Tôhoku Math. J.
43 (1991), 275–288

DISTRIBUTION FORMULA FOR TERMINAL SINGULARITIES ON THE MINIMAL RESOLUTION OF A QUASI-HOMOGENEOUS SIMPLE $K3$ SINGULARITY

Dedicated to Professor Ryosuke Nakagawa on his sixtieth birthday

KIMIO WATANABE

(Received May 1, 1990, revised December 13, 1990)

Introduction. Let (X, x) be a germ of a normal isolated singularity of dimension three and let $\sigma: Y \to X$ be a minimal (partial) resolution, i.e., a relatively minimal model of a resolution. The singularity (X, x) is called a simple $K3$ singularity if it is quasi-Gorenstein and if the exceptional set of Y consists of a single normal $K3$ surface D. Here we call D a normal $K3$ surface if the minimal resolution of D is a $K3$ surface. Y may still have finitely many terminal singularities $\{y_i\}$ along D.

When a simple $K3$ singularity is defined by a quasi-homogeneous polynomial of type (p, q, r, s), the minimal (partial) resolution of the singularity is given by the so-called *α-blow-up* (see Reid [R, p. 297]). In this case, the terminal singularities $\{y_i\}$ along the exceptional set are all cyclic terminal singularities, and the minimal resolution is unique (see Tomari [T, Corollary 4]).

In this paper, we obtain a simple formula describing the distribution of terminal singularities of the minimal resolution in terms of the type (p, q, r, s) of the quasi-homogeneous defining polynomial for the simple $K3$ singularity:

$$24-\sum\left(r_i-\frac{1}{r_i}\right)=\frac{(p+q+r+s)}{pqrs}(pq+pr+ps+qr+qs+rs)\,,$$

where r_i is the index of the terminal singularity y_i (compare Theorem 4.4 and [KT, Theorem 9, p. 360]).

For the simple $K3$ singularity (X, x) we define integers by

$$c_m(X, x):=\dim_{\boldsymbol{C}}\frac{\Gamma(Y, \mathcal{O})}{\Gamma(Y, \mathcal{O}(-(m+1)D))}\,,$$

and the Poincaré series

$$P(t; X, x):=\sum_{m=0}^{\infty}c_m(X, x)t^m\,,$$

which is a formal power series in an indeterminate t. By the Riemann-Roch theorem for normal isolated singularities (Watanabe [W3]), the Poincaré series can be expressed in terms of the intersection numbers of the exceptional set on a good resolution $\rho: M \to Y$.

1. Definition of simple $K3$ singularities. In this section, we recall known results and basic definitions together with examples.

DEFINITION 1.1 (Reid [R]). A germ (X, x) of a normal singularity is said to be a terminal (resp. canonical) singularity if the following two conditions are satisfied:

(i) There is an integer $r>0$ such that the multiple rK_X of the canonical divisor K_X is a Cartier divisor (the smallest such r being called the index of (X, x)).

(ii) Let $\pi: M \to X$ be an arbitrary resolution, and let $E_1, \cdots, E_n$ be the exceptional divisors. Then $rK_M = \pi^*(rK_X) + \sum_i a_i E_i$ with all $a_i > 0$ (resp. $a_i \geq 0$).

DEFINITION 1.2. If X is a normal analytic space, a partial resolution of the singularity (X, x) consists of a normal analytic space Y and a proper analytic map $\sigma: Y \to X$ such that σ is biholomorphic on the inverse image of the set R of regular points of X and that $\pi^{-1}(R)$ is dense in Y.

DEFINITION 1.3. A partial resolution $\sigma: Y \to X$ of the singularity (X, x) is a minimal resolution if the singularities of Y are terminal, and the canonical divisor K_Y of Y is numerically effective with respect to σ (see [KMM, p. 291]).

By Mori [M, Theorem 0.3.12, (i)], there exists a minimal resolution of a normal three-dimensional isolated singularity.

DEFINITION 1.4. A normal compact complex surface S is said to be a normal $K3$ surface if the following three equivalent (see, e.g., Umezu [U]) conditions are satisfied:

(1) Its minimal resolution is a $K3$ surface.

(2) $\omega_S \simeq \mathcal{O}_S$, and S is birational to a $K3$ surface.

(3) $\omega_S \simeq \mathcal{O}_S$, $H^1(S, \mathcal{O}_S) = 0$ and its singularities are at worst rational double points.

DEFINITION 1.5 ([W1]). For each positive integer m, the m-genus of a normal isolated singularity (X, x) in an n-dimensional analytic space is defined to be

$$\delta_m(X, x) = \dim_{\mathbf{C}} \Gamma(X - \{x\}, \mathcal{O}(mK)) / L^{2/m}(X - \{x\}),$$

where K is the canonical line bundle on $X - \{x\}$, and $L^{2/m}(X - \{x\})$ is the set of all holomorphic m-ple n-forms on $X - \{x\}$ which are $L^{2/m}$-integrable at x. Let $\pi: (M, E) \to (X, x)$ be a resolution of the singularity (X, x). Then

$$\begin{aligned} \delta_1(X, x) &= \dim_{\mathbf{C}} \Gamma(M - E, \mathcal{O}(K)) / \Gamma(M, \mathcal{O}(K)) = \dim_{\mathbf{C}} H_c^1(M, \mathcal{O}(K)) \\ &= \dim_{\mathbf{C}} H^{n-1}(M, \mathcal{O}) = p_g(X, x), \end{aligned}$$

where $p_g(X, x)$ is the geometric genus, and the subscript c represents compact support.

