DISTRIBUTION OF THE RATIO OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE TO THE VARIANCE

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1. Introduction. Let $x_1, \cdots, x_n$ be variables representing n successive observations in a population which obeys a distribution law

$$ce^{-(x-\xi)^2/2\sigma^2}\,dx, \qquad \left(c = \frac{1}{\sigma\sqrt{2\pi}}\right),$$

i.e. which is normal, with the mean ξ and the standard deviation σ. For the sample we define as usual the mean,

$$\bar{x} = \frac{1}{n}\sum_{\mu=1}^{n} x_\mu,$$

the variance,

$$s^2 = \frac{1}{n}\sum_{\mu=1}^{n}(x_\mu - \bar{x})^2,$$

and also the mean square successive difference

$$\delta^2 = \frac{1}{n-1}\sum_{\mu=1}^{n-1}(x_{\mu+1} - x_\mu)^2.$$

The reasons for the study of the distribution of the mean square successive difference δ^2, in itself as well as in its relationship to the variance s^2, have been set forth in a previous publication[2], to which the reader is referred. The distribution of δ^2, and in particular its moments, were also studied there. The present paper is devoted to the investigation of the ratio

$$\eta = \frac{\delta^2}{s^2}.$$

A comparison of the observed value of η with that distribution is particularly suited as a basis of the judgment whether the observations $x_1, \cdots, x_n$ are independent or whether a trend exists. (Cf. sections 1 and 2, loc. cit.[2])

The moments of η have already been determined by J. D. Williams by a

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[1] Also Scientific Advisory Committee of the Ballistic Research Laboratory, Aberdeen Proving Ground.

[2] John von Neumann, R. H. Kent, H. R. Bellinson, B. I. Hart, "The mean square successive difference," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 153–162.

different method.[3] Williams' results have been checked by W. J. Dixon at the suggestion of S. S. Wilks, whose stimulating interest has been largely responsible for the undertaking of the series of papers on δ^2 and $\frac{\delta^2}{s^2}$. The present rather exhaustive discussion, however, brings out several other essential characteristics of this statistic, and provides the key to some very effective computational methods. It is further hoped that the reader will find that the mathematical methods used and the generalizations indicated have an interest of their own.

From the latter point of view the final results of sections 5 and 7, concerning the distribution of values of quadratic and of Hermitian forms, may deserve special attention.

2. Diagonalization of the quadratic forms and replacement by a spherical mean. Since δ^2 and s^2 are unchanged when we replace each x_μ by $x_\mu - \xi$, we may assume $\xi = 0$. Then the distribution law of x is

$$ce^{-x^2/2\sigma^2}\,dx, \qquad \text{and that of } x_1, \cdots, x_n \text{ is } \prod_{\mu=1}^{n} ce^{-x_\mu^2/2\sigma^2}\,dx_\mu,$$

i.e.

$$c^n e^{-\sum_{\mu=1}^{n} x_\mu^2/2\sigma^2}\,dx_1 \cdots dx_n.$$

Any linear orthogonal transformation of the $x_1, \cdots, x_n$ leaves $\sum_{\mu=1}^{n} x_\mu^2$ and $dx_1 \cdots dx_n$ unchanged, hence the above distribution law will likewise be left unchanged. Thus, we may subject the two quadratic forms δ^2, s^2 to any simultaneous linear, orthogonal transformation.

Consider one carrying $x_1, \cdots, x_n$ into, say $x_1', \cdots, x_n'$, which brings the quadratic form $(n - 1)\delta^2$ into the diagonal form, say $\sum_{\mu=1}^{n} A_\mu x_\mu'^2$. Such a transformation does not affect the characteristic values of the quadratic forms[4], and these characteristic values are obviously $A_1, \cdots, A_n$ in the case of $\sum_{\mu=1}^{n} A_\mu x_\mu'^2$. Consequently $A_1, \cdots, A_n$ are the characteristic values of the original quadratic form $(n - 1)\delta^2$. We shall determine them as such in the next section.

Clearly we always have $(n - 1)\delta^2 \geqq 0$, hence all $A_\mu \geqq 0$. Some A_μ may

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[3] J. D. Williams, "Moments of the ratio of the mean square successive difference to the mean square difference in samples from a normal universe," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 239–241. Cf. also L. C. Young, "On randomness in orded sequences," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 293–300.

[4] For the properties of matrices and quadratic forms cf. e.g.: J. H. M. Wedderburn, Lectures on Matrices, *Amer. Math. Soc. Colloquium Publications*, Vol. 17, New York, 1934. In the present context cf. mainly Chapters II and VI.

equal 0 say k $(= 0, 1, \cdots, n)$ of them, which we can arrange to be $A_{n-k+1}, \cdots, A_n$.

$(n-1)\delta^2 = 0$ is thus equivalent to $x'_1 = \cdots = x'_{n-k} = 0$, i.e. to $n-k$ independent conditions. On the other hand this amounts obviously to $x_1 = \cdots = x_n$, and these are $n-1$ independent conditions. So $k = 1$ and consequently $A_1, \cdots, A_{n-1} > 0$, $A_n = 0$. And our linear orthogonal transformation must carry the x-vectors with $x_1 = \cdots = x_n$ into the x'-vectors with $x'_1 = \cdots = x'_{n-1} = 0$. Among the former, $\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}$ has the length 1; among the latter only $0, \cdots, 0, \pm 1$ have. Hence these correspond to each other. Now the scalar (inner) product of two vectors is an orthogonal invariant, that of a vector $x_1, \cdots, x_n$ with $\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}$ is $\sqrt{n}\bar{x}$, that of a vector $x'_1, \cdots, x'_n$ with $0, \cdots, 0, \pm 1$ is $\pm x'_n$, hence

$$\sqrt{n}\,\bar{x} = \pm x'_n.$$

Put $x_\mu = \bar{x} + u_\mu$. Then clearly $\sum_{\mu=1}^{n} u_\mu = 0$. Hence

$$\sum_{\mu=1}^{n} x_\mu^2 = n\bar{x}^2 + \sum_{\mu=1}^{n} u_\mu^2 = x_n'^2 + ns^2.$$

Owing to the orthogonality, the left-hand side is equal to $\sum_{\mu=1}^{n} x_\mu'^2$, therefore

$$ns^2 = \sum_{\mu=1}^{n-1} x_\mu'^2.$$

Remembering that $A_n = 0$, we also have

$$(n-1)\delta^2 = \sum_{\mu=1}^{n-1} A_\mu x_\mu'^2.$$

Consequently

$$\eta = \frac{\delta^2}{s^2} = \frac{n}{n-1} \frac{\sum_{\mu=1}^{n-1} A_\mu x_\mu'^2}{\sum_{\mu=1}^{n-1} x_\mu'^2}.$$

The distribution law is, as we know, the same in $x'_1, \cdots, x'_n$ as in $x_1, \cdots, x_n$, namely

$$c^n e^{-\sum_{\mu=1}^{n} x_\mu'^2/2\sigma^2}\, dx'_1 \cdots dx'_n.$$

Thus $x'_1, \cdots, x'_n$ are independent. η depends on $x'_1, \cdots, x'_{n-1}$ only, hence we may disregard x'_n altogether, and use the distribution law of the $x'_1, \cdots, x'_{n-1}$,

$$c^{n-1} e^{-\sum_{\mu=1}^{n-1} x_\mu'^2/2\sigma^2}\, dx'_1 \cdots dx'_{n-1}.$$

With respect to $x_1', \cdots, x_{n-1}'$ we may now state that the $x_1', \cdots, x_{n-1}'$ distribution of η can be obtained by determining first the distribution of η over every spherical surface

$$\sum_{\mu=1}^{n-1} x_\mu'^2 = r^2$$

and then averaging these distributions with the weights $\psi(r)\,dr$, where $\psi(r)\,dr$ is the probability of the spherical shell from r to $r + dr$ with respect to our original $x_1', \cdots, x_{n-1}'$ distribution law. (It happens to be $c'e^{-r^2/2\sigma^2}r^{n-2}\,dr$, but this is immaterial.)

Since the $x_1', \cdots, x_{n-1}'$ distribution law is obviously spherically symmetric in these variables, the first-mentioned distributions over the spherical surfaces are readily obtained by assigning each piece of the surfaces in question its own relative, $n - 2$-dimensional area as weight.

Since η is a homogeneous function of $x_1', \cdots, x_{n-1}'$ of order zero, these spherical surface distributions of η are the same for all r. Consequently we can replace all these r by, say $r = 1$, and the subsequent averaging over the r may be omitted altogether.

Finally, since we restrict ourselves to $r = 1$, i.e. to the spherical surface

$$\sum_{\mu=1}^{n-1} x_\mu^2 = 1,$$

the denominator of η may be omitted and we have

$$\eta = \frac{n}{n-1} \sum_{\mu=1}^{n-1} A_\mu x_\mu'^2.$$

We sum up, writing again $x_1, \cdots, x_{n-1}$ for $x_1', \cdots, x_{n-1}'$, then the desired distribution of η is that of

$$\eta = \frac{n}{n-1} \sum_{\mu=1}^{n-1} A_\mu x_\mu^2,$$

where the point $x_1, \cdots, x_{n-1}$ is uniformly distributed over the spherical surface

$$\sum_{\mu=1}^{n-1} x_\mu^2 = 1.$$

Here $A_1, \cdots, A_{n-1}$ are all positive, and together with 0 they are the characteristic values of the quadratic form

$$(n-1)\delta^2 = \sum_{\mu=1}^{n-1} (x_{\mu+1} - x_\mu)^2$$

$$= x_1^2 + 2\sum_{\mu=2}^{n-1} x_\mu^2 + x_n^2 - 2\sum_{\mu=1}^{n-1} x_\mu x_{\mu+1}.$$

3. The characteristic values A_μ ; first orientation concerning η. We have shown that there exist (counting multiplicities) precisely $n - 1$ positive roots A of the characteristic equation

$$\mathrm{Det}\begin{vmatrix} A-1 & 1 & & & & & & \\ 1 & A-2 & 1 & & & & & \\ & 1 & A-2 & 1 & & & & \\ & & 1 & \ddots & & & & \\ & & & \ddots & \ddots & 1 & & \\ & & & & 1 & A-2 & 1 & \\ & & & & & 1 & A-2 & 1 \\ & & & & & & 1 & A-1 \end{vmatrix} = 0$$

(the empty places are filled with zeros), and that these roots are the $A_1, \cdots, A_{n-1}$.

