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Research Article

Distributions of Patterns of Pair of Successes Separated by Failure Runs of Length at Least k_1 and at Most k_2 Involving Markov Dependent Trials: GERT Approach

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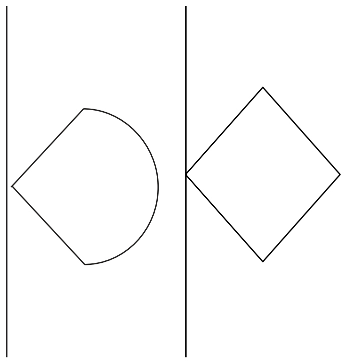
We use the Graphical Evaluation and Review Technique (GERT) to obtain probability generating functions of the waiting time distributions of 1st, and mth nonoverlapping and overlapping occurrences of the pattern $\Lambda_f^{k_1,k_2} = S\underbrace{FF\cdots F}_{k_1 \le k_f \le k_2}S$ $(k_1 > 0)$, involving homogenous Markov dependent trials. GERT besides providing visual picture of the system helps to analyze the system in a less inductive manner. Mean and variance of the waiting times of the occurrence of the patterns have also been obtained. Some earlier results existing in literature have been shown to be particular cases of these results.

1. Introduction

Probability generating functions of waiting time distributions of runs and patterns have been studied and utilized in various areas of statistics and applied probability, with applications to statistical quality control, ecology, epidemiology, quality management in health care sector, and biological science to name a few. A considerable amount of literature treating waiting time distributions have been generated, see Fu and Koutras [1], Aki et al. [2], Koutras [3], Antzoulakos [4], Aki and Hirano [5], Han and Hirano [6], Fu and Lou [7], and so forth. The books by Godbole and Papastavridis [8], Balakrishnan and Koutras [9], Fu and Lou [10] provide excellent information on past and current developments in this area.

The probability generating function is very important for studying the properties of waiting time distributions of runs and patterns. Once a potentially problem-specific statistic involving runs and patterns has been defined, the task of deriving its distribution can be very complex and nontrivial. Traditionally, combinatorial methods were used to find the exact distributions for the numbers of runs and patterns. By using the theory of recurrent events, Feller [11] obtained the probability generating function for waiting time of a success run $A = \underbrace{SS\cdots S}_{k}$ of size k in a sequence of Bernoulli trials. Fu and Chang [12] developed general method based on the finite Markov chain imbedding technique for finding the mean and probability generating functions of waiting time distributions of compound patterns in a sequence of i.i.d. or Markov dependent multistate trials. Ge and Wang [13] studied the consecutive-k-out-of-n: F system involving Markov Dependence.

Graphical Evaluation and Review Technique (GERT) has been a well-established technique applied in several areas. However, application of GERT in reliability studies has not been reported much. It is only recently that Cheng [14] analyzed reliability of fuzzy consecutive-k-out-of-n: F system using GERT. Agarwal et al. [15, 16], Agarwal & Mohan [17] and Mohan et al. [18] have also studied reliability of m-consecutive-k-out-of-n: F system and its various generalizations using GERT, and illustrated the efficiency of GERT in



(a) Deterministic output (b) Probabilistic output

Figure 1: Type of GERT nodes.

reliability analysis. Mohan et al. [19] studied waiting time distributions of 1st, and mth nonoverlapping and overlapping occurrences of the pattern $\Lambda^{k} = S\underbrace{FF\cdots F}_{k}S$, involving Markov dependent trials, using GERT. In this paper, probability generating functions of the waiting time distributions of 1st, and mth nonoverlapping and overlapping occurrences of a pattern involving pair of successes separated by a run of failures of length at least k_1 and at most k_2, $\Lambda_{f}^{k_1,k_2} = S\underbrace{FF\cdots F}_{k_1 \le k_f \le k_2}S$ $(k_1 > 0)$, involving Homogenous Markov Dependence, that is, probability that component i fails depends only upon the state of component $(i-1)$ and not upon the state of the other components, (Ge and Wang [13]) have been studied using GERT. Mean and variance of the time of their occurrences can then be obtained easily. Some earlier results existing in literature have been shown to be particular cases.

2. Notations and Assumptions

$\Lambda_{f}^{k_1,k_2} = S\underbrace{FF\cdots F}_{k_1 \le k_f \le k_2}S$: pair of successes separated by a run of failures of length at least k_1 and at most k_2, $k_1 > 0$.

X_i: indicator random variable for state of trial i, $X_i = 0$ or 1 according as trial i is success or failure.

p_0, q_0: $\Pr\{X_1 = 0\}$; $q_0 = 1 - p_0$

p_1, q_1: $\Pr\{X_i = 0 \mid X_{i-1} = 0\}$, probability that trial i is a success given that preceding trial $(i-1)$ is also a success, for $i = 2, 3, \ldots$; $q_1 = 1 - p_1$.

p_2, q_2: $\Pr\{X_i = 0 \mid X_{i-1} = 1\}$, probability that trial i is a success given that preceding trial $(i-1)$ is a failure, for $i = 2, 3, \ldots$; $q_2 = 1 - p_2$.

