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DISTRIBUTION OF ZEROS OF ORTHOGONAL POLYNOMIALS[1]

BY

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ABSTRACT. The purpose of the paper is to investigate distribution of zeros of orthogonal polynomials given by a three term recurrence relation.

Let α be a nondecreasing and bounded function on the real line such that the range of α is infinite and $x^n \in L^2(d\alpha)$ for every $n \in \mathbf{N}$. Then there exists a unique system of polynomials $\{p_n(d\alpha)\}_{n=0}^{\infty}$ such that

$$p_n(d\alpha, x) = \gamma_n(d\alpha)x^n + \cdots, \qquad \gamma_n(d\alpha) > 0,$$

and

$$\int_{-\infty}^{\infty} p_n(d\alpha, t)p_m(d\alpha, t)\, d\alpha(t) = \delta_{nm}.$$

Such a system of polynomials satisfies the three term recurrence relation

$$xp_{n-1}(d\alpha, x) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, p_n(d\alpha, x) + \alpha_{n-1}(d\alpha)p_{n-1}(d\alpha, x) + \frac{\gamma_{n-2}(d\alpha)}{\gamma_{n-1}(d\alpha)}\, p_{n-2}(d\alpha, x),$$

$n = 1, 2, \ldots,$ where $p_0(d\alpha, x) = \gamma_0, p_{-1}(d\alpha, x) = 0$ and

$$\alpha_n(d\alpha) = \int_{-\infty}^{\infty} xp_n^2(d\alpha, x)\, d\alpha(x).$$

This recurrence relation completely characterizes orthogonal polynomials in the following sense. If a system of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfies the recursion formula

$$xp_{n-1}(x) = \frac{\gamma_{n-1}}{\gamma_n}\, p_n(x) + \alpha_{n-1}p_{n-1}(x) + \frac{\gamma_{n-2}}{\gamma_{n-1}}\, p_{n-2}(x)$$

for $n = 1, 2, \ldots$ with $p_{-1} \equiv 0$, $p_0 \equiv \gamma_0$, $\gamma_n > 0$ and $\alpha_n \in \mathbf{R}$ then $\{p_n\}$ is orthogonal with respect some weight α. This beautiful result was proved by J. Favard. (See e.g. [5].) In many instances α is uniquely determined by the recurrence relation. This is the case when both $\{\gamma_{n-1}/\gamma_n\}$ and $\{|\alpha_n|\}$ are bounded sequences or, in other words, when the support of $d\alpha$ is compact.

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Let us recall that supp($d\alpha$) is always closed, therefore compactness is equivalent to boundedness.

Although a great number of results are known on specific orthogonal polynomials, the general theory of orthogonal polynomials still hides its little and big secrets. For example, the behavior of Christoffel functions is a little secret and Steklov's conjecture is a big one. There exist, however, a few classes of weight functions when one can prove nice results for the corresponding orthogonal polynomials. Such classes are S and M which are defined as follows: $\alpha \in S$ if supp($d\alpha$) $= [-1, 1]$ and $\log \alpha'(\cos\theta)$ is integrable on $[0, \pi]$. $\alpha \in M$ if the coefficients in the corresponding recurrence relations satisfy the conditions

$$\lim_{n\to\infty} \alpha_n(d\alpha) = 0, \qquad \lim_{n\to\infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} = \frac{1}{2}.$$

Let us remark that if $\alpha \in S$ then

$$\sum_{k=1}^{\infty} \left\{ \alpha_k^2(d\alpha) + \left[\frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} - \frac{1}{2} \right]^2 \right\} < \infty,$$

therefore $S \subset M$. (See [**9**].) Other examples of weights belonging to M are the Pollaczek weight [**10**] and β defined by supp($d\beta$) $= [-1, 1]$, β is absolutely continuous and

$$\beta'(x) = \exp\left\{ -\frac{1}{\sqrt{1-x^2}} \right\}$$

for $-1 < x < 1$ [**9**]. The class S has been thoroughly investigated by G. Szegö, S. Bernstein, N. Akhiezer, M. Krein, A. Kolmogorov, Ya. Geronimus, G. Freud and others during the last sixty years. (See [**5**], [**6**], [**7**] and [**10**].) Much less work has been completed concerning M. For the last few years it became known that those weights α which belong to M play an important role in applications of the theory of orthogonal polynomials since in many cases systems of orthogonal polynomials arise from difference equations. For a subclass of M, K. Case has obtained surprising results [**3**]. The whole class M has been thoroughly investigated in [**9**].

The purpose of this paper is to investigate the distribution of zeros of orthogonal polynomials corresponding to weights which belong to M. Let us recall that all the zeros of orthogonal polynomials are real. We denote them by $x_{kn}(d\alpha)$:

$$x_{1n}(d\alpha) < x_{2n}(d\alpha) < \cdots < x_{nn}(d\alpha).$$

It was G. Szegö [**10**] who proved that if $\alpha \in S$ and f is continuous on $[-1, 1]$ then

$$\sum_{k=1}^{n} f(x_{kn}(d\alpha)) = \frac{n}{\pi} \int_0^{\pi} f(\cos\theta)\, d\theta + o(n) \tag{1}$$

an $n \to \infty$. Later P. Erdös and P. Turan [4] showed that (1) still holds if the condition $\alpha \in S$ is replaced by $\operatorname{supp}(d\alpha) = [-1, 1]$ and $\alpha'(x) > 0$ for almost every $x \in [-1, 1]$. In [9] we proved (1) when $\alpha \in M$.

Before going into more details let us note that the zeros of orthogonal polynomials are eigenvalues of very specific Toeplitz matrices. The eigenvalue distribution of Toeplitz matrices has been investigated by several authors. Let us refer to [2], [7], [8], and [9].

In this paper we are going to find the meaning of $o(n)$ in (1) when α is a relatively nice weight. Furthermore, we will give conditions assuming the existence of

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n} f(x_{kn}(d\beta)) - \sum_{k=1}^{n} f(x_{kn}(d\alpha))\right]$$

and we will also calculate this limit.

Let us remark that R. Askey is largely responsible for the birth of M. Had he less conjectures (see e.g. [1, p. 46]) the class M would never have been born.

For a given weight $d\alpha$ the Christoffel function $\lambda_n(d\alpha)$ is defined by

$$\lambda_n(d\alpha, z) = \min_{\substack{\pi_n \in \mathbf{P}_n \\ \pi_n(z)=1}} \int_{-\infty}^{\infty} |\pi_n(t)|^2 \, d\alpha(t)$$

where $\mathbf{P}_n$ is the set of all algebraic polynomials of degree less than n. The numbers $\lambda_n(d\alpha, x_{kn}(d\alpha))$ are called Cotes numbers and are denoted by $\lambda_{kn}(d\alpha)$. By the Gauss-Jacobi quadrature formula for every polynomial $\pi \in \mathbf{P}_{2n}$

$$\sum_{k=1}^{n} \pi(x_{kn}(d\alpha))\lambda_{kn}(d\alpha) = \int_{-\infty}^{\infty} \pi(t)\, d\alpha(t). \tag{2}$$

The fundamental polynomials of Lagrange interpolation corresponding to $d\alpha$ are denoted by $l_{kn}(d\alpha, x)$ $(k = 1, 2, \dots, n)$. We have

$$l_{kn}(d\alpha, x) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \lambda_{kn}(d\alpha) p_{n-1}(d\alpha, x_{kn}(d\alpha)) \frac{p_n(d\alpha, x)}{x - x_{kn}(d\alpha)}. \tag{3}$$

