

## Research Article

# Distributions of the Ratio and Product of Two Independent Weibull and Lindley Random Variables

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Received 26 November 2019; Accepted 13 April 2020; Published 1 May 2020

Academic Editor: Alessandro De Gregorio

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In this paper, we derive the cumulative distribution functions (CDF) and probability density functions (PDF) of the ratio and product of two independent Weibull and Lindley random variables. The moment generating functions (MGF) and the  $k$ -moment are driven from the ratio and product cases. In these derivations, we use some special functions, for instance, generalized hypergeometric functions, confluent hypergeometric functions, and the parabolic cylinder functions. Finally, we draw the PDF and CDF in many values of the parameters.

## 1. Introduction

The distributions of ratio of random variables are widely used in many applied problems of engineering, physics, number theory, order statistics, economics, biology, genetics, medicine, hydrology, psychology, classification, and ranking and selection [1, 2]. Examples include safety factor in engineering, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, inventory ratios in economics, and Mendelian inheritance ratios in genetics [3, 4]. Also, ratio distribution involving two Gaussian random variables is used in computing error and outage probabilities [5]. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures, and aging of concrete pressure vessels [6, 7]. An important example of ratios of random variables is the stress-strength model in the context of reliability. It describes the life of a component which has a random strength and is subjected to random stress. The general numerical method is developed for computing the PDF of the sums, products, or ratio of any number of nonnegative independent random variables [8]. The ratio

and product distributions have been studied by several authors especially when independent random variables come from the same family or different families. For ratio distribution, the historical review, see [9, 10] for the normal family, [11] for Student's  $t$  family, [12] for the Weibull family, [13] for the noncentral chi-squared family, [14] for the gamma family, [15] for the beta family, [16] for the logistic family, [17] for the Frechet family, [3] for the inverted gamma family, [18] for Laplace family, [7] for the generalized-F family, [19] for the hypoexponential family, [2] for the gamma and Rayleigh families, and [20] for gamma and exponential families. For product distribution, the historical review, see [21] for  $t$  and Rayleigh families, [4] for Pareto and Kumaraswamy families, [6] for the  $t$  and Bessel families, and [22] for the independent generalized gamma-ratio family. A new distribution is introduced based on compounding Weibull and Lindley distributions; in this approach, several properties of the distribution are derived [23]. In this paper, we derived the ratio and product of cumulative distribution functions (CDF) and probability density functions (PDF) of the independent Weibull and Lindley random variables. In this derivation, we used some special functions and integrals, for instance, generalized and

confluent hypergeometric functions, parabolic cylinder function, gamma function of negative integer numbers, and some special integrals. We derived the moment generated function (MGF), m-moment, mean, and variance of the ratio random variable  $Z = X/Y$ , while we derived the CDF, PDF, and the MGF with respect to the product random variable  $XY$ . The rest of this paper is organized as follows: in Section 2, the derivation of the CDF, PDF, CGF, plots of the CDF and PDF, m-moment, mean, and variance of  $Z = X/Y$  by using some special functions and integrals are given such that  $X$  has Weibull distribution and  $Y$  has Lindley distribution. The CDF, PDF, CGF, and plots of the CDF and PDF have been given in Section 3. Finally, some conclusions are considered in Section 4. Weibull distribution  $X \sim W(a, b)$  with shape parameter  $a > 0$  and scale parameter  $b > 0$ , the PDF and CDF are defined as

$$f(x) = \frac{a}{b^a} x^{a-1} \exp\left[-\left(\frac{x}{b}\right)^a\right]. \quad (1)$$

And the CDF is

$$F(x) = 1 - \exp\left[-\left(\frac{x}{b}\right)^a\right] \quad (2)$$

such that  $X > 0$ . Lindley distribution  $Y \sim L(c)$  with shape parameter  $c > 0$ , the PDF and CDF are given as

$$f(y) = \frac{c^2}{c+1} (y+1) \exp[-cy]. \quad (3)$$

Also, the CDF is

$$F(y) = 1 - \frac{(c+cy+1)}{c+1} \exp[-cy] \quad (4)$$

such that  $Y > 0$ . In this work, we assume that the shape parameter  $a = 2$  in Weibull distribution. So, the PDF, equation (1), and CDF, equation (2), become, respectively,

