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DISTRIBUTION OF THE CANONICAL CORRELATIONS AND ASYMPTOTIC EXPANSIONS FOR DISTRIBUTIONS OF CERTAIN INDEPENDENCE TEST STATISTICS[1]

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0. Summary. The sample canonical correlations between two sets of variates are given a representation as the roots of a determinantal equation involving independent matrix variates having simple standardized distributions. This result is applied to obtain asymptotic formulas for the non-null distributions of three criteria for testing the hypothesis of independence of two sets of variates.

1. Introduction. Consider an s-variate random variable $\mathbf{x}$ distributed $N(\mathbf{0}, \Sigma)$. As is well known the distribution problem of the non-zero mean case can be reduced to that of the zero mean case. Let $X(s \times N)$ be the matrix of a sample of size N from this distribution. Further let $\mathbf{x}$ be partitioned into $\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$, where $\mathbf{x}_1$ is a q-vector and $\mathbf{x}_2$ is a p-vector, with $s = q+p$, and let X be partitioned into $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ accordingly. We assume without loss of generality $p \leqq q$. Also let $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ be the corresponding partition of the covariance matrix, i.e. Σ_{11} is $q \times q$ square. The canonical correlations of $\mathbf{x}_1$ and $\mathbf{x}_2$ are defined as the maximum correlations between a linear function of the component variates of $\mathbf{x}_1$ and that of the component variates of $\mathbf{x}_2$. Let the canonical correlations be denoted by $(\rho_1, \cdots, \rho_p)$. Then the ρ_i^2 are given by the roots of the determinantal equation

$$|\lambda\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}| = 0.$$

Analogously, the sample canonical correlations $(r_1^2, \cdots, r_p^2)$ are the roots of the determinantal equation

$$|\lambda S_{22} - S_{21}S_{11}^{-1}S_{12}| = 0 \tag{1}$$

where

$$XX' = S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

with partition corresponding to that of Σ.

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The matrix variates involved in (1) are correlated. We shall now show that the set of suitably transformed sample canonical correlations has the same distribution as that of the roots of a determinantal equation involving independent matrix variates. More precisely, we have

THEOREM. *The set of transformed canonical correlations* $(\tilde{r}_1^{\,2}, \cdots, \tilde{r}_p^{\,2})$ *is distributed like the roots of the determinantal equation*

$$|\lambda W - [(\tilde{P}T + Z_1)(\tilde{P}T + Z_1)' + Z_2 Z_2']| = 0, \tag{2}$$

where

$$\tilde{r}_i^{\,2} = r_i^{\,2}/(1-r_i^{\,2}), \qquad \tilde{\rho}_i^{\,2} = \rho_i^{\,2}/(1-\rho_i^{\,2}),$$

$$\tilde{P} = \operatorname{diag}\{\tilde{\rho}_1, \cdots, \tilde{\rho}_p\},$$

$W \sim W_p(I, N-q)$, *$T(p \times p)$ is such that* $TT' \sim W_p(I, N)$, *and $Z_1(p \times p)$ and $Z_2(p \times (q-p))$ are matrices with independent standard normal variates as their elements. All matrix variates figuring in* (2) *are independently distributed.*

When $p = 1$, this reduces to the particularly simple case of the multiple correlation:

$$\tilde{R}^2 = \{(\tilde{\rho}\chi_N + z)^2 + \chi^2_{q-1}\}/\chi^2_{N-q}, \tag{3}$$

where $\tilde{R}^2 = R^2/(1-R^2)$, R being the sample multiple correlation, $\tilde{\rho}$ is the analogous transform of the population multiple correlation, χ_k and $\chi_k^{\,2}$ are chi- and chi-square variates on k degrees of freedom, and z is a standard normal variate. Again, the variates figuring in (3) are independently distributed. This result for the multiple correlation was also given by Hodgson [5]. The relation (2) can be regarded as a generalization of a result for the simple correlation coefficient apparently due to Elfving [2] and rediscovered by Fraser and Sprott (Fraser [3]), and by Ruben [14].

PROOF. The proof of the theorem follows essentially from the argument of Constantine ([1], pages 1282–1283) in his derivation of the joint distribution of the sample canonical correlations. (The author wishes to thank the referee for pointing out this fact.) An alternative proof is outlined as follows. We may regard X as generated from $E(p \times N)$, a matrix of independent standard normal variates, by the transformation $X = \theta E$, with $\Sigma = \theta\theta'$ the (unique) lower triangular factorization of Σ. Partitioning and factorization leads to

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ \Upsilon & I \end{bmatrix} \begin{bmatrix} H_1' & 0 \\ 0 & H_2' \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

where in each partitioned square matrix the leading submatrix is of order $q \times q$, H_1 and H_2 are orthogonal and are related to $\Upsilon = (\tilde{P} \vdots 0)$ by $\theta_{22}^{-1}\theta_{21} = H_2 \Upsilon H_1'$ (see James [8] for existence and uniqueness of such a factorization with $\tilde{\rho}_1 \geqq \cdots \geqq \tilde{\rho}_p \geqq 0$), $\eta_1 = \theta_{11}H_1$, $\eta_2 = \theta_{22}H_2$, and $[E_1' \vdots E_2']$ is the corresponding partition of E'. Clearly we may equivalently consider the canonical correlations between the sets $Y_1 = \eta_1^{-1}X_1$ and $Y_2 = \eta_2^{-1}X_2$:

