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Distributions of Gamma m-Spacings

Mohamed I. Riffi [1,*]

[1]Department of Mathematics, Faculty of Sciences, The Islamic University of Gaza, Gaza, Palestine.

*** Corresponding author**
e-mail address: riffim@gmail.com

Abstract

This paper is concerned with the distributions of the m-spacings of order statistics associated with a sample from a two-parameter gamma population when the shape parameter is a positive integer. We prove that the m-spacings have finite mixture distributions with negative components, each of which in turn has a finite gamma mixture distribution. We write the probability density function of the m-spacings in closed form and provide a Mathematica code for the implementation.

Keywords:

Gamma distribution; Gamma mixture; Negative gamma mixtures; Incomplete gamma function; Order

1. Introduction:

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n and $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the corresponding order statistics, where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$. Define the m-spacings (or spacings of order m) by $\mathcal{D}_{i,n}^{(m)} = X_{(i+m)} - X_{(i)}$ for $0 \leq i < i + m \leq n$. By convention, we set $X_{(0)} = 0$ and $\mathcal{D}_{0,n}^{(1)} = X_{(1)}$.

The random variables $\mathcal{D}_{i,n}^{(m)}$ are called the m-spacings between order statistics (cf. Kamps, 1995). Order statistics and their spacings play an important role in mathematical statistics and other fields of applied probability. These spacings can be used to characterize some distributions like the uniform and exponential distributions (cf. Pyke, 1965; Ahsanullah, 1978). They are very important in statistical inference. For example, they are used in goodness-of-fit testing (cf. Beirlant, 1985; Del Pino, 1979), nonparametric entropy estimation (cf. Beirlant, 1997), and in testing for uniformity (cf. Cressie, 1976). In particular, gamma spacings play an important role in reliability, communication systems, queueing applications, and other areas in the engineering discipline.

For various properties of order statistics, the reader may refer to Arnold et al. (1992); David (1981). Spacings and their properties are deeply reviewed in Pyke (1965). One also can refer to DasGupta (2011) for more information about spacings.

This work generalizes a work done by Riffi (2016). The problem we tackle in this paper is determining the distributions of the $\mathcal{D}_{i,n}^{(m)}$,s when the population of the sample is $\mathcal{G}(\alpha, \beta)$, where $\alpha = 1,2, \ldots$ and β is any strictly positive real number. We will show that the $\mathcal{D}_{i,n}^{(m)}$'s have finite mixtures with negative components, each of these components in turn has a traditional finite gamma mixture distribution. That is, we will show that the probability density function (pdf) of $\mathcal{D}_{i,n}^{(m)}$ is given by

$$f_{\mathcal{D}_{i,n}^{(m)}}(r) = \sum_{k=0}^{m-1} \delta_k^{(m)} \mathcal{C}_k^{(m)}(r), \quad (1.1)$$

where

$$\mathcal{C}_k^{(m)}(r) = \sum_{j=0}^{(\alpha-1)(n-i-k)} w_j f_j(r|\alpha_j, \beta_j)(r), \quad (1.2)$$

$\delta_k^{(m)}$ is an integer, $f_j(r|\alpha_j, \beta_j)$ is a gamma pdf of shape and scale parameters α_j and β_j, respectively, and $w_0, w_1, \dots, w_{(\alpha-1)(n-i-k)}$ denote mixture proportions or weights that satisfy the constraint: $0 < w_j < 1$ and $\sum_j w_j = 1$.

The representation (1.1) can be helpful in studying the moments and other properties and behaviours of the m-spacings.

2. FINITE MIXTURES WITH NEGATIVE COMPONENTS

A d-dimensional random vector $X = (X_1, X_2, \dots, X_n)$ is said to follow a g-component finite mixture distribution, if its pdf $f(x)$ can be written as

$$f(x) = \sum_{i=1}^{g} w_i f_i(x), \quad (2.1)$$

where the $f_i(x)$ are densities and w_i are nonnegative quantities that sum to 1; that is,

$$0 \le w_i \le 1, \forall i = 1, \dots, g, \text{and } i = \sum_{i=1}^{g} w_i = 1. \quad (2.2)$$

The quantities $w_1, \dots, w_g$ are called the mixture proportions or weights (cf. McLachlan, 2000).

In the representation (1.1), it turns out that some of the $\delta_k^{(m)}$ are negative integers. This means that some of the components of the mixture are negative. Therefore, we adopt the notion of finite mixtures with negative components introduced in Zhang (2005) to suitably describe the distributions of $\mathcal{D}_{i,n}^{(m)}$.

If we cancel the nonnegative constraint of mixture weights in (2.2), the mixture in (1.1) becomes a finite mixture with negative components.

3. THE DISTRIBUTION OF $\boldsymbol{\mathcal{D}_{i,n}^{(m)}}$

The random variable X is said to follow a gamma distribution with shape and scale parameters $\alpha > 0$ and $\beta > 0$, respectively, written for short as $X \sim \mathcal{G}(\alpha, \beta)$, if the probability density function (pdf) of X is given by

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty, \quad (3.1)$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the gamma function at $\alpha > 0$.

The cumulative distribution function (cdf) of $X \sim \mathcal{G}(\alpha, \beta)$ at $t > 0$ is given by

$$F_X(t) = \frac{\gamma(\alpha, t/\beta)}{\Gamma(\alpha)}, \quad (3.2)$$

where $\gamma(\alpha, t/\beta) = \int_0^{t/\beta} x^{\alpha-1} e^{-x} dx$ is the lower incomplete gamma function evaluated at $(\alpha, t/\beta)$.

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a gamma distribution $\mathcal{G}(\alpha, \beta)$ with shape and scale parameters α and β, respectively.

Recall that the joint pdf of the ith and jth order statistics $X_{(i)}$ and $X_{(j)}, 1 \le i < j \le n$, is given by (cf. DasGupta, 2011)

$$f_{i,j}(x, y) = \frac{n!}{(i-1)!\,(j-1-i)!\,(n-j)!} f(x) f(y) F^{i-1}(x) \times [F(y) - F(x)]^{j-1-i} [1 - F(y)]^{n-j}, \quad (3.3)$$

for $-\infty < x < y < \infty$.

When $j = i + m$, (3.3) reduces to

$$f_{i,i+m}(x,y) = \frac{n!}{(i-1)!\,(m-1)!\,(n-i-m)!} f(x)f(y)F^{i-1}(x)[F(y) - F(x)]^{m-1} \times [1-F(y)]^{n-i-m}. \tag{3.4}$$

It follows from (3.4) that the pdf of $\mathcal{D}_{i,n}^{(m)}$ at $r > 0$ is then the integral

$$f_{\mathcal{D}_{i,n}^{(m)}}(r|\alpha,\beta) = \int_0^\infty f_{i,i+m}(x, x+r)dx. \tag{3.5}$$

