DISTRIBUTION OF SUM OF IDENTICALLY DISTRIBUTED EXPONENTIALLY CORRELATED GAMMA-VARIABLES[1]

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0. Summary. A distribution of a sum of identically distributed Gamma-variables correlated according to an "exponential" autocorrelation law $\rho_{kj} = \rho^{|k-j|}(k, j = 1, \cdots n)$ where ρ_{kj} is the correlation coefficient between the kth and jth random variables and $0 < \rho < 1$ is a given number is derived. An "approximate" distribution of the sum of these variables under the assumption that the sum itself is a Gamma-variable is given. A comparison between exact and approximate distributions for certain values of the correlation coefficient, the number of variables in the sum and the values of parameters of the initial distributions is presented.

1. Introduction and general remarks. Distribution of sum of correlated Gamma-variables has a significant interest and many applications in engineering, insurance and other areas. Besides the applications mentioned in [3] and [6] we should like to note the usefulness of this distribution in problems connected with representation of precipitation amounts. Weekly or monthly, etc., precipitation amounts are regarded as sums of daily amounts, the latter being well fitted Gamma distribution [4]. A particular solution for the case of a constant correlation between each pair of variables in the sum is presented in [3]. In this note, we shall extend this result for the case of "an exponential autocorrelation scheme" between the variables, where each one of the variables has the marginal density function given by:

$$f(x) = [\Gamma(r)\theta^r]^{-1}e^{-x/\theta}x^{r-1}, \qquad x \geqq 0 \tag{1.1}$$
$$= 0 \qquad x < 0$$

In meteorological applications, it is sometimes assumed that the sum random variable is itself a Gamma-variable and an "approximate" Gamma distribution whose first two moments are identical with the "exact" distribution of the sum is used [4], (see also, for example, [7] for an earlier application in another field). In the case of identically distributed r.v. and of stationary exponential correlation model (namely, when the correlation coefficient between the ith and the jth r.v. in the sum is given by $\rho^{|i-j|}$ for all i and j, ρ being a given positive constant) it is easy to verify [4], that the appropriate parameters r_n and θ_n of the "approximate" Gamma-variable, representing the sum r.v., are given by:

$$r_n = \left\{n^2 \Big/ \left[n + \frac{2\rho}{1-\rho}\left(n - \frac{1-\rho^n}{1-\rho}\right)\right]\right\} r \tag{1.2}$$

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and

$$\theta_n = \left\{\left[n + \frac{2\rho}{1-\rho}\left(n - \frac{1-\rho^n}{1-\rho}\right)\right] \Big/ n\right\}\theta \tag{1.3}$$

when n is the number variables in the sum and r and θ are the parameters of initial r.v. distributed according to (1.1). Since the first two moments completely determine Gamma distribution, the "approximate" distribution is unique. The purpose of this note is to compare this "approximate" distribution (which is quite easily computable and applied) with an "exact" distribution of the sum (which is much more difficult to be actually determined). We also note that, similar to the case treated in [3], the obtained "natural" exact distribution may not be unique, since the correlation scheme, in general, does not determine uniquely a multivariate Gamma distribution; this however, does not affect the final results of this paper.

2. Distribution of a sum of identically distributed exponentially correlated Gamma-variables. Consider the "characteristic function" $\phi(t_1, t_2, \cdots, t_n)$,

$$\phi(t_1, t_2, \cdots, t_n) = |I - i\theta TV|^{-r}, \tag{2.1}$$

where θ and r are positive constants, I is the $n \times n$ identity matrix, T is the $n \times n$ diagonal matrix with the elements $t_{jj} = t_j$, and V is an arbitrary $n \times n$ positive definite matrix. This characteristic function leads to a joint probability density function whose marginals are given by (1.1) and whose matrix of second moments is some positive definite matrix V^*, say.

In particular, if the elements of V are given by $v_{ij} = \rho^{|i-j|}$, $i, j = 1, \cdots, n$, ($0 < \rho < 1$ a given constant) it is easy to verify by differentiation of (2.1) that the corresponding elements of V^* will be given by $v_{ij}^* = r\theta^2\rho^{2|i-j|}$, $i, j = 1, \cdots, n$.

