DISTRIBUTIONS OF MATRIX VARIATES AND LATENT ROOTS DERIVED FROM NORMAL SAMPLES[1]

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1. Summary. The paper is largely expository, but some new results are included to round out the paper and bring it up to date.

The following distributions are quoted in Section 7.

1. Type ${}_0F_0$, exponential: (i) χ^2, (ii) Wishart, (iii) latent roots of the covariance matrix.

2. Type ${}_1F_0$, binomial series: (i) variance ratio, F, (ii) latent roots with unequal population covariance matrices.

3. Type ${}_0F_1$, Bessel: (i) noncentral χ^2, (ii) noncentral Wishart, (iii) noncentral means with known covariance.

4. Type ${}_1F_1$, confluent hypergeometric: (i) noncentral F, (ii) noncentral multivariate F, (iii) noncentral latent roots.

5. Type ${}_2F_1$, Gaussian hypergeometric: (i) multiple correlation coefficient, (ii) canonical correlation coefficients.

The modifications required for the corresponding distributions derived from the complex normal distribution are outlined in Section 8, and the distributions are listed.

The *hypergeometric functions* ${}_pF_q$ of matrix argument which occur in the multivariate distributions are defined in Section 4 by their expansions in *zonal polynomials* as defined in Section 5. Important properties of zonal polynomials and hypergeometric functions are quoted in Section 6. Formulae and methods for the calculation of zonal polynomials are given in Section 9 and the zonal polynomials up to degree 6 are given in the appendix.

The distribution of quadratic forms is discussed in Section 10, orthogonal expansions of ${}_0F_0$ and ${}_1F_1$ in Laguerre polynomials in Section 11 and the asymptotic expansion of ${}_0F_0$ in Section 12. Section 13 has some formulae for moments

2. Introduction. Two major aims in principal component analysis, multiple discriminant analysis and canonical correlation analysis are: (1) to assess the joint significance of a number of variables, and (2) to replace them with a smaller number of linear functions which contain all or most of the essential information.

The magnitudes of the effects within the population can be measured by the latent roots of certain matrices or determinantal equations depending upon the population parameters. It is important to know the sampling distributions of the estimates of them: (1) to be aware of the magnitude of possible sampling errors.

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(2) to construct exact significance tests, particularly in the case when certain population roots are acknowledged to be real and thereupon become nuisance parameters in tests of significance of further roots, and (3) to find the sensitivity or power of various tests.

3. The symmetry of a normal multivariate sample. This is the key to the sampling distributions. Suppose that the columns of a $m \times n$ matrix X are independently normally distributed with covariance matrix Σ and $E[X] = M$. The probability density,

$$(1) \qquad (2\pi)^{-\frac{1}{2}mn} |\Sigma|^{-\frac{1}{2}n} \exp[-\tfrac{1}{2} \operatorname{tr}(\Sigma^{-1}(X - M)(X - M)')] (dX),$$

is invariant under the group of transformations

$$(2) \qquad \begin{aligned} X &\to LXH & L &\varepsilon \operatorname{Gl}(m) \\ M &\to LMH & H &\varepsilon \operatorname{O}(n) \\ \Sigma &\to L\Sigma L', \end{aligned}$$

where $\operatorname{Gl}(m)$ is the group of all real $m \times m$ nonsingular matrices L, and $\operatorname{O}(n)$ is the group of all $n \times n$ orthogonal matrices H.

On account of the symmetry, the Fourier analysis of the group plays a big role in the distributional theory.

4. Hypergeometric functions. The noncentral χ^2, noncentral F and multiple correlation distributions, as found by Fisher [11] in 1928, involve Bessel and hypergeometric functions which can all be written as special cases, for particular integers p and q, of the generalized hypergeometric function

$$(3) \qquad {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}$$

where the hypergeometric coefficient $(a)_k$ is given by

$$(4) \qquad (a)_k = a(a + 1) \cdots (a + k - 1).$$

${}_pF_q$ is a function of the real or complex variable x depending upon the real or complex numbers $a_1, \cdots, a_p, b_1, \cdots, b_q$. Special cases are

$$(5) \qquad {}_0F_0(x) = e^x \qquad \text{exponential}$$

$$(6) \qquad {}_1F_0(a; x) = (1 - x)^{-a} \qquad \text{binomial series}$$

$$(7) \qquad {}_0F_1(\tfrac{1}{2}n; \tfrac{1}{4}x^2) = \int_0^\pi e^{x \cos\theta} \sin^{n-2}\theta \, d\theta \Big/ \int_0^\pi \sin^{n-2}\theta \, d\theta$$

Bessel (in the noncentral χ^2 distribution)

$$(8) \qquad {}_1F_1(a; b; x) \qquad \text{confluent (in the noncentral } F \text{ distribution)}$$

$$(9) \qquad {}_2F_1(a_1, a_2; b; x) \qquad \text{Gaussian (in the multiple correlation distribution).}$$

The corresponding multivariate distributions involve a generalization of this function to the case in which the variable x is replaced by a symmetric matrix S and ${}_pF_q$ is a real or complex valued symmetric function of the latent roots of S.

The Bessel function of matrix argument was introduced by Bochner [4] as an inverse Laplace transform of the exponential function. The general system of hypergeometric functions of matrix argument is due to Herz [18] who defined them by Laplace and inverse Laplace transforms as in Equations (28) and (29). Constantine [8] discovered the power series representation which we take as our definition.

As we are now dealing with symmetric functions of m variables, the power series can be expanded in terms of one of the types of symmetric polynomials. For any such type of basis of the symmetric polynomials, the individual homogeneous polynomials of degree k are usually indexed by partitions $\kappa = (k_1, k_2, \cdots, k_m)$, $k_1 \geqq k_2 \geqq \cdots \geqq k_m \geqq 0$, $k_1 + \cdots + k_m = k$, of k into not more than m parts. Hence whereas in the case of a single variable, we sum over all integers k, in the case of a matrix variable, we sum over all partitions κ of all integers. While in theory any basis of the symmetric polynomials would do, in practice a colossal simplification of the coefficients is achieved if certain homogeneous symmetric polynomials, $C_\kappa(S)$, called *zonal polynomials* derived from the group representation theory of $\mathrm{Gl}(m)$, are used. They will be defined in the next section.

The hypergeometric functions which appear in the distributions of the matrix variates are given by the (Constantine [8]).

DEFINITION

$$(10) \quad {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; S) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(S)}{k!}$$

$a_1, \cdots, a_p, b_1, \cdots, b_q$ are real or complex constants and the multivariate hypergeometric coefficient $(a)_\kappa$ is given by

$$(11) \quad (a)_\kappa = \prod_{i=1}^{m} (a - \tfrac{1}{2}(i-1))_{k_i},$$

where, as before,

$$(12) \quad (a)_k = a(a+1) \cdots (a+k-1).$$

The latent roots distributions involve functions of both population and sample roots, namely,

DEFINITION.

$$(13) \quad {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; S, T) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(S)C_\kappa(T)}{C_\kappa(I_m)k!}.$$

Zonal polynomials and hypergeometric functions of products ST of symmetric matrices are defined as symmetric functions of the latent roots of ST. Note that although ST may not be a symmetric matrix, its latent roots are equal to the latent roots of $S^{\frac{1}{2}} TS^{\frac{1}{2}}$, $T^{\frac{1}{2}} ST^{\frac{1}{2}}$ and TS.

5. Zonal polynomials. The definition of zonal polynomials requires a few concepts from group representation theory. Let V_k be the vector space of homogeneous polynomials $\varphi(S)$ of degree k in the $n = \frac{1}{2}m(m+1)$ different elements of the $m \times m$ symmetric matrix S. The dimension N of V_k is the number $N = (n+k-1)!/(n-1)!\,k!$ of monomials

$$\prod_{i \leqq j}^{m} s_{ij}^{k_{ij}}, \quad \text{of degree} \quad \sum_{i \leqq j}^{m} k_{ij} = k.$$

Corresponding to any congruence transformation

$$S \to LSL' \tag{14}$$

by a nonsingular $m \times m$ matrix L, we can define a linear transformation of the space V_k of polynomials $\varphi(S)$, namely

$$\varphi \to L\varphi : (L\varphi)(S) = \varphi(L^{-1} SL^{-1'}). \tag{15}$$

A subspace $V' \subset V_k$ is called *invariant* if $LV' \subset V'$ for all nonsingular matrices L. V' is called an *irreducible* invariant subspace if it has no proper invariant subspace. Thrall [39] in 1942, Theorem 3, p. 378 proved that V_k decomposes into a direct sum of irreducible invariant subspaces V_κ corresponding to each partition κ of k into not more than m parts

$$V_k = \bigoplus_{\kappa} V_\kappa . \tag{16}$$

The polynomial $(\mathrm{tr}\, S)^k \varepsilon V_k$ then has a unique decomposition

$$(\mathrm{tr}\, S)^k = \sum_{\kappa} C_\kappa(S) \tag{17}$$

into polynomials, $C_\kappa(S) \varepsilon V_\kappa$, belonging to the respective invariant subspaces.

The *zonal polynomial* $C_\kappa(S)$ is defined as the component of $(\mathrm{tr}\, S)^k$ in the subspace V_κ. It is a symmetric homogeneous polynomial of degree k in the latent roots of S.

Equation (17) holds for all m, and the zonal polynomials look the same for all m; but if the partition κ has more than m parts, the corresponding zonal polynomial $C_\kappa(S)$ will be identically zero.

