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DISTRIBUTION-FREE TOLERANCE INTERVALS FOR STOCHASTICALLY ORDERED DISTRIBUTIONS[1]

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Consider k stochastically ordered distributions with $F_{(1)} \leqq \cdots \leqq F_{(k)}$. The present paper deals with distribution-free tolerance intervals for $F_{(j)}$ based on order statistics in samples of same size from each of the k distributions. Two criteria are defined for determining such intervals. These two criteria are extensions of β-expectation tolerance intervals and β-content tolerance intervals with confidence coefficient γ used in the single population literature. A tolerance interval for the lifetime distribution of a series system is considered as an example.

1. Introduction and formulation of the problem. Confidence intervals for ordered parameters have been considered by Alam, Saxena and Tong [3] and Alam and Saxena [2], among others. This paper deals with tolerance intervals for distributions of a stochastically ordered family, such as the largest or the smallest of k distribution functions. The results obtained here have potential applications to reliability and life-testing problems for j-out-of-k systems. In particular, consider a series system of k components whose lifetime distributions are stochastically ordered. Then a tolerance interval for the lifetime distribution of the system is related to a tolerance interval on the largest of the distribution functions of the k components (see Section 4).

Consider k ($\geqq 1$) distributions with unknown continuous cdf's F_i, $i = 1, \cdots, k$, and assume that the distributions can be stochastically ordered, i.e., $F_{(1)} \leqq \cdots \leqq F_{(k)}$, where $(1), \cdots, (k)$ is a permutation of the first k positive integers. Let $X_{i1}, \cdots, X_{in}$ be a random sample from F_i, $i = 1, \cdots, k$. For a fixed j we consider tolerance intervals $I_j = I_j(X_{11}, \cdots, X_{kn})$, for the jth smallest cdf $F_{(j)}$. Let $\mathbf{F} = (F_1, \cdots, F_k)$, and let Ω denote the set of all k-tuples $\mathbf{F}$. Let $P_{(j)}(I_j)$ denote the probability coverage of I_j by $F_{(j)}$. Since I_j is a random set function depending on kn random variables $X_{11}, \cdots, X_{kn}$, $P_{(j)}(I_j)$ is itself a random variable. The following two criteria are used in the construction of tolerance intervals. These criteria are extensions of those used in the single population literature, see for example Guttman [6].

Criterion A. An interval I_j is a β-expectation tolerance interval for $F_{(j)}$ if

$$\inf_\Omega E_{\mathbf{F}}(P_{(j)}(I_j)) \geqq \beta\,. \tag{1.1}$$

Criterion B. An interval I_j is a β-content tolerance interval for $F_{(j)}$ at

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confidence level γ if

$$\inf_{\Omega} P_{\mathbf{F}}\{P_{(j)}(I_j) \geqq \beta\} \geqq \gamma \,. \tag{1.2}$$

2. Proposed procedure and the infima. Consider independent random samples of size n from each distribution. Let $Y_{i;r,n}$ denote the rth order statistic in the sample from F_i and let the ranking of $Y_{i;r,n}$'s be denoted by

$$Y_{(1);r,n} \leqq Y_{(2);r,n} \leqq \cdots \leqq Y_{(k);r,n} \,.$$

For every i define $Y_{(i);0,n} = -\infty$ and $Y_{(i);n+1,n} = +\infty$. For $i \leqq i'$ and $r \leqq s$, (with at least one strict inequality), let the tolerance intervals to be considered for $F_{(j)}$ be labelled as

$$\begin{aligned} &I_{1j}: (-\infty, Y_{(i');s,n}) \quad \text{for} \quad i' \geqq k-j+1\,; \\ &I_{2j}: (Y_{(i);r,n}, \infty) \quad \text{for} \quad i \leqq k-j+1\,; \\ &I_{3j}: (Y_{(i);r,n}, Y_{(i');s,n}) \quad \text{for} \quad i \leqq k-j+1 \leqq i'\,, \quad r \leqq s\,, \quad \text{with at least one strict inequality.} \end{aligned} \tag{2.1}$$

