1983

Distributions of the Duration and Value of Job Search with Learning

Peter Morgan

ISSN: 0318-725X
ISBN: 0-7714-0399-2

RESEARCH REPORT 8302

DISTRIBUTIONS OF THE DURATION AND VALUE OF JOB SEARCH WITH LEARNING*

by

Peter Morgan

Department of Economics
University of Western Ontario
London, Ontario N6A 5C2
Canada

and

Department of Economics
Flinders University of South Australia
Adelaide
Australia 5042

December, 1982

ABSTRACT

An expected value maximizing sequential search rule can be expressed in terms of switchpoint values. In adaptive search problems these switchpoints are ex ante indeterminate. This paper shows, for a wide class of learning procedures, that the adaptive sequential search rule can be re-expressed in terms of ex ante determinate fixed points. This allows the derivation of ex ante (and subsequent) probability functions for the duration of search and the value of the offer eventually accepted by the searcher. Some examples and comparative statics properties of these functions are presented.

1. Introduction

Recently, a number of models have been presented in which the responses of agents to the search activities of other agents sustain a stationary non-degenerate distribution of prices.[1] In a discussion of the usefulness of these models Schwartz and Wilde [22, p. 18] noted that while the models are a response to recognizing that "general equilibrium models yield competitive equilibria only by making strong assumptions about the information available to market participants", these models "make strong assumptions of their own, in particular respecting the methods by which the consumers in them become informed and the nature of consumer expectations". Schwartz and Wilde [22, p. 19] express particular concern over the strong assumption common to the above models that agents "have rational expectations in that they know the true distribution of prices in a market when they begin to search, but do not know which firms charge these prices", and advise that "a useful task, from a theoretical viewpoint, is to see what equilibria would occur in models that drop the rational expectations assumption, assume real world methods of information acquisition and relatively plausible consumer search strategies". The current paper is directed towards this goal.

Dropping the rational expectations assumption complicates the analysis of equilibrium search models considerably. The complications arise because, without this assumption, the search rules employed must include a description of how searchers learn about the offer distribution they face as they sample from it. This complicates efforts to deduce ex ante probability distributions of search length and accepted offers since the quantities (reservation values) commonly used to parameterize these distributions are ex ante unknown when searchers learn about their offer distribution as they search. This paper explains how these ex ante distributions can be calculated for a sequential search model in which searchers learn about their offer distribution. The model examines only the supply side of a labour market and does not explain how the job offer distribution is generated. The paper should be viewed as providing results useful for the construction of an equilibrium sequential search model with learning in which both sides of the labour market are modeled together.

The structure of the paper is as follows. The model is presented in the next section. Section 3 builds on the work of Kohn and Shavell [9] by establishing the existence of a sequence of ex ante determinate fixed points and showing the best sequential search rule is completely defined by these fixed points. Section 4 explains how the fixed points are used to derive the ex ante probability functions of search length and the accepted wage. Section 5 examines some special cases. Section 6 provides some comparative statics results and some concluding comments are offered in Section 7. Most of the proofs are confined to an appendix.

2. The Model

The searcher discussed in this paper wishes to maximize the expected present value (epv) of a job net of the costs of searching sequentially for the job.[2] The searcher enjoys full recall and may sample job offers from a set $N = \{1,\ldots,n\}$ of firms.[3] Firms' offers have values which, for brevity, are called wages $w \in [\underline{w},\bar{w}]$, $0 \le \underline{w} < \bar{w} < \infty$. From the searcher's viewpoint, firms' offers are i.i.d. with a c.d.f. $F(w)$. The implied p.d.f. is $f(w)$. At the outset the searcher has some initial estimate $F_0(w)$ of $F(w)$ and, as he gathers job offers $w_1, w_2, \ldots, w_j$, he can compute revisions $F_j(w|w_1,\ldots,w_j)$ of his initial estimate of $F(w)$. A variety of learning processes and dependencies of $F_j(\cdot)$ upon $w_1,\ldots,w_j$ can be postulated. The specific postulates of this model are that, for $j \geq 1$,

$$F_j(w|w_1,\ldots,w_i',\ldots,w_j) > F_j(w|w_1,\ldots,w_i'',\ldots,w_j) \ \forall w \in (\underline{w},\bar{w}) \tag{2-1}$$

if $w_i' < w_i''$, for $i = 1,\ldots,j$, and that

$$F_j(w|w_1,\ldots,w_j) \text{ is continuous w.r.t. } w_1,\ldots,w_j. \tag{2-2}$$

Loosely speaking, (2-1) means that increasing the value of an observed wage makes the searcher believe it is more likely that better wages will be observed in the future.[4] Notice that (2-1) is consistent with a wide variety of learning processes, some of which are very simple. There is no requirement that these learning processes must cause F_j to converge in probability to F.[5]

Let c_j denote the marginal cost of sampling firm j, j=1,...,n. It is assumed that $c_1,\ldots,c_n$ are independent of the order in which firms are sampled.[6] Without loss of generality, suppose the firms have been labelled with the indices 1,...,n so that

$$c_1 \leq c_2 \leq \cdots \leq c_n. \tag{2-3}$$

Let $\delta \geq 0$ be the searcher's rate of time preference and define $\beta = 1/(1+\delta)$.

Denote the best of the offered wages $w_1,\ldots,w_j$ by $w_j^{max} = \max\{w_1,\ldots,w_j\}$; $j \geq 1$. The problem is to use the stopping rule which maximizes $V_j^n(w_j^{max}, w_1,\ldots,w_j,\beta)$, the searcher's e.p.v. of continuing to search after observing $w_1,\ldots,w_j$, to deduce the ex ante probability functions for his search length and the wage he eventually accepts.

3. The Best Sequential Search Rule

For any given order in which to sample firms, the results of Kohn and Shavell [9] can be applied to express the best stopping rule in terms of switchpoint wages. In search models with learning these switchpoints are ex ante indeterminate and, consequently, are not of direct value in deriving ex ante determinate expressions for the probability distributions of the length of search and the accepted wage.[7] However, the last part of this section establishes the existence and some properties of a sequence of fixed points which can be used to derive ex ante determinate expressions for these probability distributions.

Kohn and Shavell [9, Theorem 4] show that, for any given vector $w_1,\ldots,w_j$ of wage observations,

$$w \gtreqless V_j^n(w,w_1,\ldots,w_j,\beta) \quad \text{as} \quad w \gtreqless s_j(w_1,\ldots,w_j,\beta) \tag{3-1}$$

where $s_j(\cdot)$, the "switchpoint" wage, is unique w.r.t. the given c.d.f. $F_j(w|w_1,\ldots,w_j)$, i.e., $s_j(\cdot)$ is the wage offer for which a searcher who has observed $w_1,\ldots,w_j$ is indifferent between accepting $s_j(\cdot)$ and observing w_{j+1}.[8] $s_j(\cdot)$ is the value of w solving

$$\begin{aligned} w &= V_j^n(w,w_1,\ldots,w_j,\beta) \\ &= -c_{j+1} + \beta E_{f_j}[\max\{w,w_{j+1},V_{j+1}^n(\max\{w,w_{j+1}\},w_1,\ldots,w_{j+1},\beta)\}|w_1,\ldots,w_j]. \end{aligned} \tag{3-2}$$

(3-1) shows that the stopping rule which maximizes the epv of sequential search can be expressed in the form

$$\left\{\begin{array}{l}\text{stop and accept } w_j^{\max} \\ \text{observe } w_{j+1}\end{array}\right\} \text{ as } w_j^{\max} \gtreqless s_j(w_1,\ldots,w_j,\beta). \tag{3-3}$$

This result applies to both static search problems and problems with learning.

Since (3-3) is the form of the best sequential stopping rule for any ordering of $c_1,\ldots,c_n$ it is the form of the rule for the best ordering of $c_1,\ldots,c_n$. For a static problem in which firms make i.i.d. offers, the epv maximizing order in which to sample firms is sample the least costly firm first, the next least costly second, and so on (see Weitzman [24]). This result carries over to the current search problem with learning. A proof of the following proposition is given in the appendix.

Proposition 1: When marginal search costs are independent of the order of search and firms' offers are i.i.d., the best sequential search rule requires that firms be sampled in order of smallest marginal cost.

Proposition 1, (3-2) and (3-3) completely characterize the best sequential search rule. However, the value of $s_j(w_1,\ldots,w_j,\beta)$ is unknown to the searcher until he receives w_j. Thus, although the ex ante probability that search is of length j can be expressed in terms of switchpoints as

$$\Pr\left(\bigcap_{i=1}^{j-1}\{w_i < s_i(w_1,\ldots,w_i,\beta)\} \cap \{w_j \geq s_j(w_1,\ldots,w_j,\beta)\}\right), \tag{3-4}$$

this expression is ex ante indeterminate.

The following results provide information allowing the probabilities (3-4) to be re-expressed in an ex ante determinate form. The first steps are to establish some of the properties of $V_j^n(\cdot)$ w.r.t. $w_1,\ldots,w_j$. These are then used to establish the properties of $s_j(\cdot)$ w.r.t. $w_1,\ldots,w_j$. We begin with

Lemma 1: $V_j^n(w_j^{max},w_1,\ldots,w_j,\beta)$ is continuous w.r.t. w_i, $\forall$ $i=1,\ldots,j$.

Proof: See the appendix.

$V_j^n(\cdot)$, the epv of continuing to search after $w_1,\ldots,w_j$ have been observed, is an expectation computed w.r.t. the p.d.f. $f_j(w|w_1,\ldots,w_j)$. The nature of $V_j^n(\cdot)$'s dependence on w_i thus partly depends upon the nature of the dependence of $f_j(w|w_1,\ldots,w_j)$ upon w_i. Loosely speaking, an increase in any of $w_1,\ldots,w_j$ induces the searcher to believe it is more likely that better wages will be observed in the future and raises his expectation of the present value of continuing to search. This learning effect is reinforced if the increased wage observation also increases the value of his best observed wage. Overall, therefore, the searcher's expectation of the value of continued search is an increasing function of observed wages.

Proposition 2: $V_j^n(w_j^{max},w_1,\ldots,w_j,\beta)$ is a strictly increasing function of w_i for $1 \leq i \leq j \leq n - 1$.

Proof: See the appendix.

$s_j(w_1,\ldots,w_j,\beta)$ is the wage offer needed to make the searcher indifferent between accepting the offer and continuing to search, given he has observed job offer values $w_1,\ldots,w_j$. The above proposition establishes that an increase in any of $w_1,\ldots,w_j$ increases the epv of continuing to search which, in turn, means that the wage offer needed to make the searcher indifferent between continuing to search and accepting the offer must also be larger; i.e., $s_j(w_1,\ldots,w_j,\beta)$ must be an increasing function of $w_1,\ldots,w_j$. This is depicted in Figure 1 for fixed values of $w_1,\ldots,w_{j-1}$. It is also apparent from Figure 1 that increasing w_j from w_j' to w_j'' causes $V_j^n(\cdot)$ to increase continuously from $V_j^n(w,w_1^*,\ldots,w_{j-1}^*,w_j',\beta)$ to $V_j^n(w,w_1^*,\ldots,w_{j-1}^*,w_j'',\beta)$, requiring $s_j(\cdot)$ to increase continuously from $s_j(w_1^*,\ldots,w_{j-1}^*,w_j',\beta)$ to $s_j(w_1^*,\ldots,w_{j-1}^*,w_j'',\beta)$. These conclusions are depicted in Figure 2 and stated in the following proposition.

