Distribution of Human Response Times

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We confirm that distributions of human response times have power-law tails and argue that, among closed-form distributions, the generalized inverse gamma distribution is the most plausible choice for their description. We speculate that the task difficulty tracks the half-width of the distribution and show that it is related to the exponent of the power-law tail. © 2015 Wiley Periodicals, Inc. Complexity 000: 00–00, 2015

Key Words: power-law; tails; GIGa; network; stochastic

1. INTRODUCTION

Human response time (RT) is defined as the time delay between a signal and the onset of human action. For example, one can measure time interval from a word appearing on a computer screen to when a participant pushes a keyboard button to indicate his or her response. Two well established empirical facts of RT are the power law tails of RT distributions [1] and 1/f noise of RT time series [2–4], which any theoretical description must address.

The generalized inverse gamma (GIGa) function [5,6] belongs to a family of distributions [6–8], which includes inverse gamma (IGa), lognormal (LN), gamma (Ga) and generalized gamma (GGa). The remarkable property of GIGa is its power-law tail. GIGa emerges as a steady state distribution in a number of systems, from stock volatility [6], to a network model of economy [9,10], to ontogenetic mass growth [11,12]. This common feature can be traced to a "birth-death" phenomenological model subject to stochastic perturbations (see below).

Here we argue that, among closed-form distributions, GIGa is the most plausible candidate for the description of RT distributions. GIGa has a natural scale parameter, which determines the onset of the power law tail, and two shape parameters, which determine the exponent of the tail and the behavior of the front end. As such, the GIGa framework is an extension of previous approaches, such as the "cocktail" model, [1] which effectively contains shape and scale parameters as well. Furthermore, we speculate that the difficulty of a cognitive task, within a class of tasks of progressively increased complexity, tracks the half-width and modal PDF of the RT distribution.

It must be emphasized that while the GIGa framework, theoretically motivated by the dynamics of complex

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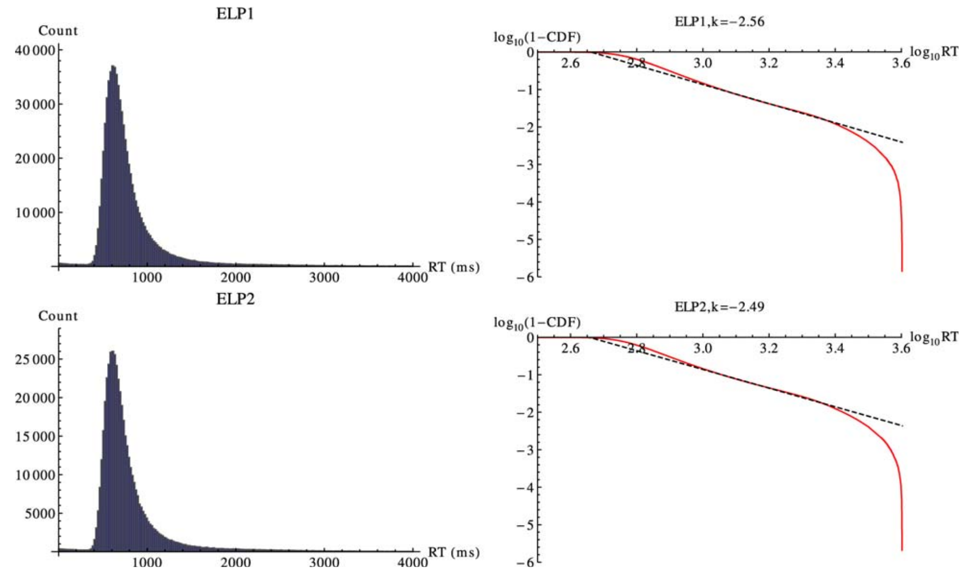
FIGURE 1



Histogram and log-log plot of ELP.

networks, [10] provides for a clear analytical description of distribution's properties as a function of its parameters, our tail fitting technique apropos power-law tails and our conjecture of the relationship between the progressive task complexity and distribution's half-width and modal PDF are not conceptually tied to the specifics of the GIGa and could be readily applied to analysis of alternative frameworks. Toward that end, we point out to a number of studies that attack similar problems with a variety of approaches [13–17].