Then

$$P_{(j)}(I_{1j}) = F_{(j)}(Y_{(i');s,n})\,, \tag{2.2}$$

$$P_{(j)}(I_{2j}) = 1 - F_{(j)}(Y_{(i);r,n})\,, \tag{2.3}$$

$$P_{(j)}(I_{3j}) = F_{(j)}(Y_{(i');s,n}) - F_{(j)}(Y_{(i);r,n})\,. \tag{2.4}$$

Some more notation used in the sequel: Let $Z_{(i),j}(r, n)$ denote the ith order statistic in a random sample of size j from a beta distribution with the pdf

$$g(z; r, n) = \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} z^{r-1}(1-z)^{n-r+1-1}\,, \qquad 0 < z < 1\,, \tag{2.5}$$

and with the cdf $G(z; r, n)$ which in the standard notation of incomplete beta functions is $I_z(r, n-r+1)$. Then the cdf of $Y_{i;r,n}$ is $G(F_i(y); r, n)$. Let $\Omega(j)$ denote the restricted set of k-tuples $\mathbf{F}$ for which $F_{(j)}$ is held fixed. Note that the subscripts i, j etc. are being used both as running subscripts and preassigned subscripts.

The following Theorems 2.1 and 2.2 give the infima for the Criteria A and B respectively.

THEOREM 2.1. (a) *For* $i' \geqq k-j+1$,

$$\inf_{\Omega} E_{\mathbf{F}}[P_{(j)}(I_{1j})] = EZ_{(i'-k+j),j}(s, n)\,.$$

(b) *For* $i \leqq k-j+1$,

$$\inf_{\Omega} E_{\mathbf{F}}[P_{(j)}(I_{2j})] = 1 - EZ_{(i),k-j+1}(r, n)\,.$$

(c) *For* $i \leqq k-j+1 \leqq i'$, $r \leqq s$ *with at least one strict inequality*,

$$\inf_{\Omega} E_{\mathbf{F}}[P_{(j)}(I_{3j})] \geqq EZ_{(i'-k+j),j}(s, n) - EZ_{(i),k-j+1}(r, n)\,.$$

THEOREM 2.2. (a) *For* $i \geqq k - j + 1$,

$$\inf_{\Omega} P_{\mathbf{F}}\{P_{(j)}(I_{1j}) \geqq \beta\} = 1 - G(G(\beta; s, n); i - k + j, j)\,.$$

(b) *For* $i \leqq k - j + 1$,

$$\inf_{\Omega} P_{\mathbf{F}}\{P_{(j)}(I_{2j}) \geqq \beta\} = G(G(1 - \beta; r, n); i, k - j + 1)\,.$$

(c) *For* $i \leqq k - j + 1 \leqq i'$ *and* $r \leqq s$ *with at least one strict inequality,*

$$\inf_{\Omega} P_{\mathbf{F}}\{P_{(j)}(I_{3j}) \geqq \beta\} > G\left(G\left(\frac{1-\beta}{2}; r, n\right); i, k - j + 1\right) - G\left(G\left(\frac{1+\beta}{2}; s, n\right); i' - k + j, j\right).$$

The following lemmas are needed for proving the theorems.

LEMMA 2.1. *Let* $\mathbf{X} = (X_1, \cdots, X_k)$ *be a random vector of* k *independent components,* X_i *with the* cdf F_i, $i = 1, \cdots, k$. *Let* $\Psi(x_1, \cdots, x_k)$ *be a nondecreasing (nonincreasing) function of* x_j *when the other components are fixed. For any* j, *let* $\mathbf{F} = (F_1, \cdots, F_{j-1}, F_j, F_{j+1}, \cdots, F_k)$ *and* $\mathbf{F}^* = (F_1, \cdots, F_{j-1}, F_j^*, F_{j+1}, \cdots, F_k)$. *Then*

$$E_{\mathbf{F}}(\Psi(\mathbf{X})) \geqq (\leqq)\, E_{\mathbf{F}^*}(\Psi(\mathbf{X}))\,,$$

if $F_j \leqq F_j^*$.

This lemma is essentially the same as a lemma of Alam and Rizvi [1] and hence the proof is omitted.