Proposition 3: For $j = 1,\ldots,n-1$ the switchpoint $s_j(w_1,\ldots,w_j,\beta)$ is a strictly increasing and continuous function of $w_1,\ldots,w_j$.

Proof: See the appendix.

Figure 1

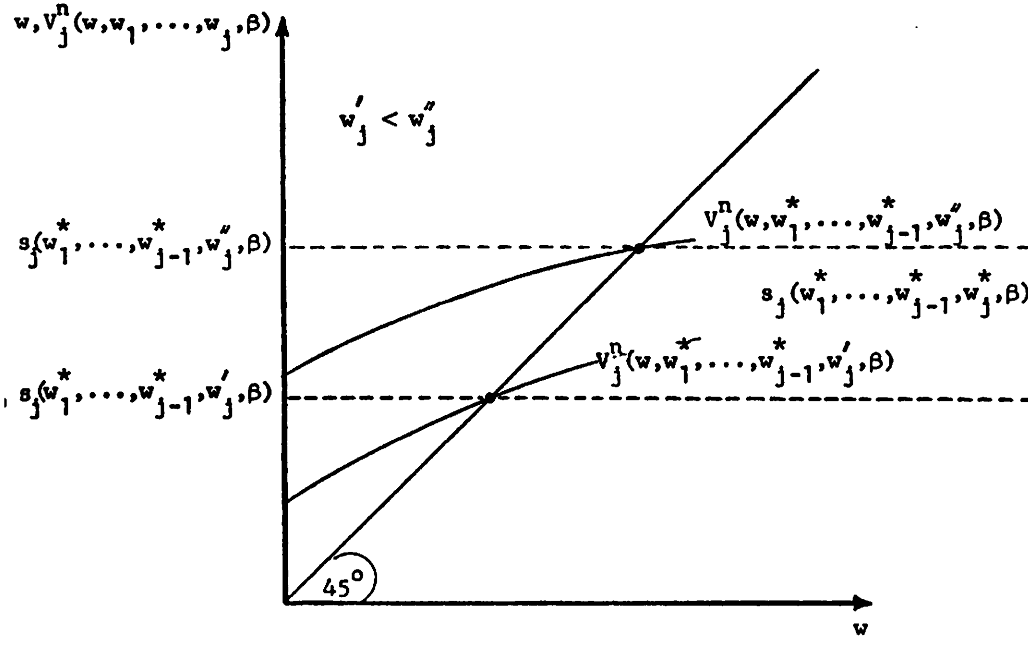
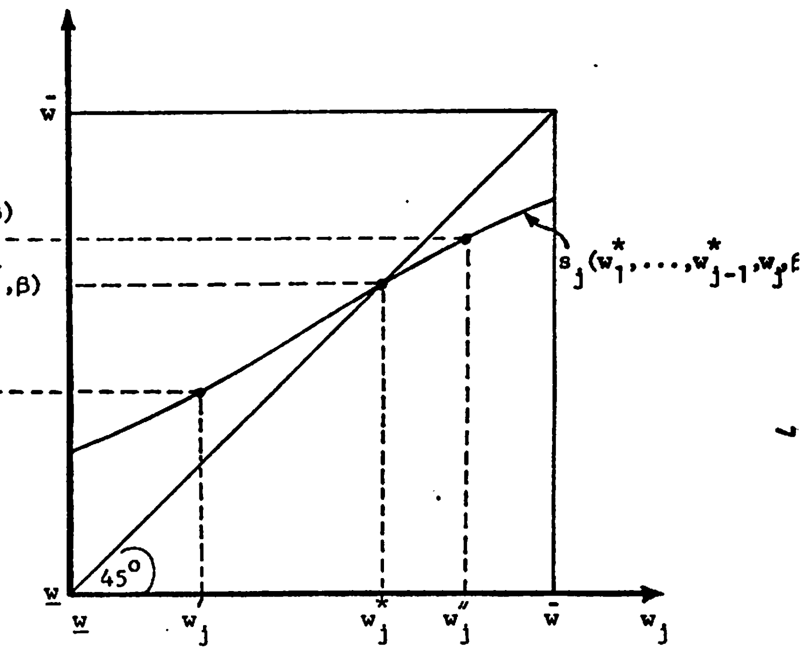


Figure 2



We are now in a position to establish the existence of the fixed points $w_1^*,\ldots,w_{n-1}^*$ which are used to derive the ex ante probability distributions of the search length and the accepted wage. To illustrate the procedure consider the searcher's stopping problem after receiving his first wage offer w_1 (a visual depiction of this problem is obtained by setting j=1 in Figure 2). His best action is to

$$\begin{Bmatrix} \text{stop and accept } w_1^{max} \\ \text{observe } w_2 \end{Bmatrix} \text{ as } w_1^{max} \equiv w_1 \gtrless s_1(w_1,\beta). \tag{3-5}$$

But Figure 2 suggests (3-5) is equivalent to the stopping rule

$$\begin{Bmatrix} \text{stop and accept } w_1^{max} \\ \text{observe } w_2 \end{Bmatrix} \text{ as } w_1 \gtrless w_1^*, \text{ where } w_1^* = s_1(w_1^*,\beta). \tag{3-6}$$

The importance difference between (3-5) and (3-6) is that the fixed point w_1^* can be evaluated ex ante, whilst the switchpoint $s_1(w_1,\beta)$ can be evaluated only once w_1 has been received. This allows us to write the probability that the search length is unity in the ex ante determinate form $\Pr(w_1 \geq w_1^*)$, rather than in the ex ante indeterminate form $\Pr(w_1 \geq s_1(w_1,\beta))$. The ex ante probabilities that search is of length $j > 1$ can all be written in terms of fixed points similar to w_1^*. This more complex task is postponed to the next section. The remainder of this section is devoted to establishing the existence of $\{w_j^*\}_{j=1}^{n-1}$.

<u>Definition</u>: $S_j(w_1,\ldots,w_j,\beta) = \max\{\underline{w},s_j(w_1,\ldots,w_j,\beta)\}$ for $j=1,\ldots,n-1$. (3-7)

An immediate consequence of Proposition 3 is

<u>Remark 1</u>: For $j=1,\ldots,n-1$, $S_j(w_1,\ldots,w_j,\beta)$ is a non-decreasing and continuous function of $w_1,\ldots,w_j$.

Notice that $S_j(w_1,\ldots,w_j,\beta)$ is not strictly increasing w.r.t. $w_1,\ldots,w_j$ even though $s_j(w_1,\ldots,w_j,\beta)$ is strictly increasing. The reason is that sufficiently high search costs or rates of time preference will lower $s_j(\cdot)$ below $\underline{w}$.

Remark 2: S_j is bounded below by $\underline{w}$ and bounded above by $\bar{w}$.

S_j is bounded below by $\underline{w}$ by definition (3-7). S_j is bounded above by $\bar{w}$ since $s_j(\cdot)$ is the value of a wage offer which makes the searcher indifferent between accepting $s_j(\cdot)$ and continuing his search. Since search costs are positive, a wage offer valued at $\bar{w}$ will always stop search; i.e.,

$$\bar{w} > s_j(\cdot) \Rightarrow \bar{w} > \max\{\underline{w}, s_j(\cdot)\} = S_j(\cdot) \tag{3-8}$$

Definition: w_1^* is a value of w_1 satisfying

$$w_1 = S_1(w_1,\beta) \tag{3-9}$$

and, for $j=2,\ldots,n-1$, w_j^* is a value of w_j satisfying

$$w_j = S_j(w_1^*,\ldots,w_{j-1}^*,w_j,\beta). \tag{3-10}$$

Theorem 1: $w_1^*,\ldots,w_{n-1}^*$ exist.

Proof: The proof begins by establishing the existence of w_1^*. Forward induction is then used to establish the existence of $w_2^*,\ldots,w_{n-1}^*$.

For $j=1$: From Remarks 1 and 2,

$$S_1 : [\underline{w},\bar{w}] \rightarrow [\underline{w},\bar{w}].$$

We can apply Brouwer's Theorem to establish the existence of w_1^* since S_1 is continuous (Remark 1) and since $[\underline{w},\bar{w}]$ is a compact, convex, non-empty set.

For $j=2,\ldots,n-1$: Given $w_1^*,\ldots,w_{j-1}^*$, Remarks 1 and 2 show

$$S_j(w_1^*,\ldots,w_{j-1}^*,w_j,\beta): \ [\underline{w},\bar{w}] \rightarrow [\underline{w},\bar{w}] \text{ and is continuous w.r.t. } w_j.$$

Applying Brouwer's Theorem proves the existence of w_j^*, completes the induction step and completes the proof. Figure 2 illustrates the result. Q.E.D.

It was noted earlier that w_1^* is ex ante determinate. Setting $j=2$ in (3-10) shows that, since w_1^* is known ex ante, so is w_2^* which, in turn, makes w_3^* ex ante determinate and so on.

In general there may be more than one fixed point solution to any of (3-9) and (3-10). If each of $w_1^*,\ldots,w_{n-1}^*$ is unique then the searcher's stopping rule possesses the reservation value property (see Rothschild [19], Rosenfield and Shapiro [18]). If one or more of $w_1^*,\ldots,w_{n-1}^*$ are non-unique then the ex ante probability functions for search length and the value of the accepted job offer can still be derived in a manner similar to that described in Section 4, but these derivations become significantly more involved. A condition sufficient for each of $w_1^*,\ldots,w_{n-1}^*$ to be unique is that $S_j(\cdot)$ always increases w.r.t. w_j at a rate of less than unity in the neighborhood of w_j^* when $(w_1,\ldots,w_{j-1}) = (w_1^*,\ldots,w_{j-1}^*)$; i.e., for all $j=1,\ldots,n-1$,