3. Brief Description of GERT and Definitions

GERT is a procedure for the analysis of stochastic networks having logical nodes (or events) and directed branches (or activities). It combines the disciplines of flow graph theory, MGF (Moment Generating Function), and PERT (Project Evaluation and Review Technique) to obtain a solution to stochastic networks having logical nodes and directed branches. The nodes can be interpreted as the states of the system and directed branches represent transitions from one state to another. A branch has the probability that the activity associated with it will be performed. Other parameters describe the activities represented by the branches.

A GERT network in general contains one of the following two types of *logical nodes* (Figure 1):

(a) nodes with *Exclusive-Or input* function and *Deterministic output* function and

(b) nodes with *Exclusive-Or input* function and *Probabilistic output* function.

Exclusive-Or Input. The node is realized when any arc leading into it is realized. However, one and only one of the arcs can be realized at a given time.

Deterministic Output. All arcs emanating from the node are taken if the node is realized.

Probabilistic Output. Exactly one arc emanating from the node is taken if the node is realized.

In this paper type (b) nodes are used.

The transmittance of an arc in a GERT network, that is, the generating function of the waiting time for the occurrence of required system state is the corresponding W-function. It is used to obtain the information of a relationship, which exists between the nodes.

If we define $W(s \mid r)$, as the conditional W function associated with a network when the branches tagged with a z are taken r times, then the equivalent W generating function can be written as follows:

$$W(s, z) = \sum_{r=0}^{\infty} W(s \mid r) z^{r}. \tag{1}$$

The function $W(0, z)$ is the generating function of the waiting time for the network realization.

Mason's Rule (Whitehouse [20], pp. 168–172). In an open flow graph, write down the product of transmittances along each path from the independent to the dependent variable. Multiply its transmittance by the sum of the nontouching loops to that path. Sum these modified path transmittances and divide by the sum of all the loops in the open flow graph yielding transmittance T as follows:

$$T = \frac{[\sum(\text{path} * \sum \text{nontouching loops})]}{\sum \text{loops}}, \tag{2}$$

where

$$\begin{aligned}\sum \text{loops} &= 1 - \left(\sum \text{first order loops}\right) \\ &\quad + \left(\sum \text{second order loops}\right) - \cdots \\ \sum &\text{nontouching loops} \\ &= 1 - \left(\sum \text{first order nontouching loops}\right) \\ &\quad + \left(\sum \text{second order nontouching loops}\right) \\ &\quad - \left(\sum \text{third order nontouching loops}\right) + \cdots\end{aligned} \tag{3}$$

For more necessary details about GERT, one can see Whitehouse [20], Cheng [14] and Agarwal et al. [15].

4. Waiting Time Distribution of the Pattern $\Lambda_f^{k_1,k_2}$

Theorem 1. *$W_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)$, the probability generating function for the waiting time distribution of the 1st occurrence of the pattern $\Lambda_f^{k_1,k_2}$ involving Homogenous Markov Dependence is given by*

$$\begin{aligned}W_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z) &= \left(q_0p_2z^2 + p_0z\left(1-q_2z\right)\right) \\ &\quad\times\left(q_1p_2{q_2}^{k_1-1}z^{k_1+1}\left(1-\left(q_2z\right)^{k_2-k_1+1}\right)\right) \\ &\quad\times\left(\left(1-q_2z\right)\left(1-q_2z-p_1z-q_1p_2z^2+q_2p_1z^2\right.\right. \\ &\qquad\left.\left. -q_1p_2{q_2}^{k_2}z^{k_2+2}+q_1p_2{q_2}^{k_1-1}z^{k_1+1}\right)\right)^{-1}.\end{aligned} \tag{4}$$

Then,

$$\begin{aligned}E\left[W\left(\Lambda_f^{k_1,k_2}\right)\right] &= E\left[\textit{minimum number of trials required}\right. \\ &\qquad\left.\textit{to obtain the pattern } \Lambda_f^{k_1,k_2}\right] \\ &= \frac{q_2\left(q_1+p_2\right)+q_1q_2^{k_1}\left(q_0+p_2\right)\left(1-q_2^{k_2-k_1+1}\right)}{q_1p_2q_2^{k_1}\left(1-q_2^{k_2-k_1+1}\right)},\end{aligned} \tag{5}$$

$$\begin{aligned}\operatorname{Var}\left[W\left(\Lambda_f^{k_1,k_2}\right)\right] &= \left.\frac{d^2W_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)}{dz^2}\right|_{z=1} \\ &\quad + E\left[W\left(\Lambda_f^{k_1,k_2}\right)\right] - \left(E\left[W\left(\Lambda_f^{k_1,k_2}\right)\right]\right)^2.\end{aligned} \tag{6}$$

The GERT network for this pattern is represented by Figure 2 where each node represents a specific state as described below:

S: initial node

1^A: a trial resulting in failure

0_i: a trial resulting in success corresponding to beginning of the ith occurrence of the pattern $\Lambda_f^{k_1,k_2}$, $i = 1,2,\ldots m$

1: a component is in failed state preceded by working component

j: jth contiguous failed trial preceded by a success trial, $j = 2,\ldots,k_1-1,k_1,k_1+1,\ldots,k_2,k_2+1$

0^i: ith occurrence of the required pattern, $i = 1,2,\ldots,m$.

