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Distribution of Roots of the Partition Function in the Complex Temperature Plane

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Fisher,[1] Abe,[2] Ono and Suzuki[3] discussed the singularities of the specific heat by observing a distribution of zeros of the partition function of the Ising model of finite size in the complex $\exp(-J/kT)$ or complex $\tanh J/kT$ plane. They expect that the zeros of the partition function of finite system in the complex temperature plane lie on simple loci. A remark is made on these arguments.

As an example the two-dimensional square lattice is considered. The partition function Z_N of the system under periodic boundary conditions consists of four parts,[4] $Z_N = Z_N^{(1)} + N_N^{(2)} + Z_N^{(3)} + Z_N^{(4)}$. The statement that *zeros of the partition function Z_N lie on two circles whose centers are at* -1 *and* $+1$, *and radii are* $\sqrt{2}$, does not hold for the total partition function Z_N but only for each of $Z_N^{(i)}$, $i=1, 2, 3$ and 4 for finite N. A sum of polynomials each of which have roots on the same locus does not necessarily have roots on that locus. For example, $Z_N^{(1)} = x^N + 1$, $Z_N^{(2)} = x^N - 1$, $Z_N^{(3)} = x^N + i$, $Z_N^{(4)} = x^N - i$. Yang-Lee's theorem[5] on the roots of the partition function in the complex e^{-H}/kT plane (H: magnetic field) holds for *total* Z_N for *any* N. This is an essential difference between roots in the complex $e^{-H/kT}$ plane and those in the complex $e^{-J/kT}$ plane. Of course this does not exclude the possibility that the above statement holds asymptotically for Z_N as N becomes sufficiently large. In order to examine this possibility, the roots of Z_N and $Z_N^{(i)}$ for finite N[6] were calculated and shown in Fig. 1. From these figures we cannot conclude whether the roots of Z_N tend to distribute on the loci stated above, or to other loci, as N becomes large. If the former were expected, the proof is required.

In conclusion, it is to be remarked that the statement in references that the zeros of the partition function of finite system lie on loci··· is generally to be replaced by that *the limiting integral is of the form of a product over zeros lying on loci···* in the case of complex temperature plane.

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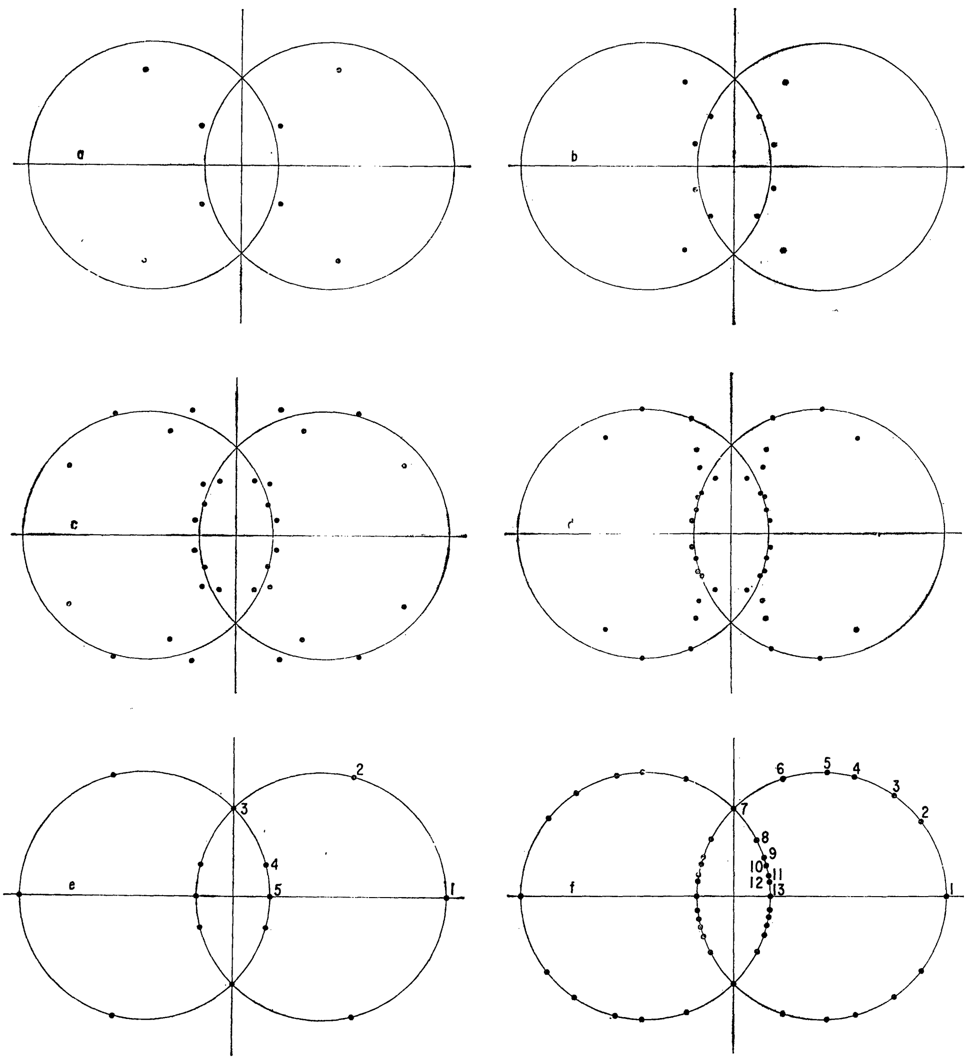