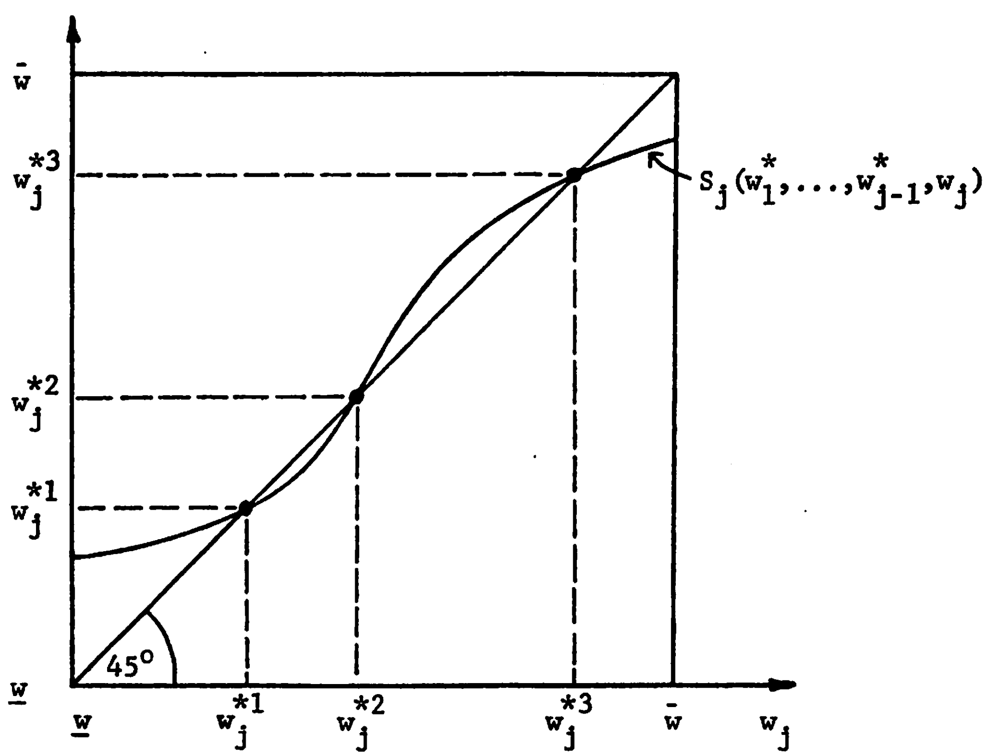
$$S_j(w_1^*,\ldots,w_{j-1}^*,w_j'',\beta) - S_j(w_1^*,\ldots,w_{j-1}^*,w_j',\beta) < w_j'' - w_j' \quad \text{for any } w_j' < w_j'' \text{ in the neighborhood of } w_j^*. \tag{3-11}$$

(3-11) is not a severe restriction. Firstly, since $\underline{w} \le S_j(\cdot) < \bar{w}$ the average rate of increase of S_j w.r.t. w_j is strictly less than unity across the interval $[\underline{w},\bar{w}]$ even if some local rates of increase are greater than unity. Secondly, (3-11) does not constrain $s_j(\cdot)$ to always increase w.r.t. w_j at a rate of less than unity since (i) S_j may not always equal s_j and (ii) (3-11) is a condition concerning only the neighbourhoods of $w_1^*,\ldots,w_j^*$, so s_j may increase w.r.t. w_j at rates in excess of unity for values of s_j below $\underline{w}$ or for $(w_1,\ldots,w_j)$ not in the neighbourhood of $(w_1^*,\ldots,w_j^*)$.[9] (3-11) quarantees uniqueness for $w_1^*,\ldots,w_j^*$ by ruling out fixed points such as w_j^{*2} in Figure 3 which violate (3-11) by requiring for their existence that $S_j(w_1^*,\ldots,w_{j-1}^*,w_j)$ increase w.r.t. w_j at a rate greater than unity in the neighborhood of w_j^{*2}. A second important implication of (3-11) is that, for $j=1,\ldots,n-1$,

$$w_j \gtreqless S_j(w_1^*,\ldots,w_{j-1}^*,w_j,\beta) \text{ as } w_j \gtreqless w_j^* . \tag{3-12}$$

We are now in a position to rewrite the searcher's best sequential search rule in terms of the ex ante determinate fixed points $w_1^*,\ldots,w_{n-1}^*$ and to derive the ex ante probability functions of the search length and the accepted wage.

Figure 3



4. The Distributions of Search Lengths and Accepted Wages

Care must be taken to distinguish the ex ante probability functions derived in this section from other probability functions of search length or accepted wages which can be defined within the context of the present search problem. Ex ante, the searcher will possess his own probability functions of search lengths or accepted wages. These are calculated using his ex ante estimate $f_o(w)$ of the true p.d.f. of wage offers from which he will sample, $f(w)$. Since typically $f_o(w) \neq f(w)$, the searcher's own ex ante probability functions will generally differ from those which state the true ex ante probabilities of search lengths and accepted wages. It is the latter true probability functions which are derived in this section. These functions must also be distinguished from the sequence of conditional probability functions for search length and the accepted wage which can be computed once one or more observations have been gathered. For example, the p.m.f. of search lengths once w_1 has been observed will usually differ from the ex ante p.m.f. of search lengths derived here.

The ex ante probability functions of search lengths and accepted wages generated by using an epv maximizing sequential search rule must take account of two complications not present in fixed-sample-size search models.[10] The first of these is that the value of the best wage offered w_j^{max} depends upon the search length j, a random variable with a distribution partly dependent upon the value of w_j^{max}. The second complication is that the searcher is not indifferent between the orders in which a given set of wage offers may be gathered. For instance, suppose firms offer wages of either \$10,000/year or \$20,000/year. A searcher who is at first offered \$10,000 may well choose to sample a second firm. Suppose the second firm's offer is \$20,000. Then the searcher will stop and accept the \$20,000 offer. This search resulted in the set of offers {\$10,000, \$20,000} being obtained in the order w_1 = \$10,000, w_2 = \$20,000. The reverse order of w_1 = \$20,000, w_2 = \$10,000 is inconsistent with epv maximization.

The searcher's initial decision is to search or not. The interesting problems are those in which at least one firm is sampled so it will be assumed throughout this section that this occurs.

Theorem 2: The ex ante p.m.f. of search lengths is

$$g(j) = \begin{cases} 1 - F(w_1^*); \ j=1 \\ \prod_{i=1}^{j-1} F(\min[w_i^*,\ldots,w_{j-1}^*]) - \prod_{i=1}^{j} F(\min[w_i^*,\ldots,w_j^*]); \ 2 \le j \le n-1 \\ \prod_{i=1}^{n-1} F(\min[w_i^*,\ldots,w_{n-1}^*]); \ j=n \end{cases}$$

The full proof of this theorem is fatiguing and is given in the appendix. To assist the reader in understanding the result, the cases for n=2,3 are now described in some detail.

(3-12) with j = 1 quickly shows that for the duopoly case of n = 2

$$g(1) = \Pr(w_1^{max} \ge S_1(w_1,\beta)) = \Pr(w_1 \ge w_1^*) = 1 - F(w_1^*) \quad (4\text{-}1)$$

and

$$g(2) = \Pr(w_1^{max} < S_1(w_1,\beta)) = \Pr(w_1 < w_1^*) = F(w_1^*). \quad (4\text{-}2)$$

The case of n = 3 is a little more complex but offers an opportunity to explain the structure of the full proof for all cases with n ≥ 3 with a minimum of fuss. The proof proceeds by deriving the cumulative probabilities that more than j wage offers are observed for j = 1,2. Beginning with j = 1 we have from (3-12) that

$$g(1) = \Pr(w_1^{max} \geq S_1(w_1,\beta)) = \Pr(w_1 \geq w_1^*) = 1 - F(w_1^*) \tag{4-3}$$

so the cumulative probability that more than one wage offer is observed is

$$g(2) + g(3) = 1 - g(1) = F(w_1^*) . \tag{4-4}$$

Moving to j = 2 we have that the cumulative probability that more than two wage offers are observed is

$$g(3) = \Pr(w_1 < w_1^*, w_2^{max} < S_2(w_1,w_2,\beta)) . \tag{4-5}$$

The essence of the proof is to show that

$$w_1 < w_1^* \text{ and } w_2^{max} < S_2(w_1,w_2,\beta) \text{ iff } w_1 < \min[w_1^*,w_2^*] \text{ and } w_2 < w_2^* . \tag{4-6}$$

The reader will find Figures 4 and 5 of visual assistance in understanding the argument establishing (4-6). In these figures w_1' and w_2' are used to denote values for w_1 and w_2 for which the searcher will observe w_3. First of all, notice that the event $w_2^{max} < S_2(w_1,w_2,\beta)$ is possible iff $S_2(w_1,w_2,\beta) > \underline{w}$ which, by definition (3-7), allows us to write

$$w_1 < w_1^* \text{ and } w_2^{max} < S_2(w_1,w_2,\beta) \Leftrightarrow w_1 < w_1^* \text{ and } w_2^{max} < s_2(w_1,w_2,\beta) \tag{4-7}$$

$$\Leftrightarrow w_1 < w_1^* \text{ and } w_1 < s_2(w_1,w_2,\beta) \text{ and } w_2 < s_2(w_1,w_2,\beta) . \tag{4-8}$$

However, Proposition 3 tells us that $s_2(\cdot)$ is strictly increasing w.r.t. w_1, so

$$w_1 < w_1^* \Leftrightarrow s_2(w_1,w_2,\beta) < s_2(w_1^*,w_2,\beta) . \tag{4-9}$$

Combining (4-8) and (4-9) gives

$$w_1 < w_1^* \text{ and } w_2^{max} < S_2(w_1,w_2,\beta) \Leftrightarrow w_1 < w_1^* \text{ and } w_1 < s_2(w_1^*,w_2,\beta) \text{ and}$$

$$w_2 < s_2(w_1^*,w_2,\beta) \ . \tag{4-10}$$

But (3-12) tells us (remember $S_2(\cdot) = s_2(\cdot)$) that

$$w_2 < s_2(w_1^*,w_2,\beta) \Leftrightarrow w_2 < w_2^* \tag{4-11}$$

and, since Proposition 3 tells us that $s_2(\cdot)$ is strictly increasing w.r.t. w_1,

$$w_2 < w_2^* \Leftrightarrow s_2(w_1^*,w_2,\beta) < s_2(w_1^*,w_2^*,\beta) \ . \tag{4-12}$$

Using (4-11) and (4-12) in (4-10) gives

$$w_1 < w_1^* \text{ and } w_2^{max} < S_2(w_1,w_2,\beta) \Leftrightarrow w_1 < w_1^* \text{ and } w_1 < s_2(w_1^*,w_2^*,\beta) \text{ and}$$

$$w_2 < s_2(w_1^*,w_2^*,\beta) \ . \tag{4-13}$$

However, by definition (3-10), $w_2^* = s_2(w_1^*,w_2^*,\beta)$ so (4-13) is

$$w_1 < w_1^* \text{ and } w_2^{max} < S_2(w_1,w_2,\beta) \Leftrightarrow w_1 < w_1^* \text{ and } w_1 < w_2^* \text{ and } w_2 < w_2^*$$

$$\Leftrightarrow w_1 < \min[w_1^*,w_2^*] \text{ and } w_2 < w_2^* \ . \tag{4-14}$$

(4-14) allows us to write (4-5) as

$$g(3) = \Pr(w_1 < \min[w_1^*,w_2^*], w_2 < w_2^*) = F(\min[w_1^*,w_2^*])F(w_2^*) \ . \tag{4-15}$$

To complete the proof for n = 3 we use (4-4) and (4-15) to write

$$g(2) = F(w_1^*) - F(\min[w_1^*,w_2^*])F(w_2^*) \ . \tag{4-16}$$

(4-3), (4-15) and (4-16) together describe g(j) for n = 3. The proof for cases with n > 3 are higher dimensional extensions of the logic presented from (4-6) to (4-16).