$$f(x) = \frac{2}{b^2} x \exp\left[-\left(\frac{x}{b}\right)^2\right]. \quad (5)$$

$$F(x) = 1 - \exp\left[-\left(\frac{x}{b}\right)^2\right]. \quad (6)$$

The generalized hypergeometric function [24] is defined as

$${}_aY_b(d_1, d_2, \dots, d_p; e_1, e_2, \dots, e_q; w) = \sum_{l=0}^{\infty} \left[ \frac{(d_1)_l (d_2)_l \dots (d_p)_l w^l}{(e_1)_l (e_2)_l \dots (e_q)_l l!} \right], \quad (7)$$

where Pochhammer symbol  $(d)_l = (\Gamma(d+l))/\Gamma(d)$  or  $(e)_l = (\Gamma(e+l))/\Gamma(e)$ . The confluent hypergeometric function [24] is defined as

$$\begin{aligned} \Psi(d, e; w) &= \frac{\Gamma(1-e)}{\Gamma(d-e+1)} [{}_1Y_1(d; e; w)] \\ &+ \frac{\Gamma(e-1)}{\Gamma(d)} w^{1-e} [{}_1Y_1(d-e+1; 2-e; w)], \end{aligned} \quad (8)$$

where  $w \in \mathfrak{R}$  and  $e \neq 0, \pm 1, \pm 2, \pm 3, \dots$ . The gamma function of the negative integer numbers [25] is given by

$$\Gamma(-r) = \frac{(-1)^r}{r!} \phi(r) - \frac{(-1)^r}{r!} \eta, \quad (9)$$

where  $\phi(r) = \sum_{l=1}^r 1/l$  and  $\eta = \lim_{n \rightarrow \infty} \sum_{l=1}^n ((1)/(l - \ln(n)))$  is Euler's constant [25] such that  $\Gamma(0) = -\eta$ . The parabolic cylinder function [24] is defined as

$$\Lambda_\nu(w) = 2^{\nu/2} \exp\left[-\frac{w^2}{4}\right] \Psi\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{w^2}{2}\right). \quad (10)$$

The first special integral [26] is given as

$$\begin{aligned} \int_0^\infty x^{\tau-1} \exp[-\rho x^2 - \sigma x] dx &= (2\rho)^{-(\tau/2)} \Gamma(\tau) \exp\left[\left(\frac{\sigma^2}{8\rho}\right)\right] \\ &\times \Lambda_{-\tau}\left(\frac{\sigma}{\sqrt{2\rho}}\right), \end{aligned} \quad (11)$$

where  $Re(\sigma) > 0$  and  $Re(\rho) > 0$ . The other special integral [26] is introduced:

$$\int_0^\infty x^{\tau-1} \Psi(d, e; x) dx = \frac{\Gamma(\tau) \Gamma(d-\tau) \Gamma(\tau-e+1)}{\Gamma(d) \Gamma(d-e+1)}. \quad (12)$$

The m-moment can be defined as

$$E(Z^m) = \int_0^\infty Z^m f(z) dz. \quad (13)$$

## 2. Distribution of the Ratio $Z = X/Y$

In this section, we derive CDF, PDF, MGF, plots of the CDF and PDF, m-moment, mean, and variance of  $Z = X/Y$  by using some special functions. Let  $X \sim W(2, b)$  and  $Y \sim L(c)$ ; then, it can be found that the CDF of  $Z = X/Y$  by

$$F(z) = \text{Pro}(Z \leq z) = \text{Pro}\left(\frac{X}{Y} \leq z\right) = \int_0^\infty F_X(z y) f(y) dy. \quad (14)$$

By substituting equations (3) and (6) in equation (14), we get

$$\begin{aligned}
 F(z) &= \int_0^\infty \left(1 - \exp\left[-\frac{z^2 y^2}{b^2}\right]\right) \frac{c^2}{c+1} (y+1) \exp[-cy] \\
 &= \frac{c^2}{c+1} \int_0^\infty (1+y) \exp[-cy] dy - \frac{c^2}{c+1} \int_0^\infty (1+y) \exp\left[-\frac{z^2 y^2}{b^2} - cy\right] dy \\
 &= \frac{c^2}{c+1} \int_0^\infty (1+y) \exp[-cy] dy - \frac{c^2}{c+1} \int_0^\infty y^{1-1} \exp\left[-\frac{z^2 y^2}{b^2} - cy\right] dy \\
 &\quad - \frac{c^2}{c+1} \int_0^\infty y^{2-1} \exp\left[-\frac{z^2 y^2}{b^2} - cy\right] dy.
 \end{aligned} \tag{15}$$