We will often make use of the following relations:

$$\lambda_n^{-1}(d\alpha, x) = \sum_{k=0}^{n-1} p_k^2(d\alpha, x) = \sum_{k=1}^{n} \frac{l_{kn}^2(d\alpha, x)}{\lambda_{kn}(d\alpha)} \tag{4}$$

for $x \in \mathbf{R}$ and

$$\int_{-\infty}^{\infty} \frac{l_{kn}^2(d\alpha, t)}{\lambda_{kn}(d\alpha)}\, d\alpha(t) = 1, \qquad \int_{-\infty}^{\infty} (t - x_{kn}(d\alpha)) \frac{l_{kn}^2(d\alpha, t)}{\lambda_{kn}(d\alpha)}\, d\alpha(t) = 0. \tag{5}$$

The expression $K_n(d\alpha, x, t)$ defined by

$$K_n(d\alpha, x, t) = \sum_{k=1}^{n-1} p_k(d\alpha, x)p_k(d\alpha, t)$$

is called Dirichlet kernel. By the Christoffel-Darboux summation formula

$$K_n(d\alpha, x, t) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \cdot \frac{p_n(d\alpha, x)p_{n-1}(d\alpha, t) - p_{n-1}(d\alpha, x)p_n(d\alpha, t)}{x - t}. \quad (6)$$

For proofs of (2)–(6) see [5].

In the following Δ will always denote a closed interval. It is an easy exercise to show that from $\operatorname{supp}(d\alpha) \subset \Delta$ the inequality

$$\gamma_{n-1}(d\alpha)/\gamma_n(d\alpha) \leqslant |\Delta|/2 \quad (7)$$

follows.

Let α be a given weight and let $g\ (\geqslant 0) \in L^1(d\alpha)$. Define β by $d\beta = g d\alpha$. If g behaves nicely then β is also a weight function. If, for example, $\operatorname{supp}(d\alpha)$ is compact and $g^{-1} \in L^\infty(d\alpha)$ then β is a weight. We will denote this weight β by $d\alpha_g$, that is $d\alpha_g = g d\alpha$.

Let $h\ (\geqslant 0)$ be defined on $[-1, 1]$ and assume that $\log h(\cos\theta)$ is integrable on $[0, \pi]$. Then the Szegö function $D(h, z)$ corresponding to h is defined by

$$D(h, z) = \exp\left\{ \frac{1}{4\pi} \int_0^{2\pi} \log h(\cos\theta) \cdot \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\, d\theta \right\}$$

for $|z| < 1$. Recall that $D(h) \in H_2$ in the unit disc, $D(h, z) \neq 0$ for $|z| < 1$,

$$\lim_{r\uparrow 1} D(h, re^{it}) = D(h, e^{it})$$

exists for almost every $t \in [0, 2\pi]$ and $|D(h, e^{it})|^2 = h(\cos t)$ for almost every $t \in [0, 2\pi]$. The argument of $D(h, e^{it})$ will be denoted by $\Gamma(h, t)$. $\Gamma(h, t)$ can be calculated by the formula

$$\Gamma(h, t) = \frac{1}{2\pi} \int_{-1}^{1} \frac{\log h(x) - \log h(y)}{x - y} \frac{\sin t}{\sqrt{1 - x^2}}\, dx$$

($y = \cos t$). Therefore $\Gamma(h, t) = -\Gamma(h, -t)$ and $\Gamma(h_1 h_2, t) = \Gamma(h_1, t) + \Gamma(h_2, t)$. If $\operatorname{supp}(d\alpha) \supset [-1, 1]$ and $\log\alpha'(\cos\theta)$ is integrable then $D(d\alpha, z)$ is defined by $D(d\alpha, z) = D(\alpha', z)$. For further properties of Szegö's function we refer to [5] and [10].

A weight function is called a Jacobi weight if it is absolutely continuous with support $[-1, 1]$ and its derivative $w = w^{(\delta,\varepsilon)}$ is defined by

$$w(x) = (1 - x)^\delta (1 + x)^\varepsilon \qquad (\delta > -1, \varepsilon > -1)$$

for $-1 < x < 1$. Let us recall a few simple properties of Jacobi polynomials which we will need later. One can explicitly compute the coefficients in the recurrence formula for Jacobi polynomials. Once the computation has been done it becomes clear that

$$\frac{\gamma_{n-1}(w)}{\gamma_n(w)} = \frac{1}{2} + O\left(\frac{1}{n^2}\right), \qquad \alpha_n(w) = O\left(\frac{1}{n^2}\right) \tag{8}$$

as $n \to \infty$. Furthermore, the inequalities

$$|p_n(w, x)| \leqslant C(1-x)^{-\delta/2-1/4}(1+x)^{-\varepsilon/2-1/4} \qquad (1 - x^2 \geqslant 1/10n) \tag{9}$$

and

$$\lambda_n^{-1}(w, x) \leqslant \begin{cases} Cn^{2\delta+2} & (1 - 1/n \leqslant x \leqslant 1), \\ Cn^{2\varepsilon+2} & (-1 \leqslant x \leqslant -1 + 1/n) \end{cases} \tag{10}$$

for $n = 1, 2, \ldots$ are satisfied with constants depending only on δ and ε. We will also use the estimate

$$\frac{p_{n+1}(w, z)}{p_n(w, z)} = z + \sqrt{z^2 - 1} + O\left(\frac{1}{n^2}\right) \tag{11}$$

$(n \to \infty)$ which holds uniformly on every compact set lying outside any ellipse with foci at -1 and 1. Here $\sqrt{z^2 - 1}$ denotes that branch which is positive for $z > 1$. For $\delta = \varepsilon = -\frac{1}{2}$, that is in case of the Chebyshev weight, we write v instead of w. All these properties of Jacobi polynomials can be found in **[10]**. In the following T_n and U_n will denote the Chebyshev polynomials of first and second kind respectively, that is

$$T_n(x) = \cos n\theta$$

and

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$$

where $x = \cos\theta$.

Now we can formulate the main result of this paper. They are stated in the following two theorems.

THEOREM 1. *Let $\alpha \in M$ and let $g > 0$ be continuous on* supp$(d\alpha)$. *Assume that f is twice continuously differentiable on $\Delta \supset$* supp$(d\alpha)$. *Then*

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n} f(x_{kn}(d\alpha_g)) - \sum_{k=1}^{n} f(x_{kn}(d\alpha))\right]$$

$$= \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi \sqrt{\frac{g(\cos\psi)}{g(\cos\theta)}}\ \frac{f(\cos\psi) - f(\cos\theta)}{(\cos\psi - \cos\theta)^2}$$
$$\cdot\left\{\cos[\Gamma(g,\psi)]\cos[\Gamma(g,\theta)](1 - \cos\psi\cos\theta) + \sin[\Gamma(g,\psi)]\sin[\Gamma(g,\theta)]\sin\psi\sin\theta\right\} d\psi\, d\theta$$

where the integral is understood in the Cauchy-Lebesgue sense.

This theorem follows immediately from Theorems 9 and 13. If α is a Jacobi weight then Theorem 1 can be improved.