Figure 4

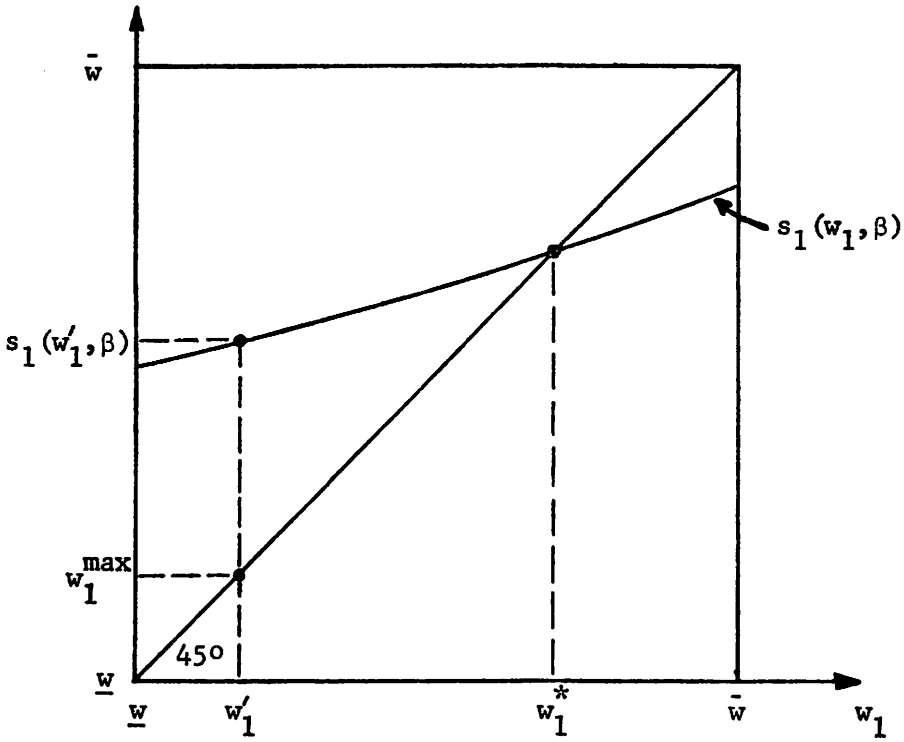
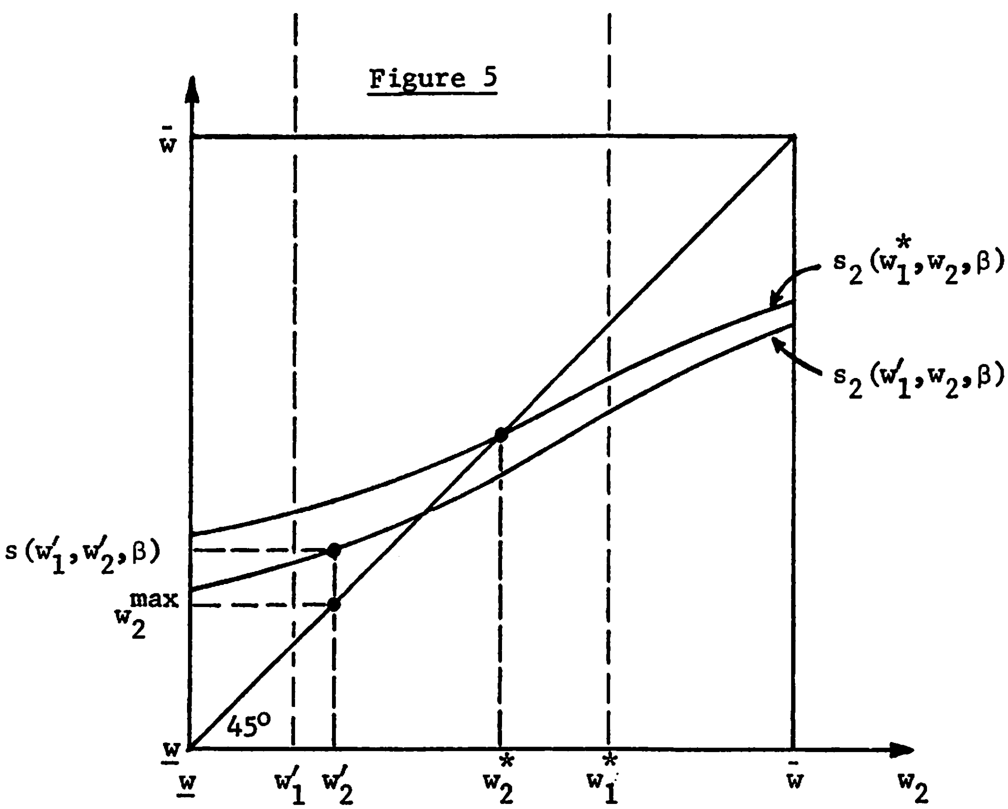


Figure 5



Obviously the c.d.f. of the wage eventually accepted by the searcher, $H(w)$, depends upon the p.m.f. of search lengths $g(j)$ in some fashion. The simplest case arises when $n = 1$ since then $g(1) = 1$ and so $H(w) = F(w)$. To illustrate the idea further consider the case of $n = 2$. Then

$$\begin{aligned} H(w) &= \Pr(w_1^{max} \le w \text{ and } j=1) + \Pr(w_2^{max} \le w \text{ and } j=2) \\ &= \Pr(w_1 \le w \text{ and } w_1 \ge w_1^*) + \Pr(w_1 \le w \text{ and } w_2 \le w \text{ and } w_1 < w_1^*) \\ &= \Pr(\min[w_1^*,w] \le w_1 \le w) + \Pr(w_1 \le \min[w_1^*,w] \text{ and } w_2 \le w) \\ &= F(w) - F(\min[w_1^*,w]) + F(\min[w_1^*,w])F(w) \\ &= F(w) - (1 - F(w))F(\min[w_1^*,w]) \ . \end{aligned} \tag{4-17}$$

The proof for the general case is given in the appendix.

<u>Theorem 3</u>: The ex ante c.d.f. of the accepted wage is

$$H(w) = \begin{cases} F(w)\,; & n=1 \\ F(w) - (1-F(w)) \sum_{j=2}^{n} \prod_{i=1}^{j-1} F(\min[w_i^*,\ldots,w_{j-1}^*,w])\,; & n \ge 2 \ . \end{cases}$$

Earlier in this section it was noted that the probability functions of search length and the accepted wage which are conditional upon some observations having been received will typically differ from the above ex ante probability functions. However, the searcher's problem after gathering some observations can be thought of as an ex ante problem commencing at the instant these observations are received. Consequently these conditional probability functions will have the same form and can be derived in the same way as the ex ante probability functions. The fixed point values will, of course, typically be different from their ex ante values.

5. Some Examples

In this section the ex ante probability functions of search length and the accepted wage are computed for a particular search problem with learning. Then the results of the previous section are related to some well-known static search problems.

Suppose a searcher is aware that job offers are distributed with the exponential p.d.f.

$$f(w|\mu) = \mu e^{-\mu w}, \text{ for } w > 0 \tag{5-1}$$

and that the searcher has a prior exponential p.d.f. over μ of

$$p_o(\mu|\beta) = \beta e^{-\beta\mu}, \text{ for } \mu > 0 . \tag{5-2}$$

The searcher's initial estimate of (5-1) is

$$f_o(w) = \int_o^\infty \mu e^{-\mu w} \beta e^{-\beta\mu} d\mu = \frac{\beta}{(\beta+w)^2} , \text{ for } w > 0 . \tag{5-3}$$

The searcher can sample either of two firms and it is assumed that the opportunity cost of search is low enough to induce him to sample at least firm 1, which will offer him w_1. When he uses w_1 to update p_o in the Bayesian manner he will obtain the posterior gamma p.d.f. (see De Groot [5, Theorem 3, p. 166]),

$$p_1(\mu|\beta,w_1) = (\beta+w_1)^2 \mu e^{-(\beta+w_1)\mu} , \text{ for } \mu > 0 , \tag{5-4}$$

giving him a revised estimate for the true offer p.d.f. (5-1) of

$$f_1(w|w_1) = \int_o^\infty \mu e^{-\mu w} (\beta+w_1)^2 \mu e^{-(\beta+w_1)\mu} d\mu = \frac{2(\beta+w_1)^2}{(\beta+w_1+w)^3}, \text{ for } w > 0 . \tag{5-5}$$

Notice that $F_1(w|w_1) = 1 - (\frac{\beta + w_1}{\beta + w_1 + w})^2$ is strictly decreasing and continuous w.r.t. $w_1 > 0$, satisfying assumptions (2-1) and (2-2).

Using (5-5) the searcher can compute the epv of continuing to search when he can recall w_1. This epv is $E_{f_1}[\max\{w_1, w_2\}|w_1] - c_2$ and, by (3-1) and (3-2),

$$w_1 \gtreqless -c_2 + E_{f_1}[\max\{w_1, w_2\}|w_1] \text{ as } w_1 \gtreqless s_1(w_1) \ , \tag{5-6}$$

where

$$s_1(w_1) = -c_2 + E_{f_1}[\max\{s_1(w_1), w_2\}|w_1] \ . \tag{5-7}$$

Using (5-5),

$$E_{f_1}[\max\{s_1(w_1), w_2\}|w_1] = s_1(\cdot)\int_0^{s_1} \frac{2(\beta+w_1)^2}{(\beta+w_1+w_2)^3}\, dw_2 + \int_{s_1}^{\infty} \frac{2(\beta+w_1)^2 w_2}{(\beta+w_1+w_2)^3}\, dw_2 \tag{5-8}$$

$$= s_1(w_1) + \frac{(\beta+w_1)^2}{\beta+w_1+s_1(w_1)} \ . \tag{5-9}$$

Using (5-7) and (5-9) to solve for $s_1(w_1)$ gives

$$s_1(w_1) = (\beta + w_1)(\beta + w_1 - c_2)/c_2 \ . \tag{5-10}$$

The searcher will therefore

$$\left\{\begin{array}{l} \text{stop and accept } w_1 \\ \\ \text{observe } w_2 \end{array}\right\} \text{ as } w_1 \gtrless (\beta + w_1)(\beta + w_1 - c_2)/c_2 \ . \tag{5-11}$$

Solving for w_1^* from (5-11) as an equality allows (5-11) to be rewritten as

$$\left\{\begin{array}{l}\text{stop and accept } w_1 \\ \\ \text{observe } w_2\end{array}\right\} \text{ as } w_1 \begin{array}{c}\in \\ \notin\end{array} [c_2 - \beta_2 - \sqrt{c_2(c_2-\beta_2)},\ c_2 - \beta_2 + \sqrt{c_2(c_2-\beta_2)}] \quad (5\text{-}12)$$

w_1^* is not unique in this case since $s_1(w_1)$ is quadratic in w_1, causing $\partial s_1(w_1)/\partial w_1 > 1$ for sufficiently large w_1.