Our numerical analysis is performed on the following data (explained in text): English Lexicon Project (ELP), Hick's Experiments (HE) and Lexical Decision Time (LDT). Two key features distinguish our approach. First, in addition to usual individual participant fitting, we perform distribution fitting on combined participants' data. While in line with individual fitting, this creates considerably less noisy sets of data. Second, we develop a procedure for fitting the tails of the distribution directly [6] and decidedly confirm that the tails of RT contain power law behavior.

This article is organized as follows. In section 2, we give a brief mathematical summary at the basis of our results. In section 3, we provide description of the experimental setup and data acquisition. In section 4, we conduct log-log tail fitting and RT distribution fitting with GIGa. In section 5, we discuss possible relation of our analysis to complexity and relative difficulty. We summarize our results in section 6.

2. MATHEMATICAL BACKGROUND

A detailed discussion of the properties of the GIGa distribution, the tail and distribution fitting and the stochastic "birth-death" model is given in a collection of Appendices in [6]. Here we present only a brief summary.

The four parameter GIGa function is given by [5]

$$\mathrm{GIGa}(x;\alpha,\beta,\gamma,\mu)=\frac{\gamma}{\beta\Gamma(\alpha)}e^{-\left(\frac{\beta}{x-\mu}\right)^{\gamma}}\left(\frac{\beta}{x-\mu}\right)^{1+\alpha\gamma} \tag{1}$$

for $x>0$ and 0 otherwise. Here μ is a shift parameter, β is a scale parameter and α and γ are shape parameters. Since the effect of the overall shift μ is trivial, we will consider a three-parameter distribution function:

$$P_{\mathrm{GIGa}}(x)=\frac{\gamma}{\beta\Gamma(\alpha)}e^{-\left(\frac{\beta}{x}\right)^{\gamma}}\left(\frac{\beta}{x}\right)^{1+\alpha\gamma} \tag{2}$$

The power-law tail of this distribution is given by $P_{\mathrm{GIGa}}(x)\propto x^{-1-\alpha\gamma}$, as $x\to\infty$.

The key properties of GIGa—the exponential front end and the power-law tail—can be gleaned from its $\gamma=1$ limit, namely the two-parameter IGa distribution PDF [5]

$$P_{\mathrm{IGa}}(x)=\frac{1}{\beta\Gamma(\alpha)}\exp\left[-\frac{\beta}{x}\right]\left(\frac{\beta}{x}\right)^{1+\alpha}. \tag{3}$$

Furthermore, as explained below, in analyzing RT distributions it is meaningful to set the mean to unity, which yields the following scaled distribution:

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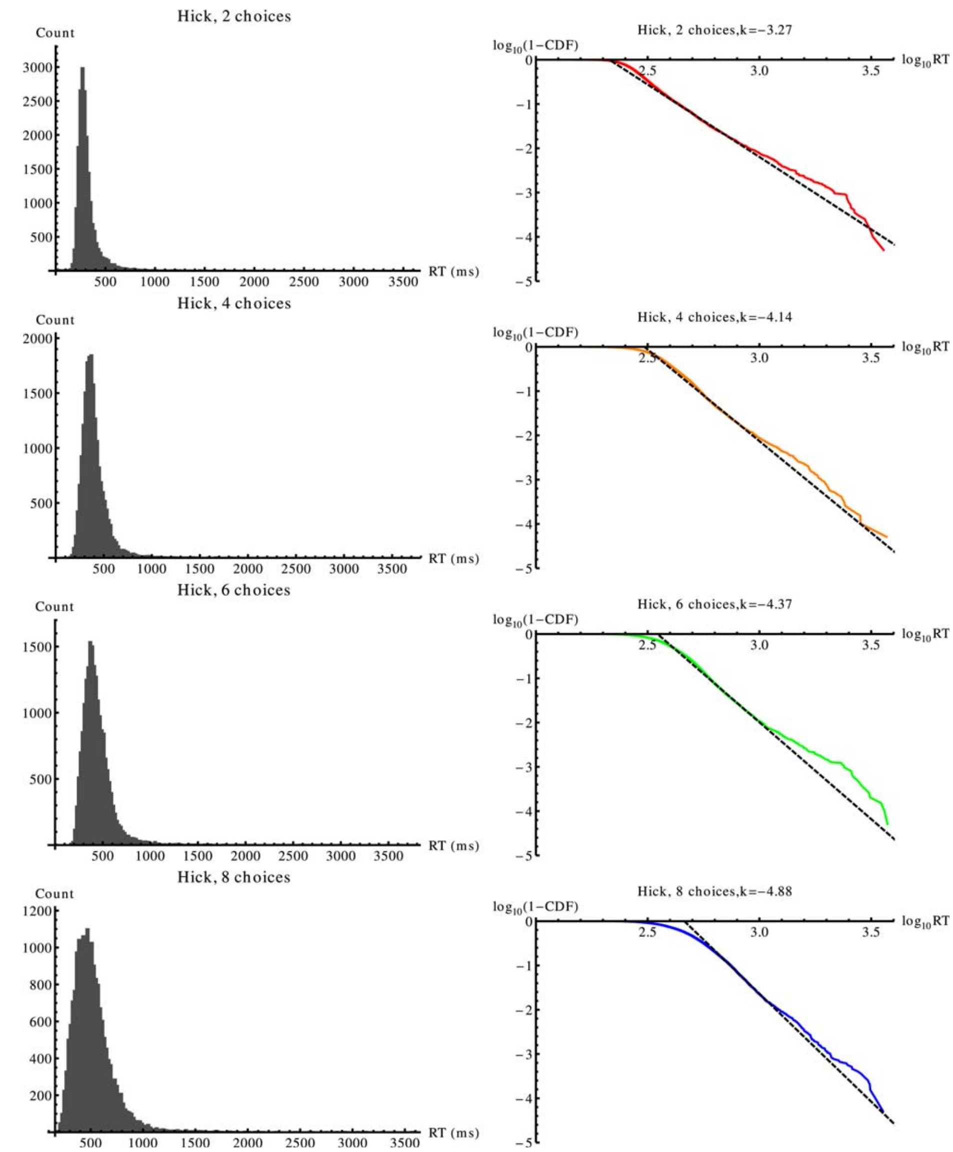
FIGURE 2