Such an A is characterized by the possibility of solving the equations

$$(A-1)x_1 + x_2 = 0, \quad x_1 + (A-2)x_2 + x_3 = 0, \quad x_2 + (A-2)x_3 + x_4 = 0,$$
$$\cdots\cdots\cdots, \quad x_{n-2} + (A-2)x_{n-1} + x_n = 0, \quad x_{n-1} + (A-1)x_n = 0,$$

in $x_1, \cdots, x_n$ not all equal to zero. Put

$$x_0 = x_1, \qquad x_{n+1} = x_n,$$

and

$$A = 2 - 2\cos\alpha,$$

then these equations become

$$x_{\mu-1} + x_{\mu+1} = 2\cos\alpha\cdot x_\mu \qquad \text{for} \qquad \mu = 1, 2, \cdots, n-1, n.$$

The last equation is satisfied by

$$x_\mu = 2\cos(\mu - \tfrac{1}{2})\alpha \qquad \text{for} \qquad \mu = 0, 1, 2, \cdots, n-1, n, n+1.$$

Now $x_0 = x_1$ is automatically fulfilled, while $x_{n+1} = x_n$ demands $\cos(n + \frac{1}{2})\alpha = \cos(n - \frac{1}{2})\alpha$. This is certainly the case when $(n + \frac{1}{2})\alpha = 2k\pi - (n - \frac{1}{2})\alpha$ (k any integer), i.e. $\alpha = \dfrac{k\pi}{n}$. For no $k = 1, \cdots, n-1$ are $x_1, \cdots, x_n$ all equal to zero $\left(\text{indeed } x_1 = 2\cos\dfrac{k\pi}{2n} > 0\right)$, hence these k give A's of the desired kind. They are

$$A = 2 - 2\cos\frac{k\pi}{n} = 4\sin^2\frac{k\pi}{2n} \qquad (k = 1, \cdots, n-1),$$

and so they are all positive and different from each other. Their number is $n - 1$. Hence they are precisely $A_1, \cdots, A_{n-1}$.

So we have shown

$$A_\mu = 2 - 2\cos\frac{\mu\pi}{n} = 4\sin^2\frac{\mu\pi}{2n} \qquad (\mu = 1, \cdots, n-1).$$

We can now reformulate the final result of the preceding section. Let us set

$$\eta = \frac{2n}{n-1}(1-\epsilon).$$

Then

$$\epsilon = \sum_{\mu=1}^{n-1} \cos\frac{\mu\pi}{n}\cdot x_\mu^2,$$

where the point $x_1, \cdots, x_{n-1}$ is uniformly distributed over the spherical surface

$$\sum_{\mu=1}^{n-1} x_\mu^2 = 1.$$

Replacement of x_μ by $x_{n-\mu}$ carries ϵ into $-\epsilon$. Therefore the distribution of ϵ is symmetric around 0. Hence the mean of ϵ is 0. The maximum of ϵ's distribution is clearly $\cos\frac{\pi}{n}$, its minimum is $\cos\frac{(n-1)\pi}{n} = -\cos\frac{\pi}{n}$. We state these facts, together with their equivalents for η.

ϵ (η)'s distribution is symmetric around its mean, which is $0\left(\frac{2n}{n-1}\right)$. The maximum of ϵ (η)'s distribution is $\cos\frac{\pi}{n}\left(\frac{2n}{n-1}\left[1+\cos\frac{\pi}{n}\right] = \frac{4n}{n-1}\cos^2\frac{\pi}{2n}\right)$, its minimum is $-\cos\frac{\pi}{n}\left(\frac{2n}{n-1}\left[1-\cos\frac{\pi}{n}\right] = \frac{4n}{n-1}\sin^2\frac{\pi}{2n}\right)$.

Thus it will be easier to obtain information concerning η by considering the distribution of ϵ, since all odd moments of ϵ are zero, etc. The investigation of ϵ instead of η was first suggested by B. I. Hart, who also found, that the first four odd moments of ϵ vanish. R. H. Kent and B. I. Hart also determined the minima and maxima of these distributions for certain small values of n.

4. Direct computation of the moments. We shall investigate the distribution law of a quantity

$$\gamma = \sum_{\mu=1}^{m} B_\mu x_\mu^2.$$

where the point $x_1, \cdots, x_m$ is equidistributed over the spherical surface

$$\sum_{\mu=1}^{m} x_\mu^2 = 1.$$

$\left(\text{Our above } \epsilon \text{ obtains by putting } m = n-1 \text{ and } B_\mu = \cos\frac{\mu\pi}{n}.\right)$

We denote the mean of any function

$$f(x_1, \cdots, x_m)$$

over the above-mentioned spherical surface (the $x_1, \cdots, x_m$ being equidistributed over it) by

$$\overline{f(x_1, \cdots, x_m)}.$$

Our primary objective is to determine the moments of this distribution

$$M_p = \overline{\gamma^p} = \overline{\left(\sum_{\mu=1}^{m} B_\mu x_\mu^2\right)^p}, \qquad (p = 0, 1, 2, \cdots).$$

Let us write Σ_m for the (m — 1-dimensional) area of the above-mentioned spherical surface (of the unit sphere in m-dimensional Euclidean space).

Now we form the function

$$f(z) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{z \sum_{\mu=1}^{m} B_\mu x_\mu^2} e^{-\sum_{\mu=1}^{m} x_\mu^2} \cdot dx_1 \cdots dx_m .$$

(This integral, as well as all others which we are going to derive from it, is obviously convergent, as long as z is sufficiently small. More precisely this is true when

$$| z | \cdot \text{Max} (| B_1 |, \cdots, | B_m |) \leqq 1.$$

We shall use them only in the neighborhood of $z = 0$.) Now clearly

$$\begin{aligned}
\left\{\frac{d^p}{dz^p} f(z)\right\}_{z=0} &= \left\{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\sum_{\mu=1}^{m} B_\mu x_\mu^2\right)^p e^{z \sum_{\mu=1}^{m} B_\mu x_\mu^2} e^{-\sum_{\mu=1}^{m} x_\mu^2} dx_1 \cdots dx_m\right\}_{z=0} \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\sum_{\mu=1}^{m} B_\mu x_\mu^2\right)^p e^{-\sum_{\mu=1}^{m} x_\mu^2} dx_1 \cdots dx_m \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} M_p \left(\sum_{\mu=1}^{m} x_\mu^2\right)^p e^{-\sum_{\mu=1}^{m} x_\mu^2} dx_1 \cdots dx_m \\
&= \int_0^{\infty} M_p r^{2p} e^{-r^2} \Sigma_m r^{m-1} dr \\
&= \Sigma_m M_p \int_0^{\infty} e^{-r^2} r^{2p+m-1} dr \text{ [5]} \\
&= \tfrac{1}{2} \Sigma_m M_p \int_0^{\infty} e^{-u} u^{p+\frac{1}{2}m-1} du \\
&= \tfrac{1}{2} \Sigma_m M_p \Gamma\left(p + \frac{m}{2}\right).
\end{aligned}$$

[5] Introduce the new integration variable $u = r^2$.

On the other hand

$$f(z) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\sum_{\mu=1}^{m} (1-B_\mu z)x_\mu^2} dx_1 \cdots dx_m$$
$$= \prod_{\mu=1}^{m} \int_{-\infty}^{\infty} e^{-(1-B_\mu z)x_\mu^2} dx_\mu \quad {}^{6}$$
$$= \prod_{\mu=1}^{m} \tfrac{1}{2}(1 - B_\mu z)^{-\frac{1}{2}} \cdot 2 \int_0^{\infty} e^{-u} u^{-\frac{1}{2}} du$$
$$= \prod_{\mu=1}^{m} \tfrac{1}{2}(1 - B_\mu z)^{-\frac{1}{2}} \cdot 2\Gamma(\tfrac{1}{2})$$
$$= \Gamma(\tfrac{1}{2})^m \mathfrak{P}(z)^{-\frac{1}{2}},$$

where

$$\mathfrak{P}(z) = \prod_{\mu=1}^{m} (1 - B_\mu z).$$

Thus

$$\tfrac{1}{2}\Sigma_m M_p \Gamma\left(p + \frac{m}{2}\right) = \Gamma\left(\frac{1}{2}\right)^m \left\{\frac{d^p}{dz^p} \mathfrak{P}(z)^{-\frac{1}{2}}\right\}_{z=0}.$$

For $p = 0$ this becomes, since $M_0 = 1$, $\mathfrak{P}(0) = 1$,

$$\frac{1}{2} \Sigma_m \Gamma\left(\frac{m}{2}\right) = \Gamma\left(\frac{1}{2}\right)^m.$$

Dividing the former equation by the latter gives, since

$$\frac{\Gamma\left(p + \frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} = \frac{m}{2}\left(\frac{m}{2} + 1\right) \cdots \left(\frac{m}{2} + p - 1\right),$$

$$M_p = \frac{1}{\frac{m}{2}\left(\frac{m}{2} + 1\right) \cdots \left(\frac{m}{2} + p - 1\right)} \left\{\frac{d^p}{dz^p} \mathfrak{P}(z)^{-\frac{1}{2}}\right\}_{z=0}.$$

In order to make a practical use of the above formula, we compute

$$\ln (\mathfrak{P}(z)^{-\frac{1}{2}}) = -\tfrac{1}{2} \sum_{\mu=1}^{m} \ln (1 - B_\mu z)$$
$$= -\tfrac{1}{2} \sum_{\mu=1}^{m} \sum_{l=1}^{\infty} -\frac{1}{l} B_\mu^l z^l$$
$$= \sum_{l=1}^{\infty} \frac{1}{2l} \left(\sum_{\mu=1}^{m} B_\mu^l\right) z^l.$$

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[6] Introduce the new integration variable $u = (1 - B_\mu z)r^2$.