There are in all k_2+5 nodes designated as $S, 1^A, 0_1, 1, 2, \ldots, k_1-1, k_1, k_1+1, \ldots, k_2-1, k_2, k_2+1$ and 0^1 representing specific states of the system, that is, a sequence of homogenous Markov dependent trials. GERT network can be summarized as follows. If the first trial results in failure then state 1^A is reached from S with conditional probability q_0 otherwise state 0_1 with conditional probability p_0. Further, if the preceding trial is failure (state 1^A) followed by a contiguous success trial then state 0_1 is reached from state 1^A with conditional probability p_2 otherwise it continues to move in state 1^A with conditional probability q_2. State 0_1 represents first occurrence of a success trial (may or may not be preceded by failed trials) and if next contiguous trial(s) is (are) also success then it continues to move in state 0_1 with conditional probability p_1 otherwise it moves in state 1 with conditional probability q_1. Again if the next contiguous trial results in failure then system state 2 occurs with conditional probability q_2 otherwise state 0_1 with conditional probability p_2. Similar procedure is followed till k_1 contiguous failed trials occur preceded by a success trial. Now, if the next contiguous trial is failure then state k_1+1 is reached with conditional probability q_2 otherwise state 0^1, first occurrence of required pattern. However, if the system is in state k_1+1 and next contiguous trial is failure system moves to state k_1+2 with conditional probability q_2. Again a similar procedure is followed till node k_2. However, if there occur k_2-k_1+1 contiguous failed trials after state k_1 then the system moves to node k_2+1 and continues to move in that state until a success trial occurs at which it moves to state 0_1 and again a similar procedure is followed for the remaining trials till state 0^1, first occurrence of the required pattern. Details on the derivation of the theorem are given in the appendix.

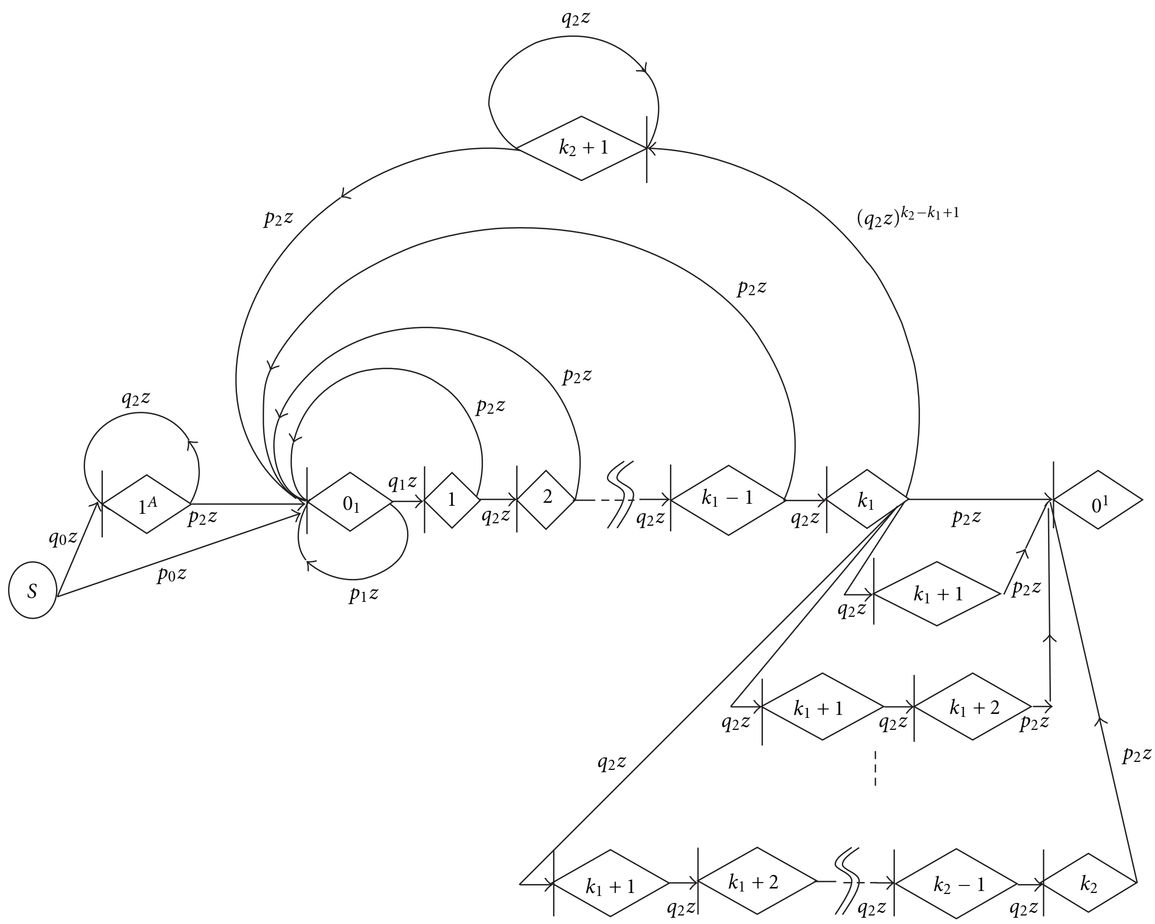


FIGURE 2: GERT network representing 1st occurrence of the pattern $\Lambda_f^{k_1,k_2} = S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S$.