The m-genus δ_m is finite and does not depend on the choice of a Stein neighborhood

X.

DEFINITION 1.6 ([W1]). A singularity (X, x) is said to be purely elliptic if $\delta_m(X, x) = 1$ for every positive integer m.

When X is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities, i.e., there exists a nowhere-vanishing holomorphic 2-form on $X - \{x\}$ (see Ishii [I2]). In higher dimension, however, purely elliptic singularities are not always quasi-Gorenstein (see [WY]).

In the following, we assume that (X, x) is quasi-Gorenstein. Let $\pi: (M, E) \to (X, x)$ be a good resolution. Then

$$K_M = \pi^* K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} m_j E_j, \quad \text{with } m_i \geq 0,\ m_j > 0,\ I \cap J = \varnothing,$$

where $E = \bigcup E_i$ is the decomposition of the exceptional set E into irreducible components. Ishii [I1] defined the essential part of the exceptional set E as $E_J = \sum_{j \in J} m_j E_j$, and showed that if (X, x) is purely elliptic, then $m_j = 1$ for all $j \in J$.

DEFINITION 1.7 (Ishii [I1]). A quasi-Gorenstein purely elliptic singularity (X, x) is of $(0, i)$-type if $H^{n-1}(E_J, \mathcal{O})$ consists of the $(0, i)$-Hodge component $H^{0,i}(E_J)$, where

$$\boldsymbol{C} \simeq H^{n-1}(E_J, \mathcal{O}) = \mathrm{Gr}_F^0 H^{n-1}(E_J) = \bigoplus_{i=0}^{n-1} H^{0,i}(E_J)$$

in the sense of Deligne's canonical mixed Hodge structure.

EXAMPLE 1.8. Consider the singularity x of the affine cone over an abelian surface. Then (X, x) is a purely elliptic singularity of $(0, 2)$-type, which is a quasi-Gorenstein singularity, but not Gorenstein singularity.

DEFINITION 1.9. A three-dimensional singularity (X, x) is a simple $K3$ singularity if the following two equivalent (Watanabe-Ishii [WI]) conditions are satisfied:

(1) (X, x) is a Gorenstein purely elliptic singularity of $(0, 2)$-type.

(2) (X, x) is quasi-Gorenstein and the exceptional divisor D is a normal $K3$ surface for any minimal resolution $\sigma: (Y, D) \to (X, x)$.

DEFINITION 1.10. Suppose that $(r_0, r_1, \cdots, r_n)$ are fixed rational numbers. A polynomial $f(z_0, z_1, \cdots, z_n)$ is said to be quasi-homogeneous of weight $(r_0, r_1, \cdots, r_n)$ if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $i_0 r_0 + i_1 r_1 + \cdots + i_n r_n = 1$.

Let d denote the smallest positive integer so that $r_0 d = q_0$, $r_1 d = q_1, \cdots, r_n d = q_n$ are integers. Then

$$f(t^{q_0} z_0, t^{q_1} z_1, \cdots, t^{q_n} z_n) = t^d f(z_0, z_1, \cdots, z_n)$$

and f is said to be of type $(q_0, q_1, \cdots, q_n; d)$.

EXAMPLE 1.11. Let $f(x, y, z, w)$ be a quasi-homogeneous polynomal of type $(p, q, r, s; h)$ with $p+q+r+s=h$, and suppose $f(x, y, z, w)=0$ defines an isolated singularity at the origin in $\boldsymbol{C}^4$. Then the origin is a simple $K3$ singularity.

REMARK 1.12. For a simple $K3$ singularity, we have $p_g(X, x)=1$.

EXAMPLE 1.13. In the notation of Example 1.11, take the weighted projective space $\boldsymbol{P}(p, q, r, s)$ with weighted homogeneous coordinates (x, y, z, w) and the hypersurface $S\subset \boldsymbol{P}^4(p, q, r, s)$ given by $f(x, y, z, w)=0$. Then S is a normal $K3$ surface.

2. Poincaré series of simple $K3$ singularities. Let (X, x) be a simple $K3$ singularity. Consider a composite of partial resolutions $(M, E)\xrightarrow{\rho}(Y, D)\xrightarrow{\sigma}(X, x)$, where σ is a minimal resolution and ρ is a good resolution. Let E_0 be the proper transform of D.

Thanks to the existence of minimal resolutions we get the following basic lemma:

Let $A=\sum a_iA_i$ be a $\boldsymbol{Q}$-divisor on M, written as a sum of distinct prime divisors. We define the round-up $\lceil A\rceil$ of A to be the divisor $\sum b_iA_i$, where b_i is the smallest integer $\geq a_i$.

LEMMA 2.1. *For any nonnegative integer m*

$$\frac{\Gamma(M, \mathcal{O})}{\Gamma(M, \mathcal{O}(-(m+1)E_0))}\simeq\frac{\Gamma(Y, \mathcal{O})}{\Gamma(Y, \mathcal{O}(-(m+1)D))}\simeq\frac{\Gamma(M-E, \mathcal{O}(K+\lceil mL\rceil))}{\Gamma(M, \mathcal{O}(K+\lceil mL\rceil))},$$

where $L=\rho^*K_Y$.