Zonal polynomials, denoted by $Z_\kappa(S)$ because they are given a different normalizing constant, are listed up to $k = 6$ in the appendix. General methods of calculating them will be described in Section 8. By comparing Equation (17) with Equation (28) in James [27], one sees that

$$C_\kappa(S) = [\chi_{[2\kappa]}(1)\, 2^k k!/(2k)!]\, Z_\kappa(S), \tag{18}$$

where $\chi_{[2\kappa]}(1)$ is the dimension of the representation $[2\kappa]$ of the symmetric group on $2k$ symbols.

It is found by substituting $(2\kappa) = (2k_1, \cdots, 2k_p)$ for $\kappa = (k_1, \cdots, k_p)$ in the well-known formula (Weyl [41] p. 213. Theorem (7.7.B)) that

(19) $$\chi_{[\kappa]}(1) = k! \prod_{i<j}^{p} (k_i - k_j - i + j) \Big/ \prod_{i=1}^{p} (k_i + p - i)!$$

From Equation (38) of Constantine [8], namely

(20) $$Z_\kappa(I_m) = 2^k(\tfrac{1}{2}m)_\kappa,$$

we have the value of the zonal polynomial at the unit matrix;

(21) $$C_k(I_m) = 2^{2k}k!(\tfrac{1}{2}m)_\kappa \prod_{i<j}^{p} (2k_i - 2k_j - i + j) \Big/ \prod_{i=1}^{p} (2k_i + p - i)!$$

Note that if $m = 1$, Equation (17) which defines the zonal polynomials becomes $x^k = C_{(k)}(x)$. Thus zonal polynomials of a matrix variable are analogous to powers of a single variable. For functions of latent roots of determinantal equations, group representation theory makes it clear that they are also the analogues of $\cos n\theta$ and $\sin n\theta$ in ordinary Fourier series.

Zonal polynomials have been developed by Hua [22] in other contexts and independently by the author [27].

6. Properties of zonal polynomials and hypergeometric functions. All the power series expansions of the latent roots distributions follow from 3 essential properties of zonal polynomials;

(22) $$\int_{O(m)} (\operatorname{tr}(XH))^{2k}\,(dH) = \sum_{\kappa} \frac{(\frac{1}{2})_k}{(\frac{1}{2}m)_\kappa} C_\kappa(XX'),$$

(23) $$\int_{O(m)} C_\kappa(SHTH')\,(dH) = \frac{C_\kappa(S)C_\kappa(T)}{C_\kappa(I_m)},$$

(24) $$E_{W(n)}[C_\kappa(XX')] = 2^k\,(\tfrac{1}{2}n)_\kappa C_\kappa(\Sigma),$$

where $E_{W(n)}[\ ]$ stands for the expectation with respect to the Wishart distribution (Formula (55)) on n degrees of freedom, and (dH) stands for the invariant or Haar measure on the orthogonal group $O(m)$, normalized so that the measure of the whole group is unity. The invariant measure on $O(m)$ is proportional to the area of the $\frac{1}{2}m(m-1)$ dimensional hypersurface in Euclidean m^2-space defined by the $\frac{1}{2}m(m+1)$ equations, $HH' = I_m$, in the m^2 components of H.

Property 1 is a special case of Equation (45) proved below.

Constantine [8] discovered the remarkable reproductive property 3 of the zonal polynomial under expectation taken with respect to the Wishart distribution.

Special cases of the generalized hypergeometric function of matrix argument are (cf. Equations (5), (6) and (7))

(25) $${}_0F_0(S) = e^{\operatorname{tr} S}$$

(26) $${}_1F_0(a; S) = |I - S|^{-a}$$

(27) $${}_0F_1(\tfrac{1}{2}n; \tfrac{1}{4}XX') = \int_{O(n)} e^{\operatorname{tr} XH}\,dH.$$

Equation (25) follows from (17), Equation (26) from (25) and (28) below, and Equation (27) from (22). The following integrals, which are a generalization of Laplace and inverse Laplace transforms, were used by Bochner [4] to define the Bessel function and by Herz [18] to define the hypergeometric functions:

$$(28)\quad \frac{1}{\Gamma_m(a)} \int_{S>0} e^{-\operatorname{tr} S}\, |S|^{a-\frac{1}{2}(m+1)} {}_pF_q(a_1, \cdots, a_p; b_1 \cdots b_q; ST)\, dS = {}_{p+1}F_q(a_1, \cdots, a_p, a; b_1, \cdots, b_q; T)$$

where $\Gamma_m(a)$ is defined in Equation (56), and

$$(29)\quad \frac{2^{\frac{1}{2}m(m-1)}\Gamma_m(b)}{(2\pi i)^{\frac{1}{2}m(m+1)}} \int_{\Re(T)=X_0>0} e^{\operatorname{tr} T}\, |T|^{-b} {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; T^{-1}S)\, (dT) = {}_pF_{q+1}(a_1, \cdots, a_p; b_1, \cdots, b_q, b; S).$$

Equation (28) follows from (24). For a proof of (29), see Constantine [8]. The integral is taken over all matrices $T = X_0 + iY$ for fixed positive definite X_0 and Y arbitrary real symmetric.

From Equation (23), it is seen that the hypergeometric function of two variables follows from that of one, by an average over $O(m)$.

$$(30)\quad {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; S, T) = \int_{O(m)} {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; SHTH')(dH).$$

The function of two variables clearly does not depend upon the order in which they occur, and it has the same properties of Laplace and inverse Laplace transform taken with respect to either variable

$$(31)\quad \frac{1}{\Gamma_m(a)} \int_{S>0} e^{-\operatorname{tr} S}|S|^{a-\frac{1}{2}(m+1)}\, {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; ST, U)(dS) = {}_{p+1}F_q(a_1, \cdots, a_p, a; b_1, \cdots, b_q; T, U).$$

$$(32)\quad \frac{2^{\frac{1}{2}m(m-1)}\Gamma_m(b)}{(2\pi i)^{\frac{1}{2}m(m+1)}} \int_{\Re(T)>0} e^{\operatorname{tr} T}|T|^{-b}\, {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; T^{-1}S, U)(dT) = {}_pF_{q+1}(a_1, \cdots, a_p; b_1, \cdots, b_q, b; S, U).$$

The power series for the function ${}_1F_0(a; S, T)$, which occurs in the distribution (65) of latent roots with unequal covariance matrices, may not converge for all requisite values of S and T, but the integral

$$(33)\quad {}_1F_0(a; -S, T) = \int_{O(m)} |I + SHTH'|^{-a}(dH)$$

is well defined for all $S > 0$ and $T > 0$.

The zonal polynomials can be expressed as integrals over the orthogonal group $O(m)$ of the characters $\chi_{\{2\kappa\}}(X)$ of the linear group $\mathrm{Gl}(m)$; $X \varepsilon \mathrm{Gl}(m)$;

$$\frac{C_\kappa(XX')}{C_\kappa(I_m)} = \int_{O(m)} \chi_{\{2\kappa\}}(XH)(dH), \tag{34}$$

see e.g. Helgason [17] Theorem 6.5 p. 426. The character $\chi_{\{\kappa\}}(X)$ of $\mathrm{Gl}(m)$ is given by

$$\chi_{\{\kappa\}}(X) = |(\epsilon_i^{k_j+m-j})|/|(\epsilon_i^{m-j})| \tag{35}$$

$$= |(h_{k_j-j+i})| \tag{36}$$

$$= |(a_{\hat{k}_j-j+i})| \tag{37}$$

$$= |(s_{k_j+2m-i-j})|/|(s_{2m-i-j})| \tag{38}$$

where $\epsilon_1, \cdots, \epsilon_m$ are the *latent roots* of X; h_i is the *monomial symmetric function* of them, i.e. the sum of all monomials of $\epsilon_1, \cdots, \epsilon_m$ of degree i, $h_0 = 1$, $h_i = 0$ for $i < 0$; a_i is the ith *elementary symmetric function*,

$$a_i = \sum_{\nu_1<\nu_2<\cdots<\nu_i}^{m} \epsilon_{\nu_1}\epsilon_{\nu_2}\cdots\epsilon_{\nu_i}; \quad a_0 = 1, \; a_i = 0, \text{ for } i \leqq 0; \tag{39}$$

$(\hat{k}_1, \hat{k}_2, \cdots, \hat{k}_m)$ is the conjugate partition to $\kappa = (k_1, \cdots, k_m)$, see Littlewood [29] p. 60; and

$$s_i = \sum_{\nu=1}^{m} \epsilon_\nu^i, \qquad s_0 = m. \tag{40}$$

$\chi_{\{2\kappa\}}(X)$ is found by substituting the partition $(2k_1, \cdots, 2k_m)$ for κ. If one or more parts $k_1, \cdots, k_m$ of the partition κ is odd, then

$$\int_{O(m)} \chi_{\{\kappa\}}(XH)(dH) = 0. \tag{41}$$

If $f(X)$, $X \varepsilon \mathrm{Gl}(m)$, is a symmetric function of the latent roots $\epsilon_1, \cdots, \epsilon_m$ of X, then from its expansion in characters

$$f(X) = \sum_{k=0}^{\infty} \sum_{\kappa} c_\kappa \chi_{\{\kappa\}}(X), \tag{42}$$

we can derive an expansion in zonal polynomials, for an integral of the function over $O(m)$

$$\int_{O(m)} f(XH)(dH) = \sum_{k=0}^{\infty} \sum_{\kappa} c_{2\kappa} \frac{C_\kappa(XX')}{C_\kappa(I_m)}. \tag{43}$$

Thus from the formula in Littlewood [25] Equation (6.2; 15) p. 86

$$\mathrm{tr}(X^{k_1})\,\mathrm{tr}(X^{k_2})\cdots\mathrm{tr}(X^{k_m}) = \sum_{\lambda} \chi_{[\lambda]}(\kappa)\,\chi_{\{\lambda\}}(X), \tag{44}$$

where $\chi_{[\lambda]}(\kappa)$ is the character of the representation $[\lambda]$ of the symmetric group for class $\kappa = (k_1, \cdots, k_m)$, we have

(45) $$\int_{O(m)} \operatorname{tr}(XH)^{k_1}\operatorname{tr}(XH)^{k_2} \cdots \operatorname{tr}(XH)^{k_m}(dH) = \sum_{\lambda} \chi_{[2\lambda]}(\kappa) \frac{C_\lambda(XX')}{C_\lambda(I_m)},$$

where the summation is over all partitions $\lambda = (\lambda_1, \cdots, \lambda_m)$ of $\frac{1}{2}k$ if k is even, and

(46) $$\int_{O(m)} \operatorname{tr}(XH)^{k_1} \cdots \operatorname{tr}(XH)^{k_m}(dH) = 0$$

if k is odd. The statistically important Equation (22) is a special case of this.