Theorem 2. $W^{no}_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)_m$, *the probability generating function for the waiting time distribution of the mth nonoverlapping occurrence of the pattern* $\Lambda_f^{k_1,k_2}$ *involving Homogenous Markov Dependence is given by*

$$\begin{aligned}
&W^{no}_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)_m \\
&= \left(q_0p_2z^2 + p_0z\left(1-q_2z\right)\right) \\
&\quad\times\left(p_2q_1{q_2}^{k_1-1}z^{k_1+1}\left\{1-\left(q_2z\right)^{k_2-k_1+1}\right\}\right)^m \\
&\quad\times\left(p_1z\left(1-q_2z\right)+q_1p_2z^2\right)^{m-1}\left(1-q_2z\right)^{-m} \\
&\quad\times\left(1-q_2z-p_1z+q_2p_1z^2-q_1p_2z^2\right. \\
&\qquad\left.+q_1p_2q_2^{k_1-1}z^{k_1+1}-\ q_1p_2{q_2}^{k_2}z^{k_2+2}\right)^{-m}.
\end{aligned}\tag{7}$$

Then,

$$\begin{aligned}
&E\left[W^{no}\left(\Lambda_f^{k_1,k_2}\right)\right] \\
&= E\left[\text{minimum number of trials required to obtain}\right. \\
&\qquad\left.\text{the mth nonoverlapping occurrence of pattern } \Lambda_f^{k_1,k_2}\right] \\
&= q_1q_2^{k_1}\left(1-q_2^{k_2-k_1+1}\right) \\
&\quad\times\left(p_1-p_0+m\left(q_1+p_2\right)\right)+m\ q_2\left(q_1+p_2\right) \\
&\quad\times\left(q_1p_2q_2^{k_1}\left(1-q_2^{k_2-k_1+1}\right)\right)^{-1},
\end{aligned}\tag{8}$$

and $\mathrm{Var}[W^{no}(\Lambda_f^{k_1,k_2})]$ *can be obtained by applying* (6).

The GERT network for $m = 2$ (say) nonoverlapping occurrence of the pattern $\Lambda_f^{k_1,k_2}$ is represented by Figure 3. Each node represents a specific state as described earlier in Figure 2.

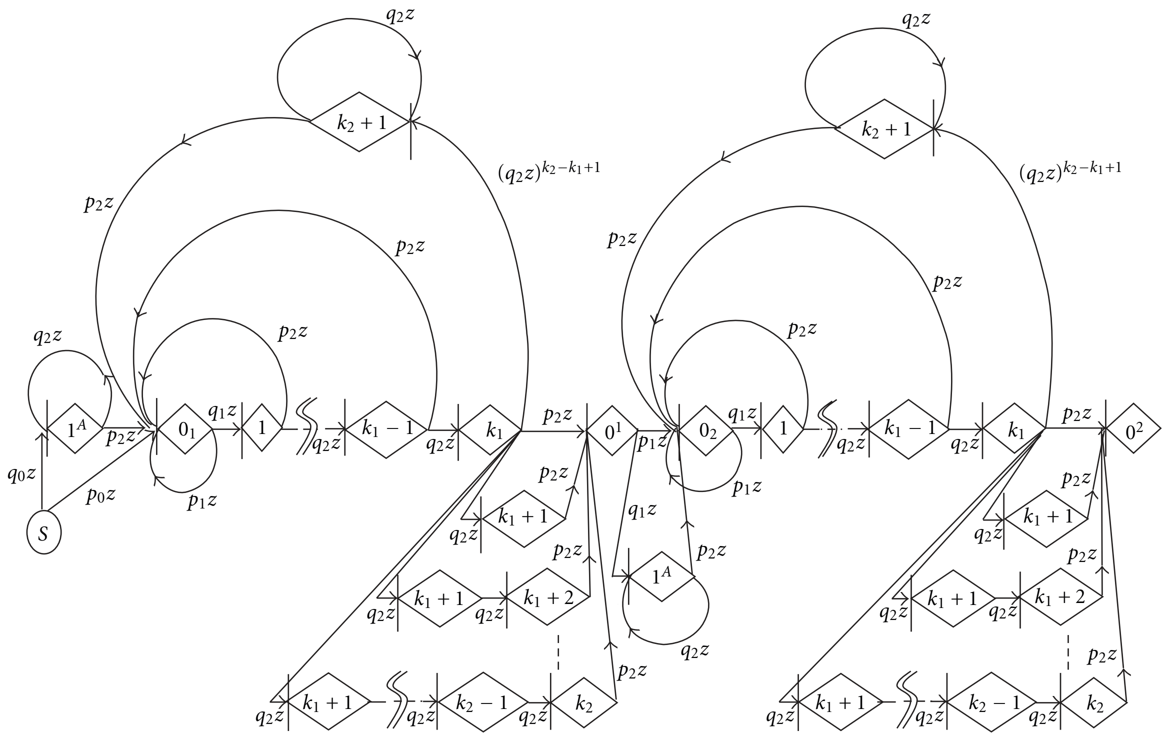


FIGURE 3: GERT network representing the mth ($m = 2$, here) nonoverlapping occurrence of the pattern $\Lambda_f^{k_1,k_2} = S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S$.