Histogram and log-log plot of Hick's experiment.

$$P_{\mathrm{IGa}}^{\mathrm{Scaled}}(x)=\frac{(\alpha-1)^{\alpha}\exp\left(-\frac{\alpha-1}{x}\right)}{\Gamma(\alpha)x^{1+\alpha}}. \quad (4)$$

so that a single parameter describes both the shape and scale of the distribution.

GIGa distribution is a steady state distribution of the stochastic "birth-death" model, described by the equation

$$dx=c_1x^{1-\gamma}dt-c_2xdt+\sigma xdW, \quad (5)$$

where dW is the Wiener noise and constants c_1, c_2 and σ are simply related to the parameters of GIGa. Many natural and social phenomena can be modeled by it so that x can alternatively stand for such additive quantities as volatility variance [6], wealth [10] mass of a species [11], and so forth, and cognitive RTs here. Since the first and second terms describe growth and decay, in cognitive phenomena they can be interpreted as various competing processes. The third, stochastic, term is the one that changes the otherwise deterministic approach, characterized by the saturation to a final value of the quantity, with the probabilistic distribution of the values—as it were, GIGa in the steady-state limit.

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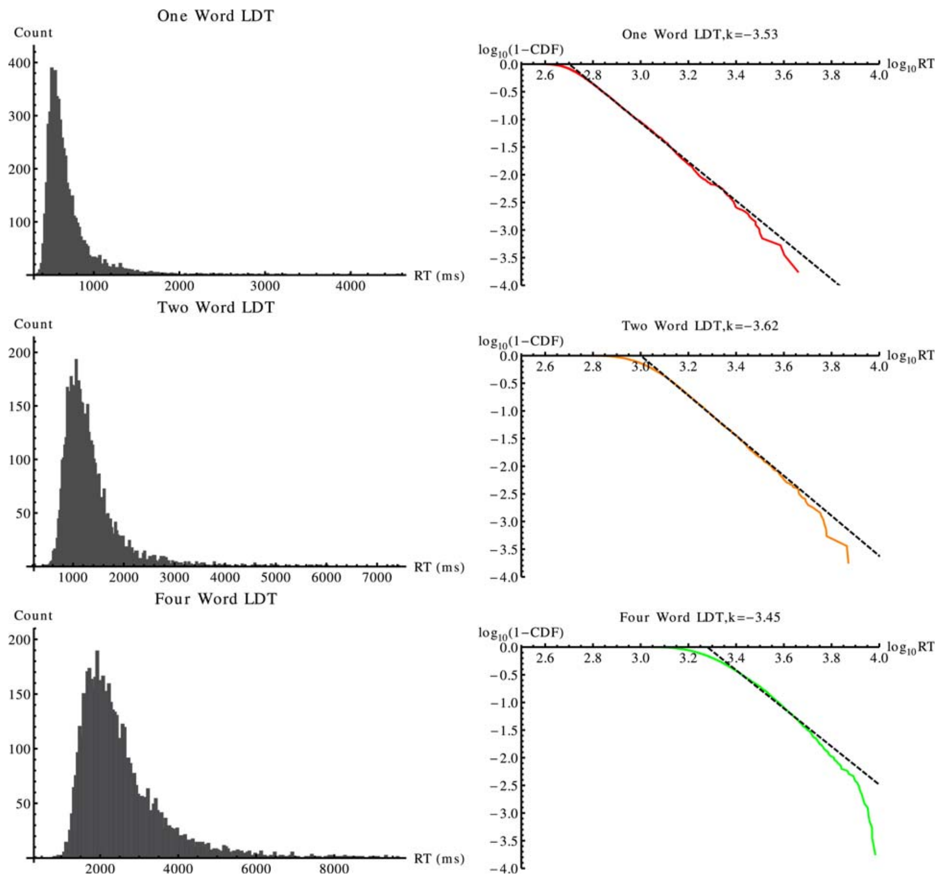
FIGURE 3