THEOREM 2. *Let $w = w^{(\delta,\varepsilon)}$ be a Jacobi weight. Let $g \geqslant 0$ be such that $g^{\pm 1}$ belongs to L^∞ on $[-1, 1]$. Suppose that f is twice continuously differentiable on $[-1, 1]$. Then*

$$\sum_{k=1}^{n} f(x_{kn}(gw)) = \frac{2n+\delta+\varepsilon+1}{2\pi}\int_0^\pi f(\cos\theta)\,d\theta - \frac{2\delta+1}{4}f(1) - \frac{2\varepsilon+1}{4}f(-1) + \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi \sqrt{\frac{g(\cos\psi)}{g(\cos\theta)}}\;\frac{f(\cos\psi)-f(\cos\theta)}{(\cos\psi-\cos\theta)^2} \cdot\{\cos[\Gamma(g,\psi)]\cos[\Gamma(g,\theta)](1-\cos\psi\cos\theta) + \sin[\Gamma(g,\psi)]\sin[\Gamma(g,\theta)]\sin\psi\sin\theta\}\,d\psi\,d\theta + o(1)$$

as $n \to \infty$ where the integral is defined in the Cauchy-Lebesgue sense.

Theorem 2 is a direct consequence of Theorems 9, 14 and 17.

As mentioned earlier, in [9] we investigated orthogonal polynomials corresponding to weights which belong to M. In particular, the following six lemmas were proved in [9]. These lemmas play a key role in proving the results of this paper.

LEMMA 3. *Let $\alpha \in M$. Then the support of $d\alpha$ can be written as the union of $[-1, 1]$ and B where B is denumerable, bounded and the only possible points of accumulation of B are -1 and 1. Furthermore, if Q is the set of all zeros of all $p_n(d\alpha)$ then all the points of accumulation of Q belong to* $\operatorname{supp}(d\alpha)$.

LEMMA 4. *Let $\alpha \in M$. If D is any domain in the complex plane such that $\bar{D} \cap \operatorname{supp}(d\alpha) = \varnothing$ then*

$$\lim_{n\to\infty} \frac{p_n(d\alpha, z)}{p_{n+1}(d\alpha, z)} = \frac{1}{z+\sqrt{z^2-1}}$$

uniformly for $z \in D$. If $x \in \operatorname{supp}(d\alpha)\setminus[-1, 1]$ then

$$\lim_{n\to\infty} \frac{p_n(d\alpha, x)}{p_{n+1}(d\alpha, x)} = x+\sqrt{x^2-1}\,.$$

LEMMA 5. *Let $\alpha \in M$ and let l be a fixed integer. If f is bounded on* $\operatorname{supp}(d\alpha)$ *and continuous on $[-1, 1]$ then*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(t)p_n(d\alpha, t)p_{n+l}(d\alpha, t)\,d\alpha(t) = \frac{1}{\pi}\int_{-1}^{1} f(t)T_l(t)\,\frac{dt}{\sqrt{1-t^2}}\,.$$

LEMMA 6. *Let $\alpha \in M$ and $l \in \mathbf{Z}$. Assume that f is bounded on $\Delta \supset$ supp$(d\alpha)$ and Riemann integrable on $[-1, 1]$. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) f(x_{kn}(d\alpha)) p_{n-1}(d\alpha, x_{kn}(d\alpha)) p_{n+l}(d\alpha, x_{kn}(d\alpha))$$
$$= -\operatorname{sign} l \, \frac{2}{\pi} \int_{-1}^{1} f(t) U_{|l|-1}(t) \sqrt{1-t^2}\, dt.$$

LEMMA 7. *Let*

$$\sum_{k=1}^{\infty} \left[|\alpha_k(d\alpha)| + \left| \frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} - \frac{1}{2} \right| \right] < \infty.$$

Then for every $x \in (-1, 1)$ $(x = \cos\theta)$

$$\left| |p_n(d\alpha, x) - e^{i\theta} p_{n-1}(d\alpha, x)| - \left(\frac{2}{\pi} \frac{\sqrt{1-x^2}}{\alpha'(x)} \right)^{1/2} \right|$$
$$< \sum_{k=n+1}^{\infty} |p_k(d\alpha, x) - 2x p_{k-1}(d\alpha, x) + p_{k-2}(d\alpha, x)|.$$

LEMMA 8. *Let $\alpha \in M$. Let g (> 0) be such that $g^{\pm 1}$ is bounded on* supp$(d\alpha)$ *and g is Riemann integrable on $[-1, 1]$. Then $\alpha_g \in M$. Furthermore, if $K \subset C \cup \{\infty\} \setminus$ supp$(d\alpha)$ is an arbitrary closed set then*

$$\lim_{n\to\infty} \frac{p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = D\left(g, z - \sqrt{z^2 - 1}\right)^{-1}$$

uniformly for $z \in K$.

Now we will prove a number of auxiliary results.

THEOREM 9. *Let $\alpha \in M$. If f is twice continuously differentiable on $\Delta \supset$* supp$(d\alpha)$ *then*

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} f(x_{kn}(d\alpha)) - \int_{-\infty}^{\infty} f(t) \lambda_n^{-1}(d\alpha, t)\, d\alpha(t) \right]$$
$$= \frac{1}{2\pi} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\, dt - \frac{f(1)}{4} - \frac{f(-1)}{4}. \tag{12}$$

PROOF. Let

$$L_n(f) = \sum_{k=1}^{n} f(x_{kn}) - \int_{\infty}^{\infty} f(t) \lambda_n^{-1}(t)\, d\alpha(t).$$

First we will show that

$$|L_n(f)| < \frac{|\Delta|^2}{8} \max_{t\in\Delta} |f''(t)|. \tag{13}$$

Using (4) and (5) we obtain

$$L_n(f) = \sum_{k=1}^{n} \int_{-\infty}^{\infty} [f(x_{kn}) - f(t)] \frac{l_{kn}^2(t)}{\lambda_{kn}}\, d\alpha(t).$$

By Taylor's formula

$$L_n(f) = \sum_{k=1}^{n} f'(x_{kn}) \int_{-\infty}^{\infty} (x_{kn} - t) \frac{l_{kn}^2(t)}{\lambda_{kn}}\, d\alpha(t)$$

$$- \frac{1}{2} \sum_{k=1}^{n} f''(y_{kn}) \int_{-\infty}^{\infty} (t - x_{kn})^2 \frac{l_{kn}^2(t)}{\lambda_{kn}}\, d\alpha(t)$$

where y_{kn} are some points belonging to Δ. Here the first sum on the right-hand side equals 0. Furthermore by (3)

$$\int_{-\infty}^{\infty} (t - x_{kn})^2 \frac{l_{kn}^2(t)}{\lambda_{kn}}\, d\alpha(t) = \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \lambda_{kn} p_{n-1}^2(x_{kn}).$$

Hence

$$|L_n(f)| \leqslant \frac{1}{2}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \max_{t\in\Delta} |f''(t)|.$$

Now (13) follows from (7). Inequality (13) and linearity of L_n suggest that (12) has to be checked only when f is of the form $f(t) = t^N$ $(N = 0, 1, \ldots)$. If $N = 0$ then (12) is clearly true. Let $N \geqslant 1$ be fixed. Then $-L_n(f)$ can be written as

$$- L_n(f) = \sum_{j=0}^{N-1} \sum_{k=1}^{n} x_{kn}^j \int_{-\infty}^{\infty} t^{N-1-j}(t - x_{kn}) \frac{l_{kn}^2(t)}{\lambda_{kn}}\, d\alpha(t),$$

that is by (3)

$$- L_n(f) = \frac{\gamma_{n-1}}{\gamma_n} \sum_{j=0}^{N-1} \sum_{k=1}^{n} p_{n-1}(x_{kn}) x_{kn}^j \int_{-\infty}^{\infty} t^{N-1-j} p_n(t) l_{kn}(t)\, d\alpha(t).$$