By using the first special integral in equation (11) to calculate the integral  $(c^2)/(c+1) \int_0^\infty y^{1-1} \exp[-(z^2 y^2)/(b^2) - cy] dy$  and the integral  $(c^2)/(c+1) \int_0^\infty y^{2-1} \exp[-(z^2 y^2)/(b^2) - cy] dy$ , one can obtain

$$\begin{aligned}
 \frac{c^2}{c+1} \int_0^\infty y^{1-1} \exp\left[-\frac{z^2 y^2}{b^2} - cy\right] dy &= \left(\frac{2z^2}{b^2}\right)^{-(1/2)} \exp\left[\frac{b^2 c^2}{8z^2}\right] \Lambda_{-1}\left(\frac{bc}{\sqrt{2}z}\right) \\
 &= \frac{b}{2z} \Psi\left(\frac{1}{2}, \frac{1}{2}; \frac{b^2 c^2}{4z^2}\right),
 \end{aligned} \tag{16}$$

$$\begin{aligned}
 \frac{c^2}{c+1} \int_0^\infty y^{2-1} \exp\left[-\frac{z^2 y^2}{b^2} - cy\right] dy &= \left(\frac{2z^2}{b^2}\right)^{-1} \exp\left[\frac{b^2 c^2}{8z^2}\right] \Lambda_{-2}\left(\frac{bc}{\sqrt{2}z}\right) \\
 &= \frac{b^2}{4z^2} \Psi\left(1, \frac{1}{2}; \frac{b^2 c^2}{4z^2}\right).
 \end{aligned} \tag{17}$$

We substitute equations (16) and (17) in equation (15):

$$\begin{aligned}
 F(z) &= 1 - \frac{c^2}{c+1} \frac{b}{2z} \Psi\left(\frac{1}{2}, \frac{1}{2}; \frac{b^2 c^2}{4z^2}\right) \\
 &\quad - \frac{b^2}{4z^2} \Psi\left(1, \frac{1}{2}; \frac{b^2 c^2}{4z^2}\right).
 \end{aligned} \tag{18}$$

Equation (18) represents the CDF of  $Z = X/Y$ , where  $X \sim W(2, b)$  and  $Y \sim L(c)$  by the confluent hypergeometric function  $\Psi(d, e; w)$ . We can analysis equation (18) as the following picture:

$$\begin{aligned}
 F(z) &= 1 - \frac{c^2}{c+1} \frac{b}{2z} \sqrt{\pi} \left[ {}_1Y_1\left(0.5; 0.5; \frac{b^2 c^2}{4z^2}\right) \right. \\
 &\quad - \left. \frac{(bc)}{2z} \left[ {}_1Y_1\left(1; \frac{3}{2}; \frac{b^2 c^2}{4z^2}\right) \right] \right. \\
 &\quad - \left. \frac{b^2}{2z^2} \left[ {}_1Y_1\left(1; 0.5; \frac{b^2 c^2}{4z^2}\right) \right] \right. \\
 &\quad - \left. \sqrt{\pi} \left(\frac{bc}{z}\right) \left[ {}_1Y_1\left(\frac{3}{2}; \frac{3}{2}; \frac{b^2 c^2}{4z^2}\right) \right] \right].
 \end{aligned} \tag{19}$$

Equation (19) is CDF of  $Z$  by the generalized hypergeometric function  ${}_pY_q(d_1, d_2, \dots, d_p; e_1, e_2, \dots, e_q; w)$ .