Histogram and log-log plot of one, two, and four word LDT.

3. DATA ACQUISITION

3.1. Data Sources and Description

ELP data is from the English Lexicon Project [18,19]. HE and LDT data was collected under the supervision of J. G. Holden.

ELP—studies pronunciation latencies to visually presented words; participants sampled from six different Universities [18,19]. Data: Two sessions, 470 participants each: session 1 (ELP1), 1500 trials; session 2 (ELP2), 1030 trials.

Hick's Choice RT Experiment (HE)—given a stimulus selected from a finite set of stimuli, participants try to respond with an action from a set of actions corresponding to this set of stimuli. Original HE is described in [20]. Data: 11 participants completed 1440 trials of 2, 4, 6, and 8 options, approximately 16,000 combined datapoints for each condition.

LDT—given a combination of letters, participants had to determine whether it was a word or a nonword. Data: Three groups of 60 participants completed 100 word and 100 nonword trials of 1, 2, and 4 word LDT respectively, only the correct word trials are depicted, approximately 6000 datapoints for each group.

4. DATA ANALYSIS

4.1. Data Preprocessing

To enhance our efforts to understand the distribution's tail behavior, we combined all participants' data from each experiment into a single distribution.

4.2. Tail Fitting

This section is motivated by the question of whether RT distributions express power-law tails. In this regard, a general technical problem arises of whether we can reliably ascertain the power-law behavior of a distribution's tail. Toward this end, we developed a method, described in Appendix H of [6], which is based on log-log fitting of

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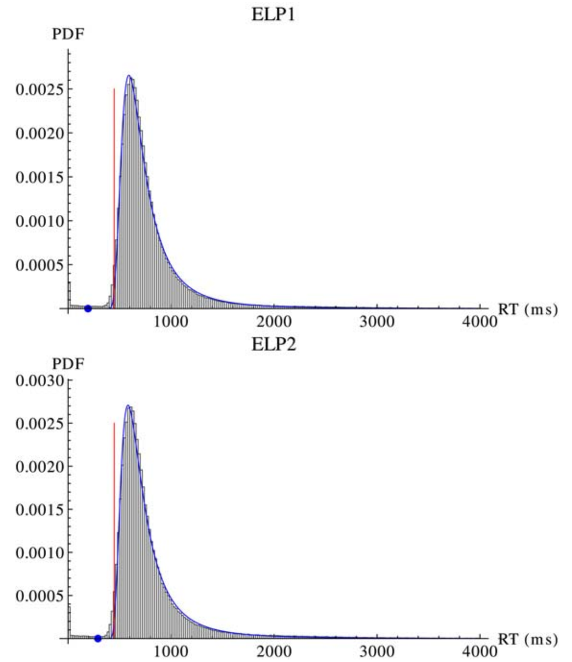
FIGURE 4



GIGa fitting of ELP. ELP1: GIGa(0.73, 396, 3.69) with $\alpha\gamma=2.7$. ELP2: GIGa(1.04, 345, 2.33) with $\alpha\gamma=2.4$. The *p*-values are both 0.