Expanding $t^{N-1-j}p_n(t)$ into Fourier series in $p_l(t)$ and using orthogonality relations for p_l we obtain

$$t^{N-1-j}p_n(t) = \sum_{l=n-N+1+j}^{n+N-1-j} \int_{-\infty}^{\infty} x^{N-1-j} p_n(x) p_l(x)\, d\alpha(x) p_l(t).$$

Therefore

$$- L_n(f) = \frac{\gamma_{n-1}}{\gamma_n} \sum_{j=0}^{N-1} \left[\sum_{l=n-N+1+j}^{n+N-1-j} \int_{-\infty}^{\infty} x^{N-1-j} p_n(x) p_l(x)\, d\alpha(x) \right.$$

$$\left. \cdot \sum_{k=1}^{n} p_{n-1}(x_{kn}) x_{kn}^j \int_{-\infty}^{\infty} p_l(t) l_{kn}(t)\, d\alpha(t) \right].$$

Using again orthogonality relations this expression for $-L_n(f)$ reduces to

$$-L_n(f) = \frac{\gamma_{n-1}}{\gamma_n} \sum_{j=0}^{N-2} \left[\sum_{l=n-N+1+j}^{n-1} \int_{-\infty}^{\infty} x^{N-1-j} p_n(x) p_l(x)\, d\alpha(x) \cdot \sum_{k=1}^{n} p_{n-1}(x_{kn}) x_{kn}^j \int_{-\infty}^{\infty} p_l(t) l_{kn}(t)\, d\alpha(t) \right].$$

Because l is always less than n and the degree of l_{kn} is $n-1$ we can apply the Gauss-Jacobi quadrature formula in the last integral. We get

$$-L_n(f) = \frac{\gamma_{n-1}}{\gamma_n} \sum_{j=0}^{N-2} \left[\sum_{l=n-N+1+j}^{n-1} \int_{-\infty}^{\infty} x^{N-1-j} p_n(x) p_l(x)\, d\alpha(x) \cdot \sum_{k=1}^{n} \lambda_{kn} p_{n-1}(x_{kn}) p_l(x_{kn}) x_{kn}^j \right].$$

Finally, changing l into $n-l$ we obtain

$$-L_n(f) = \frac{\gamma_{n-1}}{\gamma_n} \sum_{j=0}^{N-2} \left[\sum_{l=1}^{N-1-j} \int_{-\infty}^{\infty} x^{N-1-j} p_n(x) p_{n-l}(x)\, d\alpha(x) \cdot \sum_{k=1}^{n} \lambda_{kn} p_{n-1}(x_{kn}) p_{n-l}(x_{kn}) x_{kn}^j \right].$$

Lemmas 5 and 6 show that $\lim_{n\to\infty} L_n(f)$ exists and equals

$$-\frac{1}{\pi^2} \sum_{j=0}^{N-2} \sum_{l=1}^{N-1-j} \int_{-1}^{1} x^{N-1-j} T_l(x) \frac{dx}{\sqrt{1-x^2}} \int_{-1}^{1} t^j U_{l-1}(t) \sqrt{1-t^2}\, dt. \tag{14}$$

Instead of calculating this expression let us note that it is independent of $\alpha \in M$. Therefore if we calculate $\lim_{n\to\infty} L_n(f)$ ($f(t) = t^N$) for one particular weight $\alpha \in M$ we will know the value of (14). Let us choose $\alpha = v$ to be the Chebyshev weight. Then $\lambda_{kn}(v) = \pi/n$ for $k = 1, 2, \ldots, n$. Hence by Gauss-Jacobi quadrature formula

$$\sum_{k=1}^{n} x_{kn}(v)^N = \frac{n}{\pi} \int_{-1}^{1} t^N \frac{dt}{\sqrt{1-t^2}}$$

for $2n > N$. Furthermore, by a simple calculation

$$\lambda_n^{-1}(v, x) = (1/\pi)\left[n - \tfrac{1}{2} + \tfrac{1}{2} U_{2n-2}(x)\right].$$

Consequently

$$\int_{-1}^{1} t^N \lambda_n^{-1}(v, t) v(t)\, dt = \frac{2n-1}{2\pi} \int_{-1}^{1} t^N \frac{dt}{\sqrt{1-t^2}} + \frac{1}{2\pi} \int_{-1}^{1} t^N \frac{U_{2n-2}(t)}{1-t^2} \sqrt{1-t^2}\, dt.$$

Hence in this case

$$L_n(f) = \frac{1}{2\pi}\int_{-1}^{1} t^N \frac{dt}{\sqrt{1-t^2}} - \frac{1}{2\pi}\int_{-1}^{1} t^N \frac{U_{2n-2}(t)}{1-t^2}\sqrt{1-t^2}\ dt$$

($f(t) = t^N$, $2n > N$). If N is odd $L_n(f) = 0$. If N is even then note that $t^N - t^{N-2}$ can be divided by $1 - t^2$. Therefore

$$L_n(f) = \frac{1}{2\pi}\int_{-1}^{1} t^N \frac{dt}{\sqrt{1-t^2}} - \frac{1}{2\pi}\int_{-1}^{1} \frac{U_{2n-2}(t)}{1-t^2}\sqrt{1-t^2}\ dt.$$

But

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{U_{2n-2}(t)}{1-t^2}\sqrt{1-t^2}\ dt = \frac{1}{2\pi}\int_{0}^{\pi} \frac{\sin(2n-1)t}{\sin t}\ dt = \frac{1}{2}.$$

Hence we obtain that (14) equals

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{t^N}{\sqrt{1-t^2}}\ dt - \frac{(1)^N}{4} - \frac{(-1)^N}{4}.$$

Therefore (12) holds if f is a polynomial.

It is easy to see that Theorem 9 is much stronger than Szegö's result on the distribution of zeros of orthogonal polynomials corresponding to weights which belong to S. (See [**10**, Chapter 12.7].) Actually, Theorem 9 implies the following result which improves Theorem 5.3 of [**9**].

THEOREM 10. *Let $\alpha \in M$ and let f be continuously differentiable on $\Delta \supset$* supp($d\alpha$). *Then*

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n} f(x_{kn}(d\alpha)) - \sum_{k=1}^{n-1} f(x_{k,n-1}(d\alpha))\right] = \frac{1}{\pi}\int_{-1}^{1} f(t)\frac{dt}{\sqrt{1-t^2}}. \tag{15}$$

PROOF. If f'' is continuous then (15) follows from Lemma 5 and (12). Thus the theorem holds if we can show that

$$\left|\sum_{k=1}^{n} f(x_{kn}(d\alpha)) - \sum_{k=1}^{n-1} f(x_{k,n-1}(d\alpha))\right| \leq C\cdot\left[\max_{x\in\Delta}|f(x)| + \max_{x\in\Delta}|f'(x)|\right] \tag{16}$$

where C is independent of f and n. But this is an immediate consequence of the mean value theorem and the fact that the zeros of $p_n(d\alpha)$ and $p_{n-1}(d\alpha)$ interlace. We have

$$\left|\sum_{k=1}^{n} f(x_{kn}(d\alpha)) - \sum_{k=1}^{n-1} f(x_{k,n-1}(d\alpha))\right|$$
$$= \left|\sum_{k=1}^{n-1}\left[f(x_{kn}(d\alpha)) - f(x_{k,n-1}(d\alpha))\right] + f(x_{nn}(d\alpha))\right|$$
$$< \max_{x\in\Delta}|f'(x)| \sum_{k=1}^{n-1} |x_{kn}(d\alpha) - x_{k,n-1}(d\alpha)| + \max_{x\in\Delta}|f(x)|$$
$$= \max_{x\in\Delta}|f'(x)|\left[\sum_{k=1}^{n} x_{kn}(d\alpha) - \sum_{k=1}^{n-1} x_{k,n-1}(d\alpha) - x_{nn}(d\alpha)\right] + \max_{x\in\Delta}|f(x)|.$$