The PDF of  $Z = X/Y$  such that  $X \sim W(2, b)$  and  $Y \sim L(c)$  can be defined as

$$f(z) = \text{Pro}(Z = z) = \text{Pro}\left(\frac{X}{Y} = z\right) = \int_0^\infty y f_X(zy) f_Y(y) dy. \tag{20}$$

By substituting equations (3) and (5), one can obtain

$$f(z) = \text{Pro}(Z = z) = \int_0^\infty \frac{2}{b^2} z y^2 \exp\left[-\left(\frac{zy}{b}\right)^2\right] \tag{21}$$

$$\begin{aligned}
 &\frac{c^2}{c+1} (y+1) \exp(-cy) dy, \\
 f(z) &= \frac{2zc^2}{b^2(c+1)} \int_0^\infty (y^3 + y^2) \exp\left[-\frac{z^2 y^2}{b^2} - cy\right] dy \\
 &= \frac{2zc^2}{b^2(c+1)} \left[ \int_0^\infty y^{3-1} \exp\left[-\frac{z^2 y^2}{b^2} - cy\right] dy \right. \\
 &\quad \left. + \int_0^\infty y^{4-1} \exp\left[-\frac{z^2 y^2}{b^2} - cy\right] dy \right].
 \end{aligned} \tag{22}$$

By using the first special integral in equation (11) to compute the integrals  $\int_0^\infty y^{3-1} \exp[-((z^2 y^2)/(b^2)) - cy] dy$  and  $\int_0^\infty y^{4-1} \exp[-((z^2 y^2)/(b^2)) - cy] dy$ , one can obtain

$$f(z) = \frac{2zc^2}{b^2(c+1)} \left[ 2^{-2} \frac{b^3}{z^3} \Psi\left(\frac{3}{2}, \frac{1}{2}; \frac{b^2c^2}{4z^2}\right) + \frac{3b^4}{8z^4} \Psi\left(2, \frac{1}{2}; \frac{b^2c^2}{4z^2}\right) \right]. \quad (23)$$

Equation (23) is PDF of  $Z = X/Y$ , where  $X \sim W(2, b)$  and  $Y \sim L(c)$  by the confluent hypergeometric function. Figure 1 is the plot of CDF in equation (18).

Figure 2 is the plot of PDF in equation (23).

By substituting equation (8) in equation (23), we have

$$\begin{aligned} f(z) &= \frac{2zc^2}{b^2(c+1)} 2^{-2} \frac{b^3}{z^3} \sqrt{\pi} \left[ {}_1Y_1\left(\frac{3}{2}; \frac{1}{2}; \frac{b^2c^2}{4z^2}\right) \right] \\ &\quad - 4 \frac{cb}{\sqrt{2z}} \left[ {}_1Y_1\left(2; \frac{3}{2}; \frac{b^2c^2}{4z^2}\right) \right] \\ &\quad + \frac{2zc^2}{b^2(c+1)} \frac{3b^4}{8z^4} \frac{4}{3} \left[ {}_1Y_1\left(2; \frac{1}{2}; \frac{b^2c^2}{4z^2}\right) \right] \\ &\quad - 2\sqrt{\pi} \frac{cb}{\sqrt{2z}} \left[ {}_1Y_1\left(\frac{5}{2}; \frac{1}{2}; \frac{b^2c^2}{4z^2}\right) \right]. \end{aligned} \quad (24)$$

Equation (24) is PDF of  $z$ -generalized hypergeometric function. We put equation (23) in equation (13) to get the following:

$$\begin{aligned} E(Z^m) &= \frac{2}{b(c+1)} \left[ \int_0^\infty Z^m \frac{b^2c^2}{4z^2} \Psi\left(\frac{3}{2}, \frac{1}{2}; \frac{b^2c^2}{4z^2}\right) dz \right] \\ &\quad + \frac{3}{c+1} \left[ \int_0^\infty Z^{m-1} \frac{b^2c^2}{4z^2} \Psi\left(2, \frac{1}{2}; \frac{b^2c^2}{4z^2}\right) dz \right] \\ &= \frac{-(cb/2)^{m+1}}{b(c+1)} \left[ \int_0^\infty \omega^{((1-m)/2)-1} \Psi\left(\frac{3}{2}, \frac{1}{2}; \omega\right) d\omega \right] \\ &\quad - \frac{3}{2(c+1)} \left(\frac{cb}{2}\right)^m \left[ \int_0^\infty \omega^{(2-m/2)-1} \Psi\left(2, \frac{1}{2}; \frac{b^2c^2}{4z^2}\right) d\omega \right]. \end{aligned} \quad (25)$$