Theorem 3.1. The pdf of $\mathcal{D}_{i,n}^{(m)}$ is a finite weighted sum of gamma pdfs with rate parameters j and scale parameters $\beta/(k+n-i-m+1)$, for $j = 1,2,\dots,m-1$, $i = 0,1,\dots,n-1$, and $k = 0,1,\dots,m-1$; i.e.,

$$\begin{aligned} f_{\mathcal{D}_{i,n}^{(m)}}(r) &= \sum_{a_1+\cdots+a_{\alpha+1}=i-1} \; \sum_{b_1+\cdots+b_\alpha=n-i-m} \; \sum_{\substack{c_1+\cdots+c_\alpha+ \\ d_1+\cdots+d_\alpha=m-1}} \; \sum_{h=0}^{\eta_b+\eta_d+\alpha-1} \delta \\ &\times \binom{i-1}{a_1,a_2,\dots,a_{\alpha+1}} \binom{n-i-m}{b_1,b_2,\dots,b_\alpha} \binom{m-1}{c_1,\dots,c_\alpha,d_1,\dots,d_\alpha} \\ &\times \binom{\eta_b+\eta_d+\alpha-1}{h} \frac{(-1)^{\gamma_a+\gamma_d}\Gamma(\eta_a+\eta_c+h+\alpha)}{\theta_a\theta_b\theta_c\theta_d(\gamma_a+n-i+1)^{\eta_a+\eta_c+h+\alpha}} \\ &\times r^{\eta_b+\eta_d+\alpha-h-1}\beta^{-\eta_b-\eta_d+h+\alpha}e^{-(\gamma_d+n-i-m+1)r/\beta}, \end{aligned} \tag{3.6}$$

where

$$\gamma_a = \sum_{\iota=2}^{\alpha+1} a_\iota, \eta_a = \sum_{\iota=3}^{\alpha+1}(\iota-2)a_\iota, \theta_a = \prod_{\iota=2}^{\alpha}(\iota-1)!^{a_{\iota+1}}, \tag{3.7}$$

$$\gamma_b = \sum_{\iota=1}^{\alpha} b_\iota, \eta_b = \sum_{\iota=2}^{\alpha}(\iota-1)b_k, \theta_b = \prod_{\iota=2}^{\alpha}(\iota-1)!^{b_\iota}, \tag{3.8}$$

$$\gamma_c = \sum_{\iota=1}^{\alpha} c_\iota, \eta_c = \sum_{\iota=2}^{\alpha}(\iota-1)c_k, \theta_c = \prod_{\iota=2}^{\alpha}(\iota-1)!^{c_\iota}, \tag{3.9}$$

$$\gamma_d = \sum_{\iota=1}^{\alpha} d_\iota, \eta_d = \sum_{\iota=2}^{\alpha}(\iota-1)d_\iota, \theta_d = \prod_{\iota=2}^{\alpha}(\iota-1)!^{d_\iota}, \quad \text{and} \tag{3.10}$$

$$\delta = n!\,\beta^{-2\alpha}/(\Gamma(\alpha)^2(m-1)!\,(n-i-m)!\,(i-1)!). \tag{3.11}$$

Proof. We use the following expansion of $\gamma(\alpha, x)$, namely,

$$\gamma(\alpha, x) = \Gamma(\alpha)\left[1 - e^{-x}\sum_{i=0}^{\alpha-1}\frac{x^i}{i!}\right] \tag{3.12}$$

when $\alpha = 1,2,\ldots$ (cf. Temme, 1994; Gradshteyn et al., 2000; Gautshi, 2003). Then we expand sums by using the Multinomial Theorem, namely, for a positive integer k and a non-negative integer n,

$$(x_1 + x_2 + \cdots + x_k)^m = \sum_{b_1+b_2+\cdots+b_k=m}\binom{m}{b_1, b_2, \ldots, b_k}\prod_{j=1}^{k} x_j^{b_j},$$

and then we use integration by parts and mathematical induction.

Hence, for $X \sim \mathcal{G}(\alpha, \beta)$, the cdf of X is

$$\begin{aligned} F_X(x) &= \frac{\gamma(\alpha, x/\beta)}{\Gamma(\alpha)} = 1 - e^{-x/\beta}\sum_{i=0}^{\alpha-1}\frac{(x/\beta)^i}{i!} \quad \text{and} \\ 1 - F_X((x+r)/\beta) &= \frac{\gamma(\alpha, (x+r)/\beta)}{\Gamma(\alpha)} = e^{-(x+r)/\beta}\sum_{i=0}^{\alpha-1}\frac{((x+r)/\beta)^i}{i!}. \end{aligned} \tag{3.13}$$

Substituting in (3.4), we get for $i = 1,2,\ldots,n-1$

$$\begin{aligned} f_{i,i+m}(x, x+r) &= \delta x^{\alpha-1}(x+r)^{\alpha-1} e^{-\frac{(m-1)x}{\beta}} e^{-\frac{2x+r}{\beta}} e^{-\frac{(n-i-m)(x+r)}{\beta}} \\ &\times \left(1 - e^{-\frac{x}{\beta}}\sum_{k=0}^{\alpha-1}\frac{\left(\frac{x}{\beta}\right)^k}{k!}\right)^{i-1}\left(\sum_{k=0}^{\alpha-1}\frac{\left(\frac{x+r}{\beta}\right)^k}{k!}\right)^{n-i-m} \\ &\times \left(\sum_{k=0}^{\alpha-1}\frac{\left(\frac{x}{\beta}\right)^k}{k!} - e^{-r/\beta}\sum_{k=0}^{\alpha-1}\frac{\left(\frac{x+r}{\beta}\right)^k}{k!}\right)^{m-1}, \end{aligned} \tag{3.14}$$

where $\delta = n!\,\beta^{-2\alpha}/(\Gamma(\alpha)^2(m-1)!\,(n-i-m)!\,(i-1)!)$.

$$
\begin{aligned}
S_1 &= \left(1 - e^{-\frac{x}{\beta}} \sum_{k=0}^{\alpha-1} \frac{\left(\frac{x}{\beta}\right)^k}{k!}\right)^{i-1} \\
&= \sum_{a_1+a_2+\cdots+a_{\alpha+1}=i-1} \binom{i-1}{a_1, a_2, \dots, a_{\alpha+1}} (1)^{a_1} (-e^{-x/\beta})^{a_2} (-\frac{x}{\beta} e^{-x/\beta})^{a_3} \cdots \\
&\times \left(-\frac{x^{\alpha-1}}{(\alpha-1)!\,\beta^{\alpha-1}} e^{-x/\beta}\right)^{a_{\alpha+1}} \\
&= \sum_{a_1+a_2+\cdots+a_{\alpha+1}=i-1} (-1)^{a_2+a_3+\cdots+a_{\alpha+1}} \binom{i-1}{a_1, a_2, \dots, a_{\alpha+1}} \\
&\times \frac{e^{-(a_2+a_3\cdots+a_{\alpha+1})x/\beta}}{(1!)^{a_3}(2!)^{a_4}(3!)^{a_5}\cdots((\alpha-1)!)^{a_{\alpha+1}}} \frac{x^{a_3+2a_4+\cdots+(\alpha-1)a_{\alpha+1}}}{\beta^{a_3+2a_4+\cdots+(\alpha-1)a_{\alpha+1}}} \\
&= \sum_{a_1+a_2+\cdots+a_{\alpha+1}=i-1} \binom{i-1}{a_1, a_2, \dots, a_{\alpha+1}} \frac{(-1)^{\gamma_a} x^{\eta_a} e^{-\gamma_a x/\beta}}{\theta_a \beta^{\eta_a}},
\end{aligned} \tag{3.15}
$$

where

$$
\gamma_a = \sum_{\iota=2}^{\alpha+1} a_\iota, \eta_a = \sum_{\iota=3}^{\alpha+1} (\iota-2) a_\iota, \theta_a = \prod_{\iota=2}^{\alpha} (\iota-1)!^{a_{\iota+1}}. \tag{3.16}
$$

