

## DISTRIBUTION OF ITERATES OF FIRST ORDER DIFFERENCE EQUATIONS

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An invariant measure which is absolutely continuous with respect to Lebesgue measure is constructed for a particular first order difference equation that has an extensive biological pedigree. In a biological context this invariant measure gives the density of the population whose growth is governed by the difference equation. Further asymptotically universal results are obtained for a class of difference equations.

### 1. Introduction

First order difference equations expressed in the form

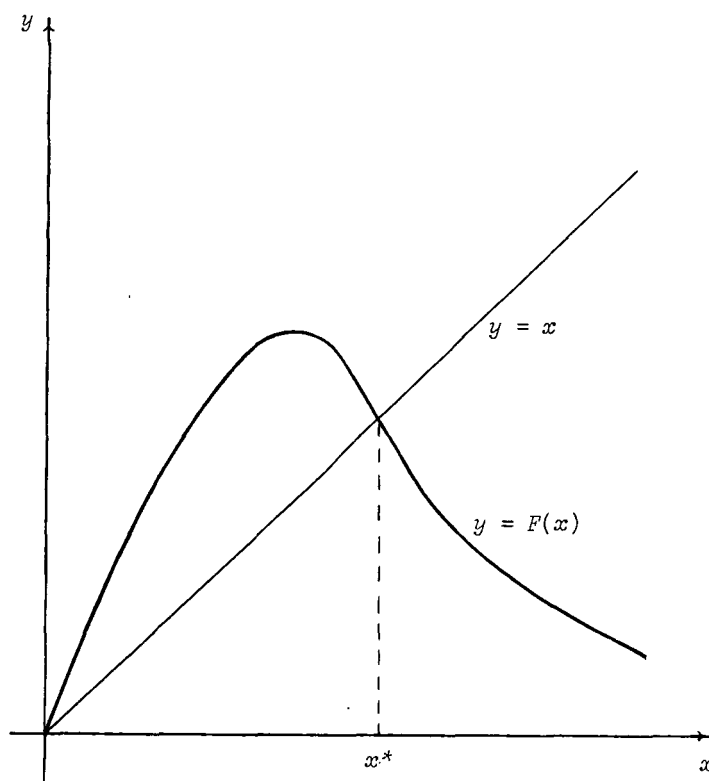
$$(1.1) \quad x_{t+1} = F(x_t)$$

have been used to model a variety of situations in the social and biological sciences [4]. In modelling the growth of populations with separate generations for example,  $x_t$  in (1.1) represents the size of the population in the  $t$ th generation and  $F(x)$  typically has the form shown in Figure 1 (see p. 134). For such functions  $F$ , the detailed dynamics of (1.1) may be quite exotic with bifurcating harmonics or cycles and eventually completely chaotic behaviour as the slope of  $F$  at the non-trivial fixed point  $x^*$  becomes large and negative [4].

The microscopic behaviour of difference equations has been the subject

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FIGURE 1. Typical form of  $F(x)$ .

of much recent work and many interesting results have been found, including various universal phenomena such as the order of appearance of cycles [5] and the behaviour of successive intervals of stability of bifurcating harmonics [1].

In this paper we will be concerned rather with macroscopic behaviour of difference equations and in particular with the existence and form of averages

$$(1.2) \quad \langle f \rangle = \lim_{n \rightarrow \infty} n^{-1} \sum_{t=1}^n f(x_t),$$

where the  $x_t$ ,  $t = 1, 2, \dots$ , are obtained from (1.1). This is a simple form of the ergodic problem and it is well known that if  $f$  is integrable,  $\langle f \rangle$  exists and is independent of  $x_0$  for almost all  $x_0$ . Moreover, if  $F$  maps a finite set  $E$  onto itself (and we will see that this covers

almost all cases of interest) there exists a measure  $\rho$  which is invariant with respect to  $F$  such that  $\rho(E) = 1$  and

$$(1.3) \quad \langle f \rangle = \int_E f d\rho .$$

This type of problem, namely the existence of "ensemble averages" such as (1.3), and invariant measures  $\rho$ , was studied some thirty or more years ago by such people as Kac [2] and Ulam and von Neumann [6]. Two particular examples are:

1. For

$$(1.4) \quad F(x) = \begin{cases} 2x & , \quad 0 < x < \frac{1}{2} , \\ 2(1-x) & , \quad \frac{1}{2} < x < 1 , \end{cases}$$

$$d\rho(y) = dy \quad (\text{Lebesgue measure}),$$

or, in other words, the distribution of iterates is uniform on the interval  $(0, 1)$  [2].

2. For

$$(1.5) \quad F(x) = 4x(1-x) ,$$

$$d\rho(y) = d(2\pi^{-1} \arcsin(2y-1)) .$$

In this case the density of iterates is given by

$$\rho'(y) = \pi^{-1} [y(1-y)]^{-\frac{1}{2}}$$

on the interval  $(0, 1)$  [6].

The general question of the existence of invariant measures  $\rho$  for which

$$(1.6) \quad d\rho(y) = \rho'(y) dy$$

has received only scant attention since the above two examples. Recently, however, there has been renewed interest in such problems. Lasota and Yorke [3] for example proved that such measures exist for piecewise continuous  $F$  such that  $\inf |F'| > 1$ . More recently, Jacobson (Private Communication) has proved that for the generalization

$$F(x) = \lambda x(1-x)$$

of (1.5), the set of  $\lambda$  for which an absolutely continuous measure exists has positive measure.

While such existence theorems are valuable, it is, nevertheless, useful to have explicit examples of invariant measures at hand. This is particularly true in the ecological context where  $\rho'(y)$  represents the density of iterates for a population whose growth is governed by difference equations (1.1). When an invariant measure which is absolutely continuous with respect to Lebesgue measure exists, one is not generally interested in the microscopic details of the population generation by generation; rather one would be interested in the statistics of the population. It is precisely this sort of information which can be calculated from the invariant measure.

The particular, and probably the most usual case, where difference equations (1.1) possess "limit cycles" is, of course, included in this description, but in the context of invariant measures the "ergodic problem" posed above is both trivial and uninformative. In particular, if the difference equation (1.1) has a " $p$ -cycle"  $x_0, x_1, x_2, \dots, x_{p-1}$ ,  $x_p = x_0$ , the "expectation" value

$$(1.7) \quad \langle f \rangle = p^{-1} \sum_{t=1}^p f(x_t)$$

and the appropriate invariant measure is the "atomic measure"

$$(1.8) \quad d\rho(y) = p^{-1} \sum_{t=1}^p \delta(y - x_t) dy,$$

where  $\delta(x)$  is the Dirac delta-function.

In this paper we present, in the following section, the general problem of constructing absolutely continuous measures for "endomorphisms"  $F$  in (1.1) (of a set  $E$  onto itself), followed in Section 3 by an explicit example of such a function in the biological literature. The relevance of these results for more general situations is discussed in the final section.

## 2. Construction of invariant measures

For functions  $F$  of the form shown in Figure 1, that is, with a

single maximum at  $\bar{x}$ , strictly increasing on  $[0, \bar{x})$ , strictly decreasing on  $(\bar{x}, \infty)$  and  $F(0) = 0$ , it is clear from Figure 2 that if  $x_t$  is in region  $I$ ,  $x_{t+1}$  will also be in  $I$ . Moreover, it is equally clear that if  $x_0$  is not in  $I$ ,  $x_t$  will be in  $I$  for some (small)  $t$ .