PROOF. Since $\Gamma(M, \mathcal{O}_M(-(m+1)E_0))\simeq\Gamma(Y, \mathcal{O}_Y(-(m+1)D))$, it suffices to show that $\Gamma(Y, \mathcal{O}_Y(-(m+1)D))$ can be identified with $\Gamma(M, \omega_M(\lceil -\rho^*mD\rceil))$ by $f\mapsto f\omega$. For any $f\in\Gamma(Y, \mathcal{O}_Y(-(m+1)D))$, we have $f\omega\in\Gamma(M, \rho^*\omega_Y(-mD))$. Therefore $f\omega\in\Gamma(M, \omega_M(\lceil -\rho^*mD\rceil))$, because $\rho^*\omega_Y=\omega_M(-\Delta)$ for some $\Delta\geq 0$.

Conversely, any $\eta\in\Gamma(M, \omega_M(\lceil -\rho^*mD\rceil))$ has a zero of order at least m at E_0. Then the holomorphic function $f=\eta/\omega$, on M, has a zero of order at least $m+1$ at E_0.

q.e.d.

We now defined the Poincaré series associated with a simple $K3$ singularity. We then compute the series as an application of the following result in [W3].

DEFINITION 2.2. Let (X,x) be a normal three-dimensional isolated singularity, and suppose that X is a sufficiently small Stein neighborhood of x. Let $\pi: (M, E)\rightarrow(X, x)$ be a resolution. Then, for any line bundle F on M, the Euler-Poincaré characteristic can be defined as

$$\chi(M, \mathcal{O}(F))=\dim_{\boldsymbol{C}}\frac{\Gamma(M-E, \mathcal{O}(F))}{\Gamma(M, \mathcal{O}(F))}+\dim H^1(M, \mathcal{O}(F))-\dim H^2(M, \mathcal{O}(F)).$$

Under a certain condition, $\chi(M, \mathcal{O}(F))$ depends only on the first Chern class of F.

THEOREM 2.3 ([W3]). *Let A be an integral divisor whose support is contained in the exceptional set E. Define the intersection number of $c_2(M)$ with $A=\sum a_iE_i$ to be*

$$c_2(M)\cdot A=\sum a_i\{c_2(E_i)+c_1(E_i)c_1(N_{E_i})\}\ ,$$

where N_{E_i} is the normal bundle of E_i in M. Then

$$\chi(M, \mathcal{O}([A]))=-\frac{1}{6}A^3+\frac{1}{4}A^2K_M-\frac{1}{12}A(c_2(M)+K_M^2)$$
$$+\dim H^1(M, \mathcal{O})-\dim H^2(M, \mathcal{O})\ .$$

THEOREM 2.4 ([W3]). *In the same notation as above, if (X, x) is quasi-Gorenstein, then*

$$2\left\{p_g(X, x)-\frac{-K_M\cdot c_2(M)}{24}\right\}=\dim_{\boldsymbol{C}}H^1(M, \mathcal{O})\ .$$

For the simple K3 singularity (X, x) we define integers by

$$c_m(X, x):=\dim_{\boldsymbol{C}}\frac{\Gamma(Y, \mathcal{O})}{\Gamma(Y, \mathcal{O}(-(m+1)D))}\ ,$$

and the Poincaré series

$$P(t; X, x):=\sum_{m=0}^{\infty}c_m(X, x)t^m\ ,$$

which is a formal power series in an indeterminate t.

In our case it is moreover possible to prove that $H^i(M, \mathcal{O}(F))$ vanish for all $i>0$. Then, using Theorem 2.3 of Riemann-Roch type, we obtain

PROPOSITION 2.5. *Let $L=\rho^*K_Y$. Then*

$$c_m(X, x)=-\frac{1}{6}(\lceil mL\rceil^3)-\frac{1}{4}(\mathrm{K}\lceil \mathrm{m}L\rceil^2)-\frac{1}{12}\lceil mL\rceil(c_2(M)+K^2)+1\ .$$

PROOF. K_Y is σ-nef and σ-big, since $\sigma: (Y, D)\to(X, x)$ is a minimal resolution; then $m\rho^*K_Y$ is also $\sigma\circ\rho$-nef and $\sigma\circ\rho$-big for any nonnegative integer m. Hence $H^i(M, \mathcal{O}(K_M+\lceil m\rho^*K_Y\rceil))=0$ for $i>0$ by the Kawamata-Viehweg vanishing theorem (for example, see [KMM, p. 306]). Therefore by Theorem 2.3 we have

$$\dim_{\boldsymbol{C}}\frac{\Gamma(M-E, \mathcal{O}(K+\lceil mL\rceil))}{\Gamma(M, \mathcal{O}(K+\lceil mL\rceil))}$$
$$=-\frac{1}{6}(K+\lceil mL\rceil)^3+\frac{1}{4}(K+\lceil mL\rceil)^2K-\frac{1}{12}(K+\lceil mL\rceil)(c_2+K^2)$$
$$+\dim H^1(M, \mathcal{O})-\dim H^2(M, \mathcal{O})$$

$$= -\frac{1}{6}(\lceil mL\rceil^3) - \frac{1}{4}(K\lceil mL\rceil^2) - \frac{1}{12}\lceil mL\rceil(c_2+K^2) - \frac{1}{12}Kc_2$$

$$+\dim H^1(M, \mathcal{O}) - \dim H^2(M, \mathcal{O}) .$$

On the other hand, a simple $K3$ singularity is purely elliptic and Cohen-Macaulay, so $p_g(X, x) = h^2(M, \mathcal{O}) = 1$ and $h^1(M, \mathcal{O}) = 0$. Thus

$$-\frac{1}{12}Kc_2 + \dim H^1(M, \mathcal{O}) - \dim H^2(M, \mathcal{O}) = 1 ,$$

by Theorem 2.4. We are done by Lemma 2.1. q.e.d.