Further integral representations of hypergeometric functions are given by Herz [18], namely of ${}_1F_1$ as the moment generating function of the multivariate beta distribution

(47) $${}_1F_1(a; b; S) = \frac{\Gamma_m(b)}{\Gamma_m(a)\Gamma_m(b-a)} \int_0^I e^{\operatorname{tr} ST} |T|^{a-\frac{1}{2}(m+1)} |I - T|^{b-a-\frac{1}{2}(m+1)} (dT),$$

and ${}_2F_1$ as the Laplace transform of this function

(48) $$\begin{aligned} &{}_2F_1(a_1, a_2; b; S) \\ &= \frac{\Gamma_m(b)}{\Gamma_m(a_1)\Gamma_m(b-a_1)} \int_0^I |I - ST|^{-a_2} |T|^{a_1-\frac{1}{2}(m+1)} |I - T|^{b-a_1-\frac{1}{2}(m+1)} (dT). \end{aligned}$$

The hypergeometric functions of matrix argument satisfy some of the Kummer relations. Herz [18] gives

(49) $${}_2F_1(a_1, a_2; b; S) = |I - S|^{-a_2}\, {}_2F_1(b - a_1, a_2; b; -S(I - S)^{-1})$$

(50) $$= |I - S|^{b-a_1-a_2}\, {}_2F_1(b - a_1, b - a_2; b; S)$$

and

(51) $${}_1F_1(a; b; S) = e^{\operatorname{tr} S}\, {}_1F_1(b - a; b; -S),$$

and also the obvious confluences

(52) $$\lim_{a_2\to\infty} {}_2F_1(a_1, a_2; b; a_2^{-1} S) = {}_1F_1(a_1; b; S)$$

(53) $$\lim_{a_1\to\infty} {}_1F_1(a_1; b; a_1^{-1} S) = {}_0F_1(b; S).$$

7. A comparison of univariate with multivariate distributions.

1. Type ${}_0F_0$, exponential.

(i) *The* χ^2 *distribution*. The sum of squares, $\chi^2 = Z_1^2 + \cdots + Z_n^2$ of n independent normal variates Z_i of mean 0 and variance 1, i.e. distributed as $N(0, 1)$, has the distribution

(54) $$\frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)} e^{-\frac{1}{2}\chi^2} (\chi^2)^{\frac{1}{2}n-1}\, d\chi^2$$

Pearson [31].

(ii) *The Wishart Distribution.* If X is an $m \times n$ matrix variate whose columns are independently normally distributed with mean vector 0 and covariance matrix Σ, i.e. $N(0, \Sigma)$, i.e. distributed as (1) with $E[X'] = 0$, then the distribution of XX' is

$$\frac{1}{2^{\frac{1}{2}mn}\Gamma_m(\frac{1}{2}n)|\Sigma|^{\frac{1}{2}n}} \exp\left[-\frac{1}{2}\operatorname{tr}(\Sigma^{-1}XX')\right] |XX'|^{\frac{1}{2}n-\frac{1}{2}(m+1)}(d(XX')) \tag{55}$$

Wishart [42],

where $\Gamma_m(a)$ is the *multivariate gamma function* defined by

$$\Gamma_m(a) = \int_{S>0} e^{-\operatorname{tr} S}|S|^{a-\frac{1}{2}(m+1)}(dS) \tag{56}$$

$$= \pi^{\frac{1}{4}m(m-1)} \sum_{i=1}^{m} \Gamma(a - \tfrac{1}{2}(i-1)). \tag{57}$$

The domain of integration is the set of all positive definite symmetric matrices, $S > 0$.

(iii) *Latent roots of the covariance matrix.* If X is distributed as in (ii), then the distribution of the latent roots $w_1, \cdots, w_m$ of XX' depends only upon the latent roots of Σ and is

$$|\Sigma|^{-\frac{1}{2}n}\, {}_0F_0(-\tfrac{1}{2}\Sigma^{-1}, W) \cdot \frac{\pi^{\frac{1}{2}m^2}}{2^{\frac{1}{2}mn}\Gamma_m(\frac{1}{2}n)\Gamma_m(\frac{1}{2}m)} |W|^{\frac{1}{2}(n-m-1)} \prod_{i<j}(w_i - w_j) \prod_i dw_i \tag{58}$$

James [25]

where $W = \operatorname{diag}(w_i)$, and ${}_0F_0$ is defined by Equation (13) for ${}_pF_q$ when $p = 0$ and $q = 0$.

The function, ${}_0F_0$, comes from the exponential factor in the Wishart distribution, viz.

$$\exp(-\tfrac{1}{2}\operatorname{tr} \Sigma^{-1} XX') = {}_0F_0(-\tfrac{1}{2}\Sigma^{-1} XX'), \tag{59}$$

and

$$\int_{O(m)} \exp(-\tfrac{1}{2} \operatorname{tr} \Sigma^{-1}HXX'H')(dH) = {}_0F_0(-\tfrac{1}{2}\Sigma^{-1}, XX'). \tag{60}$$

Fisher [12] Hsu [21] and Roy [35] gave the distribution when Σ is a scalar matrix in 1939.

2. Type ${}_1F_0$, binomial series.

(i) *Variance ratio F.* If the variates $Z_1, \cdots, Z_p$ are independent $N(0, \sigma_1^2)$ and the independent variates $Z_{p+1}, \cdots, Z_{p+n}$ are independent $N(0, \sigma_2^2)$ then the distribution of

$$F = Z_1^2 + \cdots + Z_p^2 / Z_{p+1}^2 + \cdots + Z_{p+n}^2 \tag{61}$$

depends on $\omega = \sigma_1^2/\sigma_2^2$, and is

$$(62)\qquad \frac{\Gamma(\frac{1}{2}(p+n))}{\Gamma(\frac{1}{2}p)\Gamma(\frac{1}{2}n)} \frac{[(p/n)(F/\omega)]^{\frac{1}{2}p-1}}{[1+(p/n)(F/\omega)]^{\frac{1}{2}(p+n)}} \frac{p}{n\omega}\, dF$$

Fisher [10].

(ii) *Latent roots when* $\Sigma_1 \neq \Sigma_2$. Suppose that the $m \times p$ matrix variate X with $p > m$ has independent columns distributed as $N(0, \Sigma_1)$ and Y is an independent $m \times n$ matrix variate with columns independently distributed as $N(0, \Sigma_2)$, i.e. XX' and YY' are independently distributed as in (55). Then the distribution of the latent roots f_i of

$$(63)\qquad |XX' - fYY'| = 0$$

depends on the latent roots ω_i of

$$(64)\qquad |\Sigma_1 - \omega\Sigma_2| = 0$$

and is, with $F = \operatorname{diag}(f_i)$, $\Omega = \operatorname{diag}(\omega_i)$

$$(65)\qquad \begin{aligned} &|\Omega|^{-\frac{1}{2}p}\, {}_1F_0[\tfrac{1}{2}(p+n); -\Omega^{-1}, F] \\ &\cdot \frac{\pi^{\frac{1}{2}m^2}\Gamma_m(\frac{1}{2}(p+n))}{\Gamma_m(\frac{1}{2}p)\Gamma_m(\frac{1}{2}n)\Gamma_m(\frac{1}{2}m)} |F|^{\frac{1}{2}p-\frac{1}{2}(m+1)} \prod_{i<j} (f_i - f_j) \prod df_i \end{aligned}$$

Constantine (unpublished),

where ${}_1F_0$ is defined by Equation (33).

3. Type ${}_0F_1$, Bessel.

(i) *Noncentral* χ^2. If variates Z_i are independent $N(0, 1)$ then the distribution of $\chi^2 = (Z_1 + \omega^{\frac{1}{2}})^2 + Z_2^2 + \cdots + Z_n^2$ is

$$(66)\qquad e^{-\frac{1}{2}\omega}\, {}_0F_1(\tfrac{1}{2}n; \tfrac{1}{4}\omega\chi^2) \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)} e^{-\frac{1}{2}\chi^2}(\chi^2)^{\frac{1}{2}n-1}\, d(\chi^2)$$

Fisher [11].