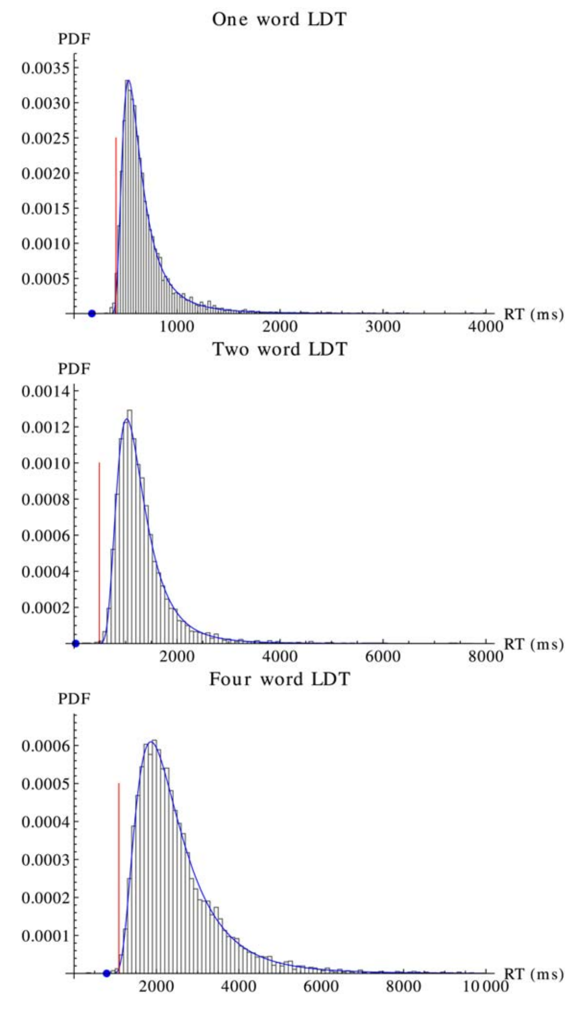
(1-CDF). It proved reliable in distinguishing power-law (fat tails) from even heavy-tail distributions, such as LN.

In Figures 1–3, we show the results of such fitting specifically for RT experiments, where RT is measured in milliseconds. With the exception of LDT, trials for most of the tasks timed out by 4 or 5 s. The latter has the potential to distort RT distributions and especially their slow tails, as manifested by downward bending of log-log plots[1]. In contrast, the maximum RT for LTD is approximately 10 s and, as seen in Figures 3 and 5, the log-log plots are closer to straight lines and GIGa fit is good. It must be made clear that it is reliability with which we can identify power-law tails that motivated us to seek a suitable power-law distribution—GIGa in this instance.

4.3. GIGa Distribution Fitting

In Figures 4–6, we show GIGa distribution fitting of RT. In the figures, the distance from the origin to the blue dot is rightward shift of GIGa distribution. The RTs to the left of the red lines are censored from the fitting of GIGa distribution. α, β, γ, the cut and shift parameters are all found by minimizing the chi-squared test statistic as follows. We choose the cut and shift parameters, find α, β, γ through maximum likelihood estimation and compute the chi-squared test statistic. We repeat this process for another

FIGURE 5



GIGa fitting of one, two, four word LDT. One word LDT: GIGa(0.754, 357, 3.96) with $\alpha\gamma=3.0$. Two word LDT: GIGa(1.96, 1424, 2.39) with $\alpha\gamma=4.7$. Four word LDT: GIGa(25.1, 7.37×10^{6}, 0.376) with $\alpha\gamma=9.4$. The *p*-values are 0.97, 0.82, and 0.87 respectively.

[1] *The scaling behavior observed in natural systems is always bounded within an interval. However, precision with which the power law exponent is evaluated depends on the size of the interval [6]. We use a "3-decade rule" to claim scaling behavior in these performances, that is, that observed quantity changes by at least three orders of magnitude. Nonetheless, the manner in which the scaling behavior decays may be of scientific interest, warranting extensions of time-out times in future experiments and temporal decay models.*