COROLLARY 2.6. *Let r be the least common multiple of the indices of the terminal singularities along D. Then c_{kr} is a polynominal of degree three in k*:

$$c_{kr} = -\frac{1}{6}(rL)^3k^3 - \frac{1}{4}K(rL)^2k^2 - \frac{1}{12}(rL)(c_2+K^2)k + 1 ,$$

where $L = \rho^* K_Y$.

DEFINITION 2.7. Let $f(t) := \sum_{m=0}^{\infty} c_m t^m$ be a formal power series. We define the r-invariant part of $f(t)$ to be

$$\frac{1}{r}\{f(t) + f(\omega t) + \cdots + f(\omega^{r-1}t)\} = \sum_{k=0}^{\infty} c_{kr}t^{kr} ,$$

where ω is a primitive r-th root of unity.

From Corollary 2.6 we obtain the r-invariant part of the Poincaré series of simple $K3$ singularities.

PROPOSITION 2.8.

$$\sum_{k=0}^{\infty} c_{kr}t^{kr} = \frac{-r^3L^3}{(1-t^r)^4} - \frac{-4r^3L^3 + r^2KL^2}{2}\cdot\frac{1}{(1-t^r)^3}$$
$$+\frac{-14r^3L^3 + 9r^2KL^2 - r(c_2L + K^2L)}{12}\cdot\frac{1}{(1-t^r)^2}$$
$$-\frac{-2r^3L^3 + 3r^2KL^2 - r(c_2L + K^2L) - 12}{12}\cdot\frac{1}{1-t^r} ,$$

where $L^3 = (1/r^3)(rL)^3$.

PROOF. It follows immediately from the equality

$$\sum_{k=0}^{\infty}(ak^3 + bk^2 + ck + d)t^k = \frac{6a}{(1-t)^4} - \frac{2(6a-b)}{(1-t)^3} + \frac{7a-3b+c}{(1-t)^2} - \frac{a-b+c-d}{1-t} .$$

3. Arithmetic Poincaré series of simple $K3$ singularities defined by a quasi-homogeneous polynomial. Let $f(x_1, x_2, x_3, x_4)$ be a quasi-homogeneous polynomial of type $(p_1, p_2, p_3, p_4; p)$. Suppose that f defines a simple $K3$ singularity (X, x) at the origin, i.e., f defines an isolated singularity at the origin and $p_1+p_2+p_3+p_4=p$, i.e., $(1, 1, 1, 1)$ is contained in the interior of the Newton boundary of f (see [W2]). Yonemura [Y] (see also Fletcher [F]) classified such quadruples of integers, which have the special properties:

LEMMA 3.1 (Yonemura [Y]). *Let p_1, p_2, p_3, p_4 and p be positive integers such that* $\gcd(p_1, p_2, p_3, p_4)=1$. *We denote by Δ the convex hull of $\{v\in \boldsymbol{Z}_0^4 \mid \sum_{i=1}^4 v_i p_i=p\}$ in $\boldsymbol{R}_0^4$, and suppose that* $(1, 1, 1, 1)\in \operatorname{Int}\Delta$. *Then*

(1) $p_1+p_2+p_3+p_4=p$;

(2) $\gcd(p_i, p_j, p_k)=1$ *for any distinct, i, j and k*;

(3) $a_{ij}:=\gcd(p_i, p_j)$ *divides p.*

PROOF. (1) Since $(1, 1, 1, 1)\in\Delta$, we have $p_1+p_2+p_3+p_4=p$.

(2) Suppose not. Then there would exist p_1, p_2 and p_3 such that $\gcd(p_1, p_2, p_3)=d>1$. Since $\gcd(p_1, p_2, p_3, p_4)=1$, we have $\gcd(p_4, d)=1$, and hence $\gcd(p, d)=1$.

Thus, for any (v_1, v_2, v_3, v_4) such that $\sum_{i=1}^4 v_i p_i=p$, the inequality $v_4\geq 1$ holds; indeed, if there is a 4-tuple $(v_1, v_2, v_3, 0)$ with $p=v_1p_1+v_2p_2+v_3p_3$, then we have $d\mid p$.

Therefore

$$\Delta\subset\{(x_1, x_2, x_3, x_4)\in \boldsymbol{R}^4 \mid x_4\geq 1\},$$

and so

$$(1, 1, 1, 1)\in \operatorname{Int}\Delta\subset\{(x_1, x_2, x_3, x_4)\in \boldsymbol{R}^4 \mid x_4>1\},$$

which is a contradiction.