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Upsilon & I \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Upsilon & I \end{bmatrix} D$$

with $D_1 = H_1'E_1$, $D_2 = H_2'E_2$. The factorizations $Y = T(Y)O(Y)$, $D = T(D)O(D)$, where T is lower traingular and O semi-orthogonal lead to

$$\begin{bmatrix} T_{11}(Y) & 0 \\ T_{21}(Y) & T_{22}(Y) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Upsilon & I \end{bmatrix} \begin{bmatrix} T_{11}(D) & 0 \\ T_{21}(D) & T_{22}(D) \end{bmatrix}. \tag{4}$$

Note that the elements of D are independent standard normal variates and $T(D)$ is a Bartlett's decomposition of DD'. The theorem with T lower triangular then follows by equating elements of (4) and noting that $(\tilde{r}_1^2, \cdots, \tilde{r}_p^2)$ are given by the roots of the determinantal equation $|\lambda T_{22}(Y)T_{22}'(Y) - T_{21}(Y)T_{21}'(Y)| = 0$. Since the conditional distribution of the $\tilde{r}_i^2$ given T depends on T only through the latent roots of $\tilde{P}^2TT'$ (see e.g. James [9], noncentral mean case), and the density of T can be put in the form $f(TT')\,dTT'$ (see e.g. Fraser [4], pages 154–156), only $TT' = W_p(I, N)$ is involved in the distribution and so T may be any factorization of $W_p(I, N)$.

2. Asymptotic expansions for distributions of certain independence test criteria.

2.1. *Introduction*. Consider the test of the hypothesis H that the variates $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent. This is equivalent to testing H: $\rho_1 = \cdots = \rho_p = 0$ against the alternative K: not all the ρ_i's are zero. We consider three criteria for this test

(a) Wilks' likelihood ratio statistic (Wilks, [18])

$$W = \Pi(1-r_i^2) = \Pi 1/(1+\tilde{r}_i^2)$$

(b) Hotelling's T_0^2 (Pillai [11])

$$U = n\sum \tilde{r}_i^2$$

(c) Pillai's V (Pillai [11])

$$V = n\sum r_i^2 = n\sum \tilde{r}_i^2/(1+\tilde{r}_i^2),$$

with $n = N-q$, N being the number of observations. These are analogues to criteria proposed for testing the general linear hypothesis. For these latter cases the distribution problem reduces to that of the roots of the determinantal equation (see e.g. Roy [13])

$$|cW - [(K+Z_1)(K+Z_1)' + Z_2Z_2']| = 0, \tag{5}$$

where W, Z_1 and Z_2 have the same meaning as in Section 1, and

$$K = \operatorname{diag}\{\lambda_1^{\frac{1}{2}}, \cdots, \lambda_p^{\frac{1}{2}}\},$$

the λ_i's being the characteristic roots of the noncentrality matrix. For our purpose we may assume that the number of the hypothesis degrees of freedom q is equal to or greater than p. Further let $\Lambda = \operatorname{diag}\{\lambda_1, \cdots, \lambda_p\}$. Then by comparing (2) and (5) we have the following

LEMMA. *Let $s = s(c_1, \cdots, c_p)$ be a scalar function of the c_i's whose density function is $f(s, \Lambda)$, and let $s^* = s(\tilde{r}_1^2, \cdots, \tilde{r}_p^2)$. Then the density function of s^* is given by*

$$g(s^*, \tilde{P}) = \int_{S \geqq 0} f(s^*, \tilde{P}S\tilde{P}) h(S)\, dS, \tag{6}$$

where

$$h(S) = [2^{p(n+q)/2}\Gamma_p((n+q)/2)]^{-1}|S|^{(n+q-p-1)/2} \operatorname{etr}(-\tfrac{1}{2}S)$$

with

$$\Gamma_p(a) = \pi^{p(p-1)/4} \prod_{i=1}^p \Gamma(a - \tfrac{1}{2}(i-1))$$

is the density function of $W_p(I, n+q)$, $n+q$ being the number of observations from $N(0, \Sigma)$.

The proof is immediate. Conditional on $\tilde{P}T$ being fixed, the density function of s^* is $f(s^*, \tilde{P}S\tilde{P})$, with $S = TT'$, as is obvious from (2) and (5). Multiplying this by the density element of S and integrating we obtain (6).

Note that the method in the Lemma can equally well be applied when f and g are respectively the characteristic functions of s and s^*. In fact, in our applications we shall work on the characteristic functions.

As the sample size tends to infinity, the alternatives of interest, as far as power consideration is concerned, are the ones near the hypothesis. We shall assume that the $n\tilde{\rho}_i^2$ are finite, and put $\alpha = n\tilde{P}^2$, $\alpha_i = n\tilde{\rho}_i^2$, so that the α_i's are finite.