Suppose that, in our example, $\mu = 0.01$, $c_2 = 10$ and $\beta = 0.01$. Then (5-12) is

$$\left\{\begin{array}{l}\text{stop and accept } w_1 \\ \\ \text{observe } w_2\end{array}\right\} \text{ as } w_1 \begin{array}{c}\in \\ \notin\end{array} [-0.005,\ 19.985] \quad (5\text{-}13)$$

The true ex ante p.m.f. of search lengths is therefore

$$g(j) = \begin{cases}\int_o^{19.985} 0.01e^{-0.01w}dw = 1 - e^{-0.19985} = 0.181;\ j=1 \\ \\ \int_{19.985}^{\infty} 0.01e^{-0.01w}dw = e^{-0.19985} = 0.819;\ j=2\ .\end{cases} \quad (5\text{-}14)$$

The searcher's own ex ante estimate of $g(j)$ will differ from (5-14) since his estimate will be based upon $f_o(w)$ and $f_1(w|w_1)$, which differ from $f(w)$ (c.f.(5-1), (5-3) and (5-5)). The true ex ante c.d.f. of the accepted wage is

$$H(w) = F(\min\{w,19.985\}) + F(w)(F(w) - F(\min\{w,19.985\})) \quad (5\text{-}15)$$

$$= (1 - e^{-0.01\min\{w,19.985\}})e^{-0.01w} + (1 - e^{-0.01w})^2\ . \quad (5\text{-}16)$$

We now use the results of the previous section to derive the ex ante probability functions of search length and the accepted offer for some well-known static search problems. In static search problems the switchpoints $s_j(\cdot)$ are independent of past observations since the searcher is fully informed, and thus does not learn, about the offer p.d.f. f(w). Hence, from (3-8) and (3-9),

$$w_j^* \equiv S_j(w_1,\ldots,w_j) = \max\{\underline{w}, s_j(w_1,\ldots,w_j)\},\ \forall\, w_1,\ldots,w_j;\ j\geq 1\ . \tag{5-17}$$

Weitzman [24] examined a model in which all offers were independently distributed with known p.d.f.'s and proved that the best order in which to sequentially sample firms is in the order of non-increasing individual reservation value. In the notation of the current model, Weitzman's rule yields

$$s_j(\cdot) = z_j,\ \text{for } j=1,\ldots,n \tag{5-18}$$

where z_j is the individual reservation value of the j^{th} firm, and that firms will be sampled in the order $1,\ldots,n$ satisfying

$$z_1 \geq z_2 \geq \ldots \geq z_n\ . \tag{5-19}$$

Substituting (5-17), (5-18) and (5-19) into Theorems 2 and 3 shows that the ex ante p.m.f. of search lengths is

$$g(j) = \begin{cases} 1 - F(z_1);\ j=1 \\ F(z_{j-1})^{j-1} - F(z_j)^j;\ 2 \leq j \leq n-1 \\ F(z_{n-1})^{n-1};\ j=n \end{cases} \tag{5-20}$$

and the c.d.f. of the accepted offer is

$$H(w) = \begin{cases} F(w);\ n=1 \\ F(w) + (1 - F(w)) \sum_{j=2}^{n} F(\min[z_{j-1}, w])^{j-1};\ n \geq 2 \end{cases} \quad (5\text{-}21)$$

(5-20) has also been derived by Reinganum [17, appendix]. In the simplest case of all where $c_1 = \ldots = c_n$, the individual reservation values are all equal, reducing (5-20) and (5-21) to

$$g(j) = \begin{cases} 1 - F(z);\ j=1 \\ F(z)^{j-1}(1 - F(z));\ 2 \leq j \leq n-1 \\ F(z)^{n-1};\ j=n \end{cases} \quad (5\text{-}22)$$

and

$$H(w) = \begin{cases} F(w);\ n=1 \\ F(w) - (1 - F(w)) \sum_{j=2}^{n} F(\min[z,w])^{j-1};\ n \geq 2 \end{cases} \quad (5\text{-}23)$$

Letting $n \to \infty$ shows the limit of (5-22) is the infinite sequence of Bernouilli trials with parameter F(z) discussed by McCall [12]. The accompanying limit of (5-23) is

$$H(w) = \frac{F(w) - F(\min[z,w])}{1 - F(\min[z,w])} \quad . \quad (5\text{-}24)$$

6. Comparative Statics

This section examines the effects upon $H(w)$ and $g(j)$ of changes to discount rates and marginal search costs. The common intuition on these effects is that increases in the searcher's rate of time preference or marginal search costs reduce the amount of search undertaken and, thereby, reduce the average value of accepted job offers and increase wage dispersion. The current model confirms the first two of these predictions but shows the third is not always true.

Since the probability functions of search length and the accepted offer are parameterized by $w_1^*,\ldots,w_{n-1}^*$ the first step in establishing the comparative statics properties of these probability functions is to determine how $w_1^*,\ldots,w_{n-1}^*$ depend upon $c_2,\ldots,c_n$ and β. Changes in c_1 do not affect $w_1^*,\ldots,w_{n-1}^*$ because it is assumed that at least firm 1 is always sampled, giving c_1 the character of a fixed cost. Sufficiently large rates of time preference or marginal search costs will make the epv of continued search so small as to generate a value for $w_1^* = \underline{w}$. In these search problems only the first firm is sampled and it is a trivial matter to show that small changes to any of $c_2,\ldots,c_n$ or β have no effect upon $H(w)$ or $g(j)$. While all of the comparative statics results of this section are stated as weak inequalities in recognition of this possibility, the remainder of this section is concerned with the more interesting class of search problems in which β and $c_2,\ldots,c_n$ generate fixed point values larger than $\underline{w}$ for at least w_1^*. To simplify and abbreviate the presentation of the comparative statics results it is assumed that the switchpoints $s_j(\cdot)$ are differentiable.

Lemma 2: w_j^* is a non-increasing function of $c_2,\ldots,c_n$ and β, for $j=1,\ldots,n-1$.

Proof: Let α be a general parameter used to denote any of $c_2,\ldots,c_n$ or β. We begin with $j=1$. By definition (3-9),

$$\frac{\partial w_1^*}{\partial \alpha} = \frac{\partial s_1/\partial\alpha}{1-\partial s_1/\partial w_1} \tag{6-1}$$

(3-11) requires

$$0 < 1 - \frac{\partial s_1}{\partial w_1} \leq 1 \ . \tag{6-2}$$

s_1 is a non-increasing function of $c_2,\ldots,c_n$, β so

$$\frac{\partial w_1^*}{\partial \alpha} \leq 0, \text{ for } \alpha \in \{c_2,\ldots,c_n,\beta\} \ . \tag{6-3}$$

We now proceed by forward induction. For any $j=2,\ldots,n-1$, assume

$$\frac{\partial w_i^*}{\partial \alpha} \leq 0, \text{ for } i=1,\ldots,j \ . \tag{6-4}$$

By definition (3-10),

$$\frac{\partial w_{j+1}^*}{\partial \alpha} = \frac{\dfrac{\partial s_{j+1}}{\partial \alpha} + \sum\limits_{i=1}^{j} \dfrac{\partial s_{j+1}}{\partial w_i} \cdot \dfrac{\partial w_i^*}{\partial \alpha}}{1 - \dfrac{\partial s_{j+1}}{\partial w_{j+1}}} \tag{6-5}$$

(6-4) and Proposition 3 together imply

$$\frac{\partial s_{j+1}}{\partial w_i} \cdot \frac{\partial w_i^*}{\partial \alpha} \leq 0 \ , \text{ for } i=1,\ldots,j. \tag{6-6}$$

(3-11) requires

$$0 < 1 - \frac{\partial s_{j+1}(w_1^*,\ldots,w_j^*,w_{j+1})}{\partial w_{j+1}} \leq 1 \ . \tag{6-7}$$

$s_{j+1}(\cdot)$ is a non-increasing function of $c_{j+2},\ldots,c_n,\beta$ which, with (6-5), (6-6) and (6-7), establishes

$$\frac{\partial w^*_{j+1}}{\partial \alpha} \le 0, \text{ for } \alpha \in \{c_{j+2},\ldots,c_n,\beta\} \ . \qquad (6\text{-}8)$$

This completes the induction step and the proof.

Q.E.D.

The above result immediately yields the following comparative statics results.

Proposition 4: For each $k = 1,\ldots,n-1$, the ex ante probability that more than k firms are sampled is a non-increasing function of $c_2,\ldots,c_n$ and β.

Proof: Appendix equation (A-28) is

$$\sum_{j=k+1}^{n} g(j) = \prod_{j=1}^{k} F(\min[w^*_j,\ldots,w^*_k]), \qquad (A\text{-}28)$$

showing $\sum_{j=k+1}^{n} g(j)$ is a non-decreasing function of $w^*_1,\ldots,w^*_k$. The result now follows immediately from Lemma 2.

Q.E.D.

Corollary 1: The expected search length is a non-increasing function of $c_2,\ldots,c_n$ and β.

Proof: $$E[j] = \sum_{j=1}^{n} jg(j) = \sum_{j=1}^{n} j\Big(\sum_{x=j}^{n} g(x) - \sum_{x=j+1}^{n} g(x)\Big) = \sum_{j=1}^{n} \sum_{x=j}^{n} g(x) \ . \qquad (6\text{-}9)$$

The result follows immediately from Proposition 4.

Q.E.D.

Proposition 5: $H(w)$ is a non-decreasing function of $c_2,\ldots,c_n$ and β.[11]

Proof: The result is trivially true for $n=1$ since then $H(w) = F(w)$, which is independent of $c_2,\ldots,c_n$ and β. For $n \ge 2$, note from Theorem 3 that $H(w)$

is non-increasing w.r.t. $w_1^*,\ldots,w_{n-1}^*$. The result now follows from Lemma 2.

Q.E.D.

Corollary 2: The expected value of the accepted wage is a non-increasing function of $c_2,\ldots,c_n$ and β.

Proof: The result follows immediately from Proposition 5 and the fact that

$$E_h[w] = \int_{\underline{w}}^{\bar{w}} w \frac{\partial H(w)}{dw} dw = \bar{w} - \int_{\underline{w}}^{\bar{w}} H(w)dw \ . \tag{6-10}$$

Q.E.D.

The comparative statics results obtained so far are comforting confirmations, in the context of a sequential search model with learning, of results obtained from static sequential search models. However, the intuition that changes in $c_2,\ldots,c_n$ or β which lead to increases in the probabilities of longer sequential search also lead to reduced wage dispersion is not generally true. While formal arguments can be given in support of this assertion, it is simpler to proceed by means of a counter-example to the above intuition.

The commonly employed measures of wage dispersion are the variance and the coefficient of variation of the p.d.f. of the accepted wage. Suppose n=3 and that the probability function of firms' offers is

$$f(w) = \begin{cases} 0.2; & w=\$20000 \\ 0.8; & w=\$10000 \ . \end{cases} \tag{6-11}$$

Also suppose that c_2, c_3 and β are such that $w_1^* = w_2^* = \$10000$. Then the probability functions of search length and the accepted wage are

$$g(j) = \begin{cases} 1.0; & j=1 \\ 0.0; & j=2,3 \end{cases} \quad \text{and} \quad h(w) = \begin{cases} 0.2; & w=\$20000 \\ 0.8; & w=\$10000 \end{cases} . \tag{6-12}$$

The implied means, variances and coefficients of variation are

$$\begin{aligned} E_g[j] &= 1.0 & \qquad & E_h[w] = \$12000 \\ var_g(j) &= 0.0 & \text{and} \qquad & var_h(w) = 16 \times 10^6 \\ cv_g(j) &= 0.0 & \qquad & cv_h(w) = 0.222 \end{aligned} \tag{6-13}$$

Now suppose a reduction in c_2 results in $w_1^* = \$12000$, $w_2^* = \$10000$. Then (6-12) alters to

$$g(j) = \begin{cases} 0.8; & j=1 \\ 0.2; & j=2 \\ 0.0; & j=3 \end{cases} \quad \text{and} \quad h(w) = \begin{cases} 0.36; & w=\$20000 \\ 0.64; & w=\$10000 \end{cases} \tag{6-14}$$

and (6-13) alters to

$$\begin{aligned} E_g[j] &= 1.2 & \qquad & E_h[w] = \$13600 \\ var_g(j) &= 0.16 & \text{and} \qquad & var_h(w) = 23.04 \times 10^6 \\ cv_g(j) &= 0.33 & \qquad & cv_h(w) = 0.353 \end{aligned} \tag{6-15}$$

Notice that reducing c_2 has increased both the variances and the coefficients of variation of both g(j) and h(w). A factor important in determining the directions of these changes is the direction of the skewness of the offer p.d.f. f(w). It may be that the endogenously determined equilibrium offer p.d.f.'s of sequential search equilibrium models with learning do not allow some of the above results concerning changes to the extent of price dispersion since they are obtained in a model with an exogenously given offer p.d.f. Until this question is resolved, however, some caution should be exercised when arguing that reducing information costs necessarily reduces variance or coefficient of variation measures of price dispersion.