(3) Suppose not. Then there would exist a_{12} such that $a_{12}\nmid p$. Therefore any element $v=(v_1, v_2, v_3, v_4)$ in $\{v\in \boldsymbol{Z}_0^4 \mid \sum_{i=0}^4 v_i p_i=p\}$ satisfies either $v_3\neq 0$ or $v_4\neq 0$, for otherwise, $p=v_1p_1+v_2p_2$ for some v_1 and v_2, and $a_{12}\mid p$, which is a contradiction.

Consider the hyperplane $H=\{x_3+x_4=2\}$ through $(1, 1, 1, 1)$. Since $(1, 1, 1, 1)\in$ $\operatorname{Int}\Delta$,

$$\{x_3+x_4>2\}\cap\{\Delta\cap \boldsymbol{R}^4\}\neq\varnothing$$

and

$$\{x_3+x_4<2\}\cap\{\Delta\cap \boldsymbol{Z}^4\}\neq\varnothing,$$

so there exist $v=(v_1, v_2, v_3, v_4)\in\Delta\cap \boldsymbol{Z}^4$ such that $v_3+v_4<2$. Therefore we have a point of the form

$$v=(v_1, v_2, 1, 0) \quad\text{or}\quad v=(v_1, v_2, 0, 1).$$

Let the point be of the form $v=(v_1, v_2, 1, 0)$. Then

$$\nu_1 p_1 + \nu_2 p_2 = p - p_3 .$$

Thus $a_{12} | p - p_3$, i.e., $a_{12} | p_1 + p_2 + p_4$, so $a_{12} | p_4$. Since $\gcd(a_{12}, p_4) = 1$, we have $a_{12} = 1$, a contradiction. q.e.d.

DEFINITION 3.2. Let $S = \boldsymbol{C}[x_1, x_2, \cdots, x_n]$ be the polynomial ring in n variables over $\boldsymbol{C}$. Introduce a filtration $\{F^k(S)\}_{k \geq 0}$ on S by putting degrees on each monomials as $\deg(x_i) = p_i$ for $1 \leq i \leq n$, and induce a filtration $\{F^k(R)\}_{k \geq 0}$ on $R = S/(f)$ by $F^k(R) = F^k(S)R$ for $k \geq 0$. For the graded ring $R = S/(f)$ we define integers

$$d_m(R) := \dim_{\boldsymbol{C}} R/F^k(R) ,$$

and the arithmetic Poincaré series

$$P_A(t : X, x) := \sum_{m=0}^{\infty} d_m(R) t^m .$$

Now consider the Poincaré series of a simple $K3$ singularity (X, x) defined by a quasi-homogeneous polynomial $f(x, y, z, w)$ of type $(p, q, r, s; h)$. Then the arithmetic Poincaré series of the simple $K3$ singularity is given as

$$P_A(t; X, x) = \frac{1 - t^h}{(1 - t^p)(1 - t^q)(1 - t^r)(1 - t^s)} \cdot \frac{1}{1 - t} .$$

REMARK 3.3. This definition is different from the ordinary one. For example, Stanley [S] uses the arithmetic Poincaré series for a graded ring $\boldsymbol{C}[x, y, z, w]/(f(x, y, z, w))$ of type $(p, q, r, s; h)$ given by

$$\frac{1 - t^h}{(1 - t^p)(1 - t^q)(1 - t^r)(1 - t^s)} .$$

EXAMPLE 3.4. Let $f(x, y, z, w) = x^2 + y^3 + z^7 + w^{42}$. The type of this quasi-homogeneous polynomial is (21, 14, 6, 1; 42). Let ϕ_k be the cyclotomic polynomial of degree k. Then

$$\frac{1 - x^{42}}{(1 - x^{21})(1 - x^{14})(1 - x^6)(1 - x^1)} \cdot \frac{1}{(1 - x)}$$

$$= \frac{\phi_{42}\phi_{21}\phi_{14}\phi_7\phi_6\phi_3\phi_2\phi_1}{(\phi_{21}\phi_7\phi_3\phi_1)(\phi_{14}\phi_7\phi_2\phi_1)(\phi_6\phi_3\phi_2\phi_1)(\phi_1)} \cdot \frac{1}{\phi_1} = \frac{\phi_{42}}{\phi_7\phi_3\phi_2\phi_1^4} .$$

Lemma 3.5. *Let σ_i be the i-th elementary symmetric polynomial in p, q, r and s. Then the Poincaré series $P_A(t; X, x)$ has the following expression in terms of the partial fractional expansion*:

$$g(t) = \frac{\sigma_1}{\sigma_4}\left(\frac{1}{(1-t)^4} + \left(-\frac{3}{2}\right)\frac{1}{(1-t)^3} + \frac{\sigma_2 + 6}{12}\frac{1}{(1-t)^2} + \left(-\frac{\sigma_2}{24}\right)\frac{1}{1-t}\right) + \sum_i \frac{\alpha_i}{t - \beta_i}$$

such that

$$\frac{\sigma_1\sigma_2}{24\sigma_4}+\sum_i \alpha_i=1 \quad and \quad \frac{\sigma_1\sigma_2}{24\sigma_4}-\sum_i \frac{\alpha_i}{\beta_i}=1\,,$$

where β_i is a pole different from 1, *and α_i is the residue of $g(t)$ at $t=\beta_i$.*