The asymptotic expansions for the non-null distributions of W, U, V in the *linear hypothesis case* are available in the literature (Sugiura and Fujikoshi [17], Itô [7], Lee [10]). Itô's expansion for U was given to the order n^{-1} only while the other expansions were worked out to the order n^{-2}. We shall later supply the term of order n^{-2} for the expansion for U. It is then straightforward to apply (6) to the characteristic functions of these statistics in the asymptotic form to derive the corresponding characteristic functions of the statistics for testing independence. Inversion will then give the distribution functions.

When (6) is applied to the asymptotic expansions of the characteristic functions of the three criteria for testing the linear hypothesis, we find that we need the following integration results. It is straightforward to obtain

$$\begin{aligned}
&\int \operatorname{etr}\{it\alpha S/n(1-2it)\}h(S)\, dS = |I - 2it\alpha/n(1-2it)|^{-(n+q)/2} \\
&= \operatorname{etr}\{it\alpha/(1-2it)\}\{1 + (4n)^{-1}[\operatorname{tr}(2it\alpha/(1-2it))^2 + 2q \operatorname{tr}(2it\alpha/(1-2it))] \\
&\quad + (96n^2)^{-1}[24q \operatorname{tr}(2it\alpha/(1-2it))^2 + 12q^2 \operatorname{tr}^2(2it\alpha/(1-2it)) \\
&\quad + 16 \operatorname{tr}(2it\alpha/(1-2it))^3 + 12q \operatorname{tr}(2it\alpha/(1-2it)) \operatorname{tr}(2it\alpha/(1-2it))^2 \\
&\quad + 3 \operatorname{tr}^2(2it\alpha/(1-2it))^2] + O(n^{-3})\},
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
&\int \operatorname{tr}(\tilde{P}^2 S) \operatorname{etr}\{it\alpha S/n(1-2it)\}h(S)\,dS \\
(8)\qquad &= |I-2it\alpha/n(1-2it)|^{-(n+q)/2}(1+q/n)\operatorname{tr}\alpha\{I-2it\alpha/n(1-2it)\}^{-1} \\
&= \operatorname{etr}\{it\alpha/(1-2it)\}\{\operatorname{tr}\alpha+(4n)^{-1}[4q\operatorname{tr}\alpha+4\cdot 2it\operatorname{tr}\alpha^2/(1-2it) \\
&\quad +2q\cdot 2it\operatorname{tr}^2\alpha/(1-2it)+(2it)^2\operatorname{tr}\alpha\operatorname{tr}\alpha^2/(1-2it)^2]+O(n^{-2})\}.
\end{aligned}$$

Now consider a matrix $A = (a_{ij})$ distributed $W_p(\Omega, n)$, with $\Omega = \operatorname{diag}\{\omega_1, \cdots, \omega_p\}$. It is easy to verify that $E\{a_{ii}^2\} = n(n+2)\omega_i^2$ and $E\{a_{ij}^2\} = n\omega_i\omega_j$. From this it is then easy to obtain, by expanding the first trace in the integrand, the result

$$\begin{aligned}
&\int \operatorname{tr}(\tilde{P}^2 S\tilde{P}^2 S) \operatorname{etr}\{it\alpha S/n(1-2it)\}h(S)\,dS \\
&= |I-2it\alpha/n(1-2it)|^{-(n+q)/2}\{\operatorname{tr}\alpha^2+(1/n)[(2q+1)\operatorname{tr}\alpha^2+\operatorname{tr}^2\alpha \\
(9)\qquad &\quad +2\cdot 2it\operatorname{tr}\alpha^3/(1-2it)]+O(n^{-2})\} \\
&= \operatorname{etr}\{it\alpha/(1-2it)\}\{\operatorname{tr}\alpha^2+(4n)^{-1}[4(2q+1)\operatorname{tr}\alpha^2+4\operatorname{tr}^2\alpha+8\cdot 2it\operatorname{tr}\alpha^3/(1-2it) \\
&\quad +2q\cdot 2it\operatorname{tr}\alpha\operatorname{tr}\alpha^2/(1-2it)+(2it)^2\operatorname{tr}^2\alpha^2/(1-2it)^2]+O(n^{-2})\}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\int \operatorname{tr}(\tilde{P}^2 S)^3 \operatorname{etr}\{it\alpha S/n(1-2it)\}h(S)\,dS = \operatorname{etr}\{it\alpha/(1-2it)\}\operatorname{tr}\alpha^3+O(n^{-1}), \\
(10)\qquad &\int \operatorname{tr}(\tilde{P}^2 S)\operatorname{tr}(\tilde{P}^2 S)^2 \operatorname{etr}\{it\alpha S/n(1-2it)\}h(S)\,dS = \operatorname{etr}\{it\alpha/(1-2it)\}\operatorname{tr}\alpha\operatorname{tr}\alpha^2 \\
&\quad +O(n^{-1}), \\
&\int (\operatorname{tr}(\tilde{P}^2 S)^2)^2 \operatorname{etr}\{it\alpha S/n(1-2it)\}h(S)\,dS = \operatorname{etr}\{it\alpha/(1-2it)\}(\operatorname{tr}\alpha^2)^2 \\
&\quad +O(n^{-1}).
\end{aligned}$$