LEMMA 2.2. *Let* X *be any order statistic in a sample of size* n *from a distribution* $F(x; \theta)$ *which belongs to a stochastically increasing family* $\{F(x; \theta), \theta \in \Omega\}$, i.e., $F(x, \theta') \leqq F(x, \theta)$ *for all* $\theta, \theta' \in \Omega$ *such that* $\theta < \theta'$. *Then* $E_\theta(X)$, *if it exists, is a nondecreasing function of* θ.

PROOF. It is sufficient to note that X is a nondecreasing function of the unordered observations of the sample. Then Lemma 2.1 applies.

PROOF OF THEOREM 2.1.

Part (a). Since $F_{(j)}(Y_{(i');s,n})$ is a nondecreasing function in each of the $Y_{1;s,n}, \cdots, Y_{k;s,n}$, from Lemma 2.1

$$\inf_{\Omega(j)} E_{\mathbf{F}}[F_{(j)}(Y_{(i');s,n})] = E_{\mathbf{F}^{1j}}[F_{(j)}(Y_{(i');s,n})]\,,$$

where $\mathbf{F}^{1j}$ has j components equal to $F_{(j)}$ and the rest are equal to unity. For the configuration $\mathbf{F}^{1j}$, $F_{(j)}(Y_{(i');s,n}) = 0$ if $i' \leqq k - j$. For $i' \geqq k - j + 1$, the distribution of $Y_{(i');s,n}$ under $\mathbf{F}^{1j}$ is the distribution of $(i' - k + j)$th order statistic in a sample of size j from a population with the cdf $G(F_{(j)}(y)_{;s,n})$. So

$$\begin{aligned} \inf_{\Omega(j)} E_{\mathbf{F}}[F_{(j)}(Y_{(i');s,n})] &= E_{\mathbf{F}^{1j}}[F_{(j)}(Y_{(i');s,n})] \\ &= \textstyle\int_0^1 z\, d[G(G(z; s, n); i' - k + j, j)] \\ &= EZ_{(i-k+j),j}(s, n). \end{aligned} \tag{2.6}$$

Since (2.6) is free of $F_{(j)}$, infimum over $\Omega(j)$ is also the infimum over Ω.

Proof of (b) follows in a similar manner. For proof of (c) note from (2.4) that

$$\inf_\Omega E_{\mathbf{F}}[P_{(j)}(I_{3j})] \geqq \inf_\Omega E_{\mathbf{F}}[F_{(j)}(Y_{(i');s,n})] - \sup_\Omega E_{\mathbf{F}}[F_{(j)}(Y_{(i);r,n})] . \tag{2.7}$$

Now using (a) and (b) in the two terms on the right side of (2.7), the result is obtained.

PROOF OF THEOREM 2.2.

Part (b). Define an indicator variable

$$\begin{aligned} T_y &= 1 \qquad \text{if} \quad F_{(j)}(Y_{(i);r,n}) \leqq y , \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then

$$P_{\mathbf{F}}\{F_{(j)}(Y_{(i);r,n}) \leqq y\} = E_{\mathbf{F}}(T_y) .$$

As a function of $Y_{1;r,n}, \cdots, Y_{k;r,n}$, clearly T_y is a nonincreasing function of each of them. Now from Lemma 2.1, $\inf_{\Omega(j)} E_{\mathbf{F}}(T_y)$ is obtained when all the $G(F_i)$'s are as small as possible. Since $F_{(j)}$ is fixed for $\mathbf{F}$ in $\Omega(j)$, the infimum is obtained at $\mathbf{F}^{0j}$ in which $k-j+1$ components are equal to $F_{(j)}$ and the rest are zero. For $i > k-j+1$, $Y_{(i);r,n} = +\infty$ for the configuration $\mathbf{F}^{0j}$ and consequently infimum of $E_{\mathbf{F}}(T)$ over $\Omega(j)$ is zero. For $i \leqq k-j+1$, the distribution of $Y_{(i);r,n}$ under $\mathbf{F}^{0j}$ is the distribution of the ith order statistic in a sample of size $k-j+1$ from a population with the cdf $G(F_{(j)}(y); r, n)$. So

$$\begin{aligned} \inf_{\Omega(j)} E_{\mathbf{F}}(T_y) &= P_{\mathbf{F}^{0j}}\{Y_{(i);r,n} \leqq F_{(j)}^{-1}(y)\} \\ &= \textstyle\int_0^{G(y;r,n)} d[G(u; i, k-j+1)] \\ &= G(G(y; r, n); i, k-j+1) . \end{aligned} \tag{2.8}$$

Since (2.8) is free of $F_{(j)}$, the infimum over $\Omega(j)$ is also the infimum over Ω. Since $P_{\mathbf{F}}(P_{(j)}(I_{2j}) \geqq \beta) = E_{\mathbf{F}}(T_{1-\beta})$, the result follows.