PROOF. By Lemma 3.1, the Poincaré series has only simple poles except $t=1$, hence it has the desired expansion. Thus it suffices to show only the latter half of the lemma. Since $p+q+r+s=h$, the residue of the meromorphic form $g(t)dt$ at infinity is

$$\operatorname{Res}\left(\frac{1-t^h}{(1-t^p)(1-t^q)(1-t^r)(1-t^s)}\cdot\frac{1}{(1-t)}dt;\infty\right)$$

$$=\operatorname{Res}\left(\frac{1-\left(\frac{1}{u}\right)^h}{\left(1-\left(\frac{1}{u}\right)^p\right)\left(1-\left(\frac{1}{u}\right)^q\right)\left(1-\left(\frac{1}{u}\right)^r\right)\left(1-\left(\frac{1}{u}\right)^s\right)}\cdot\frac{1}{\left(1-\frac{1}{u}\right)}d\left(\frac{1}{u}\right);\infty\right)$$

$$=\operatorname{Res}\left(\frac{u^h-1}{(u^p-1)(u^q-1)(u^r-1)(u^s-1)}\cdot\frac{u}{(u-1)}\cdot\frac{du}{-u^2};\infty\right)=-1\,.$$

Thus the sum of the other residues is 1, and so

$$\frac{\sigma_1\sigma_2}{24\sigma_4}+\sum_i \alpha_i=1\,.$$

Since $1=c_0=g(0)$,

$$\frac{\sigma_1\sigma_2}{24\sigma_4}-\sum_i \frac{\alpha_i}{\beta_i}=1\,.$$

q.e.d.

As a consequence of this lemma, one can easily calculate the r-invariant part of $P_A(t, X, x)$:

PROPOSITION 3.6.

$$\sum_{k=0}^{\infty} c_{kr}t^{kr}=\frac{\sigma_1}{\sigma_4}\left(\frac{r^3}{(1-t^r)^4}-\frac{4r^3-r^2}{2}\cdot\frac{1}{(1-t^r)^3}+\frac{14r^3-9r^2+(\sigma_2+1)r}{12}\cdot\frac{1}{(1-t^r)^2}\right.$$
$$\left.-\left\{\frac{2r^3-3r^2+(\sigma_2+1)r}{12}-\frac{\sigma_2}{24}\right\}\frac{1}{1-t^r}\right)+\sum_\lambda\frac{(\beta_\lambda)^{r-1}\cdot\alpha_\lambda}{t^r-(\beta_\lambda)^r}$$

i.e.,

$$c_{kr}=\frac{\sigma_1}{\sigma_4}\left\{\frac{1}{6}(kr)^3+\frac{1}{4}(kr)^2+\frac{\sigma_2+1}{12}(kr)\right\}+1\,.$$

PROOF. Denote temporarily the r-invariant part of a formal power series $f(t)\in \boldsymbol{C}[[t]]$ by r-inv$[f(t)]$. Then

$$r\text{-inv}\left[\frac{1}{1-t}\right]=r\text{-inv}\left[\sum_{n=0}^{\infty}t^n\right]=\sum_{n=0}^{\infty}(t^r)^n=\frac{1}{1-t^r}\,,$$

$$r\text{-inv}\left[\frac{1}{(1-t)^2}\right]=r\text{-inv}\left[\sum_{n=0}^{\infty}(n+1)t^n\right]=\sum_{n=0}^{\infty}(nr+1)t^{nr}=r\sum_{n=0}^{\infty}n(t^r)^n+\sum_{n=0}^{\infty}(t^r)^n$$

$$=\frac{rt^r}{(1-t^r)^2}+\frac{1}{1-t^r}\,,$$

$$r\text{-inv}\left[\frac{2}{(1-t)^3}\right]=r\text{-inv}\left[\sum_{n=0}^{\infty}(n+1)(n+2)t^n\right]=\sum_{n=0}^{\infty}(nr+1)(nr+2)t^{nr}$$

$$=r^2\sum_{n=0}^{\infty}n^2(t^r)^n+3r\sum_{n=0}^{\infty}n(t^r)^n+2\sum_{n=0}^{\infty}(t^r)^n$$

$$=r^2\cdot\frac{t^r(t^r+1)}{(1-t^r)^3}+3r\cdot\frac{t^r}{(1-t^r)^2}+\frac{2}{1-t^r}\,,$$

$$r\text{-inv}\left[\frac{6}{(1-t)^4}\right]=r\text{-inv}\left[\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)t^n\right]=\sum_{n=0}^{\infty}(nr+1)(nr+2)(nr+3)t^{nr}$$

$$=r^3\sum_{n=0}^{\infty}n^3(t^r)^n+11r^2\sum_{n=0}^{\infty}n^2(t^r)^n+6r\sum_{n=0}^{\infty}n(t^r)^n+6\sum_{n=0}^{\infty}(t^r)^n$$

$$=r^3\cdot\frac{t^r(t^{2r}+4t^r+1)}{(1-t^r)^4}+11r^2\cdot\frac{t^r(t^r+1)}{(1-t^r)^3}+6r\cdot\frac{t^r}{(1-t^r)^2}+\frac{6}{1-t^r}\,.$$

The rest part of the proof easily follows from these equalities.