Write

$$\alpha_l = \frac{1}{2l}\sum_{\mu=1}^{m} B_\mu^l,$$

then

$$\mathfrak{P}(z)^{-\frac{1}{2}} = e^{\alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots}$$

$$= 1 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 + \cdots,$$

and so

$$M_p = \frac{1 \cdot 2 \cdots p}{\frac{m}{2}\left(\frac{m}{2}+1\right)\cdots\left(\frac{m}{2}+p-1\right)}\beta_p .$$

Clearly

$$\beta_1 = \alpha_1 ,$$

$$\beta_2 = \alpha_2 + \tfrac{1}{2}\alpha_1^2 ,$$

$$\beta_3 = \alpha_3 + \alpha_1\alpha_2 + \tfrac{1}{6}\alpha_1^3 ,$$

$$\beta_4 = \alpha_4 + \tfrac{1}{2}\alpha_2^2 + \alpha_1\alpha_3 + \tfrac{1}{2}\alpha_1^2\alpha_2 + \tfrac{1}{24}\alpha_1^4 .$$

In our application (cf. above)

$$B_{m+1-\mu} = -B_\mu .$$

This has the consequence that

$$\alpha_1 = \alpha_3 = \alpha_5 = \cdots = 0.$$

Thus the z functions we compute contain only even powers of z and consequently

$$\beta_1 = \beta_3 = \beta_5 = \cdots = 0,$$

$$M_1 = M_3 = M_5 = \cdots = 0,$$

and

$$\beta_2 = \alpha_2 ,$$

$$\beta_4 = \alpha_4 + \tfrac{1}{2}\alpha_2^2 ,$$

$$\beta_6 = \alpha_6 + \alpha_2\alpha_4 + \tfrac{1}{6}\alpha_2^3 ,$$

$$\beta_8 = \alpha_8 + \tfrac{1}{2}\alpha_4^2 + \alpha_2\alpha_6 + \tfrac{1}{2}\alpha_2^2\alpha_4 + \tfrac{1}{24}\alpha_2^4 .$$

As mentioned before, we actually have $m = n - 1$ and $B_\mu \equiv \cos \frac{\mu\pi}{n}$. Consequently

$$\begin{aligned}\alpha_l &= \frac{1}{2l}\sum_{\mu=1}^{n-1}\left\{\cos\frac{\mu\pi}{n}\right\}^l = \frac{1}{2l}\sum_{\mu=1}^{n-1}\left\{\tfrac{1}{2}(e^{i\mu\pi/n} + e^{-i\mu\pi/n})\right\}^l \\ &= \frac{1}{2^{l+1}l}\sum_{\mu=1}^{n-1}\sum_{k=0}^{l}\binom{l}{k} e^{i(2k-l)\mu\pi/n}\,{}^{7} \\ &= \frac{1}{2^{l+1}l}\sum_{k=0}^{l}\binom{l}{k}\sum_{\mu=1}^{n-1} e^{i2\pi\mu(k-\frac{1}{2}l)/n} \\ &= \frac{1}{2^{l+1}l}\sum_{k=0}^{l}\binom{l}{k}\left\{\sum_{\mu=0}^{n-1} e^{i2\pi\mu(k-\frac{1}{2}l)/n} - 1\right\}.\end{aligned}$$

The inner sum has obviously these values

$$\sum_{\mu=0}^{n-1} e^{i2\pi\mu(k-\frac{1}{2}l)/n} \begin{cases} = n & \text{if } k - \tfrac{1}{2}l \text{ is divisible by } n \\ = 0 & \text{otherwise.}\end{cases}$$

Also

$$\sum_{k=0}^{l}\binom{l}{k}\cdot(-1) = -2^l.$$

Consequently

$$\alpha_l = \frac{n}{2^{l+1}l}\sum_k{}'\binom{l}{k} - \frac{1}{2l},$$

where $\sum_k{}'$ extends over those $k = 0, \cdots, l$, for which $k - \frac{1}{2}l$ is divisible by n.

Let us now determine the k occurring in the following sum (as above, $k - \frac{1}{2}l$ is divisible by n) $\sum_k{}'$. $k = \frac{1}{2}l$ is clearly one of them. All others are of the form $k = \frac{1}{2}l \pm hn$, $h = 1, 2, \cdots$. The term contributed is the same for $+$ and for $-$, since

$$\binom{l}{\frac{1}{2}l + hn} = \binom{l}{\frac{1}{2}l - hn}.$$

So we have

$$\alpha_l \begin{cases} = 0, & \text{for } l \text{ odd,} \\ = \dfrac{1}{2l}\left\{\dfrac{n}{2^l}\left[\dbinom{l}{\frac{1}{2}l} + 2\displaystyle\sum_{h=1,2,\cdots}\dbinom{l}{\frac{1}{2}l - hn}\right] - 1\right\}, & \text{for } l \text{ even.}\end{cases}$$

[7] As pointed out above, we need to consider only the even l.

The number of terms which the sum $\sum_{h=1,2,\cdots}$ contributes depends on the comparative sizes of l and n. The number is clearly

$$\begin{aligned} &0 \text{ for } && \tfrac{1}{2}l < n, \\ &1 \text{ for } && n \leqq \tfrac{1}{2}l < 2n, \\ &2 \text{ for } && 2n \leqq \tfrac{1}{2}l < 3n, \\ &\cdots\cdots\cdots\cdots\cdots . \end{aligned}$$

Explicit formulae follow:[8]

$$\alpha_1 = \alpha_3 = \alpha_5 = \alpha_7 = \alpha_9 = \cdots = 0,$$

$$\alpha_2 = \frac{n-2}{8}, \qquad (0 \text{ for } n = 1),$$

$$\alpha_4 = \frac{3n-8}{64}, \qquad (0 \text{ for } n = 1, 2),$$

$$\alpha_6 = \frac{5n-16}{192}, \qquad \left(0 \text{ for } n = 1, 2; \frac{1}{384}, n = 3\right),$$

$$\alpha_8 = \frac{35n-128}{2048}, \quad \left(0 \text{ for } n = 1, 2; \frac{1}{2048}, n = 3; \frac{1}{128}, n = 4\right).$$

$$\beta_1 = \beta_3 = \beta_5 = \beta_7 = \beta_9 = \cdots = 0,$$

$$\beta_2 = \frac{n-2}{8}, \qquad (0 \text{ for } n = 1),$$

$$\beta_4 = \frac{n^2 + 2n - 12}{128}, \qquad (0 \text{ for } n = 1, 2),$$

$$\beta_6 = \frac{n^3 + 12n^2 + 8n - 168}{3072}, \qquad \left(0 \text{ for } n = 1, 2; \frac{5}{1024}, n = 3\right),$$

$$\beta_8 = \frac{n^4 + 28n^3 + 212n^2 - 64n - 3696}{98304}, \quad \left(0 \text{ for } n = 1, 2; \frac{35}{32768}, n = 3; \frac{35}{2048}, n = 4\right).$$

$$M_1 = M_3 = M_5 = M_7 = M_9 = \cdots = 0,$$

$$M_2 = \frac{8}{(n-1)(n+1)} \cdot \beta_2 = \frac{n-2}{(n-1)(n+1)}, \qquad (0 \text{ for } n = 1),$$

[8] The author wishes to express his thanks to Miss B. I. Hart for her kind help in carrying out these computations.

$$M_4 = \frac{384}{(n-1)(n+1)(n+3)(n+5)} \cdot \beta_4 = \frac{3(n^2+2n-12)}{(n-1)(n+1)(n+3)(n+5)},$$
$$(0 \text{ for } n = 1, 2),$$

$$M_6 = \frac{46080}{(n-1)(n+1)(n+3)(n+5)(n+7)(n+9)} \cdot \beta_6$$
$$= \frac{15(n^3+12n^2+8n-168)}{(n-1)(n+1)(n+3)(n+5)(n+7)(n+9)},$$
$$(0 \text{ for } n = 1, 2; \tfrac{5}{1024}, n = 3).$$

$$M_8 = \frac{10321920}{(n-1)(n+1)(n+3)(n+5)(n+7)(n+9)(n+11)(n+13)} \cdot \beta_8$$
$$= \frac{105(n^4+28n^3+212n^2-64n-3696)}{(n-1)(n+1)(n+3)(n+5)(n+7)(n+9)(n+11)(n+13)},$$
$$(0 \text{ for } n = 1, 2; \tfrac{35}{32768}, n = 3; \tfrac{112}{21879}, n = 4).$$

We conclude this section by obtaining asymptotic formulae for the distribution of ϵ when $n \to \infty$.