Theorem 3. *$W^o_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)_m$, the probability generating function for the waiting time distribution of the mth overlapping occurrence of the pattern $\Lambda_f^{k_1,k_2}$ involving Homogenous Markov Dependence is given by*

$$\begin{aligned}
&W^o_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)_m \\
&= \left(q_0p_2z^2 + p_0z\left(1-q_2z\right)\right) \\
&\quad\times\left(q_1p_2q_2^{k_1-1}z^{k_1+1}\left\{1-\left(q_2z\right)^{k_2-k_1+1}\right\}\right)^m \\
&\quad\times\left(\left(1- q_2z\right)\left(1 - q_2z-p_1z- q_1p_2z^2+q_2p_1z^2\right.\right. \\
&\qquad\left.\left.-q_1p_2q_2^{k_2}z^{k_2+2}+ q_1p_2{q_2}^{k_1-1}z^{k_1+1}\right)\right)^{-m}.
\end{aligned} \tag{9}$$

Then,

$$\begin{aligned}
&E\left[W^o\left(\Lambda_f^{k_1,k_2}\right)\right] \\
&= E\left[\textit{minimum number of trials required to obtain}\right. \\
&\qquad\left.\textit{the mth overlapping occurrence of pattern } \Lambda_f^{k_1,k_2}\right] \\
&= \frac{q_1q_2^{k_1}\left(1-q_2^{k_2-k_1+1}\right)\left(q_0+p_2\right)+mq_2\left(q_1+p_2\right)}{q_1p_2q_2^{k_1}\left(1-q_2^{k_2-k_1+1}\right)}.
\end{aligned} \tag{10}$$

and Var $[W^o(\Lambda_f^{k_1,k_2})]$ *can be obtained by applying* (6).

The GERT network for mth ($m = 2$, say), overlapping occurrence of the pattern $\Lambda_f^{k_1,k_2}$ is represented by Figure 4. Each node represents a specific state as described earlier in Figure 2.

Particular Cases. (i) For $k_1 = k_f = k_2 = k-2$, $p_0 = p_1 = p_2 = p$, and $q_0 = q_1 = q_2 = q$, in the pattern $\Lambda_f^{k_1,k_2}$, that is, for a run of at most $k-2$ failures bounded by successes, the probability generating function becomes

$$\begin{aligned}
&W_{S\underbrace{FF\cdots F}_{k-2}S}(0,z) \\
&= \frac{(pz)^2\left(1-(qz)^{k-1}\right)}{(1-qz)\left(1-z+\left(pz\left(1-(qz)^{k-1}\right)\right)\right)},
\end{aligned} \tag{11}$$

verifying the results of Koutras [3, Theorem 3.2].

(ii) If $k_1 = 1$, $k_f = k_2 = k$, that is, pair of successes are separated by a run of failures of length at least one and at most k, that is, $S\underbrace{FF\cdots F}_{1\le k}S$ then (4), (7), and (9), respectively, become

$$\begin{aligned}
&W_{S\underbrace{FF\cdots F}_{1\le k}S}(0,z) \\
&= \frac{\left(q_0p_2z^2+p_0z\left(1-q_2z\right)\right)\left(p_2q_1z^2\left(1-\left(q_2z\right)^k\right)\right)}{\left(1-q_2z\right)\left(1-q_2z-p_1z+q_2p_1z^2-q_1p_2{q_2}^kz^{k+2}\right)}.
\end{aligned}$$