(ii) *Noncentral Wishart.* If the $m \times n$ matrix variate X has independent normally distributed columns with covariance Σ and $E[X] = M$, i.e. if X is distributed as in (1), then the distribution of XX' is

$$(67)\qquad \begin{aligned} &\exp[-\tfrac{1}{2}\operatorname{tr}(\Sigma^{-1}MM')]\, {}_0F_1(\tfrac{1}{2}n; \tfrac{1}{4}\Sigma^{-1}MM'\Sigma^{-1}(XX')) \\ &\cdot \frac{1}{2^{\frac{1}{2}mn}\Gamma_m(\frac{1}{2}n)|\Sigma|^{\frac{1}{2}n}} \exp[-\tfrac{1}{2}\operatorname{tr}(\Sigma^{-1}XX')]|XX'|^{\frac{1}{2}(n-m-1)}(d(XX')). \end{aligned}$$

T. W. Anderson [1] gave the distribution for rank $(M) = 2$ in 1946 and Weibull [40] for rank $(M) = 3$ in 1953. James [23], [24] gave a method of calculating the coefficients in 1955 and the zonal polynomial expansion in [27] 1961.

(iii) *Noncentral means with known covariance.* If X is distributed as in (ii) and w_i are the latent roots of $|XX' - w\Sigma| = 0$, then the distribution of $W = \operatorname{diag}$

(w_i) depends only upon $\Omega = \operatorname{diag}(\omega_i)$ where ω_i are the latent roots of $|MM' - \omega\Sigma| = 0$, and is

$$(68)\quad e^{-\frac{1}{2}\operatorname{tr}\Omega}\, {}_0F_1(\tfrac{1}{2}n; \tfrac{1}{4}\Omega, W) \cdot \frac{\pi^{\frac{1}{2}m^2}}{2^{\frac{1}{2}mn}\Gamma_m(\frac{1}{2}n)\Gamma_m(\frac{1}{2}m)}\, e^{-\frac{1}{2}\operatorname{tr} W}|W|^{\frac{1}{2}(n-m-1)} \prod_{i<j} (w_i - w_j) \prod dw_j .$$

James [27].

4. Type ${}_1F_1$ confluent hypergeometric.

(i) *Noncentral F on p and n degrees of freedom.* If variates $Z_1, \cdots, Z_p, Z_{p+1}, \cdots, Z_n$ are independent $N(0, 1)$, then

$$(69)\quad F = \frac{((Z_1 + \omega^{\frac{1}{2}})^2 + Z_2^2 + \cdots + Z_p^2)/p}{Z_{p+1}^2 + \cdots + Z_{p+n}^2)/n}$$

is distributed as

$$(70)\quad e^{-\frac{1}{2}\omega}\, {}_1F_1(\tfrac{1}{2}(p + n); \tfrac{1}{2}p; \tfrac{1}{2}\omega(1 + [(p/n)F]^{-1})^{-1}) \cdot \frac{\Gamma(\frac{1}{2}(p + n))}{\Gamma(\frac{1}{2}p)\Gamma(\frac{1}{2}n)} \frac{((p/n)F)^{\frac{1}{2}p-1}}{(1 + (p/n)F)^{\frac{1}{2}(p+n)}} \frac{p}{n}\, dF$$

Fisher [11].

(ii) *Noncentral multivariate F.* If the matrix variate X is $m \times p$ and Y is $m \times n$ with $p \leqq m \leqq n$, if the columns are all independently normally distributed with covariance Σ, and if $E[X] = M$, $E[Y] = 0$, then the distribution of

$$(71)\quad F = X'(YY')^{-1} X$$

depends upon $\Omega = M' \Sigma^{-1} M$, and is

$$(72)\quad e^{-\frac{1}{2}\operatorname{tr}\Omega}\, {}_1F_1(\tfrac{1}{2}(p + n); \tfrac{1}{2}m; \tfrac{1}{2}\Omega(I + F^{-1})^{-1}) \cdot \frac{\Gamma_p(\frac{1}{2}(p + n))}{\Gamma_p(\frac{1}{2}m)\Gamma_p(\frac{1}{2}(p + n - m))} \frac{|F|^{\frac{1}{2}(m-p-1)}}{|I + F|^{\frac{1}{2}(p+n)}} \prod_{i \leqq j} df_{ij} .$$

D. G. Kabe [28] has given the distribution for Ω rank 2, and Sitgreaves [37] for $p = 2$ in connection with the distribution of the classification statistic, for the numerical evaluation of which, see Sitgreaves [38].

The case $p = 1$ is Hotelling's [18a] T^2.

(iii) *Noncentral latent roots.* If X and Y are as in (ii) then the distribution of the latent roots f_i of $|XX' - fYY'| = 0$ depends upon the population latent roots ω_i of $|MM' - \omega\Sigma| = 0$ and is, for $p \geqq m$,

$$(73)\quad e^{-\frac{1}{2}\operatorname{tr}\Omega}\, {}_1F_1(\tfrac{1}{2}(p + n); \tfrac{1}{2}p; \tfrac{1}{2}\Omega, (I + F^{-1})^{-1}) \cdot \frac{\Gamma_m(\frac{1}{2}(p + n))\pi^{\frac{1}{2}m^2}}{\Gamma_m(\frac{1}{2}p)\Gamma_m(\frac{1}{2}n)\Gamma_m(\frac{1}{2}m)} \frac{|F|^{\frac{1}{2}(p-m-1)} \prod_{i<j} (f_i - f_j)}{|I + F|^{\frac{1}{2}(p+n)}}\, df_1 \cdots df_m$$

Constantine [8].

Fisher [12] Hsu [21] and Roy [35] gave the distribution for $\Omega = 0$ in 1939, Roy [35a] for one non-zero root in 1942, T. W. Anderson [1] for 2 non-zero roots in 1946, and Bartlett [3] in 1946 gave a method of calculating the first few terms.

For $p \leqq m$, the distribution of the non-zero roots $f_1, \cdots, f_p$ is

$$e^{-\frac{1}{2}\operatorname{tr}\Omega}\ {}_1F_1(\tfrac{1}{2}(p+n);\ \tfrac{1}{2}m;\ \tfrac{1}{2}\Omega,\ (I+F^{-1})^{-1})$$

$$(74)\qquad \cdot \frac{\Gamma_p(\frac{1}{2}(p+n))\pi^{\frac{1}{2}p^2}}{\Gamma_p(\frac{1}{2}m)\Gamma_p(\frac{1}{2}(p+n-m))\Gamma_p(\frac{1}{2}p)} \frac{|F|^{\frac{1}{2}(m-p-1)}}{|I+F|^{\frac{1}{2}(p+n)}} \prod_{i<j}^{p} (f_i - f_j)\, df_1 \cdots df_p .$$

5. Type ${}_2F_1$ Gaussian hypergeometric.

(i) *Multiple correlation coefficient.* The multiple correlation coefficient R between variates y and $x_1, \cdots, x_q$, calculated from a sample of $N = n + 1$ observations, is distributed as

$$(1-\rho^2)^{\frac{1}{2}n}\ {}_2F_1(\tfrac{1}{2}n, \tfrac{1}{2}n;\ \tfrac{1}{2}q;\ \rho^2R^2)$$

$$(75)\qquad \cdot \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}(n-q))\Gamma(\frac{1}{2}q)} (R^2)^{\frac{1}{2}q-1}(1-R^2)^{\frac{1}{2}(n-q)-1}\, d(R^2)$$

Fisher [11].

(ii) *Canonical correlation coefficients.* The distribution of the canonical correlation coefficients $r_1^2, \cdots, r_p^2$ between variates $y_1, \cdots, y_p$ and $x_1, \cdots, x_q$ calculated from a sample of $N = n + 1$ observations depends on the population canonical correlation coefficients $\rho_1, \cdots, \rho_p$ and is, with $R = \operatorname{diag}(r_i)$, $P = \operatorname{diag}(\rho_i)$

$$|I - P^2|^{\frac{1}{2}n}\ {}_2F_1(\tfrac{1}{2}n, \tfrac{1}{2}n;\ \tfrac{1}{2}q;\ P^2, R^2)$$

$$(76)\qquad \cdot \frac{\Gamma_p(\frac{1}{2}n)\pi^{\frac{1}{2}p^2}}{\Gamma_p(\frac{1}{2}(n-q))\Gamma_p(\frac{1}{2}q)\Gamma_p(\frac{1}{2}p)} |R^2|^{\frac{1}{2}(q-p-1)}|I - R^2|^{\frac{1}{2}(n-q-p-1)}$$

$$\cdot \prod_{i<j} (r_i^2 - r_j^2) \prod dr_i^2$$

Constantine [8].

Fisher [11] Hsu [21] and Roy [35] found the distribution when $P = 0$ in 1939. Bartlett [3] in 1947 found the distribution for 1 non-zero root and a method of calculating the first few terms of the expansion of the general distribution.

From the confluences (52) (53), we have the following limiting distribution of $nR^2 = W$ as $n \to \infty$ such that $0 < nP^2 = \Omega < \infty$,

$$e^{-\frac{1}{2}\operatorname{tr}\Omega}\ {}_0F_1(\tfrac{1}{2}q;\ \tfrac{1}{4}\Omega,\ W)$$

$$(77)\qquad \cdot \frac{1}{2^{\frac{1}{2}pq}\Gamma_p(\frac{1}{2}q)\Gamma_p(\frac{1}{2}p)} e^{-\frac{1}{2}\operatorname{tr} W} |W|^{\frac{1}{2}q-\frac{1}{2}(p+1)} \prod_{i<j}^{p} (w_i - w_j)\, dw_1 \cdots dw_p .$$

It is the distribution (68) of non central means with known covariance for q d.f. and noncentrality Ω. For the case $p = 1$, Fisher [11] showed that the limiting form of the distribution of the multiple correlation coefficient is noncentral χ^2.