Similarly,

$$
\begin{aligned}
S_2 &= \left(\sum_{k=0}^{\alpha-1} \frac{\left(\frac{(x+r)}{\beta}\right)^k}{k!} \right)^{n-i-m} \\
&= \sum_{b_1+b_2+\cdots+b_\alpha=n-i-m} \binom{n-i-m}{b_1, b_2, \ldots, b_\alpha} (1)^{b_1} \left(\frac{(x+r)}{1!\,\beta}\right)^{b_2} \\
&\times \left(\frac{(x+r)^2}{2!\,\beta^2}\right)^{b_3} \cdots \left(\frac{(x+r)^{\alpha-1}}{(\alpha-1)!\,\beta^{\alpha-1}}\right)^{b_\alpha} \\
&= \sum_{b_1+b_2+\cdots+b_\alpha=n-i-m} \binom{n-i-m}{b_1, b_2, \ldots, b_\alpha} \\
&\times \frac{(x+r)^{b_2+2b_3+\cdots+(\alpha-1)b_\alpha}}{(1!)^{b_2}(2!)^{b_3}(3!)^{b_4}\cdots((\alpha-1)!)^{b_\alpha}} \frac{1}{\beta^{b_2+2b_3+\cdots+(\alpha-1)b_\alpha}} \\
&= \sum_{b_1+b_2+\cdots+b_\alpha=n-i-m} \binom{n-i-m}{b_1, b_2, \ldots, b_\alpha} \frac{(x+r)^{\eta_b}}{\theta_b \beta^{\eta_b}},
\end{aligned} \tag{3.17}
$$

where

$$
\eta_b = \sum_{\iota=2}^{\alpha} (\iota-1) b_\iota, \theta_b = \prod_{\iota=2}^{\alpha} (\iota-1)!^{b_\iota}. \tag{3.18}
$$

Similarly,

$$
\begin{aligned}
S_3 &= \sum_{c_1+\cdots+c_\alpha+d_1+\cdots+d_\alpha=m-1} \binom{m-1}{c_1, c_2, \ldots, c_\alpha, d_1, d_2, \ldots, d_\alpha} \\
&\times (1)^{c_1} \left(\frac{x/\beta}{1!}\right)^{c_2} \cdots \left(\frac{(x/\beta)^{\alpha-1}}{(\alpha-1)!}\right)^{c_\alpha} \\
&\times \left(-e^{-r/\beta}\right)^{d_1} \left(-\frac{e^{-r/\beta}\left(\frac{x+r}{\beta}\right)}{1!}\right)^{d_2} \cdots \left(-\frac{e^{-r/\beta}\left(\frac{x+r}{\beta}\right)^{\alpha-1}}{(\alpha-1)!}\right)^{d_\alpha} \\
&= \sum_{c_1+\cdots+c_\alpha+d_1+\cdots+d_\alpha=m-1} \binom{m-1}{c_1, c_2, \ldots, c_\alpha, d_1, d_2, \ldots, d_\alpha} \\
&\times \frac{(-1)^{\gamma_d} x^{\eta_c} e^{-\gamma_d r/\beta} (x+r)^{\eta_d}}{\theta_c \beta^{\eta_c} \theta_d \beta^{\eta_d}},
\end{aligned} \tag{3.19}
$$

where

$$
\eta_c = \sum_{\iota=2}^{\alpha} (\iota-1) c_\iota, \theta_c = \prod_{\iota=2}^{\alpha} (\iota-1)!^{c_\iota}, \gamma_d = \sum_{\iota=1}^{\alpha} d_\iota, \tag{3.20}
$$

$$\eta_d = \sum_{\iota=2}^{\alpha} (\iota - 1)d_\iota, \theta_d = \prod_{\iota=2}^{\alpha} (\iota - 1)!^{d_\iota}.$$

In the same way, we can write

$$(x + r)^{\eta_b+\eta_d+\alpha-1} = \sum_{h=0}^{\eta_b+\eta_d+\alpha-1} \binom{\eta_b + \eta_d + \alpha - 1}{h} x^h r^{\eta_b+\eta_d+\alpha-h-1}. \quad (3.21)$$

Therefore,

$$\begin{aligned} f_{i,i+m}(x, x + r) &= \delta x^{\alpha-1} e^{-\frac{(m-1)x}{\beta}} e^{-\frac{2x+r}{\beta}} e^{-\frac{(n-i-m)(x+r)}{\beta}} \\ &\sum_{a_1+\cdots+a_{\alpha+1}=i-1} \sum_{b_1+\cdots+b_\alpha=n-i-m} \sum_{\substack{c_1+\cdots+c_\alpha+ \\ d_1+\cdots+d_\alpha=m-1}} \sum_{h=0}^{\eta_b+\eta_d+\alpha-1} \binom{i-1}{a_1, a_2, \ldots, a_{\alpha+1}} \\ &\times \binom{n-i-m}{b_1, b_2, \ldots, b_\alpha} \binom{m-1}{c_1, \ldots, c_\alpha, d_1, \ldots, d_\alpha} \binom{\eta_b + \eta_d + \alpha - 1}{h} \\ &\times \frac{(-1)^{\gamma_a} x^{\eta_a} e^{-\gamma_a x/\beta}}{\theta_a \beta^{\eta_a}} \frac{1}{\theta_b \beta^{\eta_b}} \frac{(-1)^{\gamma_d} x^{\eta_c} e^{-\gamma_d r/\beta}}{\theta_c \beta^{\eta_c} \theta_d \beta^{\eta_d}} x^h r^{\eta_b+\eta_d+\alpha-h-1}. \end{aligned} \quad (3.22)$$

Integrating with respect to r over $(0, \infty)$, we get the pdf of $\mathcal{D}_{i,n}^{(m)}$ as

$$\begin{aligned} &f_{\mathcal{D}_{i,n}^{(m)}}(r) \\ &= \delta \sum_{a_1+\cdots+a_{\alpha+1}=i-1} \sum_{b_1+\cdots+b_\alpha=n-i-m} \sum_{\substack{c_1+\cdots+c_\alpha+ \\ d_1+\cdots+d_\alpha=m-1}} \\ &\times \sum_{h=0}^{\eta_b+\eta_d+\alpha-1} \binom{i-1}{a_1, a_2, \ldots, a_{\alpha+1}} \binom{n-i-m}{b_1, b_2, \ldots, b_\alpha} \binom{m-1}{c_1, \ldots, c_\alpha, d_1, \ldots, d_\alpha} \\ &\times \binom{\eta_b + \eta_d + \alpha - 1}{h} \frac{(-1)^{\gamma_a+\gamma_d} e^{-\gamma_d r/\beta} r^{\eta_b+\eta_d+\alpha-h-1}}{\theta_a \theta_b \theta_c \theta_d \beta^{\eta_a+\eta_b+\eta_c+\eta_d}} I(r), \end{aligned} \quad (3.23)$$

where

$$\begin{aligned} &I(r) \\ &= \int_0^\infty x^{\eta_a+\eta_c+h+\alpha-1} e^{-(m+\gamma_a-1)x/\beta} e^{-(2x+r)/\beta} e^{-(n-i-m)(x+r)/\beta} dx \\ &\quad = \Gamma(\eta_a + \eta_c + h \\ &\qquad + \alpha)(\beta/(\gamma_a + n - i + 1))^{\eta_a+\eta_c+h+\alpha} e^{-(n-i-m+1)r/\beta}. \end{aligned} \quad (3.24)$$