7. Some Comments

This paper makes two contributions to the search literature. First, it shows how an epv maximizing sequential search rule can be expressed in terms of ex ante determinate fixed points, rather than in terms of ex ante indeterminate switchpoint values. Secondly, it utilizes this advance to derive ex ante probability functions, together with some of their properties, for the duration of search and the wage which is eventually accepted. No description is offered for the generation of the wage offer distribution F(w) and, consequently, the derived c.d.f. of the accepted wage H(w) is conditional upon F(w). Nevertheless, if an (equilibrium) pair of c.d.f.'s (F(w), H(w)) exists such that F(w) implies H(w), and H(w) induces firms to offer wages with a dispersion described by F(w), then the results given here describe the extent of the dispersions in wages and search duration which will exist in this equilibrium. The question of the existence of such a dispersed price equilibrium has already been answered in the affirmative for three particular static sequential search problems (see [4], [6] and [16]) and for various static non-sequential search problems (see [2], [8], [20]). The results presented here will be helpful in answering the existence question when searchers utilize their search experiences and any of the many learning processes admitted in the current model to learn about the offer distribution they face, F(w). In this way the current paper will also assist in addressing questions about the existence and properties of rational expectations equilibria.

Naturally the form of H(w) depends partly upon the particular search rule employed by searchers. The current paper presents the form of H(w) generated by an epv maximizing sequential search rule. However, it should be possible to extend the arguments of this paper to examine the probability functions of search duration and the accepted wage generated by the more general search rules examined in [3], [7], [14], and [15].

FOOTNOTES

*I wish to thank Glenn MacDonald, Preston McAfee, John McMillan, Richard Manning, Alan Slivinski and seminar audiences at New York University and the University of Western Ontario for constructive comments. Any errors are my responsibility alone. This article was written while on sabbatical at the University of Western Ontario from the Flinders University of South Australia.

[1] Examples of these models are Axell [1], Braverman [2], Carlson and McAfee [4], Feigin and Landsberger [6], Hey [8], Reinganum [16], [17], Sandanand and Wilde [20] and Varian [23].

[2] Sequential strategies are not necessarily the strategies which yield the highest net epv of search. A class of search strategies including sequential strategies as special cases is presented in Morgan and Manning [14], Morgan [15], Gal, Landsberger and Levykson [7], and Benhabib and Bull [3].

[3] The no recall case may be dealt with in a manner similar to the analysis presented here for the full recall case. McAfee [11] has described the probability functions for search length and acceptable price offers for a static consumer search problem with no recall.

[4] More formally, (2-1) requires $F_j(w|w_1,\ldots,w_j)$ to be first-degree stochastic dominant w.r.t. each of $w_1,\ldots,w_j$.

[5] Rothschild [19] and Rosenfield and Shapiro [18] have presented search models with Bayesian learning. Kohn and Shavell [9] also examine Bayesian learning.

[6] Salop [21] formally introduced search paths into a sequential search model where firms could be sampled without pecuniary cost. Weitzman [24] completely describes the best search path when offer distributions are known and when offers and marginal sampling costs are independent across alternatives. In some search problems, marginal search costs depend on the order in which firms are sampled, e.g., if firms are visited physically then marginal costs will typically be smaller if firms within one locality are sampled rather than firms in different localities. These problems will usually be complex and difficult to solve. Nevertheless, the results presented in this paper extend to these cases. This extension has been omitted because of the additional notational and analytic complexities required and because the gain in economic understanding yielded is slight.

[7] Static search models are models in which the searcher has complete knowledge of the offer p.d.f. $f(w)$. In these models the switchpoints are ex ante determinate and can be used directly to derive the probability distributions of search length and the accepted wage.

[8] It is important to distinguish the switchpoint $s_j(w_1,\ldots,w_j,\beta)$ from the best available wage offer w_j^{max}. $s_j(\cdot)$ is the wage offer for which the searcher is indifferent between accepting $s_j(\cdot)$ and continuing to search with recall when his current estimate of $f(w)$ is $f_j(w|w_1,\ldots,w_j)$. $s_j(\cdot)$ will usually not equal w_j^{max}. For more details see [9].

[9] (3-11) implicitly places an upper bound upon the marginal information value of the j^{th} wage offer when $w_i = w_i^*$ for $i=1,\ldots,j-1$. For brevity, suppose $s_j(\cdot)$ is differentiable. (3-11) is satisfied if

$$\partial s_j(w_1^*,\ldots,w_{j-1}^*,w_j,\beta)/\partial w_j < 1. \quad \text{Since}$$

$$s_j(w_1^*,\ldots,w_{j-1}^*,w_j,\beta) = V_j^n(s_j(w_1^*,\ldots,w_{j-1}^*,w_j,\beta),w_1^*,\ldots,w_{j-1}^*,w_j,\beta)$$

$$\partial s_j(\cdot)/\partial w_j = \frac{\partial V_j^n(s_j(\cdot),w_1^*,\ldots,w_{j-1}^*,w_j,\beta)/\partial w_j}{1 - \partial V_j^n(y,w_1^*,\ldots,w_{j-1}^*,w_j,\beta)/\partial y\Big|_{y=s_j(\cdot)}}$$

The numerator of $\partial s_j(\cdot)/\partial w_j$ is the marginal informational value of w_j caused by modifying $f_j(w|w_1^*,\ldots,w_{j-1}^*,w_j)$ by a change in w_j. The denominator of $\partial s_j(\cdot)/\partial w_j$ is 1 minus the marginal value of a current best wage offer $y = s_j(w_1^*,\ldots,w_{j-1}^*,w_j,\beta)$. (3-11) is satisfied if the denominator of $\partial s_j(\cdot)/\partial w_j$ is a strict upper bound for the numerator.

[10] The fixed-sample-size case is dealt with in detail in [13].

[11] Propositions 4 and 5 can be stated as "$1 - G(j)$ and $1 - H(w)$ are both first-degree stochastic dominant w.r.t. $c_2,\ldots,c_n$ and β."

[12] I am grateful to Preston McAfee for suggesting the structure of this proof.

REFERENCES

[1] B. AXELL, Search Market Equilibrium, Scandanavian Journal of Economics 79 (1977), 20-40.

[2] A. BRAVERMAN, Consumer Search and Alternative Market Equilibria, Review of Economic Studies 47 (1980), 487-502.

[3] J. BENHABIB and C. BULL, Job Search: The Choice of Intensity, Working Paper No. 81-28 (Oct. 1981), C. V. Starr Center for Applied Economics, New York University.

[4] J. CARLSON and P. McAFEE, Discrete Equilibrium Price Dispersion, Journal of Political Economy, forthcoming, 1983.

[5] M. DE GROOT, Optimal Statistical Decisions, McGraw-Hill, New York (1970).

[6] P. FEIGIN and M. LANDSBERGER, The Existence and Properties of a Stationary Distribution for Unemployment When Job Search is Sequential, Journal of Economic Dynamics and Control 3 (1981), 329-341.

[7] S. GAL, M. LANDSBERGER and B. LEVYKSON, A Compound Strategy for Search in the Labour Market, International Economic Review 22 (1981), 597-608.

[8] J. HEY, Price Adjustment in an Atomistic Market, Journal of Economic Theory 8 (1974), 483-499.

[9] M. KOHN and S. SHAVELL, The Theory of Search, Journal of Economic Theory 9 (1974), 93-123.

[10] R. MacMINN, Job Search and the Labour Dropout Problem Reconsidered, Quarterly Journal of Economics 95 (1980), 69-87.

[11] P. McAFEE, Optimal Search With Risk Aversion and Wealth Effects, mimeographed manuscript, Department of Economics, University of Western Ontario (1982).

[12] J. J. McCALL, Economics of Information and Job Search, Quarterly Journal of Economics 84 (1970), 113-126.

[13] R. MANNING and P. MORGAN, Search and Consumer Theory, Review of Economic Studies 49 (1982), 203-216.

[14] P. MORGAN and R. MANNING, Optimal Search, Research Report No. 8219 (Sept., 1982), Department of Economics, University of Western Ontario.

[15] P. MORGAN, Search and Optimal Sample Sizes, Research Report No. 8208 (April, 1982), Department of Economics, University of Western Ontario.

[16] J. REINGANUM, A Simple Model of Equilibrium Price Dispersion, Journal of Political Economy 87 (1979), 851-858.

[17] J. REINGANUM, Nash Equilibrium Search for the Best Alternative, forthcoming, Journal of Economic Theory (1982).

[18] D. ROSENFIELD and R. SHAPIRO, Optimal Adaptive Price Search, Journal of Economic Theory 25 (1981), 1-20.

[19] M. ROTHSCHILD, Searching for the Lowest Price When the Distribution of Prices is Unknown, Journal of Political Economy 82 (1974), 689-711.

[20] A. SADANAND and L. WILDE, A Generalized Model of Pricing for Homogeneous Goods Under Imperfect Information, Review of Economic Studies 49 (1982), 229-240.

[21] S. SALOP, Systematic Job Search and Unemployment, Review of Economic Studies 40 (1973), 191-201.

[22] A. SCHWARTZ and L. WILDE, Imperfect Information, Monopolistic Competition, and Public Policy, American Economic Review 72 (1982), 18-23.

[23] H. VARIAN, A Model of Sales, American Economic Review 70 (1980), 651-659.

[24] M. WEITZMAN, Optimal Search for the Best Alternative, Econometrica 47 (1979), 641-654.

APPENDIX

This appendix contains the proofs of Lemma 1, Propositions 1, 2 and 3 and Theorems 2 and 3.

Proof of Proposition 1: Recall from (2-3) that the indices $1,\ldots,n$ are allocated to firms so that $c_1 \le c_2 \le \ldots \le c_n$. Let the index set $\{1',\ldots,n'\} = \{1,\ldots,n\}$ so that we may use $1',\ldots,n'$ to denote any arbitrary sampling order of firms 1 to n. The problem is to assign values 1 to n to the indices $1'$ to n' so as to maximize the ex ante epv of search given the sampling order $1',\ldots,n'$,

$$V_o^n(\beta|1',\ldots,n') = \int_{\underline{w}}^{\bar{w}} \max\{w_1^{max} - c_{1'}, \int_{\underline{w}}^{\bar{w}} \max\{w_2^{max} - c_{1'} - c_{2'}, \int_{\underline{w}}^{\bar{w}} \max\{w_3^{max} - c_{1'} - c_{2'} - c_{3'}, \ldots\} \cdot f_2(w_3|w_1,w_2)dw_3\} f_1(w_2|w_1)dw_2\} f_o(w_1)dw_1 \tag{A-1}$$

Since $c_{1'}$ is common to all the maxima nested in (A-1) the searcher can do no better than to set $1' = 1$, i.e., since $c_{1'}$ is incurred with certainty if search occurs $c_{1'}$ should be the smallest possible marginal cost c_1.