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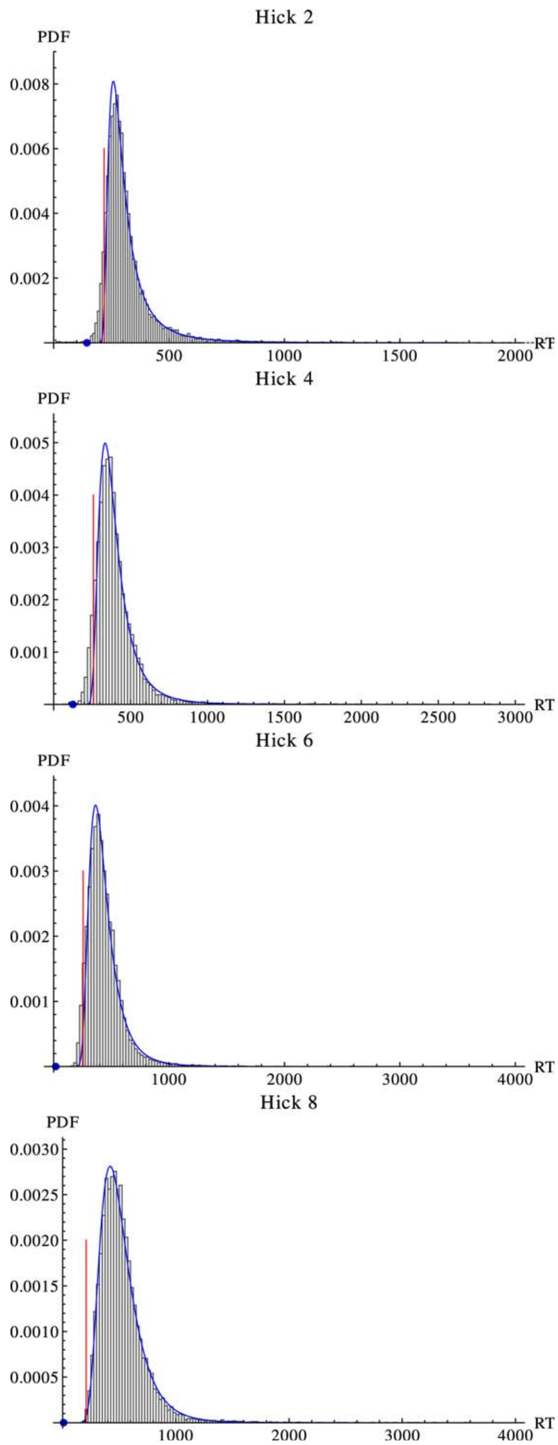
FIGURE 6



GIGa fitting of Hick's experiment. The parameters $\{\alpha, \beta, \gamma\}$ of GIGa are $\{0.731, 115, 3.41\}$, $\{1.57, 275, 2.48\}$, $\{1.64, 430, 3.07\}$, and $\{7.80, 2922, 1.10\}$ respectively. $\alpha\gamma$ are 2.5, 3.9, 5.0, and 8.6 respectively. The p-values are all 0.

group of cut and shift parameters. In the end, we obtain the parameters that minimize the chi-squared test statistic.

Visually, GIGa fitting is good, yet p-values are all zero, with the exception of LDT, which may be an artifact of very large samples. Reference [21] argues that chi-squared statistic yields poor results for goodness-of-fit—we used chi-squared statistic because, due to the cut parameter, the total number of RTs is not fixed in our parameter fitting. Lastly, in Figure 7 we show the relationship between the tail exponent parameter $\alpha\gamma$ and log-log fitted exponent parameter—with the exception of 4 LDT (which is one of the hardest tasks—see below), the correspondence is quite good. We also point out that a GIGa fit of 4 LDT is characterized by a small γ, in which case a direct tail fitting becomes less reliable [6].

We wish to emphasize that, per Figure 7, both direct tail fitting and GIGa fitting convincingly demonstrate the significant variability of the power law exponent in the tails of RT distributions. This clearly undermines the recent prediction of an α-stable distribution in [22], which constrains the power law exponent to the interval (1, 3).

Another point we wish to stress is that GIGa provides good fits to individual participants' data as well. The importance of combining the data is that it serves two key purposes: it significantly reduces the noise and, by minimizing individual variations, it better identifies the trend. The latter may be particularly important in identifying drug efficacies for cognitive problems, such as ADHD, or the nature of the problem, such as dyslexia. Another way to look at individual versus group fitting is as a choice between two averaging procedures: one involves fitting individuals' data with a distribution, such as GIGa, and then averaging thus obtained parameters over all participants, while the other involves fitting the combined data—we believe that the latter is superior.