Therefore,

$$\begin{aligned} & f_{\mathcal{D}_{i,n}^{(m)}}(r) \\ & = \sum_{a_1+\cdots+a_{\alpha+1}=i-1} \sum_{b_1+\cdots+b_\alpha=n-i-m} \sum_{\substack{c_1+\cdots+c_\alpha+ \\ d_1+\cdots+d_\alpha=m-1}} \sum_{h=0}^{\eta_b+\eta_d+\alpha-1} \delta \\ & \times \binom{i-1}{a_1,a_2,\ldots,a_{\alpha+1}} \binom{n-i-m}{b_1,b_2,\ldots,b_\alpha} \binom{m-1}{c_1,\ldots,c_\alpha,d_1,\ldots,d_\alpha} \\ & \times \binom{\eta_b+\eta_d+\alpha-1}{h} \frac{(-1)^{\gamma_a+\gamma_d}\Gamma(\eta_a+\eta_c+h+\alpha)}{\theta_a\theta_b\theta_c\theta_d(\gamma_a+n-i+1)^{\eta_a+\eta_c+h+\alpha}} \\ & \times r^{\eta_b+\eta_d+\alpha-h-1}\beta^{-\eta_b-\eta_d+h+\alpha}e^{-(\gamma_d+n-i-m+1)r/\beta}. \end{aligned} \tag{3.25}$$

Remark 3.1. For dealing with finite sums of gamma random variables, one may refer to Moschopoulos (1985); Paris (2011).

Theorem 3.2. The m-spacing $\mathcal{D}_{i,n}^{(m)}$ has a finite mixture distribution with negative components, each of which in turn has a finite gamma mixture distribution. That is,

$$f_{\mathcal{D}_{i,n}^{(m)}}(r|\beta) = \sum_{k=0}^{m-1} \delta_k^{(m)}\, \mathcal{C}_k^{(m)}(r), \tag{3.26}$$

where

$$\mathcal{C}_k^{(m)}(r) = \sum_{h=0}^{(\alpha-1)(n-i-k)} w_h\, f_h(r|\alpha_h,\beta_h), \tag{3.27}$$

$\delta_k^{(m)}$ is an integer, $f_h(r|\alpha_h,\beta_h)$ is a gamma pdf of shape and scale parameters α_h and β_h, respectively.

Proof. Let $\mathcal{M}_k^{(m)}(r) = f_{\mathcal{D}_{i,n}^{(m)}}(r)|_{\gamma_d=m-1-k}$. Then

$$\begin{aligned} & \mathcal{M}_k^{(m)}(r) \\ & = \sum_{a_1+\cdots+a_{\alpha+1}=i-1} \sum_{b_1+\cdots+b_\alpha=n-i-m} \sum_{\substack{c_1+\cdots+c_\alpha=k \\ d_1+\cdots+d_\alpha=m-1-k}} \sum_{h=0}^{\eta_b+\eta_d+\alpha-1} \delta \\ & \qquad \times \binom{i-1}{a_1,a_2,\ldots,a_{\alpha+1}} \binom{n-i-m}{b_1,b_2,\ldots,b_\alpha} \binom{m-1}{c_1,\ldots,c_\alpha,d_1,\ldots,d_\alpha} \\ & \qquad \times \binom{\eta_b+\eta_d+\alpha-1}{h} \frac{(-1)^{\gamma_a+\gamma_d}\Gamma(\eta_a+\eta_c+h+\alpha)}{\theta_a\theta_b\theta_c\theta_d(\gamma_a+n-i+1)^{\eta_a+\eta_c+h+\alpha}} \\ & \qquad \times r^{\eta_b+\eta_d+\alpha-h-1}\beta^{-\eta_b-\eta_d+h+\alpha}e^{-(n-i-k)r/\beta}. \end{aligned} \tag{3.28}$$

Note that $\max(\eta_b) = (\alpha-1)(n-i-m)$ and, when $\gamma_d = m-1-k$, $\max(\eta_d) = (\alpha-1)(m-1-k)$, because the maximum value of η_d in this case is attained at $d_1 = d_2 \cdots = d_{\alpha-1} = 0$ and

$d_\alpha = m-1-k$. In this case $\eta_d = (\alpha-1)d_\alpha = (\alpha-1)(n-i-k)$. This implies that h ranges from 0 to $(\alpha-1)(n-i-k)$; i.e., $h = 0,1,\dots,(\alpha-1)(n-i-k)$.

By the same reasoning, for each value of k, the power of r ranges from 0 to $(\alpha-1)(n-i-k)$.

Note also that, when $\gamma_d = m-1-k$ and $a_1+\cdots+a_{\alpha+1} = i-1$, $\gamma_a = i-1-a_1$. Therefore,

$$(-1)^{\gamma_a+\gamma_d} = (-1)^{m+i-k}(-1)^{-a_1}/(-1)^2 = (-1)^{m+i-k}(-1)^{a_1}.$$

Therefore, we can write $\mathcal{M}_k^{(m)}(r)$ as

$$\begin{aligned}
&\mathcal{M}_k^{(m)}(r)\\
&= \frac{1}{r}\sum_{h=0}^{(\alpha-1)(n-i-k)}\Bigg(\sum_{a_1+\cdots+a_{\alpha+1}=i-1}\ \sum_{b_1+\cdots+b_\alpha=n-i-m}\ \sum_{\substack{c_1+\cdots+c_\alpha=k\\ d_1+\cdots+d_\alpha=m-1-k}}\delta\,.\\
&\quad\times\binom{i-1}{a_1,a_2,\dots,a_{\alpha+1}}\binom{n-i-m}{b_1,b_2,\dots,b_\alpha}\binom{m-1}{c_1,\dots,c_\alpha,d_1,\dots,d_\alpha}\\
&\quad\times\binom{\eta_b+\eta_d+\alpha-1}{h}\frac{(-1)^{\gamma_a+\gamma_d}\Gamma(\eta_a+\eta_c+h+\alpha)}{\theta_a\theta_b\theta_c\theta_d(\gamma_a+n-i+1)^{\eta_a+\eta_c+h+\alpha}}\\
&\quad\times.\left(\frac{\beta}{r}\right)^{-\eta_b-\eta_d+h+\alpha}\Bigg)e^{-(n-i-k)r/\beta}.
\end{aligned} \tag{3.29}$$

Therefore,

$$\mathcal{M}_k^{(m)}(r) = \sum_{h=0}^{(\alpha-1)(n-i-k)} \phi_k^{(m)}(h) r^h e^{-(n-i-k)r/\beta}, \tag{3.30}$$

and

$$f_{\mathcal{D}_{i,n}^{(m)}}(r) = \sum_{k=0}^{m-1}\sum_{h=0}^{(\alpha-1)(n-i-k)} \phi_k^{(m)}(h) r^h e^{-(n-i-k)r/\beta}, \tag{3.31}$$

where $\phi_k^{(m)}(h)$ is the coefficient of $r^h e^{-(n-i-k)r/\beta}$ in (3.29) and $h = 0,1,\dots,(\alpha-1)(n-i-k)$.