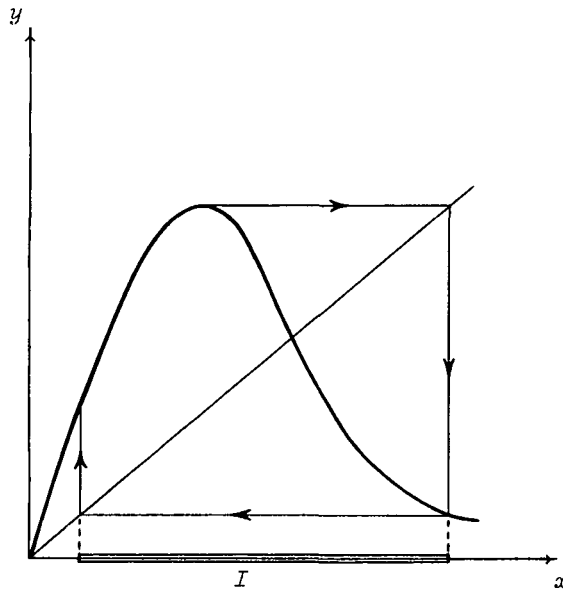


FIGURE 2. The interval of iterates.

There is no loss in generality then in restricting the function  $F$  to  $I$  and further, by an appropriate contraction, taking  $I$  to be the unit interval. Endomorphisms of the unit interval onto itself, as shown in Figure 3 (see p. 138), then include the general case depicted in Figure 1 after an appropriate change of variables.

It is clear from Figure 3 that in order for  $d\rho(y)$  to be an invariant measure (henceforth, "which is absolutely continuous with respect to Lebesgue measure") we must have

$$\begin{aligned}\rho'(y)dy &= \rho'(x_1)dx_1 + \rho'(x_2)dx_2 \\ &= \rho'(f_1(y))f_1'(y)dy - \rho'(f_2(y))f_2'(y)dy, \quad a \leq y \leq 1.\end{aligned}$$

That is,

$$(2.1) \quad \rho'(u) = \rho'(f_1(u))f_1'(u) - \rho'(f_2(y))f_2'(y), \quad a \leq u \leq 1.$$

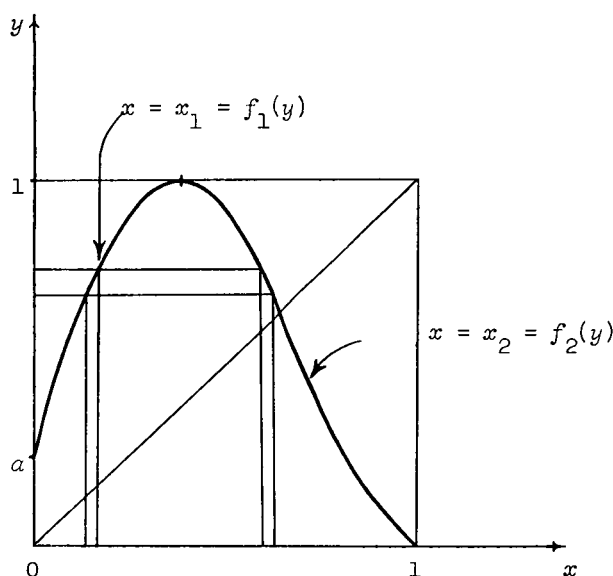


FIGURE 3. The general case after change of variables.

and similarly,

$$(2.2) \quad \rho'(y) = -\rho'(f_2(y))f_2'(y), \quad 0 \leq y \leq a.$$

(The minus signs in (2.1) and (2.2), it will be noted, are due to the fact that  $f_2'(y) < 0$ .)

From (1.2) and (1.3) we seek a solution of (2.1) and (2.2) such that

$$(2.3) \quad \lim_{n \rightarrow \infty} n^{-1} \sum_{t=1}^n f(x_t) = \int_0^1 f(y) \rho'(y) dy$$

for integrable  $f$  and almost all  $x_0$ . This clearly requires that  $\rho$  be monotone increasing and  $\rho(1) - \rho(0) = 1$ .

The Kac and Ulam and von Neumann examples of the previous section are easily seen to satisfy (2.1) and (2.2). Another example, which is relevant in Biology, is given in the following section.