In this case our formulae show that all α_l (l even) behave asymptotically like constant multiples of n. It also appears from our formulae for the β_l (l even), that

$$\beta_l = \frac{1}{(\frac{1}{2}l)!} \alpha_2^{\frac{1}{2}l} + \text{a polynomial in } \alpha_2, \alpha_4, \cdots, \alpha_{l-2} \text{ of total order} \leqq \tfrac{1}{2}l - 1.$$

Consequently $\frac{1}{(\frac{1}{2}l)!} \alpha_2^{\frac{1}{2}l}$ is the dominant term in this expression, and so we have asymptotically

$$\beta_l \sim \frac{1}{(\frac{1}{2}l)!} \alpha_2^{\frac{1}{2}l} \sim \frac{1}{(\frac{1}{2}l)!} \left(\frac{n}{8}\right)^{\frac{1}{2}l}.$$

From this

$$M_l \sim \frac{l!}{\left(\frac{n}{2}\right)^l} \beta_l \sim \frac{l!}{(\frac{1}{2}l)!} \left(\frac{1}{2n}\right)^{\frac{1}{2}l}.$$

Now the normal distribution

$$c_1 e^{-y^2/2\sigma_1^2}\, dy, \qquad \left(c_1 = \frac{1}{\sigma_1\sqrt{2\pi}}\right),$$

with the mean 0 and the standard deviation σ_1 has the moments

$$m_l = \int_{-\infty}^{\infty} y^l c_1 e^{-y^2/2\sigma_1^2}\, dy.$$

This is clearly 0 for l odd, while for l even[9]

$$m_l = \sigma_1^{l+1} c_1 \cdot 2^{\frac{1}{2}(l+1)} \int_0^\infty e^{-u}\, u^{\frac{1}{2}(l-1)}\, du$$
$$= 2^{\frac{1}{2}(l+1)} \sigma_1^{l+1} c_1 \Gamma\left(\frac{l+1}{2}\right).$$

For $l = 0$ this becomes, since $m_0 = 1$,

$$1 = 2^{\frac{1}{2}} \sigma_1 c_1 \Gamma(\tfrac{1}{2}).$$

Dividing the former equation by the latter gives, since

$$\frac{\Gamma\left(\frac{l+1}{2}\right)}{\Gamma(\frac{1}{2})} = \frac{1}{2}\cdot\frac{3}{2} \cdots \frac{l-1}{2},$$

$$m_l = 1\cdot 3 \cdots (l-1)\sigma_1^l = \frac{l!}{2^{\frac{1}{2}l}(\frac{1}{2}l)!}\sigma_1^l = \frac{l!}{(\frac{1}{2}l)!}\left(\frac{\sigma_1^2}{2}\right)^{\frac{1}{2}l}.$$

Comparing the formulae for M_l and for m_l shows that $M_l \sim m_l$ if $\frac{1}{2n} = \frac{\sigma_1^2}{2}$, $\sigma_1 = \sqrt{\frac{1}{n}}$. So we see:

For $n \to \infty$ the distribution of ϵ becomes asymptotically normal, with the mean 0 and the standard deviation $\sigma_1 = \sqrt{\frac{1}{n}}$. (The same result could be obtained by applying the general theorems of Liapounoff and others.)

5. The distribution law, general discussion. We return to the quantity γ, defined at the beginning of the preceding section, of which our ϵ is a special case. We wish to obtain direct information concerning the distribution law of this γ.

Since a permutation of the B_μ is permissible, we arrange them such that

$$B_1 \geqq B_2 \geqq \cdots \geqq B_m .$$

(In the special case $\gamma = \epsilon$, the $B_\mu = \cos \frac{\mu\pi}{n}$ are given in this arrangement.)

The distribution of γ covers obviously the interval

$$B_1 \geqq y \geqq B_m .$$

And if not $B_1 = \cdots = B_m$, i.e. if $B_1 > B_m$, which we assume to be the case, then we have obviously a continuous distribution law for γ in this interval. We denote it by $\omega(y)\, dy$.

---

[9] Introduce the new integration variable $u = \frac{y^2}{2\sigma_1^2}$.

Assume for the moment that $B_m > 0$. Then the quantity

$$\gamma^{-\frac{1}{2}m} = \left(\sum_{\mu=1}^{m} B_\mu x_\mu^2\right)^{-\frac{1}{2}m},$$

is bounded, and we can therefore form its mean value. This is the $-\frac{m}{2}$ moment of γ (cf. the beginning of the preceding section)

$$M_{-\frac{1}{2}m} = \overline{\gamma^{-\frac{1}{2}m}} = \overline{\left(\sum_{\mu=1}^{m} B_\mu x_\mu^2\right)^{-\frac{1}{2}m}}$$
$$= \int_{B_m}^{B_1} y^{-\frac{1}{2}m}\,\omega(y)\,dy.$$

With any two $a > b > 0$ $\left(\text{we shall have } \frac{a}{b} \to \infty \text{ subsequently}\right)$ form the quantity

$$t(a, b) = \int_{a^2 \geq \sum_{\mu=1}^{m} x_\mu^2 \geq b^2} \cdots \int \left(\sum_{\mu=1}^{m} x_\mu^2\right)^{-\frac{1}{2}m} dx_1 \cdots dx_m \text{ [10]}$$
$$= \int_b^a r^{-m}\cdot\Sigma_m\, r^{m-1}\,dr = \Sigma_m \int_b^a \frac{dr}{r}$$
$$\doteq \Sigma_m \ln \frac{a}{b}.$$

Consider next

$$s(a, b) = \int_{a^2 \geq \sum_{\mu=1}^{m} \frac{1}{B_\mu} x_\mu^2 \geq b^2} \cdots \int \left(\sum_{\mu=1}^{m} x_\mu^2\right)^{-\frac{1}{2}m} dx_1 \cdots dx_m \text{ [11]}$$
$$= \int_{a^2 \geq \sum_{\mu=1}^{m} x_\mu^2 \geq b^2} \cdots \int \left(\sum_{\mu=1}^{m} B_\mu x_\mu^2\right)^{-\frac{1}{2}m} \sqrt{\prod_{\mu=1}^{m} B_\mu}\, dx_1 \cdots dx_m$$
$$= \int_{a^2 \geq \sum_{\mu=1}^{m} x_\mu^2 \geq b^2} \cdots \int M_{-\frac{1}{2}m} \left(\sum_{\mu=1}^{m} x_\mu^2\right)^{-\frac{1}{2}m} \sqrt{\prod_{\mu=1}^{m} B_\mu}\, dx_1 \cdots dx_m$$
$$= M_{-\frac{1}{2}m} \sqrt{\prod_{\mu=1}^{m} B_\mu}\; t(a, b).$$

[10] Concerning this transformation to polar coordinates and the quantity Σ_m cf. the first part of the preceding section.

[11] Replace each variable x_μ by $\sqrt{B_\mu}\, x_\mu$.

On the other hand, a comparison of their respective integration domains makes it clear that

$$t(B_m a, B_1 b) \leqq s(a, b) \leqq t(B_1 a, B_m b).$$

Thus

$$\Sigma_m \ln \frac{B_m a}{B_1 b} \leqq M_{-\frac{1}{2}m} \sqrt{\prod_{\mu=1}^{m} B_\mu} \cdot \Sigma_m \ln \frac{a}{b} \leqq \Sigma_m \ln \frac{B_1 a}{B_m b},$$

i.e.

$$\frac{1}{\sqrt{\prod_{\mu=1}^{m} B_\mu}} \frac{\ln \frac{a}{b} - \ln \frac{B_1}{B_m}}{\ln \frac{a}{b}} \leqq M_{-\frac{1}{2}m} \leqq \frac{1}{\sqrt{\prod_{\mu=1}^{m} B_\mu}} \frac{\ln \frac{a}{b} + \ln \frac{B_1}{B_m}}{\ln \frac{a}{b}}.$$

Now let $\frac{a}{b} \to \infty$, then

$$M_{-\frac{1}{2}m} = \frac{1}{\sqrt{\prod_{\mu=1}^{m} B_\mu}}$$

obtains, i.e.

$$\overline{\gamma^{-\frac{1}{2}m}} = \int_{B_m}^{B_1} y^{-\frac{1}{2}m} \omega(y)\, dy = \frac{1}{\sqrt{\prod_{\mu=1}^{m} B_\mu}}.$$

We now drop the assumption $B_m > 0$. We consider instead a real number z with $z < B_m$. Replace each B_μ by $B_\mu - z$. Then the one with $\mu = m$ becomes > 0. And γ is obviously replaced by $\gamma - z$. Consequently our above equation is now valid in the form

$$\overline{(\gamma - z)^{-\frac{1}{2}m}} = \int_{B_m}^{B_1} (y - z)^{-\frac{1}{2}m} \omega(y)\, dy = \frac{1}{\sqrt{\prod_{\mu=1}^{m} (B_\mu - z)}}.$$

Let now z be a complex variable. The second term of the above equation is a (locally) analytical function of z, except in the (real) interval $B_1 \geqq z \geqq B_m$. The third term, too, is a (locally) analytical function of z, except at the (real) points $B_1, \cdots, B_m$. Consequently both are one-valued analytical functions of z in the simply connected domain which obtains from the complex z plane by exclusion of the (real) half line

$$z \geqq B_m.$$

Hence the equation

$$(1) \qquad \int_{B_m}^{B_1} (y - z)^{-\frac{1}{2}m} \omega(y)\, dy = \frac{1}{\sqrt{\prod_{\mu=1}^{m} (B_\mu - z)}},$$

which holds for all (real) $z < B_m$, remains true for all complex z of the above domain.[12]

We observe next that $\omega(y)$ is an analytical function of y in $B_1 \geqq y \geqq B_m$, whenever $y \neq B_1, \cdots, B_m$. This is easily established by using any multiple integral expression for $\omega(y)$ which, while hard to evaluate explicitly, puts this analyticity into evidence.[13]

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[12] $(y - z)^{-\frac{1}{2}m}$ and the factors $(B_\mu - z)^{-\frac{1}{2}}$ of $\dfrac{1}{\sqrt{\prod_{\mu=1}^{m} (B_\mu - z)}}$ are those branches of these analytical functions which are (real and) > 0 when z is (real and) $< B_\mu$. When m is even (as it will be, cf. below) the domain of analyticity is somewhat more extended, but we need not discuss this.

[13] The computation which follows gives the desired analyticity in a simple way, and also makes it clear why the analyticity fails at $y = B_1, \cdots, B_m$.

Consider the $y \neq B_1, \cdots, B_m$ in $B_1 \geqq y \geqq B_m$. The probability of $\gamma \leqq y$ is $p(y) = \int_{B_m}^{y} \omega(y)\, dy$, and we may establish its analyticity instead of that of $p'(y) = \omega(y)$.

Obviously $p(y)$ is equally the probability of $\sum_{\mu=1}^{m} B_\mu x_\mu^2 \leqq y \sum_{\mu=1}^{m} x_\mu^2$, if the $x_1, \cdots, x_m$ are equidistributed over a spherical surface $\sum_{\mu=1}^{m} x_\mu^2 = r^2$, with any given $r > 0$.