2.2. *Expansion for the distribution of W.* Applying (6) to (1.32) in Sugiura and Fujikoshi [17] with the correspondence of parameters: $p \to p$, $b \to q$, $N-s-1 \to N-q = n$, $\Omega \to \frac{1}{2}\tilde{P}S\tilde{P}$ and using (7) to (10), we obtain the characteristic function of $L = m \log W$, with $m = n+(q-p-1)/2$. (Sugiura [16] gave a similar expansion for the distribution of W. The author wishes to thank the referee for this reference.) The distribution function of L is then obtained by inversion and the result is

$$\begin{aligned}
\Pr\{L \leqq x\} = G(x \mid f, \lambda)+(1/m)\textstyle\sum_{j=0}^{3} a_{1j}G(x \mid f+2j, \lambda) \\
+(1/m^2)\textstyle\sum_{j=0}^{6} a_{2j}G(x \mid f+2j, \lambda)+O(m^{-3})
\end{aligned}$$

where $G(x \mid v, \lambda)$ denotes the distribution function of a noncentral chi-square variate on v degrees of freedom with noncentrality parameter $\lambda = \frac{1}{2}\operatorname{tr}\alpha$, $f = qp$, and

$$a_{10} = -qS_1+S_2, \qquad a_{11} = (k+q)S_1-2S_2, \qquad a_{12} = -kS_1+2S_2,$$
$$a_{13} = -S_2;$$
$$a_{20} = -\delta-qlS_1+(q+l)S_2+\tfrac{1}{2}q^2S_1{}^2-\tfrac{4}{3}S_3-qS_1S_2+\tfrac{1}{2}S_2{}^2,$$

$$a_{21} = q^2S_1-4qS_2-q(q+k)S_1{}^2+4S_3+(3q+k)S_1S_2-2S_2{}^2,$$

$$a_{22} = \delta-k(q+k)S_1+(2p+6q+3)S_2+(\tfrac{1}{2}l^2+3qk+1)S_1{}^2-8S_3 -(4q+3k)S_1S_2+4S_2{}^2,$$

$$a_{23} = k^2S_1-(3p+5q+5)S_2-(k^2+qk+2)S_1{}^2+\tfrac{32}{3}S_3 +(3q+4k)S_1S_2-5S_2{}^2,$$

$$a_{24} = (3k+1)S_2+(\tfrac{1}{2}k^2+1)S_1{}^2-8S_3-(q+3k)S_1S_2+4S_2{}^2,$$

$$a_{25} = \tfrac{8}{3}S_3+kS_1S_2-2S_2{}^2, \qquad a_{26} = \tfrac{1}{2}S_2{}^2,$$

with $S_j = \operatorname{tr}(\tfrac{1}{2}\alpha)^j$, $k = (q+p+1)/2$, $l = (q-p-1)/2$, $\delta = qp(q^2+p^2-5)/48$.

2.3. *Expansion for the distribution of U: linear hypothesis case.* As has been remarked, the asymptotic expansion of the distribution of U was given by Itô to the order n^{-1} only. We now use Itô's method to add the term of order n^{-2} to his expansion. (Siotani [15] derived this term by a different method. The author wishes to thank the referee for this reference.) Now the characteristic function of U was given by Hsu [6]

$$(11) \quad \phi(t) = (\tfrac{1}{2}n)^{-\frac{1}{2}qp}[\Gamma_p(\tfrac{1}{2}(n+q))/\Gamma_p(\tfrac{1}{2}n)]\int(2\pi)^{-\frac{1}{2}qp}\left|I+(1/n)C\right|^{-\frac{1}{2}(n+q)} \cdot \operatorname{etr}\{itC+(2it)^{\frac{1}{2}}KX'\}\,dX,$$

where n and q are respectively the error and hypothesis degrees of freedom, $C = XX'$, $K(p\times q)$ has $(\lambda_1{}^{\frac{1}{2}}, \cdots, \lambda_p{}^{\frac{1}{2}})$ along the diagonal and zeros elsewhere, and the domain of integration is the $p\times q$ real matrices (see also Itô [7]). For our purpose we take $q \geqq p$. The constant on the right of (11) can be expanded using Stirling's formula:

$$(12) \quad \begin{aligned}&(\tfrac{1}{2}n)^{-\frac{1}{2}qp}\Gamma_p(\tfrac{1}{2}(n+q))/\Gamma_p(\tfrac{1}{2}n)\\ &= 1+(4n)^{-1}qp(q-p-1)\\ &\quad+(96n^2)^{-1}qp[3qp^3-2(3q^2-3q+4)p^2+3(q^3-2q^2+5q-4)p-8q^2+12q+4]\\ &\quad+O(n^{-3}).\end{aligned}$$

The determinant part in the integrand can be expanded asymptotically:

$$(13) \quad \begin{aligned}\left|I+(1/n)C\right|^{-\frac{1}{2}(n+q)} = \operatorname{etr}(-\tfrac{1}{2}C)\{1&+(4n)^{-1}(\operatorname{tr}C^2-2q\operatorname{tr}C)+(96n^2)^{-1}\\ &\cdot[12q^2\operatorname{tr}^2C+24q\operatorname{tr}C^2-16\operatorname{tr}C^3-12q\operatorname{tr}C\operatorname{tr}C^2\\ &+3\operatorname{tr}^2C^2]+O(n^{-3})\}.\end{aligned}$$