The proof of (a) follows in a similar manner. For (c) note that

$$\begin{aligned} &P_{\mathbf{F}}\{P_{(j)}(I_{3j}) \geqq \beta\} \\ &\quad \geqq P_{\mathbf{F}}\left\{F_{(j)}(Y_{(i);r,n}) \leqq \frac{1-\beta}{2} \text{ and } 1 - F_{(j)}(Y_{(i');s,n}) \leqq \frac{1-\beta}{2}\right\} \\ &\quad > P_{\mathbf{F}}\left\{F_{(j)}(Y_{(i);r,n}) \leqq \frac{1-\beta}{2}\right\} - P_{\mathbf{F}}\left\{F_{(j)}(Y_{(i');s,n}) \leqq \frac{1+\beta}{2}\right\} . \end{aligned} \tag{2.9}$$

Now using (a) and (b) in the two terms of (2.9), the result (c) is obtained.

3. Choices for *i*, *i′*, *r* and *s*.

CRITERION A. For the intervals I_{1j}, I_{2j} and I_{3j} it is desirable that i be as large as possible and i' be as small as possible in order to keep the intervals as "small" as possible. So in view of Theorem 2.1, take $i = k - j + 1 = i'$. It is also desirable that the value of s be as small as possible and the value of r be as large as possible.

Since the family of beta distributions (2.5) indexed by r, for each n, is a stochastically increasing family, $EZ_{(i),j}(r, n)$ is, by Lemma 2.2, an increasing function of r. So to satisfy (1.1) for the interval $(-\infty, Y_{(k-j+1);s,n})$ for $F_{(j)}$, choose the smallest s such that $EZ_{(1),j}(s, n) \geqq \beta$, where β lies between 0 and $EZ_{(1),j}(n, n) = \Gamma(1 + 1/n)\Gamma(j + 1)/\Gamma(j + 1 + 1/n)$. The sample size n can be made large enough to accomodate any assigned value β in (0, 1). For the interval $(Y_{(k-j+1);r,n}, \infty)$ for $F_{(j)}$ to satisfy (1.1), choose the largest r satisfying $1 - EZ_{(k-j+1);k-j+1}(r, n) \geqq \beta$, where β lies between 0 and $1 - EZ_{(k-j+1),k-j+1}(1, n) = \Gamma(1 + 1/n)\Gamma(k - j + 2)/\Gamma(k - j + 2 + 1/n)$.

Now consider $(Y_{(k-j+1);r,n}, Y_{(k-j+1);s,n})$ as a two-sided β-expectation tolerance interval for $F_{(j)}$. For s and r, choose the smallest s (say s_0) and the largest r (say r_0) so that

$$EZ_{(1),j}(s_0, n) \geqq \frac{1 + \beta}{2}, \tag{3.1}$$

and

$$EZ_{(k-j+1),k-j+1}(r_0, n) \leqq \frac{1 - \beta}{2}. \tag{3.2}$$

Then,

$$EZ_{(1),j}(s_0, n) - EZ_{(k-j+1),k-j+1}(r_0, n) \geqq \beta,$$

where β lies between 0 and $\min_{t=j,k-j+1}\{2\Gamma(1 + 1/n)\Gamma(t + 1)/\Gamma(t + 1 + 1/n) - 1\}$. The following relations are helpful to determine r_0 and s_0.

$$EZ_{(i),j}(r, n) + EZ_{(j-i+1),j}(n - r + 1, n) = 1, \qquad i \leqq j, \tag{3.3}$$

$$\frac{r}{n + 1} \leqq EZ_{(j),j}(r, n) \leqq \frac{r}{n + 1} + \left(\frac{r(n - r + 1)}{(n + 1)^2(n + 2)(2j - 1)}\right)^{\frac{1}{2}} (j - 1). \tag{3.4}$$