5. TASK DIFFICULTY

In Figure 8, we plot the power law exponent from the best fit GIGa above as a function of their half-width. With the exception of Hick 6, there is a clear tracking between the two (notice that by eye HE PDFs seemingly show decrease of modal PDF and increase of PDF half-width with the increase of Hick's number). We speculate that the half width of the distribution and modal PDF would be a natural measure of a task difficulty, at least for a series of tasks of progressing complexity within a particular class of tasks.

This conjecture is easily analyzed in terms of the GIGa distribution—a potentially plausible description of RT distributions, given its connection to complex networks. In Appendix A of [6], it is explained that due to GIGa's scaling property, it is sufficient to consider the $\gamma=1$ case, that is,

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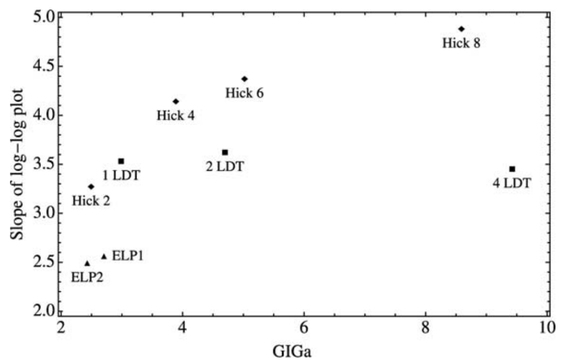
FIGURE 7



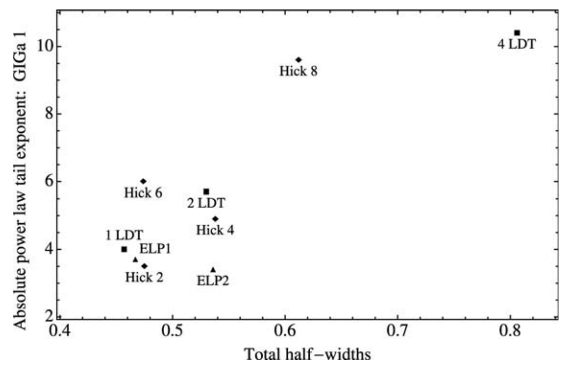
Best fit GIGa $\alpha\gamma$ versus log-log fitted tail exponent; triangles: ELP, squares: LDT, diamonds: HE.

IGa given by Eq. (3). Furthermore, we can eliminate one more parameter by setting mean to unity. In general, this is done to discard a simple stretching of a distribution by a constant. In RT, specifically, the reason is that in some cognitive tasks the mean may not be a good indicator of difficulty since an easy cognitive task may require a more idiosyncratic response and vice versa[2].

For such IGa, given by Eq. (4), a single parameter α then defines both scale and shape, that is, modal PDF and the half width, which are roughly inversely proportional to each other, are directly related to the exponent of the power law tail (for exact analytical relationship, see Appendix A of [6]). As shown in Figure 9, IGa undergoes a transition from small α to large α, characterized by a minimum/maximum in modal PDF/half-width in Figure 10.

[2]*More precisely, there are indeed three parameters in the GIGa model—one "scale" and two "shape" parameters, all of which actually affect the form of the PDF (the fourth parameter describes the overall shift of the PDF along the abscissa). The scale parameter uniformly stretches/compresses—rescales—the PDF, while the shape parameters reshape both the front end and the tail, the latter via the value of the tail exponent. One of the three parameters is eliminated by fixing the mean of the PDF to unity, which is equivalent to fixing the scale parameter. For a specific cognitive task, this eliminates the effect of "time dilation" [12], where the only difference between the PDFs is rescaling with the mean RT, indicating the same neurophysiological process. The other parameter can be approximately factored out through the scaling property explained in Appendix A of [6]. So in effect, to understand the behavior of GIGa, it is sufficient to consider variations of just one parameter, while to perform the actual fitting one needs all of them*

FIGURE 8

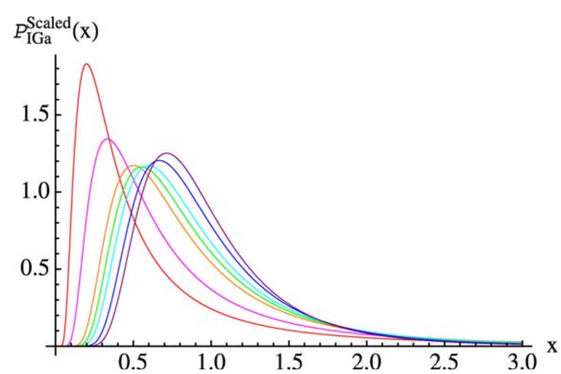


Best fit GIGa absolute power law tail exponent $\alpha\gamma+1$ versus its half width; triangles: ELP, squares: LDT, diamonds: HE.