In fact,

$$\phi_k^{(m)}(h) = \frac{1}{r} \sum_{a_1+\cdots+a_{\alpha+1}=i-1} \sum_{b_1+\cdots+b_\alpha=n-i-m} \sum_{\substack{c_1+\cdots+c_\alpha=k \\ d_1+\cdots+d_\alpha=m-1-k}} \delta$$
$$\times \binom{i-1}{a_1, a_2, \dots, a_{\alpha+1}} \binom{n-i-m}{b_1, b_2, \dots, b_\alpha} \binom{m-1}{c_1, \dots, c_\alpha, d_1, \dots, d_\alpha} \tag{3.32}$$
$$\times \binom{\eta_b + \eta_d + \alpha - 1}{h} \frac{(-1)^{\gamma_a+\gamma_d}\Gamma(\eta_a + \eta_c + h + \alpha)\beta^{-\eta_b-\eta_d+h+\alpha}}{\theta_a\theta_b\theta_c\theta_d(\gamma_a + n - i + 1)^{\eta_a+\eta_c+h+\alpha}}.$$

Let $\delta_k^{(m)} = \int_0^\infty \mathcal{M}_k^{(m)}(r)dr$. We will see in the next section that $\delta_k^{(m)}$ is a nonzero integer for all $m = 1,2,\dots,n-1$ and $k = 0,1,\dots,m-1$.

Now, let $\mathcal{C}_k^{(m)}(r) = \mathcal{M}_k^{(m)}(r)/\delta_k^{(m)}$, then we have

$$f_{\mathcal{D}_{i,n}^{(m)}}(r) = \sum_{k=0}^{m-1} \delta_k^{(m)} \mathcal{C}_k^{(m)}(r), \tag{3.33}$$

where

$$\mathcal{C}_k^{(m)}(r) = \sum_{h=0}^{(\alpha-1)(n-i-k)} \phi_k^{(m)}(h) r^h e^{-(n-i-k)r/\beta}/\delta_k^{(m)}. \tag{3.34}$$

It's clear from the definition of $\mathcal{C}_k^{(m)}$ that $\int_0^\infty \mathcal{C}_k^{(m)}(r)dr = 1$ and that $\mathcal{C}_k^{(m)}(r)$ has a gamma mixture distribution, because for each fixed k, $\phi_k^{(m)}(h)r^h e^{-(n-i-k)r/\beta}/\delta_k^{(m)}$ is a gamma pdf with shape parameter $\alpha_h = h+1$ and scale parameter $\beta_h = \beta/(n-i-k)$, where $h = 0,1,\dots,(\alpha-1)(n-i-k)$. The weights (or proportions) of the mixture are obtained by integrating each term of (refsum) with respect to r on the positive real line. They turn to be

$$w_h = h!\left(\frac{\beta}{n-i-k}\right)^{h+1} \phi_k^{(m)}(h)/\delta_k^{(m)}. \tag{3.35}$$

□

1. COMPUTING $\boldsymbol{\phi_k^{(m)}}$

Although it is very tedious to deal with $\phi_k^{(m)}$ directly, we can write it in a way that enables us to concisely and easily calculate the ratio $\phi_k^{(m)}/\phi_k^{(\ell)}$, for $k < \ell \le m$. This can be achieved by making use of the assumption that $\mathcal{M}_k^{(m)}$, and consequently both of $\mathcal{C}_k^{(m)}$ and $\phi_k^{(m)}$, is obtained by letting $\gamma_d = m-1-k$.

So, let $\gamma_d = m-1-k$ in (3.19). Then

$$S_3 = (-1)^{m-1-k} e^{-(m-1-k)r/\beta} \sum_{\substack{c_1+\cdots+c_\alpha=k \\ d_1+\cdots+d_\alpha=m-1-k}} \binom{m-1}{c_1,\dots,c_\alpha,d_1,\dots,d_\alpha}$$
$$\times \prod_{j=1}^{\alpha}\left(\frac{x/\beta}{(j-1)!}\right)^{c_j} \prod_{j=1}^{\alpha}\left(\frac{(x+r)/\beta}{(j-1)!}\right)^{d_j}$$
$$= (-1)^{m-1-k} e^{-(m-1-k)r/\beta} \sum_{\substack{c_1+\cdots+c_\alpha=k \\ d_1+\cdots+d_\alpha=m-1-k}} \frac{(m-1)!}{k!\,(m-1-k)!}\frac{k!}{c_1!\cdots c_\alpha!}$$
$$\times \prod_{j=1}^{\alpha}\left(\frac{x/\beta}{(j-1)!}\right)^{c_j} \frac{(m-1-k)!}{d1!\cdots d_\alpha!}\prod_{j=1}^{\alpha}\left(\frac{(x+r)/\beta}{(j-1)!}\right)^{d_j} \quad (4.1)$$
$$= (-1)^{m-1-k} e^{-(m-1-k)r/\beta} \binom{m-1}{k}\left(\sum_{j=0}^{\alpha-1}\frac{(x/\beta)^j}{j!}\right)^k$$
$$\times \left(\sum_{j=0}^{\alpha-1}\frac{((x+r)/\beta)^j}{j!}\right)^{m-1-k}.$$

Going back to the joint pdf of $X_{(i+m)}$ and $X_{(i)}$, we can write it as

$$f_{i,i+m}(x,x+r) = (-1)^{m-1-k}\delta x^{\alpha-1}(x + r)^{\alpha-1} e^{-(n-i+1)x/\beta} e^{-(n-i-k)r/\beta}$$
$$\times \binom{m-1}{k}\left(1-e^{x/\beta}\sum_{j=0}^{\alpha-1}\frac{(x/\beta)^j}{j!}\right)^{i-1} \quad (4.2)$$
$$\times \left(\sum_{j=0}^{\alpha-1}\frac{((x+r)/\beta)^j}{j!}\right)^{n-i-1-k}\left(\sum_{j=0}^{\alpha-1}\frac{(x/\beta)^j}{j!}\right)^{k}.$$

Therefore,

$$\begin{aligned}
&f_{i,i+m}(x, x+r)\\
&= (-1)^{m-1-k}\delta e^{-(n-i-k)r/\beta}\binom{m-1}{k}\sum_{a_1+\cdots+a_{\alpha+1}=i-1}\ \sum_{b_1+\cdots+b_\alpha=n-i-1-k}\\
&\times\sum_{c_1+\cdots+c_\alpha=k}\ \sum_{h=0}^{\eta_b+\alpha-1}\binom{i-1}{a_1, a_2, \dots, a_{\alpha+1}}\binom{n-i-1-k}{b_1, b_2, \dots, b_\alpha}\\
&\times\binom{k}{c_1, c_2, \dots, c_\alpha}\binom{\eta_b+\alpha-1}{h}\frac{(-1)^{\gamma_a}r^{\eta_b+\alpha-h-1}}{\theta_a\theta_b\theta_c\beta^{\eta_a+\eta_b+\eta_c}}\\
&\times x^{\eta_a+\eta_c+h+\alpha-1}e^{-(a+n-i+1)x/\beta}.
\end{aligned} \tag{4.3}$$

Note also that when $\gamma_d = m-1-k$ and $a_1+\cdots+a_{\alpha+1} = i-1$, $\gamma_a = i-1-a_1$. In this case we have

$$(-1)^{\gamma_a+m-1-k} = (-1)^{m+i-k}(-1)^{-a_1}/(-1)^2 = (-1)^{m+i-k}(-1)^{a_1}.$$

Finally, $\mathcal{M}_k^{(m)}(r)$ is the integral of $f_{i,i+m}(x, x+r)$ with respect to x over $(0, \infty)$.