Now by (4) and (5)

$$\sum_{k=1}^{n} x_{kn}(J\alpha) = \sum_{k=1}^{n} \int_{-\infty}^{\infty} t\,\frac{l_{kn}^2(d\alpha, t)}{\lambda_{kn}(d\alpha)}\,d\alpha(t)$$
$$= \int_{-\infty}^{\infty} t\lambda_n^{-1}(d\alpha, t)\,d\alpha(t)$$
$$= \int_{-\infty}^{\infty} tp_{n-1}^2(d\alpha, t)\,d\alpha(t) + \int_{-\infty}^{\infty} t\lambda_{n-1}^{-1}(d\alpha, t)\,d\alpha(t)$$
$$= \alpha_{n-1}(d\alpha) + \sum_{k=1}^{n-1} \int_{-\infty}^{\infty} t\,\frac{l_{k,n-1}^2(d\alpha, t)}{\lambda_{k,n-1}(d\alpha)}\,d\alpha(t)$$
$$= \alpha_{n-1}(d\alpha) + \sum_{k=1}^{n-1} x_{k,n-1}(d\alpha).$$

Therefore (16) is satisfied with C depending on $\operatorname{supp}(d\alpha)$.

LEMMA 11. *Let $\alpha \in M$ and let l be a fixed integer. Suppose that $g > 0$ is continuous on* $\operatorname{supp}(d\alpha)$. *Then for every function f which is continuous on* $[-1, 1]$ *and bounded on* $\operatorname{supp}(d\alpha)$ *the limit relation*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} p_n(d\alpha, t)p_{n+l}(d\alpha_g, t)f(t)\,d\alpha(t)$$
$$= \frac{1}{\pi}\int_0^{\pi} \frac{f(\cos\theta)}{\sqrt{g(\cos\theta)}}\cos\left[\Gamma(g, \theta) + l\theta\right]d\theta \tag{17}$$

holds.

PROOF. First let f be an entire function. Let Q be the set of all those points x for which $p_n(d\alpha, x) = 0$ for some n. By Lemma 3, $\bar{Q}\setminus[-1, 1]$ is denumerable. Therefore we can choose a sequence $\{r_m\}_{m=1}^{\infty}$ such that $r_m \downarrow 1$ as $m \to \infty$ and, for every $m = 1, 2, \ldots$, $\Gamma_m \cap \bar{Q} = \varnothing$ where the ellipse Γ_m is defined by

$$\Gamma_m = \left\{ z: z = \frac{1}{2}\left(r_m e^{i\theta} + \frac{1}{r_m e^{i\theta}} \right), 0 \leq \theta < 2\pi \right\}.$$

Now fix m and let $z \in \Gamma_m$. Because $(p_n(d\alpha, t) - p_n(d\alpha, z))/(t - z)$ is a polynomial of degree less than n in t we have

$$\int_{-\infty}^{\infty} p_n(d\alpha, t)\, \frac{p_n(d\alpha, t) - p_n(d\alpha, z)}{t - z}\, d\alpha(t) = 0,$$

that is

$$\int_{-\infty}^{\infty} \frac{p_n(d\alpha, t)}{z - t}\, d\alpha(t) = \frac{1}{p_n(d\alpha, z)} \int_{-\infty}^{\infty} \frac{p_n^2(d\alpha, t)}{z - t}\, d\alpha(t).$$

Multiply here both sides by $f(z)p_{n+l}(d\alpha_g, z)$ and integrate over Γ_m. Using Cauchy's formula we obtain

$$\int_{-\tau_m}^{\tau_m} p_n(d\alpha, t) p_{n+l}(d\alpha_g, t) f(t)\, d\alpha(t)$$

$$= \int_{-\infty}^{\infty} p_n^2(d\alpha, t) \left\{ \frac{1}{2\pi i} \int_{\Gamma_m} \frac{f(z) p_{n+l}(d\alpha_g, z)}{p_n(d\alpha, z)(z - t)}\, dz \right\} d\alpha(t) \qquad (18)$$

where $\tau_m = \frac{1}{2}(r_m + r_m^{-1})$. Here we used the fact that for $|t| > \tau_m$ the function $f(z)p_{n+l}(d\alpha, z)(z - t)^{-1}$ is analytic inside Γ_m. Now we can apply Lemmas 4, 5, and 8. We obtain that the right side in (18) converges as $n \to \infty$ and its limit equals

$$\frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1 - t^2}} \cdot \left\{ \frac{1}{2\pi i} \int_{\Gamma_m} D\left(g, z - \sqrt{z^2 - 1}\right)^{-1} \left(z + \sqrt{z^2 - 1}\right)^l f(z) \frac{dz}{z - t} \right\} dt.$$

But

$$\frac{1}{\pi} \int_{-1}^{1} \frac{1}{(z - t)\sqrt{1 - t^2}}\, dt = \frac{1}{\sqrt{z^2 - 1}}.$$

Hence the previous expression can be written as

$$\frac{1}{2\pi i} \int_{\Gamma_m} D\left(g, z - \sqrt{z^2 - 1}\right)^{-1} \left(z + \sqrt{z^2 - 1}\right)^l f(z) \frac{dz}{\sqrt{z^2 - 1}}.$$

By Lemma 5

$$\lim_{n \to \infty} \int_{|t| > 1} p_n^2(d\beta, t)\, d\beta(t) = 0 \qquad (19)$$

whenever $\beta \in M$. Therefore if f is an entire function then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} p_n(d\alpha, t) p_{n+l}(d\alpha_g, t) f(t)\, d\alpha(t)$$
$$= \frac{r_m^l}{2\pi} \int_0^{2\pi} D\left(g, \frac{1}{r_m e^{i\theta}}\right)^{-1} e^{il\theta} f\left(\frac{r_m e^{i\theta} + r_m^{-1} e^{-i\theta}}{2}\right) d\theta.$$

Let us recall that $\Gamma(g, \theta)$ is an odd function of θ. Consequently, letting $m \to \infty$ and using Lebesgue's dominated convergence theorem we easily obtain (17). Applying continuity arguments we see that (17) remains valid whenever f is continuous on $\Delta \supset \operatorname{supp}(d\alpha)$. The general case follows immediately from (19).

LEMMA 12. *Let $\alpha \in S$, $g \geqslant 0$ and $l \in \mathbf{Z}$ be given. Assume that $\alpha_g \in S$ and $f/\sqrt{g} \in L^\infty(-1, 1)$. Then* (17) *holds.*

PROOF. It is an almost direct consequence of the definition of Szegö's function $D(d\beta, z)$ that from $\beta \in S$

$$\lim_{n\to\infty} \int_0^{\pi} \left| \sqrt{\sin t \beta'(\cos t)}\; p_n(d\beta, \cos t) - \sqrt{\frac{2}{\pi}} \cos\left[nt + \Gamma(v^{-1} d\beta, t)\right] \right|^2 dt = 0$$

follows. (See e.g. [**6**, Chapter 9].) Therefore

$$\lim_{n\to\infty} \int_{-1}^{1} p_n^2(d\beta, x) d\left[\beta_s(x) + \beta_j(x)\right] = 0$$

must also be true since the $L^2_{d\beta}$ norm of $p_n(d\beta)$ equals 1 for every n. Now the lemma follows from the previous two formulae and from the Riemann-Lebesgue lemma.

