamiltonian cycle H containing the vertices of X in order such that $(\gamma_i - \epsilon)n \leq d_H(x_i, x_{i+1}) \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$.*

Proof: By Lemma 4 a hamiltonian cycle H with the vertices of X_t in order and $d_H(x_i, x_{i+1}) \geq \epsilon n$ for all $x_i \in X_t$ and for some $\epsilon > 0$ results. Now applying Lemmas 2 and 3, the desired result follows.

Again sharpness of the minimum degree bound is trivial. For sharpness of the connectivity bound, consider the graph in Figure 6. Notice each path we connect must use a vertex of K hence $|K| \geq \frac{3t}{2}$.
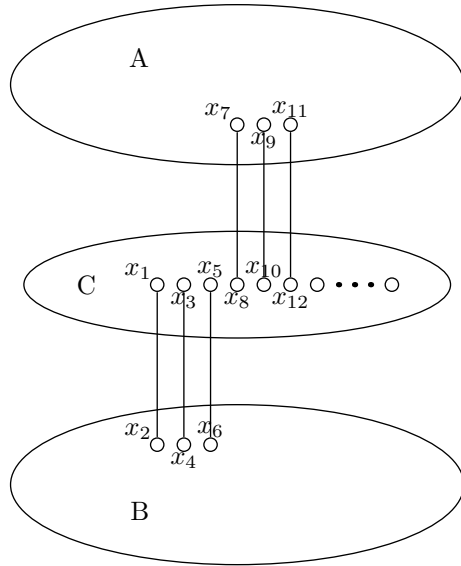

Figure 6: The graph G.

This completes the proof of Theorem 7. □

4 Further Results

Using only slight modifications to the above lemmas, we also prove the theorems below. A survey of similar results may be found in [4]. In 1995, Ota [6] found a sharp lower bound on σ_2 for any set of t chosen vertices to be contained in a common cycle of G.

Theorem 8 *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+t-1}{2}$. For every set $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a spanning collection $\mathscr{C}$ of vertex disjoint cycles C_i with*

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$x_i \in C_i$ such that $(\gamma_i - \epsilon)n \leq |C_i| \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$. Furthermore, this condition is sharp.

The conditions in this theorem are sharp because of the following example. Let $G_1 = K_t + (K_{(n-t)/2} \cup K_{(n-t)/2})$ when $n - t$ divisible by 2. Clearly $\delta(G_1) = \frac{n+t}{2} - 1$. If we choose S to be the vertices of the K_t and we choose $\gamma_1, \gamma_2, \ldots, \gamma_t$ so there is no subset $I_0 \subset [t]$ of the index set such that $\frac{1}{2} - \epsilon tn \leq \sum_{i \in I_0} \gamma_i \leq \frac{1}{2} + \epsilon tn$ then this graph cannot contain the desired collection of cycles.

Theorem 9 Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+\lceil 1.5t \rceil - 1}{2}$. For every set $X = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\} \subseteq V(G)$ of $2t$ vertices, there exists a spanning collection $\mathscr{P}$ of vertex disjoint paths $P_i = (x_i, \ldots, y_i)$ such that $(\gamma_i - \epsilon)n \leq |P_i| \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$. Furthermore, this condition is sharp.

The sharpness of Theorem 9 is given by the following construction. Let $a = \left\lceil \frac{n - \lceil frac3t2 - 1 \rceil}{2} \right\rceil$ and $b = \left\lceil \frac{n - \lceil frac3t2 - 1 \rceil}{2} \right\rceil$ and let $A = K_a$, $B = K_b$ and $K = K_{\lceil frac3t2 - 1 \rceil}$. Let $G_2 = K + (A \cup B)$. Suppose $\gamma_i = \frac{1}{t}$ let $x_i \in K$ and let $y_i \in A$ for $1 \leq i \leq t$. If we suppose n has the correct parity, it can be shown that $\delta(G_2) = \frac{n + \lceil frac3t2 \rceil - 1}{2} - 1$ but there cannot exist a spanning collection of paths of length approximately $\frac{n}{t}$ from x_i to y_i for $1 \leq i \leq t$.

The following is an easy corollary to Theorem 9. The sharpness is also given by the same example as above.

Corollary 10 Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+\lceil 1.5t \rceil - 1}{2}$. For every set $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$ and $Y \subseteq V(G)$ with $t \leq |Y| \leq \frac{n}{8}$, there exists a spanning collection $\mathscr{P}$ of vertex disjoint paths $P_i = (x_i, \ldots, y_i)$ where $y_i \in Y$ such that $(\gamma_i - \epsilon)n \leq |P_i| \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$. Furthermore, this condition is sharp.