The identity (3.3) is easy to prove. The first inequality of (3.4) follows from Lemma 2.1 and the proof of the second inequality of (3.4) can be found in David [5], page 47. The values from the table in the appendix give strong indication that the upper bound given by (3.4) is quite close to the true value. For illustration suppose $n = 30$, $k = 3$, $j = 1$ and $\beta = .8$. Then $EZ_{(1),1}(s, 30) \geqq .9$ gives $s = 28$, and $EZ_{(3),3}(r, n) \leqq .1$ gives $r = 1$. Then the exact β value is .8446. If r is chosen to be 2 then the exact β value is .8011. Working with the bounds, if $r = 2$ and $s = 28$, then $\beta = .7998$.

For the asymptotic behavior of the tolerance intervals of Theorem 2.1, the following lemma is needed.

LEMMA 3.1. *If $r/n = \lambda + O(1/n)$, $0 < \lambda < 1$, then for any fixed i and j ($i \leqq j$) and large n,*

$$EZ_{(i),j}(r, n) \cong \lambda + \left(\frac{\lambda(1 - \lambda)}{n}\right)^{\frac{1}{2}} E(Z_{(i),j}),$$

where $Z_{(i),j}$ is the ith *order statistic in a sample of size j from the standard normal distribution Φ.*

PROOF. Let $Z_{1,j}(r, n), \cdots, Z_{j,j}(r, n)$ be a random sample of size j from a beta distribution (2.5). Define

$$Y_{i,j}(r, n) = \frac{(Z_{i,j}(r, n) - \lambda)n^{\frac{1}{2}}}{(\lambda(1-\lambda))^{\frac{1}{2}}} . \tag{3.5}$$

Let $H_n(y; r, n)$ denote the common cdf of $Y_{i,j}(r, n)$. Then the cdf of $Y_{(i),j}(r, n)$ is $G(H_n(y; r, n); i, j)$. Since $H_n(y; r, n) \to \Phi(y)$ as $n \to \infty$ where $\Phi(y)$ is the standard normal cdf, $G(H_n(y; r, n); i, j) \to G(\Phi(y); i, j)$ as $n \to \infty$. Further we have

$$\begin{aligned} E|Y_{(i),j}(r,n)|^2 &= \frac{n}{\lambda(1-\lambda)} E(Z_{(i),j}(r, n) - \lambda)^2 \\ &\leqq \frac{n}{\lambda(1-\lambda)} \frac{j!}{(i-1)!(j-i)!} E(Z_{i,j}(r, n) - \lambda)^2 . \end{aligned}$$

Since $r/n = \lambda + O(1/n)$,

$$E(Z_{i,j}(r, n) - \lambda)^2 = \frac{r(r+1)}{(n+2)(n+1)} + \lambda^2 - \frac{2r\lambda}{n+1} = O(1/n) .$$

So, there exists a number M such that $\sup_n E|Y_{(i),j}(r, n)|^2 \leqq M$. Using Theorem 4.5.2 of Chung [4], $\lim_{n\to\infty} EY_{(i),j}(r, n) = EZ_{(i),j}$. Hence

$$\begin{aligned} EZ(_{(i),j}(r, n) &= \lambda + \left(\frac{\lambda(1-\lambda)}{n}\right)^{\frac{1}{2}} EY_{(i),j}(r. n) \\ &\cong \lambda + \left(\frac{\lambda(1-\lambda)}{n}\right)^{\frac{1}{2}} EZ_{(i),j} . \end{aligned}$$

Now consider result (c) of Theorem 2.1. Let $r/n = \delta + O(1/n)$ and $s/n = \lambda + O(1/n)$, $0 < \delta < \lambda < 1$. Then an approximate lower bound for infimum of the expected probability coverage of the interval I_{3j} by $F_{(j)}$, with $i = i' = k - j + 1$ as recommended earlier, is

$$\lambda - \delta + \left(\frac{\lambda(1-\lambda)}{n}\right)^{\frac{1}{2}} EZ_{(1),j} - \left(\frac{\delta(1-\delta)}{n}\right)^{\frac{1}{2}} EZ_{(k-j+1),k-j+1} .$$

Note that when $k = 1$, and the rth and sth order statistics are used, the exact expected probability coverage is $(s - r)/(n + 1)$. Thus when n is large, the present procedure for a tolerance interval of an ordered distribution works almost as well as the procedure when only one distribution is under consideration. To choose r_0, s_0 replace the expected values in (3.1), (3.2) by their asymptotic equivalents.