REMARK 3.7. The sum of the residues of the Poincaré series of a graded simple $K3$ singularity is 1, the proof of which was suggested by M. Tomari.

In what follows we show the following proposition:

PROPOSITION 3.8. *The α-blow-up gives a minimal resolution of simple $K3$ singularities defined by a quasi-homogeneous polynomial.*

PROPOSITION 3.9. *Let $f(x_1, x_2, x_3, x_4)$ be a quasi-homogeneous polynomial of type $(p_1, p_2, p_3, p_4; p)$, and suppose that $f(x_1, x_2, x_3, x_4)=0$ defines an isolated singularity at the origin in $\boldsymbol{C}^4$. Denote by X the hypersurface $\{f=0\}$. Then there exist mutually distinct x_i and x_j such that $\{x_i=x_j=0\}\cap X$ consists of a finite number of affine curves.*

PROOF. Otherwise, the union $\bigcup_{i\neq j}\{x_i=x_j=0\}$ of planes in $\boldsymbol{C}^4$ would be contained in X, and so there are polynomials g_i $(i=1, 2, 3, 4)$ such that

$$f(x_1, x_2, x_3, x_4)=\sum x_i x_j x_k g_l\,,$$

which contradicts the assumption that $f(x_1, x_2, x_3, x_4)$ defines an isolated singularity at the origin. q.e.d

COROLLARY 3.10. *Let the notation be as above. Take the weighted projective space $\boldsymbol{P}(p_1, p_2, p_3, p_4)$ with weighted homogeneous coordinates y_1, y_2, y_3, y_4, and the hypersurface $S\subset\boldsymbol{P}^4(p_1, p_2, p_3, p_4)$ given by $f(y_1, y_2, y_3, y_4)=0$. Then there exist mutually distinct y_i and y_j such that $\{y_i=y_j=0\}\cap S$ consists of a finite number of points.*

LEMMA 3.11. *Let $f(x_1, x_2, x_3, x_4)$ be a quasi-homogeneous polynomial. Suppose that f defines a simple $K3$ singularity (X, x). Let $\sigma:(Y, D)\to(X, x)$ be a partial resolution obtained by the α-blow-up of $\boldsymbol{C}^4$. Then K_Y is numerically effective with respect to σ.*

PROOF. Let C be any curve in D. Take coordinate functions x_i and x_j as above. Then, there exist positive integers m_i and m_j such that

$$(\sigma^*x_i)=m_iD+B_i\,,\qquad (\sigma^*x_j)=m_jD+B_j\,,$$

where B_i and B_j are non-compact divisors on Y, i.e., proper transforms of (x_i) and (x_j). Since $K_Y\simeq -D$ as a $\boldsymbol{Q}$-Cartier divisor,

$$m_iC\cdot K_Y=C\{B_i-(\sigma^*x_i)\}=C\cdot B_i\,.$$

If $C\not\subset B_i$, then $m_iC\cdot K_Y\geq 0$. If $C\subset B_i$, then $C\not\subset B_j$, because $B_i\cap B_j\cap D$ consists of a finite number of points. Therefore $m_jC\cdot K_Y=C\cdot B_j\geq 0$. q.e.d.

LEMMA 3.12 (Yonemura [Y, Corollary 3.5]). *Let $f(x_1, x_2, x_3, x_4)$ be a quasi-homogeneous polynomial. Suppose that f defines a simple $K3$ singularity (X, x). Let $\sigma:(Y, D)\to(X, x)$ be the partial resolution obtained by the α-blow-up of $\boldsymbol{C}^4$. Then the singularities of Y along D are all cyclic terminal singularities.*

REMARK. Lemmas 3.11 and 3.12 are special cases of results in Tomari [T].

4. Comparison. The Poincaré series $P(t; X, x)$ and the arithmetic Poincaré series $P_A(t; X, x)$ agree (see [TW, Remark 2.4, p. 694]) as the following consequence of Proposition 3.8 shows:

PROPOSITION 4.1. $P(t; X, x)=P_A(t; X, x)$.

Then, comparing the r-invariant part of $P(t; X, x)$ (in Proposition 2.8) with the r-invariant part of $P_A(t; X, x)$ (in Proposition 3.6), we have:

THEOREM 4.2. *In the same notation as above,*

(1) $\dfrac{\sigma_1}{\sigma_4} = -(\rho^* K_Y)^3$,

(2) $\dfrac{\sigma_1}{\sigma_4}(\sigma_2+1) = -\{c_2(M)\cdot\rho^* K_Y + K_M^2\cdot\rho^* K_Y\}$.

COROLLARY 4.3.

$$-c_2(M)\cdot\rho^* K_Y = \frac{\sigma_1\sigma_2}{\sigma_4}.$$

PROOF. By the projection formula, we have $(\rho^* K_Y)^3 = K_M\cdot(\rho^* K_Y)^2 = K_M^2\cdot\rho^* K_Y$.
q.e.d.

REMARK 4.4. $r\sigma_1/\sigma_4$ is an integer, since $r^3\sigma_1/\sigma_4 = (\rho^* rK_Y)^3 = rK_M\cdot(r\rho^* K_Y)^2 = r^2K_M^2\cdot(r\rho^* K_Y)$ and $K_M^2\cdot(r\rho^* K_Y)$ is an integer.