Substitute (13) in (11) and integrate. With the help of the following integration results

$$I(\operatorname{tr}C) = qp+\tau_1,$$

$$I(\operatorname{tr}C^2) = qp(q+p+1)+2(q+p+1)\tau_1+\tau_2,$$

$$I(\operatorname{tr}^2C) = qp(qp+2)+2(qp+2)\tau_1+\tau_1{}^2,$$

$$I(\mathrm{tr}\, C^3) = qp(q^2+p^2+3pq+3q+3p+4)+3(q^2+p^2+3qp+3q+3p+4)\tau_1 + 3(q+p+2)\tau_2+3\tau_1{}^2+\tau_3,$$

$$I(\mathrm{tr}\, C\, \mathrm{tr}\, C^2) = qp(qp+4)(q+p+1)+3(qp+4)(q+p+1)\tau_1+(qp+8)\tau_2 + 2(q+p+1)\tau_1{}^2+\tau_1\tau_2,$$

$$I(\mathrm{tr}^2\, C^2) = qp[qp^3+2(q^2+q+4)p^2+(q^3+2q^2+21q+20)p+4(2q^2+5q+5)] + 4[qp^3+2(q^2+q+4)p^2+(q^3+2q^2+21q+20)p+4(2q^2+5q+5)]\tau_1 + 2[qp^2+(q^2+q+20)p+20q+32]\tau_2+4[p^2+2(q+1)p + (q^2+2q+7)]\tau_1{}^2+4(q+p+1)\tau_1\tau_2+16\tau_3+\tau_2{}^2,$$

where

$$I(g(C)) = \int g(C)(2\pi)^{-\frac{1}{2}qp}\, \mathrm{etr}\, \{-\tfrac{1}{2}(X-(2it/(1-2it))^{\frac{1}{2}}K)(X-(2it/(1-2it))^{\frac{1}{2}}K)'\}\, dX,$$
$$\tau_v = (2it/(1-2it))^v \textstyle\sum \lambda_j{}^v,$$

and also using (12), we have after inversion and much simplification

$$(14)\quad \Pr\{U \leqq x\} = G(x \mid f, \lambda)+(4n)^{-1} \textstyle\sum_{j=0}^{4} b_{1j}G(x \mid f+2j, \lambda) + (96n^2)^{-1} \sum_{j=0}^{8} b_{2j}G(x \mid f+2j, \lambda)+O(n^{-3})$$

with $\lambda = \sum \lambda_j$, and, writing σ_v for $\sum \lambda_j{}^v = \mathrm{tr}\, \Lambda^v$,

$$b_{10} = qp(q-p-1), \qquad b_{11} = 2q(\sigma_1-qp),$$

$$b_{12} = qp(q+p+1)-2(2q+p+1)\sigma_1+\sigma_2,$$

$$b_{13} = 2(q+p+1)\sigma_1-2\sigma_2, \qquad b_{14} = \sigma_2;$$

$$b_{20} = qp[3qp^3-2(3q^2-3q+4)p^2+3(q^3-2q^2+5q-4)p-8q^2+12q+4],$$

$$b_{21} = 12q^2p(q-p-1)(\sigma_1-qp),$$

$$b_{22} = -6q^2p[p^3+2p^2-3(q^2+1)p-4(2q+1)]+12q[p^3+(q+2)p^2 - (4q^2-q+3)p-4(2q+1)]\sigma_1+6q[2q\sigma_1{}^2-(p^2-qp+p-4)\sigma_2],$$

$$b_{23} = -4qp[(3q^2+4)p^2+3(q^3+q^2+8q+4)p+8(2q^2+3q+2)] + 12[-qp^3+(3q^2-2q+4)p^2+3(2q^3+q^2+9q+4)p+4(6q^2+7q+4)]\sigma_1 - 24(qp+2q^2+q+2)\sigma_1{}^2+12[qp^2-(2q^2-q+4)p-16q-8]\sigma_2 + 16\sigma_3+12q\sigma_1\sigma_2,$$

$$b_{24} = 3qp[qp^3+2(q^2+q+4)p^2+(q^3+2q^2+21q+20)p+4(2q^2+5q+5)] - 12[qp^3+(5q^2+2q+12)p^2+(4q^3+5q^2+45q+32)p + 4(6q^2+11q+9)]\sigma_1+12[p^2+2(3q+1)p+6q^2+6q+15]\sigma_1{}^2 + 12[3(q^2+6)p+36q+32]\sigma_2-12(4q+p+1)\sigma_1\sigma_2-96\sigma_3+3\sigma_2{}^2,$$

$$\begin{aligned} b_{25} = {} & 12[qp^3+2(q^2+q+4)p^2+(q^3+2q^2+21q+20)p+4(2q^2+5q+5)]\sigma_1 \\ & -24[p^2+(3q+2)p+(2q^2+3q+9)]\sigma_1{}^2 \\ & -12[qp^2+(2q^2+q+24)p+32q+40]\sigma_2 \\ & +12[3(2q+p+1)\sigma_1\sigma_2+16\sigma_3]-12\sigma_2{}^2, \end{aligned}$$