8. The complex normal distribution. Wooding [43] and Goodman [14] have studied distributions derived from a sample of n independent observations from a complex m-variate normal distribution. Writing the observations as the columns of a $m \times n$ matrix

$$Z = X + iY$$

we have the probability density

$$(78) \qquad \frac{1}{\pi^{mn}|\Sigma|^n} \exp[-\operatorname{tr} \Sigma^{-1}(Z - M)(\bar{Z} - \bar{M})'] \prod_{i=1}^{m} \prod_{j=1}^{n} dx_{ij}\, dy_{ij} .$$

The matrix of means $M = E[Z]$ is a matrix of mn complex parameters and the covariance matrix $\Sigma = n^{-1} E[(Z - M)(\bar{Z} - \bar{M})']$ is Hermitian; i.e. $\bar{\Sigma}' = \Sigma$.

The distribution (78) is symmetrical; it is invariant under the group of transformations,

$$(79) \qquad Z \to LZU \qquad L \varepsilon \operatorname{Gl}(m, C)$$

$$(80) \qquad M \to LMU \qquad U \varepsilon U(n)$$

$$(81) \qquad \Sigma \to L\Sigma\bar{L}'.$$

Gl (m, C) is the *full* linear group i.e. the group of all nonsingular $m \times m$ matrices L with complex elements, and $U(n)$ is the *unitary group* i.e. the group of all $n \times n$ complex unitary matrices U; $U\bar{U}' = I_n$.

The complex normal distribution (78) and the distributions derived from it are analogous to the real distributions given in Section 7 except that they involve the Fourier analysis of the functions of a Hermitian matrix A under congruence transformation

$$(82) \qquad A \to LA\bar{L}' \qquad L \varepsilon \operatorname{Gl}(m, C)$$

by the full linear group in place of the Fourier analysis of functions of the real symmetric matrices under congruence transformation by the real linear group.

Concepts required are:

1. The complex multivariate gamma function.

$$(83) \qquad \tilde{\Gamma}_m(a) = \int_{\bar{A}'=A>0} e^{-\operatorname{tr} A} |A|^{a-m}(dA) = \pi^{\frac{1}{2}m(m-1)} \prod_{i=1}^{m} \Gamma(a - i + 1);$$

2. The complex multivariate hypergeometric coefficient

$$(84) \qquad [a]_\kappa = \prod_{i=1}^{p} (a - i + 1)_{k_i}$$

where $\kappa = (k_1, k_2, \cdots, k_p)$ is a partition of the integer k;

3. The zonal polynomial of a Hermitian matrix, A,

$$(85) \qquad \tilde{C}_\kappa(A) = \chi_{[\kappa]}(1)\chi_{\{\kappa\}}(A)$$

where $\chi_{[\kappa]}(1)$ is the *dimension* of the representation $[\kappa]$ of the symmetric group and is given in (19), and $\chi_{\{\kappa\}}(A)$ is the *character* of the representation $\{\kappa\}$ of the linear group and is given as a symmetric function of the latent roots $\epsilon_1, \cdots, \epsilon_m$ of A by Equations (35)–(38);

4. The reproductive property of the zonal polynomial

$$\frac{1}{\tilde{\Gamma}_m(a)} \int_{\bar{A}'=A>0} e^{-\operatorname{tr} A} |A|^{a-m} \tilde{C}_\kappa(AB)(dA) = [a]_\kappa \tilde{C}_\kappa(B); \tag{86}$$

5. The hypergeometric functions

$$_p\tilde{F}_q(a_1, \cdots, a_p; b_1, \cdots, b_q; A) = \sum_{k=0}^{\infty} \sum_\kappa \frac{[a_1]_\kappa \cdots [a_p]_\kappa}{[b_1]_\kappa \cdots [b_q]_\kappa} \frac{\tilde{C}_\kappa(A)}{k!} \tag{87}$$

$$_p\tilde{F}_q(a_1, \cdots, a_p; b_1, \cdots, b_q; A, B) = \sum_{k=0}^{\infty} \sum_\kappa \frac{[a_1]_\kappa \cdots [a_p]_\kappa}{[b_1]_\kappa \cdots [b_q]_\kappa} \frac{\tilde{C}_\kappa(A)\tilde{C}_\kappa(B)}{\tilde{C}_\kappa(I_m)k!}; \tag{88}$$

6. The special cases

$$_0\tilde{F}_0(A) = e^{\operatorname{tr} A} \tag{89}$$

$$_1\tilde{F}_0(a; A) = |I - A|^{-a} \tag{90}$$

and

$$_0\tilde{F}_1(n; X\bar{X}') = \int_{U(n)} e^{\operatorname{tr}(XU+\overline{XU})} (dU) \tag{91}$$

where X is a $m \times n$, $m \leqq n$, complex matrix and (dU) is the invariant measure on the unitary group $U(n)$ normalized to make the total measure unity;

7. The splitting

$$\int_{U(n)} {}_pF_q(AUB\bar{U}')(dU) = {}_pF_q(A, B); \tag{92}$$

and finally,

8. The result that, if $f(A)(dA)$ is the probability density of a Hermitian matrix variate A, then the distribution of the diagonal matrix W of the latent roots of A, $A = UW\bar{U}'$, is

$$\int_{U(m)} f(UA\bar{U}')(dU) \frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(m)} \prod_{i<j}^{m} (w_i - w_j)^2 \, dw_1 \cdots dw_m. \tag{93}$$

By means of these results, the complex analogues of all the distributions in Section 7 can be written down practically at sight, as follows:

1. Type $_0F_0$, exponential.

(i) *Wishart.* If X is a $m \times n$ matrix of complex variates distributed in the complex multivariate normal distribution (78) with $E[X] = 0$, then $A = X\bar{X}'$

is a positive definite Hermitian matrix distributed as

$$\frac{1}{\tilde{\Gamma}_m(n)|\Sigma|^n} e^{-\operatorname{tr}\Sigma^{-1}A} |A|^{n-m}(dA) \tag{94}$$

Goodman [14].

(ii) *Roots of the complex covariance matrix.* The latent roots $\alpha_1, \cdots, \alpha_m$ of A are real and distributed as

$$|\Sigma|^{-n} {}_0\tilde{F}_0(-\Sigma^{-1}, A) \cdot \frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)} |A|^{n-m} \prod_{i<j}^{m} (\alpha_i - \alpha_j)^2 \prod_{i=1}^{m} d\alpha_i \tag{95}$$

2. Type ${}_1F_0$, binomial series.

Latent roots with unequal covariance matrices. If A and B are independent central complex Wishart variates distributed as (94) with p and n degrees of freedom and covariance matrices Σ_1 and Σ_2, then the distribution of the roots $F = \operatorname{diag}(f_i)$ of

$$|A - fB| = 0 \tag{96}$$

depends on the roots $\Omega = \operatorname{diag}(\omega_i)$ of

$$|\Sigma_1 - \omega\Sigma_2| = 0 \tag{97}$$

and is

$$|\Omega|^{-p} {}_1\tilde{F}_0(p + n; -\Omega^{-1}, F) \cdot \frac{\pi^{m(m-1)}\tilde{\Gamma}_m(p + n)}{\tilde{\Gamma}_m(p)\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)} |F|^{p-m} \prod_{i<j}^{m} (f_i - f_j)^2 \, df_1 \cdots df_m \tag{98}$$

3. Type ${}_0F_1$, Bessel.

(i) *Noncentral Wishart.* If X is a $m \times n$ matrix of normal variates distributed as (78), then the distribution of $A = X\bar{X}'$ is

$$e^{-\operatorname{tr}\Sigma^{-1}M\bar{M}'} {}_0\tilde{F}_1(n; \Sigma^{-1}M\bar{M}'\Sigma^{-1}A) \cdot \frac{1}{\tilde{\Gamma}_m(n)|\Sigma|^n} e^{-\operatorname{tr}\Sigma^{-1}A} |A|^{n-m}(dA). \tag{99}$$

(ii) *Noncentral means with known covariance.* If X is again distributed as in (78), then the distribution of the latent roots $W = \operatorname{diag}(w_i)$ of

$$|X\bar{X}' - w\Sigma| = 0 \tag{100}$$

depends on the parameters $\Omega = \operatorname{diag}(\omega_i)$,

$$|M\bar{M}' - \omega\Sigma| = 0 \tag{101}$$

and is

$$(102)\qquad e^{-\operatorname{tr}\Omega}\,{}_0\tilde{F}_1(n;\Omega,W)\cdot\frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}e^{-\operatorname{tr}W}|W|^{n-m}\prod_{i<j}^{m}(w_i-w_j)^2\,dw_1\cdots dw_m.$$