Let (V, p) be a germ of a terminal singularity of dimension three, and let $\mu: W\to V$ be a good resolution such that $\mu: W-\mu^{-1}(p) \simeq V-\{p\}$. We write $K_W = \mu^* K_V + E$ and $E = \sum_j a_j E_j$, where E_j are exceptional divisors of μ. Let

$$\Delta(V, p) := -(E\cdot c_2(W)).$$

THEOREM 4.5. *In the same notation as above,*

$$\frac{\sigma_1\sigma_2}{\sigma_4} = 24 - \sum\left\{r(y_i) - \frac{1}{r(y_i)}\right\},$$

where the summation $\sum$ is taken over all terminal quotient singular points of indices $r(y_i)$ on Y.

PROOF. From Corollary 4.3,

$$-c_2(M)\cdot K_M + c_2(M)\cdot\{K_M - \rho^* K_Y] = \frac{\sigma_1\sigma_2}{\sigma_4}$$

and so

$$-c_2(M)\cdot K_M - \sum_i \Delta(Y, y_i) = \frac{\sigma_1\sigma_2}{\sigma_4}.$$

By a result of Reid or Kawamata [K, Lemma 2.2],

$$\Delta(Y, y_i) = r(y_i) - \frac{1}{r(y_i)}.$$

Thus

$$\frac{\sigma_1\sigma_2}{\sigma_4}=24-\sum\left\{r(y_i)-\frac{1}{r(y_i)}\right\},$$

by Theorem 2.4. q.e.d.

EXAMPLE 4.6. Consider the singularity $x^2+y^3+z^7+w^{42}=0$. The minimal resolution of this singularity is unique and has three terminal singularities, which are of indices 2, 3 and 7. Then

$$\frac{42\times 545}{1764}=24-\left\{\left(2-\frac{1}{2}\right)+\left(3-\frac{1}{3}\right)+\left(7-\frac{1}{7}\right)\right\}.$$

REFERENCES

[F] A. R. FLETCHER, Plurigenera of 3-folds and weighted hypersurfaces, Thesis submitted for the degree of Doctor of Philosophy at the University of Warwick, 1988.

[I1] S. ISHII, On isolated Gorenstein singularities, Math. Ann. 270 (1985), 541–554.

[I2] S. ISHII, Two-dimensional singularities with bounded plurigenera δ_m are Q-Gorenstein singularities, Contemporary Math. 90 (1989), 135–145.

[K] Y. KAWAMATA, On the plurigenera of minimal algebraic 3-Folds with $K\sim 0$, Math. Ann. 275 (1986), 539–546.

[KMM] Y. KAWAMATA, K. MATSUDA AND K. MATSUKI, Minimal model problem, in Algebraic Geometry, Sendai, 1985 (T. Oda, ed.), Advanced Studies in Pure Math. 10, Kinokuniya, Tokyo and North-Holland, Amsterdam, New York, Oxford, 1987, 283–360.

[KT] R. KOBAYASHI AND A. N. TODOROV, Polarized period map for generalized $K3$ surfaces and the moduli of Einstein metrics, Tôhoku Math. J. 39 (1987), 341–363.

[M] S. MORI, Flip theorem and the existence of minimal models for 3-folds, J. Amer. Math. Soc. 1 (1988), 117–253.

[R1] M. REID, Canonical 3-folds, Journées de Géométrie Algébrique d'Angers (A. Beauville, ed.), Sijthoff and Noordhoff, Alphen aan den Rijn, 1980, 273–310.

[R2] M. REID, Minimal model of canonical 3-folds, in Algebraic Varieties and Analytic Varieties (S. Iitaka, ed.), Advanced Studies in Pure Math. 1, Kinokuniya, Tokyo, and North-Holland, Amsterdam, New York, Oxford, 1983, 131–180.

[S] R. STANLEY, Hilbert functions of graded algebras. Adv. in Math. 28, 1978, 57–83.

[T] M. TOMARI, On the uniqueness of minimal model of singularities, preprint, 1990.

[U] Y. UMEZU, On normal projective surfaces with trivial dualizing sheaf, Tokyo J. Math. 4 (1981), 343–354.

[W1] K. WATANABE, On plurigenera of normal isolated singularities, I, Math. Ann. 250 (1980), 65–94.

[W2] K. WATANABE, On plurigenera of normal isolated singularities, II, in Complex Analytic Singularities (T. Suwa and P. Wagrech, eds.), Advanced Studies in Pure Math. 8, Kinokuniya, Tokyo and North-Holland, Amsterdam, New York, Oxford, 1986, 671–685.

[W3] K. WATANABE, Riemann-Roch theorem for normal isolated singularities, preprint, 1989.

[WI] K. WATANABE AND S. ISHII, On simple $K3$ singularities (in Japanese), in Proc. of Conf. on Algebraic Geometry at Tokyo Metropolitan Univ., 1988 (N. Sasakura, ed.), 20–31.

[WY] K. WATANABE AND T. YONEMURA, On ring-theoretic genus p_r and plurigenera $\{\tau_m\}$ of normal isolated singularities, preprint, 1988.

[Y] T. YONEMURA, Hypersurface simple $K3$ singularities, Tôhoku Math. J. 42 (1990), 351–380.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA
IBARAKI, 305
JAPAN