Fig. 1. Roots of a) $Z_{2\times2}$, b) $Z_{3\times3}$, c) $Z_{4\times4}$, d) $Z_{5\times5}$ (by Suzuki and Kawabata), e) $Z^{(i)}_{2\times2}$, [The roots of $Z^{(1)}=Z^{(4)}$ in the first quadrant are Nos. 2, 4, those of $Z^{(2)}$ are Nos. 1, 3 (double), 5, and those of $Z^{(3)}$ are No. 3 (four fold).] and f) $Z^{(i)}_{4\times4}$. [The roots of $Z^{(1)}=Z^{(4)}$ in the first quadrant are Nos. 2, 5 (double), 6, 8, 9 (double), 12, those of $Z^{(2)}$ are Nos. 1, 4 (double), 7 (six fold) 10 (double), 13 and those of $Z^{(3)}$ are Nos.3 (double), 7 (eight fold), 11(double).]

Appendix. Number of configurations of $Z_{4\times4}$. (Errata to reference 6))
The number for 9−16+spins is the same for 8−1+spins.

No. of + −spin pairs / No. of +spins	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	32	88	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	96	256	208	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	24	256	736	576	228	0	0	0	0	0	0	0	0
5	0	0	0	0	0	192	688	1664	1248	448	128	0	0	0	0	0	0
6	0	0	0	0	0	96	704	1824	2928	1568	768	64	56	0	0	0	0
7	0	0	0	0	0	64	624	1920	3680	3136	1392	512	96	0	16	0	0
8	0	0	0	0	8	0	768	1600	4356	3264	2112	576	120	64	0	0	2
State Density	2	0	32	64	424	1728	6688	13568	20524	13568	6688	1728	424	64	32	0	2

who corrected $Z_{4\times4}$ in reference, 6) and Dr. Suzuki and Dr. Kawabata who calculated $Z_{5\times5}$, for communicating their results before publication.

1) M. E. Fisher, *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, 1964), Vol. 7c, p. 1.
2) R. Abe, Prog. Theor. Phys. **37** (1967), 1070.
3) S. Ono, Y. Karaki, M. Suzuki and C. Kawabata, Phys. Letters **24A** (1967), 703. M. Suzuki, to be published.
4) B. Kaufman, Phys. Rev. **76** (1946), 1232.
5) C. N. Yang and T. D. Lee, Phys. Rev. **87** (1952), 404, 410.
6) S. Katsura, *Proceedings of International Conference on Theoretical Physics* (Kyoto, 1953), p. 534; Prog. Theor. Phys. **11** (1954), 476; Busseiron-Kenkyu, No. 70 (1954).