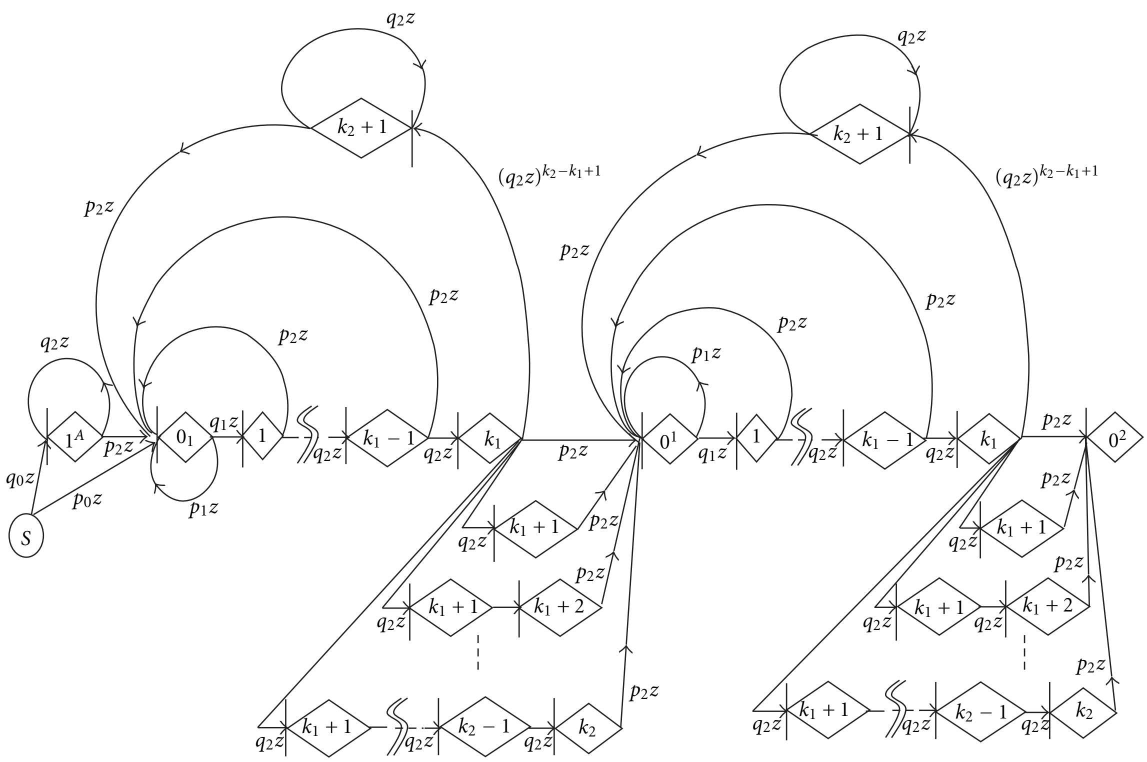


FIGURE 4: GERT network representing the mth ($m=2$, here) overlapping occurrence of the pattern $\Lambda_f^{k_1,k_2} = S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S$.

$$
\begin{aligned}
&W^{no}_{S\underbrace{FF\cdots F}_{1\le k}S} \\
&= \left(q_0p_2z^2 + p_0z\left(1-q_2z\right)\right)\left(q_1p_2z^2 + p_1z\left(1-q_2z\right)\right)^{m-1} \\
&\quad\times\left(q_1p_2z^2\left(1-\left(q_2z\right)^k\right)\right)^m\left(1-q_2z\right)^{-m} \\
&\quad\times\left(1-p_1z-q_2z+p_1q_2z^2-p_2q_1q_2^kz^{k+2}\right)^{-m}. \\
&W^{o}_{S\underbrace{FF\cdots F}_{1\le k}S}(0,z) \\
&= \frac{\left(q_0p_2z^2+p_0z\left(1-q_2z\right)\right)\left(p_2q_1z^2\left(1-\left(q_2z\right)^k\right)\right)^m}{\left(1-q_2z\right)\left(1-q_2z-p_1z+q_2p_1z^2-p_2q_1q_2{}^kz^{k+2}\right)^m}.
\end{aligned}
\tag{12}
$$

(iii) If $k_1 = k$ and $k_2 \to \infty$, then it reduces to the pattern $\Lambda^{\ge k} = S\underbrace{FF\cdots F}_{\ge k}S$, that is,

$$
\begin{aligned}
&W_{S\underbrace{FF\cdots F}_{\ge k}S}(0,z) \\
pt &= \left(q_0p_2z^2+p_0z\left(1-q_2z\right)\right)\left(p_2q_1q_2^{k-1}z^{k+1}\right) \\
&\quad\times\Big(\left(1-q_2z\right)\Big(1-q_2z-p_1z+q_2p_1z^2 \\
&\qquad -q_1p_2z^2\left(1-\left(q_2z\right)^{k-1}\right)\Big)\Big)^{-1}.
\end{aligned}
\tag{13}
$$

(iv) If $k_1 = k_f = k_2 = k$,

then (4), (7), (9) reduce to the results of Mohan et al. [19], that is,

$$
\begin{aligned}
&W_{S\underbrace{FF\cdots F}_{k}S}(0,z) \\
&= \left(q_0p_2z^2+p_0z\;\left(1\;-\;q_2z\right)\right)\left(q_1p_2q_2^{k-1}z^{k+1}\right) \\
&\quad\times\left(1-q_2z-p_1z-p_2q_1z^2+p_1q_2z^2-q_1p_2q_2^kz^{k+2}\right. \\
&\qquad\left.+q_1p_2q_2^{k-1}z^{k+1}\right)^{-1}, \\
&W^{no}_{S\underbrace{FF\cdots F}_{k}S}(0,z)_m \\
&= \left(q_0p_2z^2+p_0z\left(1-q_2z\right)\right)\left(p_2q_1q_2^{k-1}z^{k+1}\right)^m \\
&\quad\times\left[p_1z\left(1-q_2z\right)+p_2q_1z^2\right]^{m-1} \\
&\quad\times\left[1-q_2z-p_1z-q_1p_2z^2+q_2p_1z^2-q_1p_2q_2^kz^{k+2}\right.
\end{aligned}
$$

$$+q_1 p_2 q_2^{k-1} z^{k+1}\Big]^{-m},$$

$$W^{o}_{S\underbrace{FF\cdots F}_{k}S}(0,z)_m$$

$$= \left(q_0 p_2 z^2 + p_0 z\ \left(1-q_2 z\right)\right)\left(p_2 q_1 q_2^{k-1} z^{k+1}\right)^m \left(1-q_2 z\right)^{m-1}$$
$$\times \left(1 - q_2 z - p_1 z - q_1 p_2 z^2 + q_2 p_1 z^2 - q_1 p_2 q_2^{k} z^{k+2}\right.$$
$$\left. + q_1 p_2 q_2^{k-1} z^{k+1}\right)^{-m}. \tag{14}$$