Given a subgraph $H \subset G$ with $2t$ chosen vertices $X = x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t \subseteq H$, the following corollary to Theorem 9 constructs a spanning collection of vertex disjoint paths from x_i to y_i in $G \setminus H$ of lengths within the prescribed range.

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Corollary 11 *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$. For sufficiently large n, let G be a graph of order n and let $H \subset G$ with $|H| = r$. Suppose $\delta(G) \geq \frac{n+r+\lceil 0.5t \rceil - 1}{2}$. For every set $X = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\} \subseteq V(H)$ of $2t$ vertices, there exists a spanning collection $\mathscr{P}$ of vertex disjoint paths $P_i = (x_i, \ldots, y_i) \subset (G \setminus H)$ such that $(\gamma_i - \epsilon)(n - r) \leq |P_i| \leq (\gamma_i + \epsilon)(n - r)$ for all $1 \leq i \leq t$. Furthermore, this condition is sharp.*

In particular, this corollary implies that one may place a linear forest on a hamiltonian cycle in a prescribed order with a given orientation on each path and with approximately given distances between the paths of the linear forest. Similar work may be found in [2].

As before, we use the same process of 4 lemmas to prove the desired results. The first lemma is used to absorb vertices into a collection of cycles containing fixed vertices. For this section, we define the following notation. For a collection of subgraphs $\mathscr{H} = \{H_1, H_2, \ldots, H_t\}$, let $\| \mathscr{H} \| = |\cup_{i=1}^{t} V(H_i)|$.

Lemma 5 *Let $t \geq 3$ be an integer, G a graph of order $n \geq 5t$, $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$ a specified set of t vertices and $\mathscr{C}$ a set of t vertex disjoint cycles $\{C_1, C_2, \ldots, C_t\}$ such that $x_i \in C_i$ for all $1 \leq i \leq t$. If $\| \mathscr{C} \| \geq \frac{3n}{4}$ and $\delta(G) \geq \frac{n+t-1}{2}$ then either $\| \mathscr{C} \| = n$ or there exists a collection $\mathscr{C}' = \{C_1', C_2', \ldots, C_t'\}$ of vertex disjoint cycles again with each cycle C_i' containing x_i such that $\| \mathscr{C}' \| > \| \mathscr{C} \|$ and $|C_i'| \geq |C_i|$ for all $1 \leq i \leq t$.*

Proof: Let $C = \cup_{i=1}^{t} V(C_i)$, $J = G \setminus C$ and let J_0 be the smallest component of $G[J]$. By the assumed degree condition, every vertex $v \in J_0$ must satisfy $d_C(v) \geq \frac{n+t-1}{2} - (|J_0| - 1)$. By definition, $|C| = n - |J| \leq n - |J_0|$ so if $|J_0| \leq t$, it follows that $d_C(v) \geq \frac{n+t-1-2(|J_0|-1)}{2} > \frac{n-|J_0|}{2} \geq \frac{|C|}{2}$. This implies that v must be adjacent to at least one pair of vertices which are consecutive along a cycle $C_i \in \mathscr{C}$ so v may be absorbed into C_i. Therefore we may assume $t + 1 \leq |J_0| \leq \frac{n}{4}$.

Since J_0 is connected and $|J_0| \geq t + 1 \geq 4$, there exist three vertices u, v and w such that (u, v, w) is a path in J_0. We know:

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$$d_C(u), d_C(w) \geq \frac{n+t}{2} - (|J_0| - 1)$$

$$\geq \frac{n}{4} + \frac{t}{2} + 1$$

$$> \frac{|C|-t}{4} + t + 1.$$

By the pigeon hole principle, there exists a pair of vertices u' and w' adjacent to u and w respectively with $d_{C_i}(u', w') < 4$ for some cycle C_i and with no prescribed vertex x_i in the $u'w'$ path. We may replace C_i with $C_i' = (\ldots, u', u, v, w, w', \ldots)$ with $|C_i'| > |C_i|$ thereby contradicting the assumptions on C and completing the proof of Lemma 5. $\qquad \square$

The following lemma is used to absorb vertices into a collection of paths with fixed endpoints. The proof is similar to the proof of Lemma 1 above.