This opens up an interesting possibility that, depending on the magnitude of α, increase in a task difficulty may either increase or decrease the magnitude of the power law exponent (in contrast to Fig. 8 where the increase in half width is accompanied by shrinking tails, that is, increase of the power law exponent—consistent with the values of of the power-law exponents obtained from direct fitting of RT tails vis-a-vis Fig. 10).

We wish to emphasize that although GIGa gives us a particularly lucid analytical description, our conjecture is completely independent of a specific heavy- or fat-tail distribution and thus can be extended to other frameworks. Furthermore, it may be also applied to the analysis of the degree of subject's attention, and so forth, in RT trials. For instance, while the half-width/modal PDF can be reduced (if not trivially) to other GIGa parameters, the reason for

FIGURE 9



PDF of IGa distributions given by Eq. (4). From left to right, $\alpha=1.5, 2, 3, 3.48, 4, 5,$ and 6, corresponding to red, magenta, orange, green, cyan, blue, and purple lines.

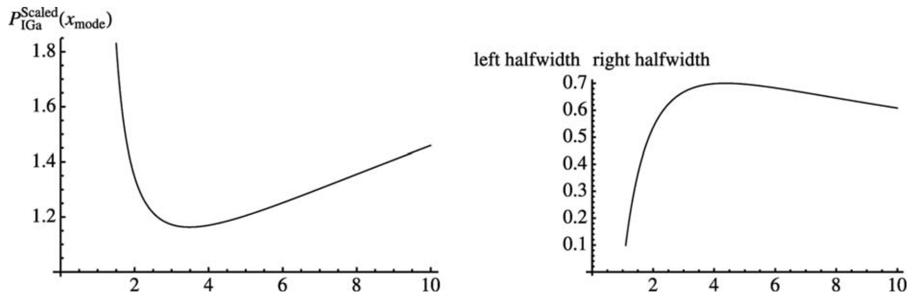


FIGURE 10

Mode and half-width of scaled IGA as a function of α.

using it is that it is easily measured for empirical distributions and that it is not tied to a specific model, such as GIGa.

Conversely, stochastic "birth-death" dynamics and complex dynamics of the generalized Bouchaud–Mezard network model [9,10] are specifically tied to GIGa and provide an appealing frameworks for thinking about cognitive processes, as well as many other natural phenomena. The latter fact was one of our chief motivators to use it for RT fitting. In a network model, the half width, the modal PDF and the power-law exponent are expressed in terms of the network connectivity, connection strength and degree of randomness (stochasticity) [10]. Relating these quantities to the empirical RT data would be of great interest.

6. SUMMARY

In conclusion, we confirmed that the tails of RT distributions exhibit power law behavior. We argued that, among closed-form distributions, GIGa is a natural candidate for fitting of measured RT as it is characterized by power law tails and motivated from a dynamical network model. While competing well in terms of goodness of fit, GIGa has an important infinitely differentiable property relative to mixture distributions and its power-law tail exponent is not limited from above, unlike alpha-stable distributions [22], and thus better describes directly fitted tails. Furthermore, since GIGa a steady-state solution of a general stochastic "birth-death" model, RT may fall into a larger class of natural, economic and social phenomena. We proposed that the task difficulty and/or subject's attention may be related to the half-width and modal PDF of the distribution and to the exponent of its power-law tail—a hypothesis that can be dissociated from a specific form of the distribution.

In future work, we hope to substantially improve on methodology of our RT measurements to have a more precise description of the distribution and its tails. We will perform a more thorough analysis of the time series and its power spectrum apropos 1/f noise and compare those with the simulations of stochastic differential equations and network models. One of the more intriguing possibilities is to improve the network model with actual connectivity and connection strength data obtained from fMRI.

Finally, we will configure a number of measurements to look for a more definitive relationship between the task difficulty and the parameters of the distribution. We will also attempt to interpret those quantities in terms of the parameters of a dynamical network, such as connectivity and connection strength. Insofar as the subject attention, this would open up an intriguing possibility of quantitatively describing ADHD drug efficacies in a control group experiment.

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