THEOREM 13. *Let $\alpha \in M$ and let $g > 0$ be continuous on* $\operatorname{supp}(d\alpha)$. *If f is twice continuously differentiable on $\Delta \supset$* $\operatorname{supp}(d\alpha)$ *then*

$$\lim_{n\to\infty} \left[\int_{-\infty}^{\infty} f(t) \lambda_n^{-1}(d\alpha_g, t)\, d\alpha_g(t) - \int_{-\infty}^{\infty} f(t) \lambda_n^{-1}(d\alpha, t)\, d\alpha(t) \right]$$
$$= \frac{1}{2\pi^2} \int_0^{\pi} \int_0^{\pi} \sqrt{\frac{g(\cos\psi)}{g(\cos\theta)}}\; \frac{f(\cos\psi) - f(\cos\theta)}{(\cos\psi - \cos\theta)^2}$$
$$\cdot \left\{ \cos[\Gamma(g, \psi)] \cos[\Gamma(g, \theta)] (1 - \cos\psi \cos\theta) \right.$$
$$\left. + \sin[\Gamma(g, \psi)] \sin[\Gamma(g, \theta)] \sin\psi \sin\theta \right\} d\psi\, d\theta \quad (20)$$

where the integral is defined in the sense of Cauchy-Lebesgue.

PROOF. We have

$$\lambda_n^{-1}(d\alpha_g, t) = \int_{-\infty}^{\infty} K_n(d\alpha_g, t) K_n(d\alpha, x, t)\, d\alpha(x)$$

and

$$\lambda_n^{-1}(d\alpha, t) = \int_{-\infty}^{\infty} K_n(d\alpha_g, x, t) K_n(d\alpha, x, t)\, d\alpha_g(x).$$

Therefore

$$\int_{-\infty}^{\infty} f(t)\lambda_n^{-1}(d\alpha_g, t)\, d\alpha_g(t) - \int_{-\infty}^{\infty} f(t)\lambda_n^{-1}(d\alpha, t)\, d\alpha(t)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)[f(x) - f(t)] K_n(d\alpha_g, x, t) K_n(d\alpha, x, t)\, d\alpha(x)\, d\alpha(t)$$

which clearly equals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)[f(x) - f(t) - f'(x)(x - t)] \cdot K_n(d\alpha_g, x, t) K_n(d\alpha, x, t)\, d\alpha(x)\, d\alpha(t)$$
$$+ \int_{-\infty}^{\infty} g(x) f'(x) \left[\int_{-\infty}^{\infty} (x - t) K_n(d\alpha_g, x, t) K_n(d\alpha, x, t)\, d\alpha(t)\right] d\alpha(x).$$

Using the Christoffel-Darboux formula and orthogonality relations we obtain

$$\int_{-\infty}^{\infty} (x - t) K_n(d\alpha_g, x, t) K_n(d\alpha, x, t)\, d\alpha(t)$$
$$= \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} p_n(d\alpha, x) p_{n-1}(d\alpha_g, x) \int_{-\infty}^{\infty} p_{n-1}(d\alpha_g, t) p_{n-1}(d\alpha, t)\, d\alpha(t).$$

Consequently, using the Christoffel-Darboux formula again we get

$$\int_{-\infty}^{\infty} f(t)\lambda_n^{-1}(d\alpha_g, t)\, d\alpha_g(t) - \int_{-\infty}^{\infty} f(t)\lambda_n^{-1}(d\alpha, t)\, d\alpha(t)$$
$$= \frac{\gamma_{n-1}(d\alpha)\gamma_{n-1}(d\alpha_g)}{\gamma_n(d\alpha)\gamma_n(d\alpha_g)} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x) \frac{[f(x) - f(t) - f'(x)(x - t)]}{(x - t)^2}$$
$$\cdot [p_n(d\alpha_g, x) p_n(d\alpha, x) p_{n-1}(d\alpha_g, t) p_{n-1}(d\alpha, t)$$
$$- p_{n-1}(d\alpha_g, x) p_n(d\alpha, x) p_n(d\alpha_g, t) p_{n-1}(d\alpha, t)$$
$$- p_n(d\alpha_g, x) p_{n-1}(d\alpha, x) p_{n-1}(d\alpha_g, t) p_n(d\alpha, t)$$
$$+ p_{n-1}(d\alpha_g, x) p_{n-1}(d\alpha, x) p_n(d\alpha_g, t) p_n(d\alpha, t)]\, d\alpha(x)\, d\alpha(t)$$
$$+ \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \int_{-\infty}^{\infty} g(x) f'(x) p_n(d\alpha, x) p_{n-1}(d\alpha_g, x)\, d\alpha(x)$$
$$\cdot \int_{-\infty}^{\infty} p_{n-1}(d\alpha_g, t) p_{n-1}(d\alpha, t)\, d\alpha(t). \tag{21}$$

Since by the conditons both α and α_g belong to M and the function

$g(x)[f(x) - f(t) - f'(x)(x - t)](x - t)^{-2}$ is continuous on $\Delta \times \Delta$ we immediately obtain from Lemma 11 that the left side in (20) exists and equals

$$\frac{1}{4\pi^2}\int_0^\pi\int_0^\pi g(\cos\psi)\frac{[f(\cos\psi) - f(\cos\theta) - f'(\cos\psi)(\cos\psi - \cos\theta)]}{(\cos\psi - \cos\theta)^2}$$

$$\cdot \frac{1}{\sqrt{g(\cos\psi)g(\cos\theta)}} \cdot \{\cos[\Gamma(g,\psi)]\cos[\Gamma(g,\theta)]$$

$$- \cos[\Gamma(g,\psi) - \psi]\cos[\Gamma(g,\theta) + \theta]$$

$$- \cos[\Gamma(g,\psi) + \psi]\cos[\Gamma(g,\theta) - \theta]$$

$$+ \cos[\Gamma(g,\psi)]\cos[\Gamma(g,\theta)]\}\, d\psi\, d\theta$$

$$+ \frac{1}{2\pi^2}\int_0^\pi \sqrt{g(\cos\psi)}\, f'(\cos\psi)\cos[\Gamma(g,\psi) - \psi]\, d\psi$$

$$\cdot \int_0^\pi \frac{1}{\sqrt{g(\cos\theta)}} \cos[\Gamma(g,\theta)]\, d\theta.$$

For simplicity, let us denote the expression between braces by $A(\theta, \psi)$. In order to finish the proof of the theorem we should show that

$$\frac{1}{4\pi^2}\int_0^\pi\int_0^\pi \sqrt{\frac{g(\cos\psi)}{g(\cos\theta)}}\, f'(\cos\psi)\, \frac{A(\theta,\psi)}{\cos\psi - \cos\theta}\, d\psi\, d\theta$$

$$= \frac{1}{2\pi^2}\int_0^\pi \sqrt{g(\cos\psi)}\, f'(\cos\psi)\cos[\Gamma(g,\psi) - \psi)]\, d\psi$$

$$\cdot \int_0^\pi \frac{1}{\sqrt{g(\cos\theta)}} \cos[\Gamma(g,\theta)]\, d\theta.$$