Lemma 6 *Let $t \geq 3$ be an integer, G a graph of order $n \geq 5t$, $X = \{x_1, y_1, x_2, y_2, \ldots, x_t, y_t\} \subseteq V(G)$ a specified set of $2t$ vertices and $\mathscr{P}$ a set of t vertex disjoint paths $\{P_1, P_2, \ldots, P_t\}$ with $P_i = (x_i, \ldots, y_i)$ for all $1 \leq i \leq t$. If $\| \mathscr{P} \| \geq \frac{3n}{4} + t$ and $\delta(G) \geq \frac{n + \lceil frac3t2 \rceil - 2}{2}$ then either $\| \mathscr{P} \| = n$ or there exists a collection $\mathscr{P}' = \{P_1', P_2', \ldots, P_t'\}$ of vertex disjoint paths again with $P_i = x_i, \ldots, y_i$ such that $\| \mathscr{P}' \| > \| \mathscr{P} \|$ and $|P_i'| \geq |P_i|$ for all $1 \leq i \leq t$. $\square$*

In all that follows, let $\gamma_1, \ldots, \gamma_t > 0$, $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$ and let $x_1, \ldots, x_t$ be a set of t prescribed vertices in G. Given a collection of cycles $\mathscr{C} = C_1, C_2, \ldots, C_t$ (or a collection of paths $\mathscr{P} = P_1, P_2, \ldots, P_t$), let $f(i) = \lceil \gamma_i n \rceil - |C_i|$ (or $f(i) = \lceil \gamma_i n \rceil - |P_i|$ respectively). Order the cycles C_i (respectively paths P_i) such that $f(i) \geq f(i+1)$. Define:

$$\mu(\mathscr{C}) = \mu(\mathscr{P}) = \sum_{i:f(i)>0} t^{f(i)}.$$

We always choose a collection of cycles $\mathscr{C}$ (or paths $\mathscr{P}$) such that $\mu(\mathscr{C})$ (respectively $\mu(\mathscr{P})$) is minimum. Notice if $|C_i| > \lceil (\gamma_i - \frac{\epsilon}{t})n \rceil$ for all i, then $|C_i| < \lceil (\gamma_i + \epsilon)n \rceil$ for all $1 \leq i \leq t$. Since we will assume the graph does not contain the desired collection of cycles, we may assume $f(1) > \frac{\epsilon}{t}n$ so $\mu(H) > t^{\lceil n\epsilon/t \rceil}$. Identical statements hold for paths as well.

Let k be the smallest integer such that $f(k) - f(k+1) > \frac{\epsilon}{t^2}n$. Since $f(1) > \frac{\epsilon}{t}n$, we know k exists and $|C_k| < \gamma_i n$. Let $\mathscr{B}$ be the collection of cycles $\{C_i\}_{i=1}^k$ and let $\mathscr{A} = \mathscr{C} \setminus \mathscr{B}$. Again, identical definitions are assumed for paths.

We now restate Lemma 2 in two forms which are appropriate to our current situation. Lemma 7 takes a collection of cycles containing chosen vertices and, given certain conditions, shows the existence of a collection which is closer, in some sense, to the desired cycle partition. Lemma 8 accomplishes the same task for paths with chosen endpoints.

Lemma 7 *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ positive real numbers having $\sum_{i=1}^t \gamma_i = 1$, $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$ and $h_1 > 0$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+t-1}{2}$. For every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, if G contains a collection of cycles $\mathscr{C} = C_1, C_2, \ldots, C_t$ with $x_i \in C_i$ such that $e(\mathscr{A}, \mathscr{B}) \geq h_1 n^2$, then either $\mu(\mathscr{C}) \leq t^{\lceil n\epsilon/t \rceil}$ or there exists a collection of cycles $\mathscr{C}'$ with $\mu(\mathscr{C}') < \mu(\mathscr{C})$.* $\qquad\square$

Lemma 8 *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ positive real numbers having $\sum_{i=1}^t \gamma_i = 1$, $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$ and $h_1 > 0$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+\lceil 1.5t \rceil -1}{2}$. For every $X = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots y_t\} \subseteq V(G)$, if G contains a collection of paths $\mathscr{P} = P_1, P_2, \ldots, P_t$ with $P_i = x_i, \ldots, y_i$ such that $e(\mathscr{A}, \mathscr{B}) \geq h_1 n^2$, then either $\mu(\mathscr{P}) \leq t^{\lceil n\epsilon/t \rceil}$ or there exists a collection of paths $\mathscr{P}'$ with $\mu(\mathscr{P}') < \mu(\mathscr{P})$.* $\square$

The proofs of these lemmas are identical to the proof of Lemma 2 except we use Lemmas 5 and 6 respectively to absorb vertices into the cycles and paths.

If there are many edges between the collections of paths, we apply Lemmas 7 or 8. Conversely, if there are very few edges between the parts of a bipartition of the $V(G)$, then we apply the following lemmas. The proofs of these lemmas are similar to the proof of Lemma 3.