An interesting particular case is when  $a$  is equal to unity. Equation (2.1) is then irrelevant and on integrating (2.2) we have

$$(2.4) \quad \rho(y) + \rho(f_2(y)) = c \quad (\text{constant}).$$

Replacing  $y$  by  $f_2(y)$  in (2.4) and subtracting the resulting equation from (2.4), we obtain

$$(2.5) \quad \rho(y) = \rho(f_2(f_2(y)))$$

and since  $\rho$  is monotonic, (2.5) implies that

$$(2.6) \quad y = f_2(f_2(y))$$

which in turn implies that on  $(0, 1)$ ,

$$(2.7) \quad f_2(y) = 1 - y.$$

The invariant measure from (2.4) is then also linear.

This result (Kac, Private Communication), which is of some interest in itself, states that the only monotonic decreasing endomorphism possessing an invariant measure is a straight line.

### 3. Invariant measure for a biological example

A particular spiked form of the difference equation (1.1) that has appeared extensively in the biological literature [4] to model population growth is

$$(3.1) \quad x_{t+1} = \begin{cases} \lambda x_t & , \quad 0 < x_t < 1 \\ \lambda x_t^{1-m} & , \quad x_t > 1 \end{cases} \quad (\lambda > 1, m > 2).$$

The maximum population size is clearly  $\lambda$  and after some small finite number of generations the population size never falls below  $\lambda^{2-m}$ . In accordance with the general situation shown in Figures 2 and 3, the transformation

$$(3.2) \quad x_t = \exp\{[(m-1)z_t - (m-2)]\log \lambda\}$$

converts (3.1) to the piecewise linear endomorphism

$$(3.3) \quad z_{t+1} = \begin{cases} z_t + (m-1)^{-1} & , \quad 0 < z_t < (m-2)(m-1)^{-1} \\ (m-1)(1-z_t) & , \quad (m-2)(m-1)^{-1} < z_t < 1. \end{cases}$$

Assuming for convenience that  $\rho(0) = 0$  and  $\rho(1) = 1$ , integration of (2.1) and (2.2) results in the functional equations

$$(3.4) \quad \rho(y) + \rho\left(1 - \frac{y}{m-1}\right) - \rho\left(y - \frac{1}{m-1}\right) = 1, \quad (m-1)^{-1} \leq y \leq 1,$$

and

$$(3.5) \quad \rho(y) + \rho\left(1 - \frac{y}{m-1}\right) = 1, \quad 0 \leq y \leq (m-1)^{-1}.$$

In the particular case where  $m$  is an integer, these equations have the solution

$$(3.6) \quad \rho(y) = \{2kym^{-1} - k(k-1)[m(m-1)]^{-1}\} : \\ (k-1)(m-1)^{-1} \leq y \leq k(m-1)^{-1}; \quad k = 1, 2, \dots, m-1\}.$$

There is no guarantee or proof that this is the only invariant measure for (3.3). Extensive numerical studies, however, confirm that  $\rho'(y)$  from (3.6) is indeed the density of iterates so that from (1.2) and (1.3), expectations for integrable functions of  $z$  are given by

$$(3.7) \quad \langle f \rangle = \lim_{n \rightarrow \infty} n^{-1} \sum_{t=1}^n f(z_t) = \int_0^1 f(y) \rho'(y) dy \\ = \sum_{k=1}^{m-1} \frac{2k}{m} \int_{(k-1)(m-1)^{-1}}^{k(m-1)^{-1}} f(y) dy.$$

The invariant measure for the original population variable  $x_t$  can be easily obtained from the change of variables (3.2). Alternatively, various population statistics can be obtained from (3.7) by an appropriate choice of the function  $f$ . For example, choosing

$$(3.8) \quad f(y) = \exp\{[(m-1)y - (m-2)] \log \lambda\}$$

one obtains from (3.7) and (3.2), that the mean population size for an ecology governed by (3.1), is given by

$$(3.9) \quad \langle x \rangle = \lim_{n \rightarrow \infty} n^{-1} \sum_{t=1}^n x_t \\ = 2\lambda^{2-m} [m(\lambda-1) \log \lambda^{m-1}]^{-1} \{m\lambda^m (1-\lambda^{-1}) + 1 - \lambda^m\}.$$