Our hypotheses concerning y imply $B_v > y > B_{v+1}$ for a suitable $v = 1, \cdots, m-1$. Consider now the expression

$$f(y) = \int \cdots \int_{\sum_{\mu=1}^{m} B_\mu x_\mu^2 \leqq y \sum_{\mu=1}^{m} x_\mu^2} e^{-\sum_{\mu=1}^{m} x_\mu^2}\, dx_1 \cdots dx_m .$$

Transforming to polar coordinates, we obtain

$$f(y) = \int_0^\infty e^{-r^2} \cdot \Sigma_m\, p(y) r^{m-1}\, dr$$

$$= \Sigma_m \int_0^\infty e^{-r^2} r^{m-1}\, dr \cdot p(y).$$

(Σ_m as before.) Hence it suffices to establish the analyticity of $f(y)$. Now on the other hand

$$f(y) = \int \cdots \int_{\sum_{\mu=1}^{v} (B_\mu - y) x_\mu^2 \leqq \sum_{\mu=v+1}^{m} (y - B_\mu) x_\mu^2} e^{-\sum_{\mu=1}^{m} x_\mu^2}\, dx_1 \cdots dx_m$$

$$= \frac{1}{\sqrt{\prod_{\mu=1}^{m} |B_\mu - y|}} \int \cdots \int_{\sum_{\mu=1}^{v} w_\mu^2 \leqq \sum_{\mu=v+1}^{m} w_\mu^2} e^{-\sum_{\mu=1}^{n} w_\mu^2 / |B_\mu - y|}\, dw_1 \cdots dw_m .$$

(We introduced the new variables $w_\mu = \sqrt{|B_\mu - y|}\, x_\mu$.) And this expression is clearly analytical in y, since $B_v > y > B_{v+1}$.

We shall need only the fact that $\omega(y)$ possesses $\frac{1}{2}m$ continuous derivatives at these places. (m will be assumed to be even, cf. below.) Its behavior at $y = B_1, \cdots, B_m$ will follow from our subsequent results in all cases where we need it.

In order to determine $\omega(y)$ from (1), as we now propose to do, it is very convenient to assume that m is even. We therefore make this assumption, and shall maintain it throughout most of what follows.

Consider a $y_0 \neq B_1, \cdots, B_m$ in $B_1 \geqq y_0 \geqq B_m$. Then $B_v > y > B_{v+1}$ for a suitable $v = 1, \cdots, m - 1$. Now put

$$z = y_0 + it \qquad (t \text{ real and } > 0),$$

form (1), take the imaginary parts of both sides, and let $t \to 0$.

Consider first the left-hand side of (1). Since $\omega(y)$ possesses $\frac{1}{2}m$ continuous derivatives at $y = y_0$, we have

$$\omega(y) = \sum_{k=0}^{\frac{1}{2}m-1} \theta_k (y - y_0)^k + e(y)(y - y_0)^{\frac{1}{2}m}$$

with a bounded $e(y)$. Clearly

$$\theta_k = \frac{1}{k!}\left\{\frac{d^k}{dy^k}\,\omega(y)\right\}_{y=y_0}$$

Thus, since $\omega(y)$ is real, all θ_k are real and $e(y)$ is also real.

Compute now the contribution of each one of the $\frac{1}{2}m + 1$ terms in the above expression for $\omega(y)$ to the imaginary part of the left-hand side of (1).

The last term, $e(y)\cdot(y - y_0)^{\frac{1}{2}m}$, gives

$$\Im \int_{B_m}^{B_1} (y - y_0 - it)^{-\frac{1}{2}m} e(y)(y - y_0)^{\frac{1}{2}m}\, dy = \Im \int_{B_m}^{B_1} \left(\frac{y - y_0}{y - y_0 - it}\right)^{\frac{1}{2}m} e(y)\, dy.$$

The integrand is uniformly bounded, and so the reality conditions cause the entire expression to $\to 0$ for $t \to 0$. Hence the contribution of this term is zero for $t \to 0$.

The other $\dfrac{m}{2}$ terms correspond to $k = 0, 1, \cdots, \dfrac{m}{2} - 1$, the k term being

$$\begin{aligned}
&\Im \int_{B_m}^{B_1} (y - y_0 - it)^{-\frac{1}{2}m}\cdot\theta_k(y - y_0)^k\cdot dy \\
&\qquad = \theta_k \Im \int_{B_m}^{B_1} \frac{(y - y_0)^k}{(y - y_0 - it)^{\frac{1}{2}m}}\, dy \\
&\qquad = \theta_k \Im \int_{B_m}^{B_1} \frac{\sum_{h=0}^{k} \binom{k}{h} (it)^h (y - y_0 - it)^{k-h}}{(y - y_0 - it)^{\frac{1}{2}m}}\, dy \\
&\qquad = \theta_k \sum_{h=0}^{k} \binom{k}{h} \Im \left\{(it)^h \int_{B_m}^{B_1} (y - y_0 - it)^{k-h-\frac{1}{2}m}\, dy\right\}.
\end{aligned}$$

The exponent $k - h - \frac{m}{2}$ in the integral is always $\leqq \left(\frac{m}{2} - 1\right) - 0 - \frac{m}{2} = -1$, and it is $= -1$ if and only if $k = \frac{m}{2} - 1$, $h = 0$. Consider first a term where this is not the case, i.e. where the exponent $k - h - \frac{m}{2} < -1$. For such a term the expression $\mathfrak{J}\{\cdots\}$ becomes

$$\mathfrak{J}(it)^h \frac{1}{k - h - \frac{m}{2} + 1} \{(y - y_0 - it)^{k-h-\frac{1}{2}m+1}\}_{y=B_m}^{y=B_1}.$$

For $t \to 0$ the last factors are bounded and real, and so the entire expression $\to 0$: for $h = 0$ because of the reality conditions, for $h > 0$ because of $(it)^h \to 0$. Thus only the term $k = \frac{m}{2} - 1$, $h = 0$ can contribute something else than zero for $t \to 0$.

Now this term is equal to

$$\theta_{\frac{1}{2}m-1}\mathfrak{J}\,\{\ln\,(y - y_0 - it)\}_{y=B_m}^{y=B_1},$$

and for $t \to 0$ this converges to

$$\theta_{\frac{1}{2}m-1}\mathfrak{J}(i\pi) = \pi\theta_{\frac{1}{2}m-1} = \frac{\pi}{\left(\frac{m}{2} - 1\right)!}\left\{\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}}\,\omega(y)\right\}_{y=y_0}.$$ [14]

Thus the imaginary part of the entire left-hand side of (1) converges for $t \to 0$ to this expression.

The right-hand side of (1) is easier to discuss. The imaginary part under consideration is now

$$\mathfrak{J}\frac{1}{\sqrt{\prod_{\mu=1}^{m}(B_\mu - y_0 - it)}} = \mathfrak{J}\prod_{\mu=1}^{m}(B_\mu - y_0 - it)^{-\frac{1}{2}}.$$

Considering[12] (its y is our $y_0 + it$), this converges for $t \to 0$ to

$$\mathfrak{J}\prod_{\mu=1}^{v}(B_\mu - y_0)^{-\frac{1}{2}}\prod_{\mu=v+1}^{m} i(y_0 - B_\mu)^{-\frac{1}{2}} = \mathfrak{J}i^{m-v}\frac{1}{\sqrt{\prod_{\mu=1}^{m}|B_\mu - y_0|}}.$$ [15]

---

[14] This evaluation $\{\ln\,(y - y_0 - it)\}_{y=B_m}^{y=B_1} \to i\pi$ is based on $t > 0$, and the fact that y moves on the real axis from B_m to B_1. It has no connection with[12].

[15] The square roots of the (real and) > 0 quantities

$$B_\mu - y_0\ (\mu = 1, \cdots, v), \qquad y_0 - B_\mu\ (\mu = v + 1, \cdots, m), \text{ and } \prod_{\mu=1}^{m}|B_\mu - y_0|$$

are taken to be > 0.

If v (hence $m - v$) is even, then this is zero. If v (hence $m - v$) is odd, then this is equal to $(-1)^{\frac{1}{2}(m-v-1)} \dfrac{1}{\sqrt{\prod_{\mu=1}^{m} | B_\mu - y_0 |}}$. Thus (1) becomes the following equation:

$$\frac{\pi}{\left(\frac{m}{2}-1\right)!}\left\{\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}}\,\omega(y)\right\}_{y=y_0} \begin{cases} = 0 & \text{if } v \text{ is even,} \\ = (-1)^{\frac{1}{2}(m-v-1)} \dfrac{1}{\sqrt{\prod_{\mu=1}^{m} | B_\mu - y_0 |}} & \text{if } v \text{ is odd.} \end{cases}$$

Simplifying this, and writing y for y_0, and also restating the definition of v gives

$$\text{(2)} \qquad \frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}}\,\omega(y) \begin{cases} = 0 & \text{if } v \text{ is even,} \\ = (-1)^{\frac{1}{2}(m-v-1)} \dfrac{\left(\frac{m}{2}-1\right)!}{\pi} \dfrac{1}{\sqrt{\prod_{\mu=1}^{m} | B_\mu - y |}} & \text{if } v \text{ is odd,} \end{cases}$$

$$B_v > y > B_{v+1}, \; v = 1, \cdots, m-1.$$

Observe finally, that if we put

$$\mathfrak{A}(y) = \prod_{\mu=1}^{m} (y - B_\mu),$$

then this product has v factors < 0 ($\mu = 1, \cdots, v$), while the others are > 0. So

$$\mathfrak{A}(y) \gtrless 0 \quad \text{for} \quad v \; \begin{matrix}\text{even} \\ \text{odd}\end{matrix},$$

and in the latter case

$$\prod_{\mu=1}^{m} | B_\mu - y | = -\mathfrak{A}(y).$$

It is clear how we may now rewrite (2).

We are now in a position to determine the behavior of $\omega(y)$ at $y = B_1, \cdots, B_m$ too, since we know how its $\frac{m}{2} - 1$-th derivative behaves in the immediate vicinity of these places. (2) shows that it is singular there, and that the nature of the singularity depends on the number of the μ, for which B_μ is equal to the y in question, i.e. on the multiplicity of this root of our polynomial $\mathfrak{A}(y)$.