Lemma 9 *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ positive real numbers having $\sum_{i=1}^t \gamma_i = 1$, $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$ and $h_1 > 0$. For sufficiently large n, let*

G be a graph of order n having $\delta(G) \geq \frac{n+t-1}{2}$. If $V(G)$ admits a partition into two sets A and B with $e(A, B) < \frac{\epsilon^2 n^2}{1600}$ then for every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a collection of cycles $\mathscr{C}$ with $\mu(\mathscr{C}) \leq t^{\lceil n\epsilon/t \rceil}$. $\qquad\square$

Lemma 10 *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$, $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$ and $h_1 > 0$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+\lceil 1.5t \rceil -1}{2}$. If $V(G)$ admits a partition into two sets A and B with $e(A, B) < \frac{\epsilon^2 n^2}{1600}$ then for every $X = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\} \subseteq V(G)$, there exists a collection of paths $\mathscr{P}$ with $\mu(\mathscr{P}) \leq t^{\lceil n\epsilon/t \rceil}$.* $\qquad\square$

The following lemmas provide a starting structure similar to Lemma 4. Lemma 11 provides a spanning collection of cycles each containing a chosen vertex and each of order a positive fraction of n. Lemma 12 provides a spanning collection of paths with the chosen endpoints each of order a positive fraction of n. Once again, the proofs of these lemmas are almost identical to the proof of Lemma 4 above.

Lemma 11 *Let $t \geq 3$ be an integer and for sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+t-1}{2}$. For every set $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a spanning collection $\mathscr{C}$ of vertex disjoint cycles C_i with $x_i \in C_i$ such that $|C_i| \geq (\frac{1}{6400t^3(1-\frac{1}{2t})})n$ for all $1 \leq i \leq t$.* $\qquad\square$

Lemma 12 *Let $t \geq 3$ be an integer and for sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+\lceil 1.5t \rceil -1}{2}$. For every set $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a spanning collection $\mathscr{C}$ of vertex disjoint cycles C_i with $x_i \in C_i$ such that $|C_i| \geq (\frac{1}{6400t^3(1-\frac{1}{2t})})n$ for all $1 \leq i \leq t$.* $\qquad\square$

Finally we prove the main results of this section.

Theorem 8: *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+t-1}{2}$. For every set $X =$*

$\{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, *there exists a spanning collection $\mathscr{C}$ of vertex disjoint cycles C_i with $x_i \in C_i$ such that $(\gamma_i - \epsilon)n \leq |C_i| \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$. Furthermore, this condition is sharp.*

Proof: First apply Lemma 11 to get a spanning collection of cycles each containing one vertex of X. Let $\mathscr{C}$ be such a collection with $\mu(\mathscr{C})$ minimum and suppose $\mu(\mathscr{C}) > t^{\lceil n\epsilon/t \rceil}$. Define $\mathscr{A}$ and $\mathscr{B}$ as above. If $e(\mathscr{A}, \mathscr{B}) \geq \frac{\epsilon^2 n^2}{1600}$, then we apply Lemma 7 to contradict the choice of $\mathscr{C}$. If $e(\mathscr{A}, \mathscr{B}) < \frac{\epsilon^2 n^2}{1600}$, then we apply Lemma 9 to construct the desired cycle system directly. $\square$

Theorem 9: *Let $t \geq 3$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \epsilon < \min\{\frac{\gamma_i}{2}\}$. For sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n + \lceil 1.5t \rceil - 1}{2}$. For every set $X = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\} \subseteq V(G)$ of $2t$ vertices, there exists a spanning collection $\mathscr{P}$ of vertex disjoint paths $P_i = (x_i, \ldots, y_i)$ such that $(\gamma_i - \epsilon)n \leq |P_i| \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$. Furthermore, this condition is sharp.*

Proof: First apply Lemma 12 to get a spanning collection of paths $x_i, \ldots, y_i$ for $1 \leq i \leq t$. Let $\mathscr{P}$ be such a collection with $\mu(\mathscr{P})$ minimum and suppose $\mu(\mathscr{P}) > t^{\lceil n\epsilon/t \rceil}$. Define $\mathscr{A}$ and $\mathscr{B}$ as above. If $e(\mathscr{A}, \mathscr{B}) \geq \frac{\epsilon^2 n^2}{1600}$, then we apply Lemma 8 to contradict the choice of $\mathscr{P}$. If $e(\mathscr{A}, \mathscr{B}) < \frac{\epsilon^2 n^2}{1600}$, then we apply Lemma 10 to construct the desired path system directly. $\square$

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