4. Type ${}_1F_1$, confluent.

(i) *Noncentral multivariate F.* If X and Y are independent $m\times p$ and $m\times n$ complex matrix variates, $p\leqq m$, whose columns are independent complex normal m-variates with covariance matrix Σ, and if $E[X]=M$ and $E[Y]=0$, then the distribution of

$$(103)\qquad F=\bar{X}'(Y\bar{Y}')^{-1}X$$

depends on the parameters

$$(104)\qquad \Omega=\bar{M}'\Sigma^{-1}M$$

and is

$$(105)\qquad e^{-\operatorname{tr}\Omega}\,{}_1\tilde{F}_1(p+n;m;\Omega(I+F^{-1})^{-1})\cdot\frac{\tilde{\Gamma}_p(p+n)}{\tilde{\Gamma}_p(m)\tilde{\Gamma}_p(p+n-m)}\frac{|F|^{m-p}}{|I+F|^{p+n}}(dF).$$

(ii) *Noncentral latent roots when $p\leqq m$.* The latent roots of F are the same as the non-zero latent roots of $|X\bar{X}'-fY\bar{Y}'|=0$ and are distributed as

$$(106)\qquad e^{-\operatorname{tr}\Omega}\,{}_1\tilde{F}_1(p+n;m;\Omega,(I+F^{-1})^{-1})\cdot\frac{\pi^{p(p-1)}\tilde{\Gamma}_p(p+n)}{\tilde{\Gamma}_p(m)\tilde{\Gamma}_p(p+n-m)\tilde{\Gamma}_p(p)}\frac{|F|^{m-p}}{|I+F|^{p+n}}\prod(f_i-f_j)^2\,df_1\cdots df_m$$

where $\Omega=\operatorname{diag}(\omega_i)$ is the diagonal matrix of latent roots of $\bar{M}'\Sigma^{-1}M$.

(iii) *Noncentral latent roots when $p\geqq m$.* The distribution of the latent roots of

$$(107)\qquad |X\bar{X}'-fY\bar{Y}'|=0$$

depends on the latent roots of

$$(108)\qquad |M\bar{M}'-\omega\Sigma|=0$$

and is

$$(109)\qquad e^{-\operatorname{tr}\Omega}\,{}_1\tilde{F}_1(p+n;p;\Omega,(I+F^{-1})^{-1})\cdot\frac{\tilde{\Gamma}_m(p+n)\pi^{m(m-1)}}{\tilde{\Gamma}_m(p)\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}\frac{|F|^{p-m}}{|I+F|^{p+n}}\prod(f_i-f_j)^2\,df_1\cdots df_m.$$

5. Type ${}_2F_1$, Gaussian hypergeometric.

Canonical correlation coefficients. The columns of $\begin{bmatrix}X\\Y\end{bmatrix}$ are n independent complex normal $(p + q)$-variates, $(p \leqq q)$, with zero means and covariance matrix

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

The latent roots $r_1^2, \cdots, r_p^2$ of

$$|X\bar{Y}'(Y\bar{Y}')^{-1}Y\bar{X}' - r_i^2 X\bar{X}'| = 0 \tag{110}$$

are real and are the squares of the canonical correlation coefficients. The distribution of $R = \text{diag}(r_i)$ depends on the population correlation coefficients $P = \text{diag}(\rho_i)$ given by

$$|\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - \rho^2\Sigma_{11}| = 0 \tag{111}$$

and is

$$\begin{aligned} &|I - P^2|^n \, {}_2\tilde{F}_1(n, n; q; P^2, R^2) \\ &\cdot \frac{\tilde{\Gamma}_p(n)\pi^{p(p-1)}}{\tilde{\Gamma}_p(n-q)\tilde{\Gamma}_p(q)\tilde{\Gamma}_p(p)} |R^2|^{q-p}|I - R^2|^{n-q-p} \prod (r_i^2 - r_j^2)^2 \, dr_1^2 \cdots dr_p^2. \end{aligned} \tag{112}$$

9. Calculation of zonal polynomials. An explicit usable formula for zonal polynomials is only available in special cases, as far as the author is aware, but the following considerations yield a general method of calculating them.

A glance at the table in the appendix for the zonal polynomials in terms of the elementary symmetric functions of the latent roots reveals a triangular arrangement of coefficients. The phenomenon can best be described in terms of *weights.* Let us order the partitions of the positive integer k lexicographically, i.e. if $\kappa = (k_1, \cdots, k_m)$ and $\lambda = (l_1, \cdots, l_m)$ are two partitions of k, then $\kappa > \lambda$ if $k_1 = l_1, \cdots, k_i = l_i, k_{i+1} > l_{i+1}$. If two monomials $\epsilon_1^{k_1} \cdots \epsilon_m^{k_m}$ and $\epsilon_1^{l_1} \cdots \epsilon_m^{l_m}$ have indices κ and λ, the former is said to be of *higher weight.* It is evident from the formula (35) for the character $\chi_{\{\kappa\}}(X)$ of Gl (m) that

THEOREM.

$$\chi_{\{\kappa\}}(X) = \epsilon_1^{k_1} \cdots \epsilon_m^{k_m} + \textit{terms of lower weight}, \tag{113}$$

where $\epsilon_1, \cdots, \epsilon_m$ *are the latent roots of the matrix* $X \varepsilon \text{Gl}(m)$.

If we put $X = \text{diag}(\epsilon_i)$ and substitute (113) in (34), we have

$$C_\kappa(XX') = c_\kappa \epsilon_1^{2k_1} \cdots \epsilon_m^{2k_m} + \textit{terms of lower weight}, \tag{114}$$

for some constant c_κ (which is actually given subsequently in Equation (132)). Hence putting $\sigma_i = \epsilon_i^2$ for the latent roots of $S = XX'$, we have the

THEOREM.

$$C_\kappa(S) = c_\kappa \sigma_1^{k_1} \cdots \sigma_m^{k_m} + \textit{terms of lower weight}. \tag{115}$$

When the zonal polynomials are expressed in terms of sums of powers of the roots

$$(116) \qquad Z_\kappa(S) = \sum_{\nu_1+2\nu_2+3\nu_3+\cdots=k} z_{\kappa\nu}\, s_1^{\nu_1} s_2^{\nu_2} \cdots ,$$

where $\nu = (1^{\nu_1}2^{\nu_2}3^{\nu_3}\cdots)$ is the partition of k consisting of ν_1 ones, ν_2 twos etc., then the coefficients satisfy orthogonality relations (James [26])

$$(117) \qquad \sum_\nu z_{\kappa\nu} z_{\lambda\nu}/z_{(k)\nu} = \delta_{\kappa\lambda} N/\chi_{[2\kappa]}(1)$$

$$(118) \qquad \sum_\mu \chi_{[2\mu]}(1) z_{\mu\kappa} z_{\mu\lambda} = \delta_{\kappa\lambda} N z_{(k)\kappa}$$

where $N = (2k)!/2^k k!$ and $z_{(k)\nu}$ is the coefficient of the top zonal polynomial $Z_{(k)}(S)$ and is given by the formula

$$(119) \qquad z_{(k)\nu} = 2^k k!/\nu_1!\nu_2!\nu_3! \cdots 2^{\nu_1}4^{\nu_2}6^{\nu_3} \cdots .$$

The reader may verify these relations for the zonal polynomials of low order given in the appendix.

A general method of calculating the zonal polynomial is as follows: (1) the elementary symmetric monomial functions of degree k are arranged according to their monomials of highest weight, (2) they are expressed in terms of the sums of powers, e.g. by the tables of David and Kendall [8a] and then (3) they are Gramm Schmidt orthogonalized relative to the orthogonality relation (117) from the bottom upwards.

The resulting polynomials are proportional to the zonal polynomials. They may be normalized to $Z_\kappa(S)$ by dividing by the coefficient of s_1^k and then $C_\kappa(S)$ is given by (18). For example, the lowest elementary symmetric function

$$(120) \qquad a_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$$

is proportional to the zonal polynomial, in fact,

$$(121) \qquad Z_{(1^k)}(S) = k!a_k .$$

The second lowest zonal polynomial $Z_{(2\,1^{k-2})}(S)$ is a linear combination of $a_1 a_{k-1}$ and a_k. By expressing these in terms of sums of powers and finding the coefficient α such that the polynomial

$$(122) \qquad a_1 a_{k-1} + \alpha a_k$$

is orthogonal to a_k relative to the orthogonality relation (117), one obtains the polynomial (122) which is proportional to $Z_{(2\,1^{k-2})}(S)$. Gramm Schmidt orthogonalization of the higher elementary symmetric monomial functions yields the zonal polynomials of higher weight.

Another general method of calculating the zonal polynomials is given in James [26].