In our actual application (to $\gamma = \epsilon$, cf. the beginning of this section) the

B_μ are pairwise different, i.e. all root multiplicities of $\mathfrak{A}(y)$ are equal to one. A further special case, which has a certain interest of its own, is when the B_μ are equal two by two, but otherwise different, i.e. all root multiplicities of $\mathfrak{A}(y)$ are equal to two. In the discussion which follows we shall therefore assume that one or the other of these two cases occurs.

In the first case $\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}}\,\omega(y)$ has on each side of a $y = B_\mu$ one of these two behaviors: It is identically zero, or it is singular, of the type $\frac{1}{\sqrt{|B_\mu - y|}}$. Thus it is at any rate integrable. Consequently $\frac{d^{\frac{1}{2}m-2}}{dy^{\frac{1}{2}m-2}}\,\omega(y)$ is continuous on each side of $y = B_\mu$, i.e. for both $y = B_\mu \pm 0$. Successive integrations give now that all $\frac{d^k}{dy^k}\,\omega(y)$, $k = 0, 1, \cdots, \frac{m}{2} - 2$, are continuous for both $y = B_\mu \pm 0$.

In the second case we have $B_1 = B_2 > B_3 = B_4 > \cdots > B_{m-1} = B_m$. So the v with $B_v > y > B_{v+1}$ is necessarily even, and $\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}}\,\omega(y)$ is identically zero for all of (2). Consequently $\frac{d^{\frac{1}{2}m-2}}{dy^{\frac{1}{2}m-2}}\,\omega(y)$ is again continuous on each side of $y = B_\mu$, i.e. for both $y = B_\mu \pm 0$. Successive integrations show again that all $\frac{d^k}{dy^k}\,\omega(y)$, $k = 0, 1, \cdots, \frac{m}{2} - 2$, are continuous for both $y = B_\mu \pm 0$.

We must therefore discuss only how much the $\frac{d^k}{dy^k}\,\omega(y)$, $k = 0, 1, \cdots, \frac{m}{2} - 2$, change from $y = B_\mu - 0$ to $y = B_\mu + 0$.

Let us return to the procedure by which we derived (2) from (1). We put again

$$z = y_0 + it \qquad (t \text{ real and } > 0)$$

and let $t \to \infty$. But we consider now (1) itself (and not merely its imaginary part), and we choose a $y_0 = B_v$.

Consider first the left-hand side of (1), always disregarding terms which stay bounded for $t \to 0$. Then we can replace the integral $\int_{B_m}^{B_1}$ of (1) by any $\int_{B_v-a}^{B_v+a}$ with any fixed $a > 0$, and this is equal to

$$\int_{B_v-a}^{B_v-0} + \int_{B_v+0}^{B_v+a}.$$

We choose this $a > 0$ so small that no $B_\mu \neq B_v$ lies between $B_v - a$ and $B_v + a$. I.e. all $\frac{d^k}{dy^k}\,\omega(y)$, $k = 0, 1, \cdots, \frac{m}{2} - 2$, are continuous from $B_v - a$ to $B_v - 0$ and also from $B_v + 0$ to $B_v + a$.

This being the case, we can evaluate the above sum of two integrals by $\frac{m}{2} - 1$ successive partial integrations. Thus we get

$$-\left\{\sum_{k=0}^{\frac{1}{2}m-2} \frac{\left(\frac{m}{2}-2-k\right)!}{\left(\frac{m}{2}-1\right)!} (y - B_v - it)^{-\frac{1}{2}m+1+k} \frac{d^k}{dy^k}\,\omega(y)\right\}_{y=B_v-a}^{y=B_v-0}$$

$$-\left\{\sum_{k=0}^{\frac{1}{2}m-2} \frac{\left(\frac{m}{2}-2-k\right)!}{\left(\frac{m}{2}-1\right)!} (y - B_v - it)^{-\frac{1}{2}m+1+k} \frac{d^k}{dy^k}\,\omega(y)\right\}_{y=B_v+0}^{y=B_v+a}$$

$$+\frac{1}{\left(\frac{m}{2}-1\right)!}\int_{B_v-a}^{B_v+a} (y - B_v - it)^{-1} \frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}}\,\omega(y)\,dy.$$

In the first two lines the $y = B_v \pm a$ terms are bounded for $t \to 0$, therefore only the $y = B_v \pm 0$ terms need be kept. Then the first two lines give

$$\sum_{k=0}^{\frac{1}{2}m-2} \frac{\left(\frac{m}{2}-2-k\right)!}{\left(\frac{m}{2}-1\right)!} (-it)^{-\frac{1}{2}m+1+k} \left\{\frac{d^k}{dy^k}\,\omega(y)\right\}_{y=B_v-0}^{y=B_v+0},$$

up to terms which stay bounded for $t \to 0$. Consider now the third line. We know that the $\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}}\,\omega(y)$ in its integrand can be majorized by $\frac{c_2}{\sqrt{|y - B_v|}}$ (for a suitable constant c_2, cf. our discussion preceding the present one). Thus the integral in question is majorized by

$$\int_{B_v-a}^{B_v+a} |y - B_v - it|^{-1} c_2 |y - B_v|^{-\frac{1}{2}}\,dy,$$

hence *a fortiori* by

$$\int_{-\infty}^{\infty} |y - B_v - it|^{-1} c_2 |y - B_v|^{-\frac{1}{2}}\,dy \text{ [16]}$$

$$= c_2 t^{-\frac{1}{2}} \int_{-\infty}^{\infty} |u - i|^{-1} |u|^{-\frac{1}{2}}\,du$$

$$= c_2 t^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{du}{\sqrt{(u^2+1)\cdot|u|}} \text{ [17]}$$

$$= c_2 \int_0^{\infty} \frac{dv}{\sqrt{v^4+1}}\, t^{-\frac{1}{2}}.$$

[16] Introduce the new integration variable $u = \frac{y - B_v}{t}$.

[17] Introduce the new integration variable $v = \sqrt{|u|}$.

Since the last integration is obviously finite, the entire expression is $O(t^{-\frac{1}{2}})$ for $t \to 0$.

Consequently the left-hand side of (1) is equal to

$$\sum_{k=0}^{\frac{1}{2}m-2} \frac{\left(\frac{m}{2}-2-k\right)!}{\left(\frac{m}{2}-1\right)!} (-it)^{-\frac{1}{2}m+1+k} \left\{\frac{d^k}{dy^k}\,\omega(y)\right\}_{y=B_v-0}^{y=B_v+0} + O(t^{-\frac{1}{2}}),$$

for $t \to 0$. $\left(\text{For } B_v = B_1 \text{ or } B_m \text{ the } \frac{d^k}{dy^k}\,\omega(y) \text{ at } y = B_v + 0 \text{ or } B_v - 0, \text{ respectively, must obviously be taken to be zero.}\right)$

Consider now the right-hand side of (1).

We first suppose the B_μ are pairwise different. The right-hand side in question is $\dfrac{1}{\sqrt{\prod_{\mu=1}^{m} (B_\mu - B_v - it)}}$, i.e. $O(t^{-\frac{1}{2}})$.

Secondly let us consider $B_1 = B_2 > B_3 = B_4 > \cdots > B_{m-1} = B_m$. So we may assume $v = 2\lambda \left(\lambda = 1, \cdots, \frac{m}{2}\right)$. The right-hand side of (1) becomes now a rational function, $\dfrac{1}{\prod_{k=1}^{\frac{1}{2}m} (B_{2k} - z)}$. (The sign is determined by[12].) So in our case it is $\dfrac{1}{\prod_{k=1}^{\frac{1}{2}m} (B_{2k} - B_{2\lambda} - it)}$, i.e. $\dfrac{(-1)^{\frac{1}{2}m-\lambda}}{\prod_{k=1}^{\lambda-1} (B_{2k} - B_{2\lambda}) \cdot \prod_{k=\lambda+1}^{\frac{1}{2}m} (B_{2\lambda} - B_{2k})} (-it)^{-1} + O(1)$.

Comparing these with our above expression gives therefore (for $t \to 0$)

$$\sum_{k=0}^{\frac{1}{2}m-2} \frac{\left(\frac{m}{2}-2-k\right)!}{\left(\frac{m}{2}-1\right)!} (-it)^{-\frac{1}{2}m+1+k} \left\{\frac{d^k}{dy^k}\,\omega(y)\right\}_{y=B_v-0}^{y=B_v+0}$$

$$= O(t^{-\frac{1}{2}}) \text{ in the first case,}$$

$$= \frac{(-1)^{\frac{1}{2}m-\lambda}}{\prod_{k=1}^{\lambda-1} (B_{2k} - B_{2\lambda}) \cdot \prod_{k=\lambda+1}^{\frac{1}{2}m} (B_{2\lambda} - B_{2k})} (-it)^{-1} + O(t^{-\frac{1}{2}}) \text{ in the second case.}$$

In this formula the left-hand side is a polynomial in $(-it)^{-1}$. Hence the $O(t^{-\frac{1}{2}})$ terms on the right-hand side must vanish, and otherwise all powers of $-it$ must have the same coefficient on both sides. Consequently

$$\frac{\left(\frac{m}{2}-2-k\right)!}{\left(\frac{m}{2}-1\right)!} \left\{\frac{d^k}{dy^k}\,\omega(y)\right\}_{y=B_v-0}^{y=B_v+0}$$

must vanish, except in the second case for the one value of k with $-\frac{m}{2} + 1 + k = -1$, i.e. $k = \frac{m}{2} - 2$. So, with this one exception, we have

$$\left\{\frac{d^k}{dy^k}\,\omega(y)\right\}_{y=B_v+0} = \left\{\frac{d^k}{dy^k}\,\omega(y)\right\}_{y=B_v-0}.$$

And in the exceptional case (second case, $v = 2\lambda$)

$$\left\{\frac{d^{\frac{1}{2}m-2}}{dy^{\frac{1}{2}m-2}}\,\omega(y)\right\}_{y=B_v-0}^{y=B_v+0} = (-1)^{\frac{1}{2}m-\lambda}\left(\frac{m}{2}-1\right)!\,\frac{1}{\prod_{k=1}^{\lambda-1}(B_{2k}-B_{2\lambda})\prod_{k=\lambda+1}^{\frac{1}{2}m}(B_{2\lambda}-B_{2k})}.$$

Thus in the first case all derivatives $\frac{d^k}{dy^k}\,\omega(y)$, $k = 0, 1, \cdots, \frac{m}{2} - 2$, are continuous even at $y = B_1, \cdots, B_m$.