$$\begin{aligned} b_{26} = {} & 12[p^2+2(q+1)p+(q^2+2q+7)]\sigma_1{}^2+6[qp^2+(q^2+q+20)p+20q+32]\sigma_2 \\ & -12(4q+3p+3)\sigma_1\sigma_2-160\sigma_3+18\sigma_2{}^2, \end{aligned}$$

$$b_{27} = 12(q+p+1)\sigma_1\sigma_2+48\sigma_3-12\sigma_2{}^2,$$

$$b_{28} = 3\sigma_2{}^2.$$

2.4. *Expansion for the distribution of U: independence case.* Straightforward application of (6) to (14) with the help of (7) to (10) yields the distribution of U for the independence case. Thus we have

$$\begin{aligned} \Pr\{U \leqq x\} = {} & G(x \mid f, \lambda')+(4n)^{-1}\textstyle\sum_{j=0}^{4} b'_{1j}G(x \mid f+2j, \lambda') \\ & +(96n^2)^{-1}\textstyle\sum_{j=0}^{8} b'_{2j}G(x \mid f+2j, \lambda')+O(n^{-3}), \end{aligned}$$

where $\lambda' = \sum \alpha_j = \operatorname{tr}\alpha$, and

$$b'_{10} = qp(q-p-1)-2q\sigma_1+\sigma_2, \qquad b'_{11} = -2q^2p+4q\sigma_1-2\sigma_2,$$

$$b'_{12} = qp(q+p+1)-2(2q+p+1)\sigma_1+2\sigma_2, \qquad b'_{13} = 2(q+p+1)\sigma_1-2\sigma_2,$$

$$b'_{14} = \sigma_2;$$

$$\begin{aligned} b'_{20} = {} & qp[3qp^3-2(3q^2-3q+4)p^2+3(q^3-2q^2+5q-4)p-8q^2+12q+4] \\ & -12q^2p(q-p-1)\sigma_1-6q(p^2-qp+p-4)\sigma_2+12q^2\sigma_1{}^2-16\sigma_3 \\ & -12q\sigma_1\sigma_2+3\sigma_2{}^2, \end{aligned}$$

$$\begin{aligned} b'_{21} = {} & -12q^3p^2(q-p-1)-24q^2(p^2-2qp+p-2)\sigma_1+12q(p^2-2qp+p-8)\sigma_2 \\ & -48q^2\sigma_1{}^2+48\sigma_3+48q\sigma_1\sigma_2-12\sigma_2{}^2, \end{aligned}$$

$$\begin{aligned} b'_{22} = {} & -6q^2p^4-12q^2p^3+18q^2(q^2+1)p^2+24q^2(2q+1)p \\ & +12q[p^3+2p^2-7(q^2+1)p-16q-8]\sigma_1-6[qp^2-(7q^2-q+8)p \\ & -40q-12]\sigma_2+24(qp+4q^2+q+1)\sigma_1{}^2-12(p+8q+1)\sigma_1\sigma_2-96\sigma_3+24\sigma_2{}^2, \end{aligned}$$

$$\begin{aligned} b'_{23} = {} & -(12q^3+16q)p^3-(12q^4+12q^3+96q^2+48q)p^2-(64q^3+96q^2+64q)p \\ & +12[-qp^3+(4q^2-2q+4)p^2+(7q^3+4q^2+31q+12)p+4(7q^2+8q+4)]\sigma_1 \\ & -48[(q^2+3)p+9q+5]\sigma_2-24(3qp+5q^2+3q+4)\sigma_1{}^2+176\sigma_3 \\ & +12(3p+11q+3)\sigma_1\sigma_2-36\sigma_2{}^2, \end{aligned}$$

$$
\begin{aligned}
b'_{24} = {} & 3q^2p^4+(6q^3+6q^2+24q)p^3+(3q^4+6q^3+63q^2+60q)p^2 \\
& +(24q^3+60q^2+60q)p-12[qp^3+(5q^2+2q+12)p^2 \\
& +(4q^3+5q^2+45q+32)p+4(6q^2+11q+9)]\sigma_1 \\
& +6[qp^2+(7q^2+q+44)p+88q+76]\sigma_2 \\
& +12[p^2+2(4q+1)p+8q^2+8q+17]\sigma_1{}^2 \\
& -12(4p+11q+4)\sigma_1\sigma_2-240\sigma_3+42\sigma_2{}^2,
\end{aligned}
$$

$$
\begin{aligned}
b'_{25} = {} & [12qp^3+24(q^2+q+4)p^2+12(q^3+2q^2+21q+20)p+48(2q^2+5q+5)]\sigma_1 \\
& -12[qp^2+(2q^2+q+24)p+32q+40]\sigma_2-24[p^2+(3q+2)p \\
& +2q^2+3q+9]\sigma_1{}^2+240\sigma_3+48(p+2q+1)\sigma_1\sigma_2-36\sigma_2{}^2,
\end{aligned}
$$

$$
\begin{aligned}
b'_{26} = {} & [6qp^2+6(q^2+q+20)p+120q+192]\sigma_2+[12p^2+24(q+1)p \\
& +12(q^2+2q+7)]\sigma_1{}^2-12(3p+4q+3)\sigma_1\sigma_2-160\sigma_3+24\sigma_2{}^2,
\end{aligned}
$$

$$b'_{27} = 48\sigma_3+12(q+p+1)\sigma_1\sigma_2-12\sigma_2{}^2,$$

$$b'_{28} = 3\sigma_2{}^2,$$

with $\sigma^\nu = \sum \alpha_j{}^\nu = \operatorname{tr} \alpha^\nu$.