The characteristic function of the distribution of the sum of the random variables whose joint distribution has the characteristic function given in (2.1) is:

$$\phi(t) = |I - i\theta tV|^{-r} \tag{2.2}$$

and (2.2) may be expressed in the form

$$\phi(t) = \prod_{j=1}^{n} (1 - i\theta\lambda_j t)^{-r}, \tag{2.3}$$

where the λ_j are the characteristic roots of the matrix V.

The distribution function of the Gamma-variable with positive parameters r and θ denoted here by $F_r(\cdot)$ is given by:

$$F_r(x) = \frac{1}{\theta^r\Gamma(r)}\int_0^x e^{-u/\theta}u^{r-1}\,du, \qquad x \geqq 0 \tag{2.4}$$
$$= 0 \qquad x < 0.$$

Let Y be the random variable whose characteristic function is given by (2.3). The Y has the same distribution as the random variable $X = \sum_{j=1}^{n} \lambda_j X_j$, where X_j, $j = 1, 2, \cdots, n$, are independent identically distributed random variables, each following a Gamma distribution with parameters r and θ as given in (2.4).

Using the well-known method developed by Robbins and Pitman in [5], (see also Box [1], Theorem 2.3), we easily obtain the distribution of the r.v. Y as follows:

$$\begin{aligned}\Pr(Y < y) &= \sum_{k=0}^{\infty} c_k F_{nr+k}(y/\lambda^*) \\ &= \sum_{k=0}^{\infty} \frac{c_k}{\theta\Gamma(nr+k)} \int_0^{y/\lambda^*} \left(\frac{u}{\theta}\right)^{nr+k-1} e^{-u/\theta}\, du,\end{aligned} \tag{2.5}$$

where $\lambda^* = \min_j \lambda_j$ and the coefficients c_k are determined by the identity,

$$\prod_{j=1}^{n} \{(\lambda_j/\lambda^*)[1 - (1 - \lambda^*/\lambda_j)z]\}^{-r} = \sum_{k=0}^{\infty} c_k z^k. \tag{2.6}$$

The upper bound on the error of truncation is also given by Robbins and Pitman [5]:

$$0 \leqq \Pr(Y < y) - \sum_{p_1}^{p_2} c_k F_{nr+k}(y/\lambda^*) \leqq 1 - \sum_{p_1}^{p_2} c_k. \tag{2.7}$$

The characteristic roots λ_j ($j = 1 \cdots n$) of the matrix $V = \{\rho^{|i-j|}\}$ can be calculated from the following formulae:

$$\lambda_j = (1 - 2\rho\cos\theta_j + \rho^2)^{-1}(1 - \rho^2), \qquad j = 1, 2, \cdots, n \tag{2.8}$$

where θ_j are the values of θ which satisfy one or the other of the equations:

$$\begin{aligned}\sin \tfrac{1}{2}(n+1)\theta &= \rho \sin \tfrac{1}{2}(n-1)\theta \quad \text{and} \\ \cos \tfrac{1}{2}(n+1)\theta &= \rho \cos \tfrac{1}{2}(n-1)\theta.\end{aligned} \tag{2.9}$$

We have been unable to find these algebraic results in the literature.[2] The outline of their derivation is as follows. The inverse of V is

$$V^{-1} = ((1 + \rho^2)I - \rho^2 A - \rho B)(1 - \rho^2)^{-1} \tag{2.10}$$

where I is the $n \times n$ identity matrix, A is the $n \times n$ matrix with elements $a_{11} = a_{nn} = 1$ and all other elements 0, and B is the $n \times n$ matrix with $b_{ij} = 1$ for $|i - j| = 1$ and all other elements 0. Hence the characteristic roots of V^{-1} are $(1 - \rho^2)^{-1}\gamma_j$, $j = 1, 2, \cdots, n$, where the γ_j are those values of γ which satisfy the equation

$$|(1 + \rho^2 - \gamma)I - \rho^2 A - \rho B| = 0. \tag{2.11}$$

Let $\Delta_n(\gamma, \rho)$ be the l.h.s. of (2.11). Expanding $\Delta_n(\gamma, \rho)$ by its minors we obtain