In the second case the same is true for $k = 0, 1, \cdots, \frac{m}{2} - 3$, but the derivative with $k = \frac{m}{2} - 2$ behaves differently for $y = B_1, \cdots, B_m$. Indeed, for $y = B_{2\lambda-1} = B_{2\lambda}\left(\lambda = 1, \cdots, \frac{m}{2}\right)$ this derivative is continuous for both $y = B_{2\lambda} \pm 0$, but it increases from $B_{2\lambda} - 0$ to $B_{2\lambda} + 0$ by

$$(-1)^{\frac{1}{2}m-\lambda}\left(\frac{m}{2}-1\right)!\,\frac{1}{\prod_{k=1}^{\lambda-1}(B_{2k}-B_{2\lambda})\prod_{k=\lambda+1}^{\frac{1}{2}m}(B_{2\lambda}-B_{2k})}.$$

$\Big($At $y = B_1 + 0$ and $B_m - 0$ the $\frac{d^k}{dy^k}\,\omega(y)$ must be thought to continue with the value zero.$\Big)$

These rules, together with (2), determine $\omega(y)$ completely.

6. First special case. We consider the first special case, where the B_μ are pairwise different. We immediately specialize further, to $\gamma = \epsilon$, i.e. $m = n - 1$, $B_\mu = \cos\frac{\mu\pi}{n}$ $(\mu = 1, \cdots, n - 1)$. (Cf. the beginning of the preceding section.) Since m must be even, n must be odd. The rules of section 5 determine $\omega(y)$; in particular all derivatives $\frac{d^k}{dy^k}\,\omega(y)$, $k = 0, 1, \cdots, \frac{n-1}{2} - 2$, are everywhere continuous, beginning and ending with zero at $y = B_1$ and B_{n-1}, respectively.

In the even intervals

$$B_2 \geqq y \geqq B_3, \qquad B_4 \geqq y \geqq B_5, \cdots, B_{n-3} \geqq y \geqq B_{n-2},$$

the derivative $\frac{d^{\frac{1}{2}(n-1)-1}}{dy^{\frac{1}{2}(n-1)-1}} \omega(y)$ is zero, i.e. $\omega(y)$ is a polynomial of degree $\frac{1}{2}(n-1)-2$. In the odd intervals

$$B_1 \geqq y \geqq B_2, \qquad B_3 \geqq y \geqq B_4, \cdots, B_{n-2} \geqq y \geqq B_{n-1},$$

we have

$$\frac{d^{\frac{1}{2}(n-1)-1}}{dy^{\frac{1}{2}(n-1)-1}} \omega(y) = \pm \frac{(\frac{1}{2}[n-1]-1)!}{\pi} \frac{1}{\sqrt{-\mathfrak{A}(y)}}$$

(the sign $\pm$ is alternating $(-1)^{\frac{1}{2}(n-1)-1}, (-1)^{\frac{1}{2}(n-1)-2}, \cdots, +$), where

$$\mathfrak{A}(y) = \prod_{\mu=1}^{n-1} \left(y - \cos \frac{\mu\pi}{n} \right).$$

Another expression for $\mathfrak{A}(y)$ may be found by the following method.

Clearly
$$\frac{\sin (n\varphi)}{\sin \varphi} = \frac{e^{in\varphi} - e^{-in\varphi}}{e^{i\varphi} - e^{-i\varphi}} = \sum_{\mu=0}^{n-1} e^{i(n-1-2\mu)\varphi}$$

is a polynomial of $\cos \varphi = \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})$ of degree $n-1$, with the highest coefficient 2^{n-1}. For $\varphi = \frac{\mu\pi}{n}$, $\mu = 1, \cdots, n-1$, $\sin (n\varphi) = 0$, $\sin \varphi \neq 0$, hence $\frac{\sin (n\varphi)}{\sin \varphi}$, as a polynomial in $\cos \varphi$, has the same roots as $\mathfrak{A}(y)$. $\mathfrak{A}(y)$ is a polynomial of degree $n-1$ with the highest coefficient 1. Consequently

$$\mathfrak{A}(\cos \varphi) = \frac{1}{2^{n-1}} \frac{\sin (n\varphi)}{\sin \varphi}.$$

This formula allows one to compute $\mathfrak{A}(y)$ quickly, examples are

$$n = 3: \mathfrak{A}(y) = y^2 - \tfrac{1}{4},$$
$$n = 5: \mathfrak{A}(y) = y^4 - \tfrac{3}{4}y^2 + \tfrac{1}{16},$$
$$n = 7: \mathfrak{A}(y) = y^6 - \tfrac{5}{4}y^4 + \tfrac{3}{8}y^2 - \tfrac{1}{64}.$$

The number of odd intervals, on which integrations must be carried out, is $\frac{1}{2}(n-1)$, but since those which are symmetric with respect to 0 require the same computations, only $\frac{1}{4}(n-1)$ or $\frac{1}{4}(n+1)$ must be considered. So there are $1, 1, 2, \cdots$ such intervals for $n = 3, 5, 7, \cdots$ respectively. The integrals are first elementary (arcsin), then elliptic, then hyperelliptic.

Numerical computations for $n = 3$ are immediate; for $n = 5, 7$ they have been carried out with considerable precision by B. I. Hart.

At $y = B_\mu$, $\frac{d^{\frac{1}{2}(n-1)-1}}{dy^{\frac{1}{2}(n-1)-1}} \omega(y)$ has a singularity of the type $\frac{1}{\sqrt{|y - B_\mu|}}$ (cf. the end of section 5), while all $\frac{d^k}{dy^k} \omega(y)$, $k = 0, 1, \cdots, \frac{1}{2}(n-1) - 2$, are continuous.

At $y = B_1$ and B_{n-1}, in particular, they are zero. Hence it follows by successive integrations that the order of vanishing of $\frac{d^k}{dy^k}\,\omega(y)$, $k = 0, 1, \cdots, \frac{1}{2}(n-1) - 2$ at $y = B_1$ and B_{n-1} is $(\frac{1}{2}(n-1) - 1) - k - \frac{1}{2} = \frac{n}{2} - 2 - k$. In particular for $k = 0$ we find that at its maximum and at its minimum $\left(B_1 \text{ and } B_{n-1}\text{, i.e. } \pm \cos\frac{\pi}{n}\right)$ the order of vanishing of $\omega(y)$ is $\frac{n}{2} - 2$.[18]

Since $\omega(y)$ has this property, and since it is obviously an even function of y, R. H. Kent has suggested approximating it by a series expansion of the form

$$\omega(y) = \sum_{h=0}^{\infty} a_h \left(\cos^2\frac{\pi}{n} - y^2\right)^{\frac{1}{2}n-2+h} \tag{3}$$

Computations by B. I. Hart, not yet published, have shown that even the use of the first four terms ($h = 0, 1, 2, 3$, the a_h being determined by the condition of normalization and by the first three even moments of the actual distribution given in section 4) give excellent approximations. The use of the formula (3) suggests itself likewise for even values of n.

7. Second special case. We consider now the second special case, where $B_1 = B_2 > B_3 = B_4 > \cdots > B_{m-1} = B_m$. This has no immediate bearing on our original problem (cf. the preceding section), but we shall nevertheless discuss it for the two following reasons. First, it is hoped that the reader will find an independent interest in the simple and complete results which can be obtained in this case. Second, there are various modifications of our original problem, which lead to this case. For example let the $x_1, \cdots, x_n$ in our original problem, as described in section 1, be complex numbers instead of real ones, replacing all squares by absolute value squares. Then one verifies easily that all characteristic values $\lambda_1, \cdots, \lambda_{n-1}$ are doubled, and so our first case goes over into our second case. (This amounts to replacing our quadratic forms by Hermitian forms, cf.[4]) It is easy to imagine two-dimensional problems where this set-up is natural.

We put $C_\lambda = B_{2\lambda-1} = B_{2\lambda}$ for $\lambda = 1, \cdots, \frac{m}{2}$, so that $C_1 > C_2 > \cdots > C_{\frac{1}{2}m}$ are the only restrictions imposed.

Every y in $B_1 \geqq y \geqq B_m$, i.e. in $C_1 \geqq y \geqq C_{\frac{1}{2}m}$, lies in an interval $C_\lambda \geqq y \geqq C_{\lambda+1}$ i.e. $B_{2\lambda} \geqq y \geqq B_{2\lambda+1}$. That is the v of (2) is always even, and so $\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}}\,\omega(y)$ is zero in every one of these intervals. Therefore $\omega(y)$ is a polynomial of degree $\frac{m}{2} - 2$ in every one of these intervals. We have already shown that $\omega(y)$ is

[18] We omit the simple discussion of $n = 3$, which must be excluded from this result.

not the same polynomial in each interval. Thus $\omega(y)$ is represented by $\frac{m}{2} - 1$ polynomials of degree $\frac{m}{2} - 2$ in the $\frac{m}{2} - 1$ intervals

$$C_1 \geqq y \geqq C_2, \quad C_2 \geqq y \geqq C_3, \cdots, C_{\frac{1}{2}m-1} \geqq y \geqq C_{\frac{1}{2}m}.$$

We could try to obtain explicit expressions for these polynomials by a direct application of the results at the close of section 5. A characterization of the distribution can, however, be obtained in a more elegant way by an indirect procedure.

Consider an arbitrary function $\mathfrak{F}(y)$. We wish to express its mean

$$\overline{\mathfrak{F}(y)} = \int_{C_{\frac{1}{2}m}}^{C_1} \mathfrak{F}(y)\omega(y)\, dy.$$

If we can do this for all $\mathfrak{F}(y)$ then the distribution is completely characterized.