2.5. *Expansion for the distribution of V.* An asymptotic expansion for the distribution of V in the linear hypothesis case has been derived by the author [10]. The result is stated as follows, with the same notation as in Subsection 2.3.

$$
\begin{aligned}
(15) \qquad \Pr\{V \leqq x\} = {} & G(x \mid f, \lambda)+(4n)^{-1}\textstyle\sum_{j=0}^{4} c_{1j}G(x \mid f+2j, \lambda) \\
& +(96n^2)^{-1}\textstyle\sum_{j=0}^{8} c_{2j}G(x \mid f+2j, \lambda)+O(n^{-3}),
\end{aligned}
$$

where

$$c_{10} = qp(q-p-1), \qquad c_{11} = 2qp(p+1)+2q\sigma_1,$$

$$c_{12} = -qp(p+q+1)+2(p+1)\sigma_1+\sigma_2, \qquad c_{13} = -2(p+q+1)\sigma_1,$$

$$c_{14} = -\sigma_2;$$

$$c_{20} = qp[3qp^3-2(3q^2-3q+4)p^2+3(q^3-2q^2+5q-4)p-8q^2+12q+4],$$

$$c_{21} = 12q^2p(q-p-1)[p(p+1)+\sigma_1],$$

$$
\begin{aligned}
c_{22} = {} & 6qp[3qp^3+(6q+8)p^2+(-q^3+7q+16)p+4q+8] \\
& +12q[-p^3+(3q-2)p^2+3(q+1)p+4]\sigma_1 \\
& +6q[-5p^2+(q-1)p+4]\sigma_2+12q^2\sigma_1{}^2,
\end{aligned}
$$

$$
\begin{aligned}
c_{23} = {} & -4qp[3qp^3+(3q^2+6q+16)p^2+(3q^2+27q+36)p+4(q^2+6q+7)] \\
& +12[3qp^3+(-q^2+6q+8)p^2+(-2q^3-q^2+3q+16)p-4q^2+8]\sigma_1 \\
& +12[4qp^2+(q+4)p+4]\sigma_2+24q(p+1)\sigma_1{}^2+16\sigma_3+12q\sigma_1\sigma_2,
\end{aligned}
$$

$$
\begin{aligned}
c_{24} = {} & 3qp[qp^3+2(q^2+q+4)p^2+(q^3+2q^2+21q+20)p+4(2q^2+5q+5)] \\
& -12[3qp^3+(3q^2+6q+16)p^2+(3q^2+27q+36)p+4(q^2+6q+7)]\sigma_1 \\
& -12[qp^2+(q^2+2)p+8q+4]\sigma_2+12[p^2-2(q-1)p-2q^2-2q-1]\sigma_1{}^2 \\
& +12(p+1)\sigma_1\sigma_2+3\sigma_2{}^2, \\
c_{25} = {} & 12[qp^3+2(q^2+q+4)p^2+(q^3+2q^2+21q+20)p+4(2q^2+5q+5)]\sigma_1 \\
& -12[qp^2+(q+12)p+4(q+4)]\sigma_2-24[p^2+(q+2)p+q+3]\sigma_1{}^2 \\
& -48\sigma_3-12(p+2q+1)\sigma_1\sigma_2, \\
c_{26} = {} & 6[qp^2+(q^2+q+20)p+20q+32]\sigma_2+12[p^2+2(q+1)p+(q^2+2q+7)]\sigma_1{}^2 \\
& -12(p+1)\sigma_1\sigma_2-16\sigma_3-6\sigma_2{}^2, \\
c_{27} = {} & 12(p+q+1)\sigma_1\sigma_2+48\sigma_3, \\
c_{28} = {} & 3\sigma_2{}^2.
\end{aligned}
$$

Direct application of (6) to (15), with the help of (7) to (10), yields the distribution of V for the independence case. This is, with the same notation as in Subsection 2.4.