[2] *Note added in proof:* Professor J. S. Frame of Michigan State University has sent the authors a derivation of the characteristic roots of the matrix V which is similar to, but simpler than, the one given here.

$$\Delta_n(\gamma, \rho) = (1-\gamma)^2 D_{n-2}(\gamma, \rho) - 2\rho^2(1-\gamma)D_{n-3}(\gamma, \rho) + \rho^4 D_{n-4}(\gamma, \rho), \tag{2.12}$$

where $D_n(\gamma, \rho)$ is the determinant of the $n \times n$ matrix U,

$$U = (1 + \rho^2 - \gamma)I - \rho B. \tag{2.13}$$

Expanding $D_n(\gamma, \rho)$ by its minors we get the following difference equation

$$D_n(\gamma, \rho) - (1 + \rho^2 - \gamma)D_{n-1}(\gamma, \rho) + \rho^2 D_{n-2}(\gamma, \rho) = 0, \tag{2.14}$$

and, is it as well known, (see e.g. [2]), this difference equation has the solution

$$D_n(\gamma, \rho) = \rho^n \sin (n + 1)\theta / \sin \theta, \tag{2.15}$$

where $1 + \rho^2 - \gamma = 2\rho \cos \theta$.

Substituting (2.15) in (2.12) and equating $\Delta_n(\gamma, \rho)$ to zero we obtain the equation

$$(1-\gamma)^2 \rho^{n-2} \frac{\sin (n-1)\theta}{\sin \theta} - 2(1-\gamma)\rho^{n-1} \frac{\sin (n-2)\theta}{\sin \theta} + \rho^n \frac{\sin (n-3)\theta}{\sin \theta} = 0. \tag{2.16}$$

Using the complex representation of the sine function we get after multiplying (2.16) by $\sin \theta$ (excluding the value of θ for which $\sin \theta = 0$),

$$e^{\frac{1}{2}i(n-3)\theta}((1-\gamma)e^{i\theta} - \rho) = \pm e^{-\frac{1}{2}i(n-3)\theta}((1-\gamma)e^{-i\theta} - \rho), \tag{2.17}$$

and after a few algebraic manipulations the equation (2.17) reduces to

$$\sin \tfrac{1}{2}(n + 1)\theta = \rho \sin \tfrac{1}{2}(n - 1)\theta, \tag{2.18}$$

and

$$\cos \tfrac{1}{2}(n + 1)\theta = \rho \cos \tfrac{1}{2}(n - 1)\theta. \tag{2.19}$$

If $\theta_j, j = 1, 2, \cdots, n$, are the values of θ which satisfy one or the other of the equations (2.18) and (2.19), then the characteristic roots of the matrix V^{-1} are given by $\gamma_j = (1 - 2\rho \cos \theta_j + \rho^2)(1 - \rho^2)^{-1}$, and the characteristic roots of V are $\lambda_j = \gamma_j^{-1} = (1 - 2\rho \cos \theta_j + \rho^2)^{-1}(1 - \rho^2), j = 1, 2, \cdots, n$.

Apparently it is not possible to express the solutions of (2.18) and (2.19) as explicit functions of n and ρ. However, for $0 \leqq \rho \leqq 1$, it can be verified that the solutions of (2.18) are located one in each of the intervals $\{[(2k-1)/n]\pi, 2k\pi/(n+1)\}$, $k = 1, 2, \cdots \frac{1}{2}n$ or $\frac{1}{2}(n-1)$ depending on whether n is even or odd; and that the solutions of (2.19) are one in each of the intervals $\{[(2k-2)/n]\pi, [(2k-1)/(n+1)]\pi, k = 1, 2, \cdots, \frac{1}{2}n$ or $\frac{1}{2}(n+1)$ depending on whether n is even or odd. We have used a convergent iterative method for finding θ_j; details of this are omitted but can be obtained from the authors.