This follows from

$$\frac{1}{\pi}\int_0^\pi \frac{\cos[\Gamma(g,\theta) + l\theta]}{\sqrt{g(\cos\theta)}}\, \frac{d\theta}{\cos\theta - \cos\psi}$$

$$= \begin{cases} 0 & \text{if } l = 0, -1, \\ \dfrac{2}{\pi}\displaystyle\int_0^\pi \frac{\cos[\Gamma(g,\theta)]}{\sqrt{g(\cos\theta)}}\, d\theta & \text{if } l = 1, \end{cases} \tag{22}$$

for almost every $\psi \in [0, \pi]$. To prove (22) let us recall that Γ is an odd function of θ. Therefore the left-hand side of (22) equals

$$\frac{1}{2\pi}\int_0^{2\pi} D\left(g^{-1}, e^{-i\theta}\right)^{il\theta} \frac{d\theta}{\cos\theta - \cos\psi}.$$

For almost every $\psi \in [0, \pi]$ this expression can be written as

$$\lim_{r\downarrow 1} \frac{1}{\pi}\int_0^{2\pi} D\left(g^{-1}, \frac{1}{re^{i\theta}}\right)\left(re^{i\theta}\right)^{l-1} \frac{d\theta}{\left(1 - r^{-1}e^{i(\psi-\theta)}\right)\left(1 - r^{-1}e^{-i(\psi+\theta)}\right)}.$$

Using the fact that $D(g^{-1})$ is an H^2 function in the unit disc we easily obtain (22).

THEOREM 14. *Let $\alpha \in S$ and let $g \geqslant 0$ be such that $g^{\pm 1} \in L^\infty(-1, 1)$. Suppose that f is twice differentiable on* $[-1, 1]$ *and $f'' \in L^\infty(-1, 1)$. Then* (20) *holds.*

PROOF. Apply Lemma 12 to (21) and repeat the arguments used in the proof of Theorem 13.

LEMMA 15. *Let $w = w^{(\delta,\varepsilon)}$ be a Jacobi weight. Then for every entire function f the asymptotic expression*

$$\sum_{k=1}^{n} f(x_{kn}(w)) = \frac{2n + \delta + \varepsilon + 1}{2\pi}\int_0^{\pi} f(\cos\theta)\, d\theta - \frac{2\delta + 1}{4} f(1) - \frac{2\varepsilon + 1}{4} f(-1) + O\left(\frac{1}{n}\right) \tag{23}$$

$(n \to \infty)$ *holds.*

PROOF. Multiplying the identity

$$\frac{p_n'(w, z)}{p_n(w, z)} = \sum_{k=1}^{n} \frac{1}{z - x_{kn}}$$

by $f(z)$ and integrating over an arbitrary ellipse Γ with foci at -1 and 1 we obtain by Cauchy's theorem that

$$\sum_{k=1}^{n} f(x_{kn}(w)) = \frac{1}{2\pi i}\int_\Gamma f(z)\frac{p_n'(w, z)}{p_n(w, z)}\, dz. \tag{24}$$

Our next step is to find a simple expression for $p_n'(w, z)$. To do this let us note that any Jacobi weight $w = w^{(\delta,\varepsilon)}$ satisfies the condition

$$\left[(1 - x^2)w(x)\right]' = \left[\varepsilon - \delta - (\delta + \varepsilon + 2)x\right]w(x).$$

Therefore, when expanding $(1 - z^2)p_n'(w, z)$ into Fourier series in $\{p_k(w, z)\}$

we get

$$(1-z^2)p_n'(w,z) = (n+\delta+\varepsilon+1)\frac{\gamma_{n-1}(w)}{\gamma_n(w)}\,p_{n-1}(w,z) + \tfrac{1}{2}\big[\delta-\varepsilon+(\delta+\varepsilon+2)\alpha_n(w)\big]p_n(w,z) - n\frac{\gamma_n(w)}{\gamma_{n+1}(w)}\,p_{n+1}(w,z). \tag{25}$$

Here the first two coefficients on the right side can be calculated by integration by parts and the third one follows from comparing leading coefficients. Putting (25) into (24) we obtain

$$\sum_{k=1}^{n} f(x_{kn}(w)) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{1-z^2}\left[(n+\delta+\varepsilon+1)\frac{\gamma_{n-1}(w)}{\gamma_n(w)}\,\frac{p_{n-1}(w,z)}{p_n(w,z)} + \tfrac{1}{2}\big[\delta-\varepsilon+(\delta+\varepsilon+2)\alpha_n(w)\big] - n\frac{\gamma_n(w)}{\gamma_{n+1}(w)}\,\frac{p_{n+1}(w,z)}{p_n(w,z)}\right]dz.$$

Now applying (8) and (11) we see that

$$\sum_{k=1}^{n} f(x_{kn}(w)) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{1-z^2}\left[\frac{n+\delta+\varepsilon+1}{2(z+\sqrt{z^2-1})} + \frac{1}{2}(\delta-\varepsilon) - \frac{n}{2}\left(z+\sqrt{z^2-1}\right)\right]dz + O\left(\frac{1}{n}\right),$$

that is,

$$\sum_{k=1}^{n} f(x_{kn}(w)) = \frac{2n+\delta+\varepsilon+1}{4\pi i}\int_\Gamma \frac{f(z)}{\sqrt{z^2-1}}\,dz + \frac{1}{4\pi i}\int_\Gamma \frac{f(z)}{1-z^2}\big[(\delta+\varepsilon+1)z+\delta-\varepsilon\big]\,dz + O\left(\frac{1}{n}\right).$$

Here the first integral can easily be computed by putting $z = \frac{1}{2}(re^{i\theta} + r^{-1}e^{-i\theta})$ $(0 < \theta < 2\pi)$ and letting $r \to 1$. Using

$$\frac{2}{1-z^2} = \frac{1}{1-z} + \frac{1}{1+z}$$

we can also calculate the second integral by Cauchy's theorem. Thus (23) is true.

For $\delta > -\frac{1}{2}$ and $\varepsilon > -\frac{1}{2}$ the following two results were proved in [2]. Our approach to the problem is different from that in [2].

LEMMA 16. *Let $w = w^{(\delta,\varepsilon)}$ be a Jacobi weight. Then there exists a positive constant $C = C(\delta, \varepsilon)$ such that for every $n = 1, 2, \ldots$ and $x \in [-1, 1]$ the inequality*

$$\left|(1-x^2)w(x)\lambda_n^{-1}(w,x) - \frac{2n+\delta+\varepsilon+1}{2\pi}\sqrt{1-x^2}\right| < C \tag{26}$$

holds.