$$
\Pr\{V \leqq x\} = G(x \mid f, \lambda') + (4n)^{-1} \textstyle\sum_{j=0}^{4} c'_{1j} G(x \mid f+2j, \lambda') + (96n^2)^{-1} \sum_{j=0}^{8} c'_{2j} G(x \mid f+2j, \lambda') + O(n^{-3}),
$$

where

$$
\begin{aligned}
& c'_{10} = qp(q-p-1)-2q\sigma_1+\sigma_2, \qquad c'_{11} = 2qp(p+1)+4q\sigma_1-2\sigma_2, \\
& c'_{12} = -qp(q+p+1)+2(p+1)\sigma_1+2\sigma_2, \qquad c'_{13} = -2(q+p+1)\sigma_1, \\
& c'_{14} = -\sigma_2;
\end{aligned}
$$

$$
\begin{aligned}
c'_{20} = {} & qp[3qp^3-2(3q^2-3q+4)p^2+3(q^3-2q^2+5q-4)p-8q^2+12q+4] \\
& -12q^2p(q-p-1)\sigma_1-6q(p^2-qp+p-4)\sigma_2+12q^2\sigma_1{}^2-16\sigma_3 \\
& -12q\sigma_1\sigma_2+3\sigma_2{}^2, \\
c'_{21} = {} & 12q^2p^2(p+1)(q-p-1)-24q^2(2p^2-qp+2p-2)\sigma_1 \\
& +12q(2p^2-qp+2p-8)\sigma_2-48q^2\sigma_1{}^2+48q\sigma_1\sigma_2+48\sigma_3-12\sigma_2{}^2, \\
c'_{22} = {} & 6qp[3qp^3+(6q+8)p^2+(-q^3+7q+16)p+4q+8] \\
& +12q[-p^3+(6q-2)p^2+(q^2+6q+7)p+8]\sigma_1 \\
& -6[11qp^2-(q^2-7q-8)p-24q+4]\sigma_2-24(qp-2q^2+q-1)\sigma_1{}^2 \\
& -96\sigma_3+12(p-6q+1)\sigma_1\sigma_2+24\sigma_2{}^2, \\
c'_{23} = {} & -4qp[3qp^3+(3q^2+6q+16)p^2+(3q^2+27q+36)p+4q^2+24q+28] \\
& +12[3qp^3+(-2q^2+6q+8)p^2+(-3q^3-2q^2-q+16)p-8q^2-4q+8]\sigma_1 \\
& +12[6qp^2+(q^2+3q+12)p+4q+12]\sigma_2+24q(3p+q+3)\sigma_1{}^2+80\sigma_3 \\
& +36(q-p-1)\sigma_1\sigma_2-24\sigma_2{}^2,
\end{aligned}
$$

$$
\begin{aligned}
c'_{24} = {} & 3qp[qp^3+2(q^2+q+4)p^2+(q^3+2q^2+21q+20)p+4(2q^2+5q+5)] \\
& -12[3qp^3+(3q^2+6q+16)p^2+(3q^2+27q+36)p+4q^2+24q+28]\sigma_1 \\
& -6[3qp^2+(3q^2+q+12)p+32q+20]\sigma_2 \\
& +12[p^2-2(2q-1)p-4q^2-4q-3]\sigma_1{}^2+48\sigma_3+12(4p+3q+4)\sigma_1\sigma_2+6\sigma_2{}^2,
\end{aligned}
$$

$$
\begin{aligned}
c'_{25} = {} & 12[qp^3+2(q^2+q+4)p^2+(q^3+2q^2+21q+20)p+4(2q^2+5q+5)]\sigma_1 \\
& -12[qp^2+(q+12)p+4q+16]\sigma_2-24[p^2+(q+2)p+q+3]\sigma_1{}^2-96\sigma_3 \\
& -24(p+2q+1)\sigma_1\sigma_2+12\sigma_2{}^2,
\end{aligned}
$$

$$
\begin{aligned}
c'_{26} = {} & 12[p^2+2(q+1)p+q^2+2q+7]\sigma_1{}^2+6[qp^2+(q^2+q+20)p+20q+32]\sigma_2 \\
& -12(p+1)\sigma_1\sigma_2-16\sigma_3-12\sigma_2{}^2,
\end{aligned}
$$

$$c'_{27} = 12(p+q+1)\sigma_1\sigma_2+48\sigma_3,$$

$$c'_{28} = 3\sigma_2{}^2.$$

2.6. *Numerical comparison.* Pillai and Jayachandran [12] have computed powers of the independence test criteria considered here for the special case of $p = 2$. Their results are based on exact expressions in infinite series. The table below will give an idea of the accuracies of the asymptotic approximations presented here.

TABLE 1

Accuracies of asymptotic approximations to powers of W, U and V for $p = 2$

				W		U		V	
q	n	$\rho_1{}^2$	$\rho_2{}^2$	exact	approx.	exact	approx.	exact	approx.
3	63	.1	0	.447	.449	.452	.452	.445	.447
3	63	.05	.05	.434	.436	.432	.432	.440	.440
5	63	.15	0	.574	.572	.580	.578	.562	.569
7	63	.05	.05	.305	.306	.303	.304	.308	.308
13	63	.1	0	.239	.240	.254	.249	.228	.209
3	83	.05	.001	.292	.292	.293	.293	.292	.291
7	83	.05	.001	.201	.201	.203	.204	.200	.199

The significance level is 5 per cent. The significant points are taken from [12] and so are the figures in the columns under "exact". The asymptotic approximations are thus seen to work quite well over the range of the parameters tabulated, with the exception of the approximation for V when $n = 63$ and $q = 13$. Apparently the approximation for V becomes more inaccurate with increasing q than those for W and U.

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