Once w_1 is received the searcher's problem is to assign values 2 to n to the indices $2'$ to n' so as to maximize

$$V_1^n(w_1,\beta|2',\ldots,n') = \int_{\underline{w}}^{\bar{w}} \max\{w_2^{max} - c_{2'}, \int_{\underline{w}}^{\bar{w}} \max\{w_3^{max} - c_2' - c_3', \ldots\} f_2(w_3|w_1,w_2)dw_3\} f_1(w_2|w_1)dw_2$$

Repeating the above argument shows the searcher should choose $2' = 2$. Continuing on in this manner completes the result.

Q.E.D.

Proof of Lemma 1:[12] The proof is by backward induction.

$$V^n_{n-1}(w^{max}_{n-1},w_1,\ldots,w_{n-1},\beta) = -c_n + \beta\int_{\underline{w}}^{\bar{w}} \max[w^{max}_{n-1},w_n] f_{n-1}(w_n|w_1,\ldots,w_{n-1})\,dw_n$$

$$= -c_n + \beta\bar{w} - \beta\int_{w^{max}_{n-1}}^{\bar{w}} F_{n-1}(w_n|w_1,\ldots,w_{n-1})\,dw_n \qquad \text{(A-2)}$$

w^{max}_{n-1} and $F_{n-1}(w_n|w_1,\ldots,w_{n-1})$ are bounded and continuous functions of $w_1,\ldots,w_{n-1}$ (see (2-2)) so Lesbesgue's Dominated Convergence Theorem can be applied to (A-2) to prove $V^n_{n-1}(w^{max}_{n-1},w_1,\ldots,w_{n-1},\beta)$ is continuous w.r.t. $w_1,\ldots,w_j$.

Now assume

$$V^n_{j+1}(w^{max}_{j+1},w_1,\ldots,w_{j+1},\beta) \text{ is continuous w.r.t. } w_1,\ldots,w_{j+1} . \qquad \text{(A-3)}$$

$$V^n_j(w^{max}_j,w_1,\ldots,w_j,\beta) = -c_{j+1} + \beta\int_{\underline{w}}^{\bar{w}} \max[w^{max}_j,w_{j+1},V^n_{j+1}(\max[w^{max}_j,w_{j+1}],w_1,\ldots,w_{j+1},\beta)] \,.$$

$$f_j(w_{j+1}|w_1,\ldots,w_j)\,dw_{j+1} \qquad \text{(A-4)}$$

w^{max}_j and (by (A-3)) $V^n_{j+1}(\cdot)$ are continuous w.r.t. $w_1,\ldots,w_j$ so

$$\max[w^{max}_j,w_{j+1},V^n_{j+1}(\cdot)] \text{ is continuous w.r.t. } w_1,\ldots,w_j . \qquad \text{(A-5)}$$

$[\underline{w},\bar{w}]$ is a compact interval. Let $\{A^i\}^n_{i=1}$ be a sequence of pairwise disjoint sub-intervals such that $\bigcup_{i=1}^{n} A^i = [\underline{w},\bar{w}]$. Denote the upper and lower bounds of A^i by b^i and a^i respectively, i.e., $A^i = [a^i,b^i]\ \forall\ i=1,\ldots,n$. Define $\{x^i(w_{j+1})\}^n_{i=1}$ as a sequence of simple functions such that

$$x^i(w_{j+1}) = \max[w_j^{max}, a^i, V_{j+1}^n(\max[w_j^{max}, a^i], w_1, \ldots, w_j, a^i)], \ \forall w_{j+1} \in A^i,$$

$$\forall \ i=1,\ldots,n. \qquad \text{(A-6)}$$

Then (A-4) can be written as

$$V_j^n(w_j^{max}, w_1, \ldots, w_j, \beta) = -c_{j+1} + \beta \lim_{n\to\infty} \sum_{i=1}^{n} x^i [F(b_i | w_1, \ldots, w_j)$$

$$- F(a_i | w_1, \ldots, w_j)] \qquad \text{(A-7)}$$

(A-5) ensures (A-6) is continuous w.r.t. $w_1, \ldots, w_j$. (2-2) assumes $F(w|w_1, \ldots, w_j)$ is continuous w.r.t. $w_1, \ldots, w_j$. It follows from (A-7) that $V_j^n(w_j^{max}, w_1, \ldots, w_j, \beta)$ is continuous w.r.t. $w_1, \ldots, w_j$.

This completes the induction step and the proof.

Q.E.D.

Proof of Proposition 2: The result is first established for $j=n-1$. Backward induction is then used to establish the result for $j=1,\ldots,n-2$. For notational brevity, let $\omega_j' = (w_1, \ldots, w_i', \ldots, w_j)$ and $\omega_j'' = (w_1, \ldots, w_i'', \ldots, w_j)$ where $\underline{w} \le w_i' < w_i'' \le \bar{w}$, let $w_j^{max'} = \max\{w_1, \ldots, w_i', \ldots, w_j\}$ and $w_j^{max''} = \max\{w_1, \ldots, w_i'', \ldots, w_j\}$, and let

$$\Delta_j = V_j^n(w_j^{max''}, \omega_j'', \beta) - V_j^n(w_j^{max'}, \omega_j', \beta). \qquad \text{(A-8)}$$

For j=n-1: $$\Delta_{n-1} = -c_n + \beta \int_{\underline{w}}^{\bar{w}} \max\{w_{n-1}^{\max''}, w_n\} f_{n-1}(w_n | \omega_{n-1}'') dw_n$$

$$+ c_n - \beta \int_{\underline{w}}^{\bar{w}} \max\{w_{n-1}^{\max'}, w_n\} f_{n-1}(w_n | \omega_{n-1}') dw_n$$

$$= \beta\Big(\int_{\underline{w}}^{\bar{w}} [\max\{w_{n-1}^{\max''}, w_n\} - \max\{w_{n-1}^{\max'}, w_n\}] f_{n-1}(w_n | \omega_{n-1}') dw_n$$

$$+ \int_{\underline{w}}^{\bar{w}} \max\{w_{n-1}^{\max''}, w_n\}[f_{n-1}(w_n | \omega_{n-1}'') - f_{n-1}(w_n | \omega_{n-1}')] dw_n\Big) \quad \text{(A-9)}$$

$w_i' < w_i'' \Rightarrow w_{n-1}^{\max'} \le w_{n-1}^{\max''}$ so, from (A-9),

$$\Delta_{n-1} \ge \beta \int_{\underline{w}}^{\bar{w}} \max\{w_{n-1}^{\max''}, w_n\}[f_{n-1}(w_n | \omega_{n-1}'') - f_{n-1}(w_n | \omega_{n-1}')] dw_n \quad \text{(A-10)}$$

Integrating (A-10) by parts gives

$$\Delta_n \ge \beta \max\{w_{n-1}^{\max''}, w_n\}[F_{n-1}(w_n | \omega_{n-1}'') - F_{n-1}(w_n | \omega_{n-1}')]_{w_n=\underline{w}}^{w_n=\bar{w}}$$

$$+ \beta \int_{w_{n-1}^{\max''}}^{\bar{w}} [F_{n-1}(w_n | \omega_{n-1}') - F_{n-1}(w_n | \omega_{n-1}'')] dw_n \quad \text{(A-11)}$$

The first term of (A-11) is zero and the second term is strictly positive by (2-1), so

$\Delta_n > 0.$

This establishes the result for j=n-1. For $1 \le j \le n-2$: Assume the result is true for j+1. Expressing Δ_j in the form of (A-9) gives

$$\Delta_j = \beta(\int_{\underline{w}}^{\bar{w}}[\max\{w_j^{max''}, w_{j+1}, V_{j+1}^n(w_{j+1}^{max''}, \omega_{j+1}'', \beta)\}$$

$$- \max\{w_j^{max'}, w_{j+1}, V_{j+1}^n(w_{j+1}^{max'}, \omega_{j+1}', \beta)\}] f_j(w_{j+1}|\omega_j')dw_{j+1}$$

$$+ \int_{\underline{w}}^{\bar{w}} \max\{w_j^{max''}, w_{j+1}, V_{j+1}^n(w_{j+1}^{max''}, \omega_{j+1}'', \beta)\}[f_j(w_{j+1}|\omega_j'') - f_j(w_{j+1}|\omega_j')]dw_{j+1}) \quad \text{(A-12)}$$

$w_i' < w_i'' \Rightarrow w_j^{max'} \le w_j^{max''}$ which, with the induction hypothesis, implies that $V_{j+1}^n(w_{j+1}^{max'}, \omega_{j+1}', \beta) < V_{j+1}^n(w_{j+1}^{max''}, \omega_{j+1}'', \beta)$. The first term of (A-12) is therefore non-negative, so

$$\Delta_j \ge \beta \int_{\underline{w}}^{\bar{w}} \max\{w_j^{max''}, w_{j+1}, V_{j+1}^n(w_{j+1}^{max''}, \omega_{j+1}'', \beta)\}[f_j(w_{j+1}|\omega_j'') - f_j(w_{j+1}|\omega_j')]dw_{j+1} \quad \text{(A-13)}$$

Integrating (A-13) by parts shows

$$\Delta_j \ge \beta \int_{w_j^{max''}}^{\bar{w}} \frac{\partial}{\partial w_{j+1}}(\max\{w_{j+1}, V_{j+1}^n(w_{j+1}, \omega_j'', \beta)\}[F_j(w_{j+1}|\omega_j') - F_{j+1}|\omega_j'')]dw_{j+1}$$

$$> 0$$

by the induction hypothesis and (2-1). This completes the induction step and the proof.

Q.E.D.

<u>Proof of Proposition 3</u>: For $j = 1,\ldots,n-1$ the switchpoint $s_j(w_1,\ldots,w_j,\beta)$ is a strictly increasing and continuous function of $w_1,\ldots,w_j$.