PROOF. Fix $x \in [-1, 1]$ and let $x = \cos\theta$ $(0 \leq \theta \leq \pi)$. By the Christoffel-Darboux formula we have

$$(1-x^2)\lambda_n^{-1}(w,x) = \frac{\gamma_{n-1}(w)}{\gamma_n(w)}$$

$$\cdot \left[(1-x^2)p_n'(w,x)p_{n-1}(w,x) - (1-x^2)p_{n-1}'(w,x)p_n(w,x)\right].$$

Thus by (25)

$$\begin{aligned}(1-x^2)\lambda_n^{-1}(w,x) = {} & \frac{\gamma_{n-1}^2(w)}{\gamma_n^2(w)}(n+\delta+\varepsilon+1)p_{n-1}^2(w,x) \\ & + \frac{\gamma_{n-1}(w)}{2\gamma_n(w)}\left[\delta-\varepsilon+(\delta+\varepsilon+2)\alpha_n(w)\right]p_{n-1}(w,x)p_n(w,x) \\ & - n\,\frac{\gamma_{n-1}(w)}{\gamma_{n+1}(w)}\,p_{n-1}(w,x)p_{n+1}(w,x) \\ & - \frac{\gamma_{n-2}(w)}{\gamma_n(w)}(n+\delta+\varepsilon)p_{n-2}(w,x)p_n(w,x) \\ & - \frac{\gamma_{n-1}(w)}{2\gamma_n(w)}\left[\delta-\varepsilon+(\delta+\varepsilon+2)\alpha_{n-1}(w)\right]p_{n-1}(w,x)p_n(w,x) \\ & + (n-1)\,\frac{\gamma_{n-1}^2(w)}{\gamma_n^2(w)}\,p_n^2(w,x).\end{aligned}$$

Now applying (8) we obtain

$$(1-x^2)\lambda_n^{-1}(w,x) = \frac{n+\delta+\varepsilon}{4}\left[p_{n-1}^2(w,x) - p_{n-2}(w,x)p_n(w,x)\right]$$
$$+\frac{n}{4}\left[p_n^2(w,x) - p_{n-1}(w,x)p_{n+1}(w,x)\right]$$
$$+\frac{1}{4}\left[p_{n-1}^2(w,x) - p_n^2(w,x)\right] + O\left(\frac{1}{n}\right)\sum_{k=n-2}^{n+1} p_k^2(w,x). \tag{27}$$

Let us simplify the expressions between the brackets above. We have

$$p_{n-1}^2(w,x) - p_{n-2}(w,x)p_n(w,x)$$
$$= p_{n-1}^2(w,x) - 2xp_{n-1}(w,x)p_n(w,x) + p_n^2(w,x)$$
$$+p_n(w,x)\left[2xp_{n-1}(w,x) - p_n(w,x) - p_{n-2}(w,x)\right]. \tag{28}$$

By the recurrence formula

$$2xp_{n-1}(w,x) - p_n(w,x) - p_{n-2}(w,x)$$
$$=\left[2\frac{\gamma_{n-1}(w)}{\gamma_n(w)} - 1\right]p_n(w,x) + 2\alpha_{n-1}(w)p_{n-1}(w,x)$$
$$+\left[2\frac{\gamma_{n-2}(w)}{\gamma_{n-1}(w)} = 1\right]p_{n-2}(w,x).$$

Therefore by (8)

$$2xp_{n-1}(w,x) - p_n(w,x) - p_{n-2}(w,x) = O\left(\frac{1}{n^2}\right)\sum_{k=n-2}^{n}|p_k(w,x)|. \tag{29}$$

Using (28) we obtain

$$p_{n-1}^2(w,x) - p_{n-2}(w,x)p_n(w,x) = \left|p_n(w,x) - e^{i\theta}p_{n-1}(w,x)\right|^2$$
$$+ O\left(\frac{1}{n^2}\right)\sum_{k=n-2}^{n}|p_k(w,x)|^2. \tag{30}$$

Applying the same argument one can show that

$$p_n^2(w,x) - p_{n-1}(w,x)p_{n+1}(w,x) = \left|p_n(w,x) - e^{i\theta}p_{n-1}(w,x)\right|^2$$
$$+ O\left(\frac{1}{n^2}\right)\sum_{k=n-1}^{n+1}|p_k(w,x)|^2. \tag{31}$$

Furthermore, it is clear that

$$p_{n-1}^2(w,x) - p_n^2(w,x) = \left|p_n(w,x) - e^{i\theta}p_{n-1}(w,x)\right|^2$$
$$-2p_n(w,x)\mathrm{Re}\left[p_n(w,x) - e^{i\theta}p_{n-1}(w,x)\right]. \tag{32}$$

Putting (30)–(32) into (27) we obtain

$$(1 - x^2)\lambda_n^{-1}(w, x) = \frac{2n + \delta + \varepsilon + 1}{4} \left|p_n(w, x) - e^{i\theta}p_{n-1}(w, x)\right|^2$$

$$+ O(1)|p_n(w, x)|\,\left|p_n(w, x) - e^{i\theta}p_{n-1}(w, x)\right|$$

$$+ O\left(\frac{1}{n}\right) \sum_{k=n-2}^{n+1} |p_k(w, x)|^2.$$

By Lemma 7 and (29)

$$\left|p_n(w, x) - e^{i\theta}p_{n-1}(w, x)\right| = \left[\frac{2\sqrt{1 - x^2}}{\pi w(x)}\right]^{1/2} + O(1) \sum_{k=n-1}^{\infty} \frac{|p_k(w, x)|}{k^2}.$$

Consequently

$$(1 - x^2)w(x)\lambda_n^{-1}(w, x) = \frac{2n + \delta + \varepsilon + 1}{2\pi} \sqrt{1 - x^2}$$

$$+ O(n) \sum_{k=n-1}^{\infty} \frac{1}{k^2} \left(w(x)\sqrt{1 - x^2}\right)^{1/2} |p_k(w, x)|$$

$$+ O(n) \frac{1}{\sqrt{1 - x^2}} \left[\sum_{k=n-1}^{\infty} \frac{1}{k^2} \left(w(x)\sqrt{1 - x^2}\right)^{1/2} |p_k(w, x)|\right]^2$$

$$+ O(1)\left(w(x)\sqrt{1 - x^2}\right)^{1/2} |p_n(w, x)|$$

$$+ O(1) \frac{1}{\sqrt{1 - x^2}} \left(w(x)\sqrt{1 - x^2}\right)^{1/2} |p_n(w, x)|$$

$$\sum_{k=n-1}^{\infty} \frac{1}{k^2} \left(w(x)\sqrt{1 - x^2}\right)^{1/2} |p_k(w, x)|$$

$$+ O\left(\frac{1}{n}\right) \frac{1}{\sqrt{1 - x^2}} \sum_{k=n-2}^{n+1} w(x)\sqrt{1 - x^2}\, |p_k(w, x)|^2.$$

Using (9) we obtain that (26) is true whenever $1 - x^2 > 1/n$. If $1 - x^2 < 1/n$ then (26) follows immediately from (10).

THEOREM 17. *Let $w = w^{(\delta,\varepsilon)}$ be a Jacobi weight. Assume that $f \in L_w^1 \cap L_v^1$ and let $[f(x) - f(\pm 1)](1 \mp x)^{-1}$ be integrable. Then*

$$\lim_{n\to\infty} \left[\int_{-1}^{1} f(t)\lambda_n^{-1}(w, t)w(t)\, dt - \frac{2n + \delta + \varepsilon}{2\pi} \int_{-1}^{1} f(t) \frac{dt}{\sqrt{1 - t^2}}\right]$$

$$= -\frac{\delta}{2} f(-1) - \frac{\varepsilon}{2} (-1). \tag{33}$$

PROOF. It follows from Theorem 9 and Lemma 15 that (33) is true if f is an entire function. By continuity arguments and Lemma 16 (33) still holds if f can be written in the form $f(x) = (1 - x^2)g(x)$ where $g \in L^1$. To complete the proof let us note that if f satisfies the conditions of the theorem then f can be represented as

$$f(x) = (1 - x^2)g(x) + h(x)$$

where $g \in L^1$ and h is an entire function.

Let us finish this paper with mentioning the following problem. Try to find similar results for the eigenvalue distribution of Toeplitz matrices of the form

$$\left\{ \int_{-\infty}^{\infty} \varphi(x) p_n(d\alpha, x) p_m(d\alpha, x)\, d\alpha(x) \right\}_{n,m=0}^{N}.$$

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