5. Conclusion

In this paper we proposed Graphical Evaluation and Review Technique (GERT) to study probability generating functions of the waiting time distributions of 1st, and mth nonoverlapping and overlapping occurrences of the pattern $\Lambda_f^{k_1,k_2} = S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S$ $(k_1 > 0)$, involving homogenous Markov dependent trials. We have also demonstrated the flexibility and usefulness of our approach by validating some earlier results existing in literature as particular cases of these results.

Appendix

Proof of Theorems

From the GERT network (Figure 2), it can be observed that there are $2(k_2-k_1+1)$ paths to reach state 0^1 from the starting node S, see Table 1.

Now, to apply Mason's rule, we must also locate all the loops. However only first and second order loops exist. First order loops are given in Table 2.

However, paths at Nos. $k_2-k_1+2, k_2-k_1+3,\cdots, 2(k_2-k_1+1)$ also contain first order nontouching loop (1^A to 1^A), whose value is given by $q_2 z$. Further, first order loop No. k_1+3 forms first order nontouching loop to both paths, whose value is given by $q_2 z$.

Also second order loops corresponding to first order loops from No. 2 to No. k_1+3 are given (since first order loop no. 1, that is, 1^A to 1^A forms nontouching loop with each of the other mentioned first order loops and can be taken separately, Whitehouse [20] pp. 257) in Table 3.

Thus, by applying Mason's rule we obtain the following generating function $W_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)$ of the waiting time for the occurrence of the pattern $\Lambda_f^{k_1,k_2}$ as follows:

$$W_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)$$

$$= \left(q_0 p_2 z^2 + p_0 z\left(1-q_2 z\right)\right)\left(1-q_2 z\right)$$
$$\times \left[q_1 p_2 q_2^{k_1-1} z^{k_1+1}\left\{1+(q_2 z)+(q_2 z)^2+\cdots+(q_2 z)^{k_2-k_1}\right\}\right]$$
$$\times \left(\left(1-q_2 z\right)\left(1-q_2 z - p_1 z + q_2 p_1 z^2 - q_1 p_2 z^2\right.\right.$$
$$\left.\left. + q_1 p_2 q_2^{k_1-1} z^{k_1+1} - q_1 p_2 q_2^{k_2} z^{k_2+2}\right)\right)^{-1}, \tag{A.1}$$

which yields (4).

Similarly, for the mth nonoverlapping occurrence of the pattern (Theorem 2) once when state 0^1 is reached, that is, the first occurrence of required pattern, if the next contiguous trial results in success then state 0_2 is directly reached with conditional probability p_1 otherwise state 1^A with conditional probability q_1 and continues to move in that state with conditional probability q_2 until a success occurs at which it moves to state 0_2 with conditional probability p_2. Thus, there are two paths to reach node 0_2 from node 0^1. Now, on reaching node 0_2 again the same procedure is followed for the second occurrence of the required pattern, represented by node 0^2. Thus, for the second occurrence of the pattern node 0^1 acts as a starting node following a similar procedure as for the first occurrence.

It can be observed from the GERT network that there are $4\cdot(k_2-k_1+1)^2$ paths (in general for any m there are $2\cdot 2^{m-1}(k_2-k_1+1)^m$ paths) for reaching state 0^2 (0^m) from the starting node S. Thus, proceeding as in Theorem 1 and applying Mason's rule we obtain the following generating function $W^{no}_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)_2$ of the waiting time for the 2nd nonoverlapping occurrence of the required pattern $\Lambda_f^{k_1,k_2}$ as follows:

$$W^{no}_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)_2$$

$$= \left(q_0 p_2 z^2 + p_0 z\left(1-q_2 z\right)\right)\left(p_1 z\left(1-q_2 z\right) + q_1 p_2 z^2\right)$$
$$\times \left(p_2 q_1 q_2^{k_1-1} z^{k_1+1}\left\{1-\left(q_2 z\right)^{k_2-k_1+1}\right\}\right)^2$$
$$\times \left(1-q_2 z\right)^{-2}\left(1-q_2 z - p_1 z + q_2 p_1 z^2 - q_1 p_2 z^2\right.$$
$$\left. + q_1 p_2 q_2^{k_1-1} z^{k_1+1} - q_1 p_2 q_2^{k_2} z^{k_2+2}\right)^{-2}. \tag{A.2}$$

Proceeding similarly as above, we can obtain generating function $W^{no}_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)_m$ as given by (7).