Proof: Let $\{\omega_j^i\}_{i=1}^{\infty} = \{(w_1^i,\ldots,w_j^i)\}_{i=1}^{\infty}$ be any sequence of vectors of wage observations such that

$$\lim_{i\to\infty} \omega_j^i = \omega_j. \quad \text{(A-14)}$$

By the definition of $s_j(\cdot)$,

$$s_j(\omega_j,\beta) = V_j^n(s_j(\omega_j,\beta),\omega_j,\beta) \quad \text{(A-15)}$$

and

$$\underset{i\to\infty}{\text{limit}}\ s_j(\omega_j^i,\beta) = \underset{i\to\infty}{\text{limit}}\ V_j^n(s_j(\omega_j^i,\beta),\omega_j^i,\beta)$$

$$= V_j^n(\underset{i\to\infty}{\text{limit}}\ s_j(\omega_j^i,\beta),\omega_j,\beta) \tag{A-16}$$

since $V_j^n(\cdot)$ is continuous (Lemma 1). Kohn and Shavell [9, Corollary 2] prove $s_j(\omega_j,\beta)$ is unique w.r.t. ω_j so comparing (A-15) and (A-16) shows s_j is continuous; i.e.,

$$s_j(\omega_j,\beta) = \underset{i\to\infty}{\text{limit}}\ s_j(\omega_j^i,\beta).$$

To show $s_j(\cdot)$ is a strictly increasing function of w_i for all $i=1,\ldots,j$ let $\omega_j' = (w_1,\ldots,w_i',\ldots,w_j)$ and $\omega_j'' = (w_1,\ldots,w_i'',\ldots,w_j)$. Suppose $w_i' < w_i''$. Then by the definition of $s_j(\cdot)$ and by Proposition 2 we have

$$V_j^n(s_j(\omega_j',\beta),\omega_j'\beta) = s_j(\omega_j',\beta) < V_j^n(s_j(\omega_j',\beta),\omega_j'',\beta) \tag{A-17}$$

and

$$s_j(\omega_j'',\beta) = V_j^n(s_j(\omega_j'',\beta),\omega_j'',\beta). \tag{A-18}$$

Kohn and Shavell [9, Theorem 4] show that, for given (ω_j'',β),

$$w \gtreqless V_j^n(w,\omega_j'',\beta) \text{ as } w \gtreqless s_j(\omega_j'',\beta)\ . \tag{A-19}$$

(A-17), (A-18) and (A-19) together imply

$$s_j(\omega_j',\beta) < s_j(\omega_j'',\beta)\ .$$

Q.E.D.

<u>Proof of Theorem 2</u>: The ex ante probability that the searcher takes more than one observation is

$$\Pr(j>1) = \Pr(w_1^{max} < S_1(w_1,\beta)) = F(w_1^*)$$

by (3-12). Hence

$$\Pr(j=1) = \Pr(j>0) - \Pr(j>1) = 1 - F(w_1^*) \ . \tag{A-20}$$

The ex ante probability that the searcher takes more than $k=2,\ldots,n-1$ observations is

$$\Pr(j>k) = \Pr(\bigcap_{i=1}^{k} \{w_i^{max} < S_i(w_1,\ldots,w_i,\beta)\}) \ .$$

However, the event $\{w_i^{max} < S_i(w_1,\ldots,w_i,\beta)\}$ is possible iff $S_i(w_1,\ldots,w_i,\beta) \geq \underline{w}$ (since $w_i^{max} < \underline{w}$ is impossible) which, by definition (3-7), implies $S_i(w_1,\ldots,w_i,\beta) = s_i(w_1,\ldots,w_i,\beta)$ so that

$$\Pr(j>k) = \Pr(\bigcap_{i=1}^{k} \{w_i^{max} < s_i(w_1,\ldots,w_i,\beta)\})$$

$$= \Pr(\{w_1 < s_1(w_1,\beta)\} \cap (\bigcap_{i=2}^{k} \{\max[w_1,w_2,\ldots,w_i]$$

$$< s_i(w_1,w_2,\ldots,w_i,\beta)\})) . \tag{A-21}$$

Since $s_i(\cdot)$ is a strictly increasing function of w_1 for $i=1,\ldots,n-1$ (Proposition 3),

$$w_1 < s_1(w_1,\beta) \Leftrightarrow w_1 < w_1^* \Leftrightarrow s_i(w_1,w_2,\ldots,w_i,\beta) < s_i(w_1^*,w_2,\ldots,w_i,\beta), \ \forall i=2,\ldots,n-1 \tag{A-22}$$

From (A-21) and (A-22),

$$\Pr(j>k) = \Pr(\{w_1 < w_1^*\} \cap (\bigcap_{i=2}^{k} \{\max[w_1,w_2,\ldots,w_i] < s_i(w_1^*,w_2,\ldots,w_i,\beta)\})). \tag{A-23}$$

$$\max[w_1,w_2] < s_2(w_1^*,w_2,\beta) \Leftrightarrow \{w_1 < s_1(w_1^*,w_2,\beta)\} \cap \{w_2 < s_2(w_1^*,w_2,\beta)\} \tag{A-24}$$

$$\Leftrightarrow \{w_1 < s_1(w_1^*,w_2,\beta)\} \cap \{w_2 < w_2^*\}. \tag{A-25}$$

Since $s_i(\cdot)$ is a strictly increasing function of w_2 for $i=2,\ldots,n-1$,

$$w_2 < w_2^* \Leftrightarrow s_i(w_1^*, w_2, \ldots, w_i, \beta) < s_i(w_1^*, w_2^*, \ldots, w_i, \beta) \ \forall \ i=2,\ldots,n-1. \qquad \text{(A-26)}$$

Combining (A-23), (A-24), (A-25) and (A-26) shows

$$\begin{aligned}\Pr(j>k) &= \Pr(\{w_1 < w_1^*\} \cap \{w_1 < s_1(w_1^*, w_2^*, \beta)\} \cap \{w_2 < w_2^*\} \\ &\qquad \cap (\bigcap_{i=3}^{k} \{\max[w_1, w_2, \ldots, w_i] < s_i(w_1^*, w_2^*, \ldots, w_i, \beta)\})) \\ &= \Pr(\{w_1 < w_1^*\} \cap \{\max[w_1, w_2] < w_2^*\} \cap (\bigcap_{i=3}^{k} \{\max[w_1, w_2, \ldots, w_i] \\ &\qquad < s_i(w_1^*, w_2^*, \ldots, w_i, \beta)\}))\end{aligned}$$

Continuing in this manner shows

$$\Pr(j>k) = \Pr(\bigcap_{i=1}^{k} \{\max[w_1, \ldots, w_i] < w_i^*\}).$$

Since $\max[w_1, \ldots, w_i] < w_i^* \Leftrightarrow \bigcap_{\ell=1}^{i} \{w_\ell < w_i^*\}$,

$$\begin{aligned}\Pr(j>k) &= \Pr(\bigcap_{i=1}^{k} \bigcap_{\ell=1}^{i} \{w_\ell < w_i^*\}) \\ &= \Pr(\bigcap_{\ell=1}^{k} \bigcap_{i=\ell}^{k} \{w_\ell < w_i^*\}) \\ &= \Pr(\bigcap_{\ell=1}^{k} \{w_\ell < \min[w_\ell^*, \ldots, w_k^*]\}). \qquad \text{(A-27)}\end{aligned}$$

Since each observation w_i is i.i.d. with c.d.f. $F(w)$, (A-27) is

$$\Pr(j>k) = \prod_{\ell=1}^{k} F(\min[w_\ell^*, \ldots, w_k^*]). \qquad \text{(A-28)}$$

Hence,

$$\Pr(j=n) = \Pr(j > n-1) = \prod_{\ell=1}^{n-1} F(\min[w_\ell^*, \ldots, w_{n-1}^*]).$$

Finally, for $2 \le k \le n-2$,

$$\begin{aligned}\Pr(j=k) &= \Pr(j > k-1) - \Pr(j>k) \\ &= \prod_{\ell=1}^{k-1} F(\min[w_\ell^*, \ldots, w_{k-1}^*]) - \prod_{\ell=1}^{k} F(\min[w_\ell^*, \ldots, w_k^*]) .\end{aligned}$$

Q.E.D.

<u>Proof of Theorem 3</u>: H(w) is the cumulative probability that the searcher will accept a wage no greater than w, irrespective of the number of firms sampled by the searcher; i.e.,

$$H(w) = \sum_{k=1}^{n} \Pr(\{w_k^{max} \le w\} \cap \{j=k\}) \tag{A-29}$$

For k=1: $\Pr(\{w_1^{max} \le w\} \cap \{j=1\}) = \Pr(\{w_1 \le w\} \cap \{w_1 \ge w_1^*\})$

$$= \Pr(\min[w_1^*,w] \le w_1 \le w) = F(w) - F(\min[w_1^*,w]). \tag{A-30}$$

For k = n: $\Pr(\{w_n^{max} \le w\} \cap \{j=n\}) = \Pr(\{w_n^{max} \le w\} \cap \{j > n-1\})$

$$= \Pr(\bigcap_{i=1}^{n} \{w_i \le w\} \cap (\bigcap_{i=1}^{n-1} \{w_i < \min[w_1^*,\ldots,w_{n-1}^*]\}))$$

$$= \Pr(\{w_n \le w\} \cap (\bigcap_{i=1}^{n-1} \{w_i \le \min[w_1^*,\ldots,w_{n-1}^*,w]\}))$$

$$= F(w) \prod_{i=1}^{n-1} F(\min[w_1^*,\ldots,w_{n-1}^*,w]). \tag{A-31}$$

For $2 \le k \le n-1$: $\Pr(\{w_k^{max} \le w\} \cap \{j=k\}) = \Pr(j=k | w_k^{max} \le w)\Pr(w_k^{max} \le w)$

$$= \Pr(j>k-1 | w_k^{max} \le w)\Pr(w_k^{max} \le w) - \Pr(j>k | w_k^{max} \le w)\Pr(w_k^{max} \le w)$$

$$= \Pr(\{j>k-1\} \cap \{w_k^{max} \le w\}) - \Pr(\{j>k\} \cap \{w_k^{max} \le w\})$$

$$= \Pr(\bigcap_{i=1}^{k-1} \{w_i < \min[w_1^*,\ldots,w_{k-1}^*]\} \cap (\bigcap_{i=1}^{k} \{w_i \le w\}))$$

$$- \Pr(\bigcap_{i=1}^{k} \{w_i < \min[w_1^*,\ldots,w_k^*]\} \cap (\bigcap_{i=1}^{k} \{w_i \le w\}))$$

$$= \Pr\left(\bigcap_{i=1}^{k-1}\{w_i \le \min[w_1^*,\ldots,w_{k-1}^*,w]\} \cap \{w_k \le w\}\right)$$

$$- \Pr\left(\bigcap_{i=1}^{k}\{w_i \le \min[w_1^*,\ldots,w_k^*,w]\}\right)$$

$$= F(w)\prod_{i=1}^{k-1} F(\min[w_1^*,\ldots,w_{k-1}^*,w]) - \prod_{i=1}^{k} F(\min[w_1^*,\ldots,w_k^*,w]). \qquad \text{(A-32)}$$

Combining (A-29), (A-30), (A-31) and (A-32) produces

$$H(w) = F(w) - F(\min[w_1^*,w]) + F(w)\prod_{i=1}^{n-1} F(\min[w_1^*,\ldots,w_{n-1}^*,w])$$

$$+ \sum_{k=2}^{n-1}\left(F(w)\prod_{i=1}^{k-1} F(\min[w_1^*,\ldots,w_{k-1}^*,w])\right.$$

$$\left. - \prod_{i=1}^{k} F(\min[w_1^*,\ldots,w_k^*,w])\right)$$

$$= F(w)\left(1 + \sum_{k=2}^{n}\prod_{i=1}^{k-1} F(\min[w_1^*,\ldots,w_{k-1}^*,w])\right)$$

$$- \sum_{k=2}^{n}\prod_{i=1}^{k-1} F(\min[w_1^*,\ldots,w_{k-1}^*,w])$$

$$= F(w) - (1 - F(w))\sum_{k=2}^{n}\prod_{i=1}^{k-1} F(\min[w_1^*,\ldots,w_{k-1}^*,w]) .$$

Q.E.D.