Explicit formulae are available for the special cases in which the partition of k only has one part, $\kappa = (k)$, or for the case in which there are only two variables.

$$(123) \qquad Z_{(k)}(A) = 2^k k! \sum_{\nu_1+2\nu_2+3\nu_3+\cdots=k} \frac{s_1^{\nu_1} s_2^{\nu_2} s_3^{\nu_3}}{\nu_1 ! \nu_2 ! \nu_3 ! \cdots 2^{\nu_1} 4^{\nu_2} 6^{\nu_3} \cdots}$$

Harold Ruben [36]

$$(124) \qquad = 2^k k! \sum_{\nu_1+2\nu_2+3\nu_3+\cdots=k} \frac{(-1)^{\nu_2+\nu_4+\nu_6\cdots}(\frac{1}{2})_{\nu_1+\nu_2+\nu_3\cdots}}{\nu_1 ! \, \nu_2 ! \, \nu_3 ! \cdots} a_1^{\nu_1} a_2^{\nu_2} a_3^{\nu_3} \cdots$$

$$s_j = \sum_{i=1}^{m} \sigma_i^j \quad a_j = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le m} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_j} .$$

The second formula follows from the generating function given by Ruben and others, but is implicit in Pitman and Robbins [33].

$$(125) \qquad \prod_{j=1}^{m} (1 - \sigma_j t)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} Z_{(k)}(S) t^k / 2^k k!$$

Ruben [36] gives a recurrence relation

$$(126) \qquad b_k = (2k)^{-1} \sum_{i=0}^{k-1} s_{k-i} b_i \qquad b_0 = 1$$

$$(127) \qquad Z_{(k)}(S) = 2^k k! b_k .$$

A zonal polynomial $Z_{k_1 \cdots k_l}(\sigma_1 \cdots \sigma_l)$ in l variables corresponding to a partition of l parts contains the k_lth power

$$(\sigma_1 \sigma_2 \cdots \sigma_l)^{k_l}$$

of the determinant $(\sigma_1 \cdots \sigma_l)$ as a factor. When the factor is divided out, the quotient must be a constant multiple of $Z_{k_1-k_l \cdots k_{l-1}-k_l 0}$. The numerical multiplying constant can be determined by the fact that we know that

$$(128) \qquad Z_\kappa(I_l) = 2^k (\tfrac{1}{2} l)_\kappa .$$

Thus we have the recurrence formula

$$(129) \qquad Z_{k_1 k_2 \cdots k_l}(\sigma_1, \cdots, \sigma_l) = 2^{l k_l} \prod_{i=1}^{l} (\tfrac{1}{2} l - \tfrac{1}{2}(i-1) + k_i - k_l)_{k_l} (\sigma_1 \sigma_2 \cdots \sigma_l)^{k_l} Z_{k_1-k_l k_2-k_l \cdots k_{l-1}-k_l}(\sigma_1 \cdots \sigma_l).$$

In particular for $l = 2$,

$$(130) \qquad Z_{(k_1 k_2)}(\sigma_1 \sigma_2) = \frac{2^{k_1-k_2} k_1 ! (2k_2)!}{k_2 !} \sum_{\nu_1+2\nu_2=k_1-k_2} \frac{(-1)^{\nu_2} (\frac{1}{2})_{\nu_1+\nu_2}}{\nu_1 ! \, \nu_2 !} a_1^{\nu_1} a_2^{\nu_2+k_2}.$$

If $\kappa = (k_1, \cdots, k_p)$, then

(131)
$$Z_\kappa(S) = 2^k \prod_{l=1}^{p} \prod_{i=1}^{l} (\tfrac{1}{2}l - \tfrac{1}{2}(i-1) + k_i - k_l)_{k_l - k_{l+1}} \cdot a_1^{k_1-k_2} a_2^{k_2-k_3} \cdots a_p^{k_p} + \text{terms of lower weight}$$

and thus

(132)
$$c_\kappa = [\chi_{[2\kappa]}(1) 2^{2k} k!/(2k)!] \prod_{l=1}^{p} \prod_{i=1}^{l} (\tfrac{1}{2}l - \tfrac{1}{2}(i-1) + k_i - k_l)_{k_l - k_{l+1}}.$$

10. Distribution of quadratic forms. This is a special case of the latent roots of the covariance matrix for a sample of 1.

If $z_1, \cdots, z_m$ are independent standard normal variables and

(133)
$$x = \sigma_1^2 z_1^2 + \cdots + \sigma_m^2 z_m^2,$$

then the probability density of x is

(134)
$$\frac{1}{2^{\frac{1}{2}m}\Gamma(\frac{1}{2}m)\,|\Sigma|^{\frac{1}{2}}}\, {}_0F_0(-\tfrac{1}{2}\Sigma^{-1}, x) x^{\frac{1}{2}m-1}\, dx \qquad \Sigma = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_m^2 \end{bmatrix}$$

Pitman and Robbins, [33].

Only the zonal polynomials of top order $C_{(k)}(-\frac{1}{2}\Sigma^{-1})$ are involved in the expansion and $C_{(k)}(x) = x^k$.

11. Expansions in Laguerre polynomials. Hotelling [20] and Gurland [16] have given an expansion of the distribution of a quadratic form in Laguerre polynomials in x, and Grad and Solomon [15] have given some values using 5 terms of the expansion, along with other methods. The expansion generalizes to

(135)
$$\begin{aligned} {}_0F_0(-\Sigma^{-1}, S) &\overset{\text{def}}{=} \int_{O(m)} \exp(-\operatorname{tr} \Sigma^{-1} HSH')(dH) \\ &= |\Sigma|^{\alpha} e^{-\operatorname{tr} S} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(I - \Sigma) L_\kappa^{(\alpha - \frac{1}{2}(m+1))}(S)}{C_\kappa(I) k!} \end{aligned}$$

where $L_\kappa^{(\gamma)}(S)$ is a Laguerre polynomial of a matrix variable in the sense of Herz [18] and Constantine (unpublished).

The Laguerre polynomials of a single variable are the orthogonal polynomials relative to the χ^2 distribution. Herz [18] showed that Hankel transforms of homogeneous polynomials of different degrees of the latent roots of a symmetric matrix S are orthogonal relative to the Wishart distribution. Constantine (unpublished) defined Laguerre polynomials as Hankel transforms of zonal polynomials $C_\pi(S)$ and derived the formula

(136)
$$L_\pi^{(\gamma)}(S) = \sum_{\kappa \leq \pi} \frac{(\gamma + \frac{1}{2}(m+1))_\pi}{(\gamma + \frac{1}{2}(m+1))_\kappa} \alpha_\kappa C_\kappa(S)$$

where the coefficients α_κ are given by the expansion

$$C_\pi(I - S) = \sum_{\kappa \leqq \pi} \alpha_\kappa C_\kappa(S). \tag{137}$$

For a single variable they reduce to the classical Laguerre polynomials. Constantine strongly conjectures that they are orthogonal relative to the Wishart distribution.

A similar formula holds for the ${}_1F_1$ hypergeometric function,

$$ {}_1F_1(a; b; -\Sigma^{-1}, S) = |\Sigma|^\alpha e^{-\operatorname{tr} S} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a)_\kappa C_\kappa(I - \Sigma) L_\kappa^{(\alpha - \frac{1}{2}(m+1))}(S)}{(b)_\kappa C_\kappa(I) k!} \tag{138}$$

though in this case, it would be more natural to expand in Jacobi polynomials of a matrix variable if these could be found.

12. Asymptotic expansions. In

$$ {}_0F_0(-\tfrac{1}{2}T, W) = \int_{O(m)} \exp\left(-\tfrac{1}{2} \sum_{ij}^{m} \tau_i w_j h_{ij}^2\right) (dH) \tag{139}$$

where $T = \operatorname{diag}(\tau_i)$, $W = \operatorname{diag}(w_j)$, $H = (h_{ij})$, we substitute

$$H = e^S = I + S + S^2/2! + S^3/3! + \cdots \tag{140}$$

where $S = (s_{ij})$ is skew symmetric matrix. When the latent roots τ_i of T and w_j of W are large and well spread out, the integrand becomes negligible for all but very small values of s_{ij} and we obtain an asymptotic series for ${}_0F_0$ and thus for the likelihood function of the distribution (58) of the latent roots of the covariance matrix on n degrees of freedom, the first term of which is, with $\Sigma^{-1} = T$, $W = nC = n \operatorname{diag}(c_i)$,

$$|T|^{\frac{1}{2}n} {}_0F_0(-\tfrac{1}{2}T, nC)$$

$$= \frac{\Gamma_m(\frac{1}{2}m) 2^{\frac{1}{2}m(m-1)}}{\pi^{\frac{1}{2}m(m+1)}} \frac{\left(\prod_{i=1}^{m} \tau_i^{\frac{1}{2}n}\right) \exp\left(-\frac{1}{2}n \sum_{i=1}^{m} \tau_i c_i\right)}{\prod_{i<j}^{m} \{n(\tau_j - \tau_i)(c_i - c_j)\}^{\frac{1}{2}}} \{1 + \cdots\}. \tag{141}$$

13. Moments. If XX' is distributed in the Wishart distribution (55), then the moment generating function of $\operatorname{tr}(XX')$ is obviously

$$E_{W(n)}[e^{t \operatorname{tr}(XX')}] = |I - 2t\Sigma|^{-\frac{1}{2}n} = \prod_{i=1}^{m} (1 - 2t\sigma_i)^{-\frac{1}{2}n} \tag{142}$$

if the σ_i are the latent roots of Σ. The kth moment is the coefficient of $t^k/k!$ in this expansion. Alternatively, we have

$$|I - 2t\Sigma|^{-\frac{1}{2}n} = {}_1F_0(\tfrac{1}{2}n, 2t\Sigma) = \sum_{k=0}^{\infty} (2^k \sum_{\kappa} (\tfrac{1}{2}n)_\kappa C_\kappa(\Sigma)) t^k/k!, \tag{143}$$

yielding the kth moment as

$$E_{W(n)}[(\operatorname{tr}(XX'))^k] = 2^k \sum_\kappa (\tfrac{1}{2}n)_\kappa C_\kappa(\Sigma). \tag{144}$$

If the positive definite symmetric matrix variate S has a multivariate beta distribution,

$$\frac{\Gamma_m(u+v)}{\Gamma_m(u)\Gamma_m(v)} |S|^{u-\frac{1}{2}(m+1)} |I - S|^{v-\frac{1}{2}(m+1)} (dS), \tag{145}$$

then, from (47), the moment generating function of its trace is

$$E[e^{t \operatorname{tr} S}] = {}_1F_1(u; u+v; tI_m) \tag{146}$$

and hence the kth moment is

$$E[(\operatorname{tr} S)^k] = \sum_\kappa [(u)_\kappa/(u+v)_\kappa] C_\kappa(I_m) \tag{147}$$

where $C_\kappa(I_m)$ is given by (21).