CRITERION B. As with Criterion A, choose i as large as possible and i' as small as possible, i.e., take $i = k - j + 1 = i'$.

For fixed x and n, $G(x; s, n)$ is a nonincreasing function of s. Therefore to satisfy (1.2) for the interval $(-\infty, Y_{(k-j+1);s,n})$ for $F_{(j)}$, choose the smallest s to satisfy $1 - G(G(\beta; s, n); 1, j) \geqq \gamma$, provided γ lies between 0 and $1 - G(G(\beta; n, n); 1, j) = (1 - \beta^n)^j$. For the interval $(Y_{(k-j+1);r,n}, \infty)$ for $F_{(j)}$, choose the largest r to satisfy $G(G(1-\beta; r, n); k-j+1, k-j+1) \geqq \gamma$, provided γ lies between 0 and $G(G(1-\beta; 1, n); k-j+1, k-j+1) = (1-\beta^n)^{k-j+1}$. For the two-sided tolerance interval $(Y_{(k-j+1);r,n}, Y_{(k-j+1);s,n})$ for $F_{(j)}$ to satisfy (1.2) the following procedure is recommended for deciding the values for r and s: choose the largest r and the smallest s so that

$$G\left(\frac{1-\beta}{2}; r, n\right) \geqq \left(\frac{1+\gamma}{2}\right)^{1/(k-j+1)}$$

and

$$1-\left[1-G\left(\frac{1+\beta}{2}; s, n\right)\right]^j \leqq \frac{1-\gamma}{2},$$

provided γ is between 0 and $\min_{t=j,k-j+1}\{2(1-((1+\beta)/2)^n)^t-1\}$.

For illustration, suppose $k=3$, $j=1$, $\beta=.8$, $\gamma=.75$ and $n=50$. Then using incomplete beta function tables [8], we find $r=2$ and $s=48$. Tables by Somerville [10] or graphs by Murphy [7] can also be used. Scheffé and Tukey [9] have given a useful approximation formula for determining n from inequalities like (3.1) when other parameters are given. For large n, the normal approximation $\Phi((-r+1+nx)/(nx(1-x))^{\frac{1}{2}})$ can be used for $G(x; r, n)$.

4. Application. Consider a series systems of k independent components whose lifetime distributions F_i's are stochastically ordered. If $H_k(t)$ denotes the cdf of the lifetime of the system, then

$$H_k(t) = 1-\prod_{i=1}^k (1-F_i(t)).$$

Suppose a β-content lower tolerance bound is required for the lifetime distribution of the system, and this is to be done without testing the system as a whole. To do so take samples of size n from each of the k populations corresponding to k different components and put them on test. For each sample stop testing as soon as the rth failure is observed and note $Y_{i;r,n}$, $i=1,\cdots,k$. Then take the lower bound to be $Y_{(1);r,n}$, i.e., take the tolerance interval for the lifetime distribution of the system as $I_S=(Y_{(1);r,n},\infty)$. Then

$$\begin{aligned} P_F(P_{H_k}(I_S) \geqq \beta) &= P_F(\textstyle\prod_{i=1}^k (1-F_i(Y_{(1);r,n})) \geqq \beta) \\ &\geqq P_F(1-F_{(k)}(Y_{(1);r,n}) \geqq \beta^{1/k}) \\ &\geqq G(G(1-\beta^{1/k}; r, n); 1, 1) = G(1-\beta^{1/k}; r, n). \end{aligned} \tag{4.1}$$

The inequality in (4.1) is due to (b) of Theorem 2.2. So to have a β-content lower tolerance bound with confidence level γ, choose largest r which satisfies

$$G(1-\beta^{1/k}; r, n) \geqq \gamma.$$

For illustration, let $k=5$, $n=50$, $\beta=.7$, $\gamma=.8$; then $r=2$.