$$\begin{aligned}
&\mathcal{M}_k^{(m)}(r)\\
&= (-1)^{m+i-k}\delta e^{-(n-i-k)r/\beta}\binom{m-1}{k}\sum_{h=0}^{(\alpha-1)(n-i-k)}\ \sum_{a_1+\cdots+a_{\alpha+1}=i-1}\\
&\times\sum_{b_1+\cdots+b_\alpha=n-i-1-k}\ \sum_{c_1+\cdots+c_\alpha=k}\binom{i-1}{a_1, a_2, \dots, a_{\alpha+1}}\binom{n-i-1-k}{b_1, b_2, \dots, b_\alpha}\\
&\times\binom{k}{c_1, c_2, \dots, c_\alpha}\binom{(\alpha-1)(n-i-k)}{h}\frac{(-1)^{a_1}r^h}{\theta_a\theta_b\theta_c\beta^{h+1}}\\
&\times\Gamma(\eta_a+\eta_b+h+\alpha)(1/(\gamma_a+n-i+1))^{\eta_a+\eta_c+h+\alpha}.
\end{aligned} \tag{4.4}$$

Therefore,

$$\begin{aligned}
&\phi_k^{(m)}(h)\\
&= (-1)^{m+i-k}\delta\binom{m-1}{k}\sum_{a_1+\cdots+a_{\alpha+1}=i-1}\ \sum_{b_1+\cdots+b_\alpha=n-i-1-k}\ \sum_{c_1+\cdots+c_\alpha=k}\\
&\times\binom{i-1}{a_1, a_2, \dots, a_{\alpha+1}}\binom{n-i-1-k}{b_1, b_2, \dots, b_\alpha}\binom{k}{c_1, c_2, \dots, c_\alpha}\\
&\times\binom{(\alpha-1)(n-i-k)}{h}\frac{(-1)^{a_1}\Gamma(\eta_a+\eta_c+h+\alpha)\beta^{-h-1}}{\theta_a\theta_b\theta_c(\gamma_a+n-i+1)^{\eta_a+\eta_c+h+\alpha}}.
\end{aligned} \tag{4.5}$$

We wrote a Mathematica code in Appendix A.1 for computing $\phi_k^{(m)}(h)$.

Remark 4.1. Note that for $k = 0,1,\dots,m-1$ and $k < \ell \le m$, the quotient $\phi_k^{(m)}/\phi_k^{(\ell)}$ reduces to

$$\frac{\phi_k^{(m)}}{\phi_k^{(\ell)}} = \frac{(-1)^{m-k}\binom{m-1}{k}}{(m-1)!\,(n-i-m)!}\frac{(\ell-1)!\,(n-i-\ell)!}{(-1)^{\ell-k}\binom{\ell-1}{k}}$$
$$= (-1)^{m-\ell}\frac{(n-i-\ell)!\,(\ell-1-k)!}{(n-i-m)!\,(m-1-k)!}.$$

Note that although $\phi_k^{(m)}$ as introduced in (4.5) is a function of h, the quotient $\phi_k^{(m)}/\phi_k^{(\ell)}$ is a constant depending only on $i, k, m,$ and n. It does not depend on h appearing in (4.5).

In particular,

$$\frac{\phi_0^{(m)}}{\phi_0^{(1)}} = (-1)^{m-1}\frac{(n-i-1)!}{(n-i-m)!\,(m-1)!} = (-1)^{m-1}\binom{n-i-1}{m-1}.$$

5. SPACINGS IN TERMS OF SPACINGS OF LOWER ORDERS

In this section, we express $\mathcal{M}_k^{(m)}(r)$ of (3.30) in terms of $f_{\mathcal{D}_{i,n}^{(j)}}(r)$, where $0 \le k \le m-1$ and $1 \le j \le m$. We start with the following theorem.

Theorem 5.1. For each m, $\mathcal{M}_k^{(m)}(r)$ is a linear combination of $f_{\mathcal{D}_{i,n}^{(j)}}(r), j = 1,2,\dots,m$. That is,

$$\mathcal{M}_k^{(m)}(r) = \sum_{j=1}^{k+1}\lambda_{j,k}^{(m)} f_{\mathcal{D}_{i,n}^{(j)}}(r), \quad k = 0,1,\dots,m-1, \tag{5.1}$$

where

$$\lambda_{j,k}^{(m)} = (-1)^{m+k-1}\binom{n-i-j}{m-j}\binom{m-j}{k-j+}. \tag{5.2}$$

Proof. If we look at $\phi_k^{(m)}$ as a function of β, we can write (5.10) as

$$f_{\mathcal{D}_{i,n}^{(m)}}(r) = \frac{1}{r}\sum_{k=0}^{m-1}\sum_{h=0}^{(\alpha-1)(n-i-k)}\phi_k^{(m)}(\beta/r)e^{-(n-i-k)r/\beta}. \tag{5.3}$$

It follows from (5.3) that

$$
\begin{aligned}
f_{\mathcal{D}_{i,n}^{(1)}}(r) &= \frac{1}{r}\sum_{h=0}^{(\alpha-1)(n-i)} \phi_0^{(1)}(\beta/r)e^{-(n-i)r/\beta} \quad (k = 0 \text{ when } m = 1) \\
&= \frac{1}{r}\sum_{h=0}^{(\alpha-1)(n-i)} \left(\phi_0^{(1)}/\phi_0^{(m)}\right)\phi_0^{(m)}(\beta/r)e^{-(n-i)r/\beta} \\
&= \left(\phi_0^{(1)}/\phi_0^{(m)}\right)\frac{1}{r}\sum_{h=0}^{(\alpha-1)(n-i)} \phi_0^{(m)}(\beta/r)e^{-(n-i)r/\beta} \quad \left(\phi_0^{(1)}/\phi_0^{(m)} \text{ is free of } h\right) \\
&= \left(\phi_0^{(1)}/\phi_0^{(m)}\right)\mathcal{M}_0^{(m)}(r).
\end{aligned}
\tag{5.4}
$$

Similarly,

$$
\begin{aligned}
f_{\mathcal{D}_{i,n}^{(2)}}(r) &= \frac{1}{r}\sum_{j=0}^{(\alpha-1)(n-i)} \phi_0^{(2)}(\beta/r)e^{-(n-i)r/\beta} + \frac{1}{r}\sum_{j=0}^{(\alpha-1)(n-i-1)} \phi_1^{(2)}(\beta/r)e^{-(n-i-1)r/\beta} \\
&= \left(\phi_0^{(2)}/\phi_0^{(m)}\right)\frac{1}{r}\sum_{j=0}^{(\alpha-1)(n-i)} \phi_0^{(m)}(\beta/r)e^{-(n-i)r/\beta} \\
&\quad + \left(\phi_1^{(2)}/\phi_1^{(m)}\right)\frac{1}{r}\sum_{j=0}^{(\alpha-1)(n-i-1)} \phi_1^{(m)}(\beta/r)e^{-(n-i-1)r/\beta} \\
&= \left(\phi_0^{(2)}/\phi_0^{(m)}\right)\mathcal{M}_0^{(m)}(r) + \left(\phi_1^{(2)}/\phi_1^{(m)}\right)\mathcal{M}_1^{(m)}(r).
\end{aligned}
\tag{5.5}
$$