The noncentral moments of the generalized variance $|XX'|$ have been given by Anderson for the case of Ω of rank $\leqq 2$, and in general by Herz [18] who is quoted by Constantine [8] as

$$E_{W(n)}[|XX'|^k] = 2^{km} \frac{\Gamma_m(k+\frac{1}{2}n)}{\Gamma_m(\frac{1}{2}n)} |\Sigma|^k {}_1F_1(-k; \tfrac{1}{2}n; -\tfrac{1}{2}\Omega), \tag{148}$$

which is a polynomial for positive integral k. The variate XX' has the noncentral Wishart distribution (67) with $\Omega = M\Sigma^{-1}M'$.

The noncentral moments of the likelihood ratio statistic have been given by Constantine [8] as

$$E\left[\left(\frac{|YY'|}{|XX'+YY'|}\right)^k\right] = \frac{\Gamma_m(k+\frac{1}{2}n)\Gamma_m(\frac{1}{2}(p+n))}{\Gamma_m(\frac{1}{2}n)\Gamma_m(k+\frac{1}{2}(p+n))} {}_1F_1(k; k + \tfrac{1}{2}(p+n); -\tfrac{1}{2}\Omega) \tag{149}$$

where XX' is a noncentral Wishart variate on p degrees of freedom and noncentrality $\Omega = M\Sigma^{-1}M'$ and YY' is a central Wishart variate on n degrees of freedom.

Pillai [32] and Mijares [30] have considered moments of elementary symmetric functions of the roots of a multivariate beta variate S distributed as (145). One method of obtaining the expectation of a monomial $a_1^{\nu_1} a_2^{\nu_2} \cdots$ in the elementary symmetric functions $a_1, a_2, \cdots$ of the roots is to express it as a linear combination of zonal polynomials by solving a triangular system of equations given in the tables in the appendix and use the formula of Constantine [8]

$$E[C_\kappa(S)] = [(u)_\kappa/(u+v)_\kappa] C_\kappa(I_m) \tag{150}$$

for the expectation of a zonal polynomial when S is distributed as (145).

Constantine [8], Equations (60) and (61), gives formulae for the incomplete gamma and beta functions in the matrix case. Comparison of the noncentral Wishart distribution (67) with the integral representation given by Anderson [1] yields Anderson's integral

$$(151)\quad {}_0F_1(b; \tfrac{1}{4}XX') = \frac{\Gamma_m(b)}{\pi^{\frac{1}{2}mk}\Gamma_m(b - \frac{1}{2}k)} \int_{0<YY'<I} e^{\operatorname{tr} XY'} |I - YY'|^{b-\frac{1}{2}(k+m+1)} (dY)$$

where X and Y are $m \times k$ matrices. If $k < m$, we replace $\Gamma_k(\frac{1}{2}m)$ by $\Gamma_m(\frac{1}{2}k)$. For $m = k = 1$, Equation (151) reduces to the Poisson integral for the ordinary Bessel function as given in Erdelyi et al. [9] p. 81 Formula (7) or (10). Godement [13] has given a generalization of Bochner and Herz's Bessel functions.

14. Conclusion. For numerical evaluation of the probability density functions the power series expansions (10) and (13) of the hypergeometric functions occurring in the distributions are of very limited value. If even one root is significant, it will take a very large number of terms of the series to give values of the likelihood function accurate enough to be of use.

The importance of the preceding theory lies in the mathematical characterization of the functions involved in the distributions. When a sufficient number of the representations of the functions are known, e.g. as series, integrals, asymptotic expansions etc., together with the relations between them, then it is to be hoped that some good points of attack on the numerical values will be found, and in particular, it may be possible to calculate the likelihood functions of the parameters of the distributions.

Added in proof. I am indebted to C. G. Khatri and A. G. Constantine who independently pointed out two errors in my original version of Equation (93). This necessitated a correction of all the complex roots distributions.

APPENDIX

Zonal Polynomials

$Z_\kappa(S)$ up to order 6

Left: in terms of sums of powers of the latent roots of S. Right: in terms of elementary symmetric functions of the latent roots of S.

1st degree

	In terms of sums of powers	In terms of elementary symmetric functions	$\chi_{[2\kappa]}(1)$
$Z_{(1)}$	s_1	a_1	1

2nd degree

	In terms of sums of powers	In terms of elementary symmetric functions	$\chi_{[2\kappa]}(1)$
$Z_{(2)}$	$s_1^2 + 2s_2$	$3a_1^2 - 4a_2$	1
$Z_{(1^2)}$	$s_1^2 - s_2$	$2a_2$	2
			3

3rd degree

	In terms of sums of powers	In terms of elementary symmetric functions	$\chi_{[2\kappa]}(1)$
$Z_{(3)}$	$s_1^3 + 6s_1s_2 + 8s_3$	$15a_1^3 - 36a_1a_2 + 24a_3$	1
$Z_{(2\,1)}$	$s_1^3 + s_1s_2 - 2s_3$	$4a_1a_2 - 6a_3$	9
$Z_{(1^3)}$	$s_1^3 - 3s_1s_2 + 2s_3$	$6a_3$	5
			15

4th degree

	s_1^4	$s_1^2s_2$	s_2^2	s_1s_3	s_4	a_1^4	$a_1^2a_2$	a_2^2	a_1a_3	a_4	$\chi_{[2\kappa]}(1)$
$Z_{(4)}$	1	12	12	32	48	105	−360	144	288	−192	1
$Z_{(3\,1)}$	1	5	−2	4	−8		18	−24	−20	32	20
$Z_{(2^2)}$	1	2	7	−8	−2			24	−32	8	14
$X_{(2\,1^2)}$	1	−1	−2	−2	4				10	−16	56
$Z_{(1^4)}$	1	−6	3	8	−6					24	14
											105

5th degree

	s_1^5	$s_1^3s_2$	$s_1s_2^2$	$s_1^2s_3$	s_2s_3	s_1s_4	s_5
$Z_{(5)}$	1	20	60	80	160	240	384
$Z_{(4\,1)}$	1	11	6	26	−20	24	−48
$Z_{(3\,2)}$	1	6	11	−4	20	−26	−8
$Z_{(3\,1^2)}$	1	3	−10	2	−4	−8	16
$Z_{(2^21)}$	1	0	5	−10	−10	10	4
$Z_{(2\,1^3)}$	1	−4	−3	2	10	6	−12
$Z_{(1^5)}$	1	−10	15	20	−20	−30	24

	a_1^5	$a_1^3a_2$	$a_1a_2^2$	$a_1^2a_3$	a_2a_3	a_1a_4	a_5	$\chi_{[2\kappa]}(1)$
$Z_{(5)}$	945	−4200	3600	3600	−2880	−2880	1920	1
$Z_{(4\,1)}$		120	−288	−126	360	144	−240	35
$Z_{(3\,2)}$			72	−96	−80	144	−40	90
$Z_{(3\,1^2)}$				42	−56	−48	80	225
$Z_{(2^21)}$					40	−60	20	252
$Z_{(2\,1^3)}$						36	−60	300
$Z_{(1^5)}$							120	42
								945

Zonal Polynomials of 6th Degree

Z	s_1^6	$s_1^4s_2$	$s_1^2s_2^2$	$s_1^3s_3$	s_2^3	$s_1s_2s_3$	$s_1^2s_4$	s_3^2	s_2s_4	s_1s_5	s_6	$\chi_{[2\kappa]}(1)$
6	1	30	180	160	120	960	720	640	1440	2304	3840	1
51	1	19	48	72	−12	80	192	−64	−144	192	−384	54
42	1	12	27	16	30	24	−18	−8	108	−144	−48	275
41^2	1	9	−12	22	−12	−60	12	16	−24	−48	96	616
3^2	1	9	33	−8	−27	120	−78	136	−114	−48	−24	132
321	1	4	3	−8	−2	0	−18	−24	−4	32	16	2673
31^3	1	0	−21	4	6	12	−6	16	12	24	−48	1925
2^3	1	0	15	−20	30	−60	30	40	−60	24	0	462
2^21^2	1	−3	3	−8	−9	0	24	4	24	−24	−12	2640
21^4	1	−8	3	12	6	20	−6	−16	−36	−24	48	1485
1^6	1	−15	45	40	−15	−120	−90	40	90	144	−120	132
												10395

	a_1^6	$a_1^4a_2$	$a_1^2a_2^2$	$a_1^3a_3$	a_2^3	$a_1a_2a_3$	$a_1^2a_4$	a_3^2	a_2a_4	a_1a_5	a_6
6	10395	−56700	75600	50400	−14400	−86400	−43200	17280	34560	34560	−23040
51		1050	−3600	−1080	1440	5472	1152	−1728	−3456	−1344	2304
42			432	−576	−576	432	648	−216	576	−1008	288
41^2				270	0	−648	−288	432	384	336	−576
3^2					720	−1728	1152	1152	−1056	−384	144
321						112	−168	−168	64	256	−96
31^3							144	0	−192	−168	288
2^3								360	−480	120	0
2^21^2									120	−192	72
21^4										168	−288
1^6											720

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