We select first an $\frac{m}{2} - 1$-fold primitive function of $\mathfrak{F}(y)$, i.e. a function $\mathfrak{G}(y)$ with

$$\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \mathfrak{G}(y) = \mathfrak{F}(y).$$

Of course $\mathfrak{G}(y)$ is determined only up to an additive polynomial of degree $\frac{m}{2} - 2$ in y.

Now the above expectation value becomes

$$\overline{\mathfrak{F}(y)} = \int_{C_{\frac{1}{2}m}}^{C_1} \frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \mathfrak{G}(y)\omega(y)\, dy$$
$$= \sum_{\lambda=1}^{\frac{1}{2}m-1} \int_{C_{\lambda+1}+0}^{C_\lambda-0} \frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \mathfrak{G}(y)\omega(y)\, dy.$$

Since all $\frac{d^k}{dy^k}\omega(y)$, $k = 0, 1, \cdots, \frac{m}{2} - 2$, are continuous from $C_{\lambda+1} + 0$ to $C_\lambda - 0$ for all $\lambda = 1, \cdots, \frac{m}{2} - 1$, we can evaluate each integral of the above sum by $\frac{m}{2} - 1$ successive partial integrations. Thus the following expression obtains:

$$\sum_{\lambda=1}^{\frac{1}{2}m-1} \left\{ \sum_{k=0}^{\frac{1}{2}m-2} (-1)^k \frac{d^{\frac{1}{2}m-k-2}}{dy^{\frac{1}{2}m-k-2}} \mathfrak{G}(y) \frac{d^k}{dy^k}\omega(y) \right\}_{y=C_{\lambda+1}+0}^{y=C_\lambda-0}$$
$$+ (-1)^{\frac{1}{2}m-1} \int_{C_{\frac{1}{2}m}}^{C_1} \mathfrak{G}(y) \frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \omega(y)\, dy.$$

Considering the definition of $\mathfrak{G}(y)$ as an $\frac{m}{2} - 1$-fold primitive function, the $\frac{d^{k'}}{dy^{k'}}\mathfrak{G}(y)$, $k' = 0, 1, \cdots, \frac{m}{2} - 2$, are everywhere continuous. This corresponds to $k' = \frac{m}{2} - k - 2, k = 0, 1, \cdots, \frac{m}{2} - 2$. Hence the first line can be rewritten as

$$-\sum_{\lambda=1}^{\frac{1}{2}m} \sum_{k=0}^{\frac{1}{2}m-2} (-1)^k \left\{\frac{d^{\frac{1}{2}m-k-2}}{dy^{\frac{1}{2}m-k-2}} \mathfrak{G}(y)\right\}_{y=C_\lambda} \left\{\frac{d^k}{dy^k} \omega(y)\right\}_{y=C_\lambda-0}^{y=C_\lambda+0}.$$

(For $C_\lambda = C_1$ or $C_{\frac{1}{2}m}$ the $\frac{d^k}{dy^k} \omega(y)$ at $y = C_1 + 0$ or $C_{\frac{1}{2}m} - 0$, respectively, must obviously be taken to be zero.) Owing to the results of section 5 all terms with $k = 0, 1, \cdots, \frac{m}{2} - 3$ vanish, and the term with $k = \frac{m}{2} - 2$ gives

$$-\sum_{\lambda=1}^{\frac{1}{2}m} (-1)^{\frac{1}{2}m-2} \mathfrak{G}(C_\lambda)(-1)^{\frac{1}{2}m-\lambda} \left(\frac{m}{2} - 1\right)! \frac{1}{\prod_{k=1}^{\lambda-1} (C_k - C_\lambda) \prod_{k=\lambda+1}^{\frac{1}{2}m} (C_\lambda - C_k)}$$

$$= \sum_{\lambda=1}^{\frac{1}{2}m} (-1)^{\lambda-1} \left(\frac{m}{2} - 1\right)! \frac{1}{\prod_{k=1}^{\lambda-1} (C_k - C_\lambda) \prod_{k=\lambda+1}^{\frac{1}{2}m} (C_\lambda - C_k)} \mathfrak{G}(C_\lambda).$$

The second line vanishes, since $\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \omega(y)$ is zero everywhere, as observed above.

Finally

$$\overline{\mathfrak{F}(y)} = \sum_{\lambda=1}^{\frac{1}{2}m} (-1)^{\lambda-1} \left(\frac{m}{2} - 1\right)! \frac{1}{\prod_{k=1}^{\lambda-1} (C_k - C_\lambda) \prod_{k=\lambda+1}^{\frac{1}{2}m} (C_\lambda - C_k)} \mathfrak{G}(C_\lambda).$$

For

$$\mathfrak{B}(z) = \prod_{k=1}^{\frac{1}{2}m} (z - C_k)$$

we have

$$\left\{\frac{d}{dz} \mathfrak{B}(z)\right\}_{z=C_\lambda} = \prod_{k=1}^{\frac{1}{2}m}{}_{(k\neq\lambda)} (C_\lambda - C_k)$$

$$= (-1)^{\lambda-1} \prod_{k=1}^{\lambda-1} (C_k - C_\lambda) \prod_{k=\lambda+1}^{\frac{1}{2}m} (C_\lambda - C_k).$$

Therefore the above formula can also be written

$$\overline{\mathfrak{F}(y)} = \left(\frac{m}{2} - 1\right)! \sum_{\lambda=1}^{\frac{1}{2}m} \frac{\mathfrak{G}(C_\lambda)}{\left\{\frac{d}{dz} \mathfrak{B}(z)\right\}_{z=C_\lambda}}.$$

Observe that the right-hand side of the above formula (which can also be easily expressed in terms of determinants) is a well-known approximate expression for $\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}}\,\mathfrak{G}(y)$, as a (repeated) difference quotient of the values $\mathfrak{G}(C_\lambda)$, $\lambda = 1, \cdots, \frac{m}{2}$. It is therefore very satisfactory that this expression gives the mean of

$$\mathfrak{F}(y) = \frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}}\,\mathfrak{G}(y).$$

Appendix. We return to the normal distribution of $x_1, \cdots, x_n$ as described in section 1, and to the quantities s^2, δ^2, η given there. We denote means with respect to that distribution by $\overline{(\cdots)}$.

It was observed by B. I. Hart and mentioned by J. D. Williams[3] by comparing the known expressions for their moments, that every moment of $\eta = \frac{\delta^2}{s^2}$ is the quotient of the corresponding moments of δ^2 and of s^2. That is

$$\overline{\left(\frac{\delta^{2p}}{s^{2p}}\right)} = \frac{\overline{\delta^{2p}}}{\overline{s^{2p}}}, \qquad (p = 0, 1, 2, \cdots).$$

This indicates some kind of independence relation involving δ^2 and s^2. The considerations which follow are intended to clarify this situation.

The above relation may be written

$$\overline{s^{2p}}\;\overline{\eta^p} = \overline{s^{2p}\eta^p},$$

or, more generally,

$$\overline{s^q}\;\overline{\eta^p} = \overline{s^q\eta^p}.$$

We shall prove this by showing that s and η are statistically independent.

We can, as in section 2, make the mean $\xi = 0$, i.e. obtain the $x_1, \cdots, x_n$ distribution law

$$c^n e^{-\sum_{\mu=1}^{n} x_\mu^2/2\sigma^2}\,dx_1 \cdots dx_n.$$

And then, again as in section 2, perform a linear orthogonal transformation, carrying $x_1, \cdots, x_n$ into, say $x'_1, \cdots, x'_n$ which leaves the distribution law in its original form

$$c^n e^{-\sum_{\mu=1}^{n} x'^2_\mu/2\sigma^2}\,dx'_1 \cdots dx'_n,$$

and makes

$$s^2 = \frac{1}{n}\sum_{\mu=1}^{n-1} x'^2_\mu,$$

$$\eta = \frac{n}{n-1}\,\frac{\sum_{\mu=1}^{n-1} A_\mu x'^2_\mu}{\sum_{\mu=1}^{n-1} x'^2_\mu}.$$

Since x'_n does not occur in s^2, η we must use only the $x'_1, \cdots, x'_{n-1}$ distribution law

$$c^{n-1} e^{-\sum_{\mu=1}^{n-1} x'^2_\mu / 2\sigma^2} dx'_1 \cdots dx'_{n-1}.$$

Now we introduce polar coordinates with respect to $x'_1, \cdots, x'_{n-1}$. These consist of a radius r with

$$r^2 = \sum_{\mu=1}^{n-1} x'^2_\mu,$$

and $n - 2$ angular variables $\varphi_1, \cdots, \varphi_{n-2}$, which can be chosen in various ways, and which we need not describe more closely. At any rate

$$dx'_1 \cdots dx'_{n-1} = r^{n-2} \, dr w(\varphi_1, \cdots, \varphi_{n-2}) \, d\varphi_1 \cdots d\varphi_{n-2}$$

where we need not determine the weight function $w(\varphi_1, \cdots, \varphi_{n-2})$. Consequently the distribution law is

$$c^{n-1} e^{-r^2/2\sigma^2} r^{n-2} \, dr w(\varphi_1, \cdots, \varphi_{n-2}) \, d\varphi_1 \cdots d\varphi_{n-2}.$$

Thus the coordinate r and the coordinates $\varphi_1, \cdots, \varphi_{n-2}$ are independent of each other.

Next

$$s^2 = \frac{1}{n} r^2,$$

and η is a homogeneous function of $x'_1, \cdots, x'_{n-1}$ of degree zero, i.e. it is independent of r. So s is a function of r alone, and η is a function of $\varphi_1, \cdots, \varphi_{n-2}$ alone. Consequently s and η likewise are independent.

Added in proof:

After this manuscript was completed, Dr. T. Koopmans informed the author of several results of his own, which he obtained in connection with other statistical investigations. They have many points of contact with this investigation, and will appear in the near future in the *Annals of Mathematical Statistics*. The author wishes to express his thanks to Dr. T. Koopmans for his communications.