3. Comparison between the approximation and exact distributions of sum of identically distributed exponentially correlated Gamma-variables. In order to obtain the distribution of sum of identically distributed Gamma-variables correlated according to the stationary exponential law $(r(X_i, X_j) = \rho^{|i-j|}$, $i, j = 1, \cdots, n)$ we replace in (2.1) the matrix $V = \{v_{ij}\} = \{\rho^{|i-j|}\}$ by the matrix $\Omega = \{\omega_{ij}\} = \{\rho^{\frac{1}{2}|i-j|}\}$ so that the matrix Ω^* which corresponds to the matrix V^* will have its elements $\{\omega_{i\,j}^*\} = \{r\theta^2\rho^{|i-j|}\}$, where $r\theta^2$ is the common variance of the Gamma-variables X_i, $i = 1, \cdots, n$.

Using the UNIVAC 1105 of the Computation Center of the University of North Carolina, we computed the parameters $\{c_k\}$ and λ^* of the distribution given by (2.5) for several different values of n, ρ, θ, r and compared several percentiles of this distribution with the corresponding percentiles of the "approximate" Gamma distribution with the parameters given by (1.2) and (1.3). The results of the computations are presented in Table I. We recall that the first two moments of these two distributions are identical; (this fact has been used for checking the correctness of the numerical calculations). Owing to computational difficulties and complications especially in computing the sequences $\{c_k\}$, we cannot claim higher accuracy than the second significant figure.

Comparing these two types of distributions, we observe that the functional dependence of parameters on n follows the same pattern in both cases. The parameter r is asymptotically linear with n, while θ is independent of n. The discrepancies between the corresponding distributions are relatively small and may be considered insignificant for most of the practical purposes in the cases of small values of ρ $(\rho \leqq .3)$ even for values of n as small as 5. The computations also indicate that for higher values of ρ $(\rho = .5$ and greater) the relatively large deviations at the lower tail decline sharply with the increase of values of n. It is also seen from Table I that the approximation becomes more accurate with the increase of the values of the parameter r. The number of c_k's necessary to obtain the cumulative sum equal 1 (up to 8th significant figure) increases with both n and, even more rapidly, with ρ. For $n = 5$, $\rho = .2$ 100 c_k's are sufficient, for $n = 5$, $\rho = .5$ more than 200 are needed and for $n = 15$, $\rho = .5$ the number of the necessary c_k's is over 300.

The number of iterative steps necessary to obtain the values of the characteristic roots λ_j with accuracy up to the 6th significant digit also depends strongly on n and ρ and varies for the cases presented below between 6 and 16.

Similar but differently inspired numerical investigations were carried by Box ([1], Table I) for integer values of λ_j (with $\lambda^* \geqq 1$). Box found that the 5% and 95% significance points of the approximate distribution lay below the corresponding values of the exact one. However, in all our cases $\lambda^* < 1$; this may account for the different behaviour of the exact distribution at the lower and upper tails (Equation 2.5).

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TABLE I

Comparison between the approximate and exact distributions of sum of identically distributed exponentially correlated Gamma-variables

Eigenvalues of the matrix $\Omega = \{\rho^{\frac{1}{2}\|i-j\|}\}$					n	ρ	r	θ	Percentiles of exact distribution corresponding to the following percentiles of the approximate distribution					
									1	5	25	75	95	99
.413	.518	.751	1.247	2.073	5	.2	1.92	.5	.62	4.14	24.81	75.73	94.80	98.75
					10	.2	2.04	.5	.70	4.39	25.03	75.61	94.87	98.87
.317	.407	.664	1.203	2.439	5	.3	1.94	.5	.47	3.75	24.67	75.93	94.68	98.63
					10	.3	2.00	.5	.56	4.06	25.00	75.81	94.15	98.75
.188	.250	.429	1.000	3.138	5	.5	2.00	.5	.28	3.18	24.60	76.24	94.77	98.72
					10	.5	2.04	.5	.35	3.50	24.81	76.20		98.76
					15	.5	2.00	.5	.42	3.74	25:10	76.0		98.8
.079	.108	.196	.580	4.038	5	.75	2.00	.5	.18	3.01	23.8	76.6		98.6
					5	.75	3.00	.5	.34	3.66	24.6	76.2		98.7
					*15	.332	.151	28.1	.10	1.40	20.1	77.5	95.2	98.70

* This case corresponds to a distribution of precipitation amounts at a certain station in Israel.

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