The asymptotic form for (3.9) as  $\lambda \rightarrow \infty$  is of some interest and is given by

$$(3.10) \quad \langle x \rangle \sim 2\lambda/m \log \lambda \quad \text{as } \lambda \rightarrow \infty.$$



For fixed  $\lambda$  and  $m \rightarrow \infty$  the more general expectation value (3.7) becomes simply

$$(3.11) \quad \langle f \rangle \sim 2 \int_0^1 x f(x) dx \quad \text{as } m \rightarrow \infty.$$

The above results are, of course, only strictly valid if  $m$  is an integer even though the asymptotic forms (3.10) and (3.11) are undoubtedly valid for arbitrary real positive  $m$ .

#### 4. Statistics and asymptotic forms for more general cases

One of the difficulties in studying simple first order difference equations (1.1) is that general properties seem to require specific conditions on the function  $F$ . An appropriate condition, that seems to cover all biological cases as well as the aforementioned microscopic studies [1], [5], amounts to considering equations of the form

$$(4.1) \quad x_{t+1} = \lambda x_t g(x_t).$$

If we require that  $g(0) = 1$  and that  $g$  be monotone decreasing for  $x > 0$ , then  $\lambda$  may be interpreted as a growth parameter which gives the increase in population per generation for low populations. The monotonicity of  $g$  requires that this growth is limited by the population as the population increases. Populations whose growth is governed by such equations might therefore be called "self-limiting".

Once the function  $g$  is chosen,  $\lambda$  becomes the only adjustable parameter. We show here an asymptotic universality in the behaviour of the distribution of iterates over a wide range of  $g$ 's as  $\lambda \rightarrow \infty$ .

Following the lead from the change of variables (3.2) of the previous section, if we take logarithms of both sides of (4.1) and divide by  $\log \lambda$ , we obtain

$$(4.2) \quad y_{t+1} = 1 + y_t + \alpha^{-1} \log g(\exp[\alpha y_t]),$$

where

$$(4.3) \quad y_t = (\log x_t)(\log \lambda)^{-1} \quad \text{and} \quad \alpha = \log \lambda.$$

Assuming that  $g(x) \sim x^{-m}$  as  $x \rightarrow \infty$ , one then obtains from (4.2)

that

$$(4.4) \quad y_{t+1} \sim \begin{cases} 1 + y_t & \text{if } y_t \leq 0 \\ 1 - (m-1)y_t & \text{if } y_t > 0 \end{cases} \quad \text{as } \lambda \rightarrow \infty .$$

A simple contraction then brings this equation to the form (3.3). Comparison with (3.10) then gives the asymptotic form for the mean value for population variables governed by (4.1):

$$(4.5) \quad \lim_{n \rightarrow \infty} n^{-1} \sum_{t=1}^n x_t \sim 2\lambda/m \log \lambda \quad \text{as } \lambda \rightarrow \infty .$$

Many other statistics can of course be obtained from the invariant measure (3.6). Some asymptotic microscopic information can also be obtained from the transformed equation (4.4). In particular, it will be noted that (4.4) as an equality, and hence (4.1) asymptotically, has an  $m$ -cycle:

$$y_0 = 0, y_1 = 1, y_2 = -m+2, \dots, y_k = -m+k, \dots, y_m = 0 .$$

It is amusing to note for example that the difference equation

$$(4.6) \quad x_{t+1} = \lambda x_t \left( 1 + x_t^3 \right)^{-1} ,$$

with  $m = 3$ , only possesses a 3-cycle and hence "complete chaos" in the asymptotic  $\lambda \rightarrow \infty$  limit!

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