By induction on ℓ, it follows that

$$
\begin{aligned}
f_{\mathcal{D}_{i,n}^{(\ell)}}(r) &= \frac{\phi_0^{(\ell)}}{\phi_0^{(m)}}\mathcal{M}_0^{(m)}(r) + \frac{\phi_1^{(\ell)}}{\phi_1^{(m)}}\mathcal{M}_1^{(m)}(r) + \cdots + \frac{\phi_{\ell-1}^{(\ell)}}{\phi_{\ell-1}^{(m)}}\mathcal{M}_{\ell-1}^{(m)}(r) \\
&= \sum_{j=0}^{\ell-1}\frac{\phi_j^{(\ell)}}{\phi_j^{(m)}}\mathcal{M}_j^{(m)}(r).
\end{aligned}
\tag{5.6}
$$

Note that when $\ell = m$, (5.6) becomes

$$f_{\mathcal{D}_{i,n}^{(m)}}(r) = \mathcal{M}_0^{(m)}(r) + \mathcal{M}_1^{(m)}(r) + \cdots + \mathcal{M}_{m-1}^{(m)}(r). \quad (5.7)$$

Solving (5.6) simultaneously in $\mathcal{M}_0^{(m)}(r), \mathcal{M}_1^{(m)}(r), \ldots,$ and $\mathcal{M}_k^{(m)}(r)$ gives (5.1).

To prove (5.2), we proceed by induction on k making use of Remark ?. For $k = 0$, we have

$$\mathcal{M}_0^{(m)}(r) = \lambda_{1,0}^{(m)} f_{\mathcal{D}_{i,n}^{(1)}}(r), \quad (5.8)$$

where

$$\begin{aligned} \lambda_{1,0}^{(m)} = \frac{\phi_0^{(m)}}{\phi_0^{(1)}} &= (-1)^{m-1} \frac{(n-i-1)!}{(m-1)!\,(n-i-m)!} \\ &= (-1)^{m-1} \binom{n-i-1}{m-i} \end{aligned} \quad (5.9)$$

and $\binom{m-j}{k-j+1} = 1$ when $k = 0$ and $j = 1$. So (5.2) is true for $k = 0$.

Assume that (5.2) is true for $k = 0,1,\ldots,m-2$. We need to prove that it is also true for $k = m - 1$. Now,

$$\begin{aligned} \mathcal{M}_{m-1}^{(m)}(r) = f_{\mathcal{D}_{i,n}^{(m)}}(r) - \mathcal{M}_0^{(m)}(r) - \mathcal{M}_1^{(m)}(r) - \cdots \\ - \mathcal{M}_{m-2}^{(m)}(r). \end{aligned} \quad (5.10)$$

The coefficient of $f_{\mathcal{D}_{i,n}^{(j)}}(r)$ in $\mathcal{M}_{m-1}^{(m)}(r)$ is

$$\begin{aligned} \lambda_{j,m-1}^{(m)} &= -\left(\lambda_{j,j-1}^{(m)} + \lambda_{j,j}^{(m)} + \cdots + \lambda_{j,m-2}^{(m)}\right) \\ &= -\sum_{k=j-1}^{m-2} \lambda_{j,k}^{(m)} \\ &= \binom{n-i-j}{m-j} \sum_{k=j-1}^{m-2} (-1)^{m+k} \binom{m-j}{k-j+1} \\ &= \binom{n-i-j}{m-j} (-1)^{m+j-1} \sum_{k=0}^{m-j-1} (-1)^k \binom{m-j}{k} \\ &= (-1)^{2m-2} \binom{n-i-j}{m-j} = \binom{n-i-j}{m-j}, \end{aligned} \quad (5.11)$$

as required, where we have used the identity (cf. Boardman, 2004)

$$\sum_{h=0}^{M}(-1)^h\binom{N}{h}=(-1)^M\binom{N-1}{M} \tag{5.12}$$

with $M=m-j-1$, and $N=m-j$. Note that $\binom{m-j}{k-j+1}=1$ when $k=m-1$. ∎

Corollary 5.1. Let $\delta_k^{(m)}$ be the integral of $\mathcal{M}_k^{(m)}(r)$ with respect to r over $(0,\infty)$. Then

$$\begin{aligned}\delta_k^{(m)}&=\sum_{j=1}^{k+1}\lambda_{j,k}^{(m)}\\&=(n-i)(-1)^{k+m+1}\binom{m-1}{k}\binom{n-i-1}{m-1}/(n-i-k).\end{aligned} \tag{5.13}$$

Proof. We integrate both sides of (5.1) with respect to r over $(0,\infty)$.

$$\delta_k^{(m)}=\int_0^\infty \mathcal{M}_k^{(m)}(r)dr=\sum_{j=1}^{k+1}\int_0^\infty \lambda_{j,k}^{(m)} f_{\mathcal{D}_{i,n}^{(j)}}(r)dr=\sum_{j=1}^{k+1}\lambda_{j,k}^{(m)} \tag{5.14}$$

as required. ∎

Remark 5.1. It is clear from (5.13) that $\delta_k^{(m)}\neq 0$.

Example 5.1. Let $\alpha=2, m=3, i=2, n=7$, and $\beta>0$. We use *Mathematica* (see Appendix A.2) to implement the computations. Then

$$f_{\mathcal{D}_{2,7}^{(3)}}(r)=\mathcal{M}_0^{(3)}(r)+\mathcal{M}_1^{(3)}(r)+\mathcal{M}_2^{(3)}(r), \tag{5.15}$$

where

$$\mathcal{M}_0^{(3)}(r)=\frac{95r^5e^{-\frac{5r}{\beta}}}{49\beta^6}+\frac{14005r^4e^{-\frac{5r}{\beta}}}{1029\beta^5}+\frac{94175r^3e^{-\frac{5r}{\beta}}}{2401\beta^4}+\frac{8792600r^2e^{-\frac{5r}{\beta}}}{151263\beta^3}+\frac{281982875re^{-\frac{5r}{\beta}}}{6353046\beta^2}+\frac{88897385e^{-\frac{5r}{\beta}}}{6353046\beta},$$

$$\mathcal{M}_1^{(3)}(r)=-\frac{6400r^4e^{-\frac{4r}{\beta}}}{1029\beta^5}-\frac{84300r^3e^{-\frac{4r}{\beta}}}{2401\beta^4}-\frac{3829625r^2e^{-\frac{4r}{\beta}}}{50421\beta^3}-\frac{236910655re^{-\frac{4r}{\beta}}}{3176523\beta^2}-\frac{88897385e^{-\frac{4r}{\beta}}}{3176523\beta},\ \text{and}$$

$$\mathcal{M}_2^{(3)}(r)=\frac{74425r^3e^{-\frac{3r}{\beta}}}{14406\beta^4}+\frac{362550r^2e^{-\frac{3r}{\beta}}}{16807\beta^3}+\frac{63946145re^{-\frac{3r}{\beta}}}{2117682\beta^2}+\frac{88897385e^{-\frac{3r}{\beta}}}{6353046\beta}.$$

Note that $\delta_0^{(3)}=6, \delta_1^{(3)}=-15$, and $\delta_2^{(3)}=10$.