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APPENDIX

TABLE 1

Expected values of $Z_{(j),j}(r, n)$, upper bounds and the normal approximation: First entry is the exact value of $EZ_{(j),j}(r, n)$, the second entry is the upper bound given by (3.4) *and the third entry is the normal approximation given by Lemma* 3.1

n:	10		20				30						
r:	1	2	1	2	3	4	1	2	3	4	5	6	7
$j = 1$	.0909	.1818	.0476	.0952	.1429	.1905	.0323	.0645	.0968	.1290	.1613	.1935	.2258
	.0909	.1818	.0476	.0952	.1429	.1905	.0323	.0645	.0968	.1290	.1613	.1935	.2258
	.1000	.2000	.0500	.1000	.1500	.2000	.0333	.0667	.1000	.1333	.1667	.2000	.2333
$j = 2$	.1342	.2433	.0708	.1292	.1841	.2373	.0481	.0879	.1255	.1619	.1976	.2327	.2674
	.1388	.2461	.0738	.1314	.1859	.2388	.0503	.0896	.1269	.1632	.1988	.2339	.2685
	.1535	.2714	.0775	.1378	.1950	.2505	.0518	.0924	.1309	.1683	.2051	.2412	.2769
$j = 3$	.1621	.2793	.0861	.1496	.2079	.2633	.0586	.1021	.1422	.1806	.2178	.2541	.2898
	.1651	.2814	.0882	.1512	.2096	.2654	.0602	.1034	.1435	.1820	.2194	.2560	.2919
	.1803	.3070	.0912	.1568	.2175	.2757	.0611	.1052	.1464	.1859	.2242	.2618	.2986
$j = 4$	.1826	.3042	.0974	.1640	.2243	.2811	.0664	.1122	.1539	.1934	.2315	.2686	.3048
	.1850	.3081	.0991	.1662	.2275	.2854	.0677	.1138	.1560	.1962	.2350	.2727	.3096
	.1977	.3302	.1002	.1691	.2319	.2921	.0671	.1135	.1564	.1972	.2367	.2752	.3128
$j = 5$	.1986	.3231	.1063	.1752	.2367	.2945	.0725	.1200	.1627	.2031	.2418	.2794	.3160
	.2016	.3303	.1082	.1787	.2423	.3021	.0739	.1224	.1665	.2080	.2480	.2867	.3244
	.2103	.3471	.1066	.1780	.2429	.3040	.0714	.1196	.1637	.2055	.2458	.2849	.3231

n:	40									
r:	1	2	3	4	5	6	7	8	9	10
$j = 1$	.0244	.0488	.0732	.0976	.1220	.1463	.1707	.1951	.2195	.2439
	.0244	.0488	.0732	.0976	.1220	.1463	.1707	.1951	.2195	.2439
	.0250	.0500	.0750	.1000	.1250	.1500	.1750	.2000	.2250	.2500
$j = 2$	.0364	.0666	.0952	.1229	.1501	.1768	.2033	.2295	.2555	.2812
	.0381	.0680	.0964	.1240	.1511	.1778	.2043	.2304	.2563	.2821
	.0389	.0694	.0985	.1268	.1545	.1819	.2089	.2357	.2623	.2886
$j = 3$	.0444	.0775	.1081	.1373	.1658	.1936	.2210	.2480	.2747	.3010
	.0457	.0785	.1091	.1385	.1671	.1951	.2227	.2498	.2766	.3031
	.0459	.0792	.1102	.1401	.1693	.1977	.2259	.2535	.2809	.3079
$j = 4$	.0503	.0853	.1171	.1473	.1766	.2051	.2330	.2605	.2875	.3142
	.0514	.0865	.1187	.1495	.1792	.2082	.2366	.2645	.2919	.3190
	.0504	.0855	.1179	.1488	.1788	.2081	.2368	.2651	.2929	.3205
$j = 5$	.0551	.0913	.1240	.1549	.1847	.2137	.2400	.2697	.2971	.3241
	.0561	.0931	.1267	.1586	.1893	.2191	.2481	.2767	.3047	.3323
	.0537	.0901	.1234	.1552	.1858	.2157	.2449	.2735	.3018	.3296

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