For mth overlapping occurrence of the pattern (Theorem 3), proceeding as in Theorem 1, once when state 0^1 is reached, that is, the first occurrence of the required pattern, if the next contiguous trial results in failure then state 1 is directly reached with conditional probability q_1 otherwise it continues to move in state 0^1 with conditional probability p_1 until a failed trial occurs resulting in occurrence of state 1 with conditional probability q_1. Now, on reaching node 1 if the next contiguous trial results in failure then state 2 occurs with conditional probability q_2 otherwise state 0^1 with conditional probability p_2. Again similar procedure is followed for the second occurrence of the required pattern. Thus, for the second occurrence, node 0^1 acts as a starting

Table 1

No.	Paths	Value
1	S to 1^A to 0_1 to 1 to $2\cdots$to k_1 to 0^1	$(q_0z)(q_1z)(p_2z)^2(q_2z)^{k_1-1}$
2	S to 1^A to 0_1 to 1 to $2\cdots$to k_1 to k_1+1 to 0^1	$(q_0z)(q_1z)(p_2z)^2(q_2z)^{k_1}$
3	S to 1^A to 0_1 to 1 to $2\cdots$to k_1 to k_1+1 to k_1+2 to 0^1	$(q_0z)(q_1z)(p_2z)^2(q_2z)^{k_1+1}$
$\vdots$	$\vdots$	$\vdots$
k_2-k_1	S to 1^A to 0_1 to 1 to $2\cdots$to k_1 to…to k_2-1 to 0^1	$(q_0z)(q_1z)(p_2z)^2(q_2z)^{k_2-2}$
k_2-k_1+1	S to 1^A to 0_1 to 1 to $2\cdots$to k_1 to…to k_2-1 to k_2 to 0^1	$(q_0z)(q_1z)(p_2z)^2(q_2z)^{k_2-1}$
k_2-k_1+2	S to 0_1 to 1 to $2\cdots$to k_1 to 0^1	$(p_0z)(q_1z)(p_2z)(q_2z)^{k_1-1}$
k_2-k_1+3	S to 0_1 to 1 to $2\cdots$to k_1 to k_1+1 to 0^1	$(p_0z)(q_1z)(p_2z)(q_2z)^{k_1}$
k_2-k_1+4	S to 0_1 to 1 to $2\cdots$to k_1 to k_1+1 to k_1+2 to 0^1	$(p_0z)(q_1z)(p_2z)(q_2z)^{k_1+1}$
$\vdots$	$\vdots$	$\vdots$
$2k_2-2k_1+1$	S to 0_1 to 1 to $2\cdots$to k_1 to…to k_2-1 to 0^1	$(p_0z)(q_1z)(p_2z)(q_2z)^{k_2-2}$
$2(k_2-k_1+1)$	S to 0_1 to 1 to $2\cdots$to k_1 to…to k_2-1 to k_2 to 0^1	$(p_0z)(q_1z)(p_2z)(q_2z)^{k_2-1}$

Table 2

No.	First order loops	Value
1	1^A to 1^A	(q_2z)
2	0_1 to 0_1	(p_1z)
3	0_1 to 1 to 0_1	$(q_1z)(p_2z)$
4	0_1 to 1 to 2 to 0_1	$(q_1z)(p_2z)(q_2z)$
$\vdots$	$\vdots$	$\vdots$
k_1+1	0_1 to 1 to 2 to…to k_1-1 to 0_1	$(q_1z)(p_2z)(q_2z)^{k_1-2}$
k_1+2	0_1 to 1 to 2 to…to k_1-1 to k_1 to k_2+1 to 0_1	$(q_1z)(p_2z)(q_2z)^{k_2}$
k_1+3	k_2+1 to k_2+1	(q_2z)

Table 3

Second order loops	Value
No. 2 and No. k_1+3	$(p_1z)(q_2z)$
No. 3 and No. k_1+3	$(q_1z)(p_2z)(q_2z)$
No. 4 and No. k_1+3	$(q_1z)(p_2z)(q_2z)^2$
$\vdots$	$\vdots$
No. k_1+1 and No. k_1+3	$(q_1z)(p_2z)(q_2z)^{k_1-1}$

node following a similar procedure as for the first occurrence (except that there is only one path to reach node 1 from node 0^1).

It can be observed that in the GERT network for 2nd overlapping occurrence of the required pattern (Figure 4), there are $2\cdot(k_2-k_1+1)^2$ (in general for any m, there are $2\cdot(k_2-k_1+1)^m$) paths for reaching state 0^2 (0^m) from the starting node S.

Now pairs $(0_1,0^1)$, $(0^1,0^2)$ are identical. Thus, proceeding as in the Theorem 1 and by applying Mason's rule the generating function $W^o_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)_2$ of the waiting time for the 2nd overlapping occurrence of the required pattern is given by

$$W^o_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)_2=\frac{\left(q_0p_2z^2+p_0z\left(1-q_2z\right)\right)\left(q_1p_2q_2^{k_1-1}z^{k_1+1}\left\{1-\left(q_2z\right)^{k_2-k_1+1}\right\}\right)^2}{\left(1-q_2z\right)\left(1-q_2z-p_1z+q_2p_1z^2-q_1p_2z^2-q_1p_2{q_2}^{k_2}z^{k_2+2}+q_1p_2{q_2}^{k_1-1}z^{k_1+1}\right)^2}. \quad (A.3)$$

Similarly, proceeding as above we can obtain generating function $W^{o}_{S\underbrace{FF\cdots F}_{k_1\le k_f\le k_2}S}(0,z)_m$ as given by (9).

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