Example 5.2. By definition, the range of a random sample of size n is $\mathcal{D}_{1,n}^{(n-1)}=X_{(n)}-$

$X_{(1)}$; i.e., $i = 1$ and $m = n - 1$. For simplicity, let $\alpha = 2, n = 5, m = 4$, and $i = 1$. For example,

$$\begin{aligned}\mathcal{M}_2^{(4)}(r) &= -\frac{612r^2e^{-\frac{2r}{\beta}}}{125\beta^3} - \frac{6276re^{-\frac{2r}{\beta}}}{625\beta^2} - \frac{2832e^{-\frac{2r}{\beta}}}{625\beta} \\ &= -6\left(\frac{102r^2e^{-\frac{2r}{\beta}}}{125\beta^3} + \frac{1046re^{-\frac{2r}{\beta}}}{625\beta^2} + \frac{472e^{-\frac{2r}{\beta}}}{625\beta}\right).\end{aligned} \tag{5.16}$$

Note that $\mathcal{C}_2^{(4)}(r)$ is

$$\mathcal{C}_2^{(4)}(r) = \frac{102r^2e^{-\frac{2r}{\beta}}}{125\beta^3} + \frac{1046re^{-\frac{2r}{\beta}}}{625\beta^2} + \frac{472e^{-\frac{2r}{\beta}}}{625\beta}. \tag{5.17}$$

Here, $(\alpha - 1)(n - i - k) = (1)(5 - 1 - 2) = 2$ and $\delta_0^{(4)} = -1, \delta_1^{(4)} = 4, \delta_2^{(4)} = -6$, and $\delta_3^{(4)} = 4$. The three weights corresponding to $\mathcal{C}_2^{(4)}(r)$ are $w_0 = 236/625, w_1 = 523/1250$, and $w_2 = 51/250$. The functions $\phi_2^{(4)}(h)$ for $h = 0,1$, and 2 are $\phi_2^{(4)}(0) = -612/125\beta^3, \phi_2^{(4)}(1) = -6276/625\beta^2$, and $\phi_2^{(4)}(2) = -2832/625\beta$.
Note that $\sum_{k=0}^{4} \delta_k^{(4)} = 1$ and $\sum_{j=1}^{3} w_j = 1$.

Example 5.3. In Example 2, $\mathcal{M}_2^{(4)}(r)$ can be expressed in terms of lower spacings as

$$\mathcal{M}_2^{(4)}(r) = \lambda_{1,2}^{(4)}\mathcal{D}_{1,5}^{(1)} + \lambda_{2,2}^{(4)}\mathcal{D}_{1,5}^{(2)} + \lambda_{3,2}^{(4)}\mathcal{D}_{1,5}^{(3)}, \tag{5.18}$$

where $\lambda_{1,2}^{(4)} = -3, \lambda_{2,2}^{(4)} = -2$, and $\lambda_{3,2}^{(4)} = -1$. Here, $\delta_2^{(4)} = \lambda_{1,2}^{(4)} + \lambda_{2,2}^{(4)} + \lambda_{3,2}^{(4)} = -6$.

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Appendices

A. MATHEMATICA CODE

A.1. Computing $\phi_k^{(m)}(h)$

The following code can be used to compute $\phi_k^{(m)}(h)$ when the sample size is n, the scale parameter is *beta*, and the shape parameter is *alpha*.

```
phi[alpha_,k_, m_, i_, n_]:= Module[{A, AP, AA, B, BB, F, FF, delta, TA, TB, TC},
AP= Sum[Subscript[a, u], {u, 2, alpha + 1}];
A= Sum[(u-2)Subscript[a, u], {u, 3, alpha + 1}];
AA= Product[((u-1)!)^Subscript[a, u+1], {u, 2, alpha}];
B=Sum[(v-1)Subscript[b, v], {v, 2, alpha}];
BB= Product[((v-1)!)^Subscript[b, v], {v, 2, alpha}];
F= Sum[(w-1)Subscript[c, w], {w, 2, alpha}];
FF= Product[((w-1)!)^Subscript[c, w], {w, 2, alpha}];
delta = (-1)^(m+i-k) E^(-((( n-i-k) r)/beta))
( n!beta^(-2 alpha)/(Gamma[alpha]^2 (m-1)! (n-i-m)! (i-1)!))Binomial[m-1, k];
TA = Table[{Subscript[a, u], 0, i-1}, {u, 1, alpha+1}];
TB = Table[{Subscript[b, v],0, n-i-1-k},{v, 1, alpha}];
TC= Table[{Subscript[c, w], 0, k}, {w, 1, alpha}];
Expand[Sum[Boole[Sum[Subscript[a, u],{u, 1, alpha+1}] == i-1 &&
Sum[Subscript[b, v],{v, 1, alpha}] == n-i-1-k &&
Sum[Subscript[c, w],{w, 1, alpha}] == k] delta Binomial[B +alpha-1, h]
(((-1)^Subscript[a, 1] r^(B+alpha-h-1))/(AA BB  FF  beta^(A+B +F)))
Gamma[A+F+h+alpha](beta/(n-i+1+AP ))^(A +F +h+alpha )
Multinomial @@ Table[Subscript[a, u], {u, 1, alpha+1}]
Multinomial @@ Table[Subscript[b, v], {v, 1, alpha}]
Multinomial @@ Table[Subscript[c, w], {w, 1, alpha}],##]&@@ Union[TA, TB, TC]]]
```

A.2. Computing $f_{\mathcal{D}_{i,n}^{(m)}}(r)$

The following code can be used to compute $f_{\mathcal{D}_{i,n}^{(m)}}(r)$ when the sample size is n, the scale parameter is *beta*, and the shape parameter is *alpha*.

```
pdf[alpha_, k_, m_, i_, n_]:= Module[{A, AP, AA, B, BB, F, FF, delta, TA, TB, TC},
AP= Sum[Subscript[a, u], {u, 2, alpha + 1}];
A= Sum[(u-2)Subscript[a, u], {u, 3, alpha + 1}];
AA= Product[((u-1)!)^Subscript[a, u+1], {u, 2, alpha}];
B=Sum[(v-1)Subscript[b, v], {v, 2, alpha}];
BB= Product[((v-1)!)^Subscript[b, v], {v, 2, alpha}];
F= Sum[(w-1)Subscript[c, w], {w, 2, alpha}];
FF= Product[((w-1)!)^Subscript[c, w], {w, 2, alpha}];
delta = (-1)^(m+i-k) E^(-((( n-i-k) r)/beta))
( n!beta^(-2 alpha)/(Gamma[alpha]^2 (m-1)! (n-i-m)! (i-1)!))Binomial[m-1, k];
```

```
TA = Table[{Subscript[a, u], 0, i-1}, {u, 1, alpha+1}];
TB = Table[{Subscript[b, v],0, n-i-1-k},{v, 1, alpha}];
TC= Table[{Subscript[c, w], 0, k}, {w, 1, alpha}];
Expand[Sum[Sum[Boole[Sum[Subscript[a, u],{u, 1, alpha+1}] == i-1 &&
Sum[Subscript[b, v],{v, 1, alpha}] == n-i-1-k &&
Sum[Sum[Subscript[c, w],{w, 1, alpha}] == k] delta
Binomial[B +alpha-1, h]
(((-1)^Subscript[a, 1] r^(B+alpha-h-1))/(AA BB  FF  beta^(B -h-alpha)))
Gamma[A+F+h+alpha](n-i+1+AP )^-(A +F +h+alpha)
Multinomial @@ Table[Subscript[a, u], {u, 1, alpha+1}]
Multinomial @@ Table[Subscript[b, v], {v, 1, alpha}]
Multinomial @@ Table[Subscript[c, w], {w, 1, alpha}],##]&@@
Union[TA, TB, TC],{h, 0, (alpha -1)(n-i-k)}]]], {k, 0, m-1}]
```