By using the second special integral in equation (12), we obtain as

$$\begin{aligned} E(Z^m) &= \frac{(cb/2)^{m+1}}{b(c+1)} \left[ \frac{\Gamma(1-m/2)\Gamma(2+m/2)m\Gamma(-m/2)}{\sqrt{\pi}} \right] \\ &\quad - \frac{3}{2(c+1)} \left(\frac{cb}{2}\right)^m \left[ 4 \frac{\Gamma(2-m/2)\Gamma(2+m/2)\Gamma(3-m/2)}{3\sqrt{\pi}} \right]. \end{aligned} \quad (26)$$

Equation (27) is  $m$ -moment of  $Z$ .

Now, if  $m = 1$ , the mean of  $Z$  in equation (27) is

$$E(Z) = \frac{(cb/2)^2}{b(c+1)} [-\sqrt{\pi}\eta] - \frac{3}{2(c+1)} \left(\frac{cb}{2}\right) \left[\frac{2\sqrt{\pi}}{3}\right]. \quad (27)$$

If  $m = 2$ , the second moment in equation (27) is

$$E(Z^2) = \frac{(cb/2)^3}{b(c+1)} 4[1-\eta] - \frac{3}{2(c+1)} \left(\frac{cb}{2}\right)^2 \left[\frac{-4\eta}{3}\right]. \quad (28)$$

The variance  $V(Z)$  can be found from equations (28) and (29) as

$$\begin{aligned} V(Z) &= \frac{(cb/2)^3}{b(c+1)} 4[1-\eta] - \frac{3}{2(c+1)} \left(\frac{cb}{2}\right)^2 \left[\frac{-4\eta}{3}\right] \\ &\quad - \left[ \frac{(cb/2)^2}{b(c+1)} [-\sqrt{\pi}\eta] - \frac{3}{2(c+1)} \left(\frac{cb}{2}\right) \left[\frac{2\sqrt{\pi}}{3}\right] \right]^2. \end{aligned} \quad (29)$$

The MGF of  $Z$  can be calculated as

$$\begin{aligned} M_Z(t) &= E(\exp[tz]) \\ &= \frac{2}{b(c+1)} \left[ \int_0^\infty \exp[tz] \frac{b^2c^2}{4z^2} \Psi\left(\frac{3}{2}, \frac{1}{2}; \frac{b^2c^2}{4z^2}\right) dz \right] \\ &\quad + \frac{3}{c+1} \left[ \int_0^\infty z^{-1} \exp[tz] \frac{b^2c^2}{4z^2} \Psi\left(2, \frac{1}{2}; \frac{b^2c^2}{4z^2}\right) dz \right] \\ &= -\frac{c}{2(c+1)} \left[ \int_0^\infty \omega^{-1/2} \exp\left[\frac{cb\omega^{-1/2}t}{2}\right] \Psi\left(\frac{3}{2}, \frac{1}{2}; \omega\right) d\omega \right] \\ &\quad - \frac{3}{2(c+1)} \left[ \int_0^\infty \exp\left[\frac{cb\omega^{-1/2}t}{2}\right] \Psi\left(2, \frac{1}{2}; \omega\right) d\omega \right], \end{aligned} \quad (30)$$

where  $\omega = (b^2c^2)/(4z^2)$ . Using equations (7) and (8) to rewrite equation (31) as the following picture,

$$\begin{aligned} M_Z(t) &= \left[ \frac{2c}{(c+1)} \sum_{l=0}^\infty \frac{(2)_l}{(3/2)_l l!} - \frac{2}{c+1} \sum_{l=0}^\infty \frac{(2)_l}{(1/2)_l l!} \right] \\ &\quad \times \int_0^\infty \omega^l \exp\left[\frac{cb\omega^{-1/2}t}{2}\right] d\omega + \left[ \frac{-\sqrt{\pi}c}{2(c+1)} \sum_{l=0}^\infty \frac{(3/2)_l}{(1/2)_l l!} \right] \\ &\quad \times \int_0^\infty \omega^{l-(1/2)} \exp\left[\frac{cb\omega^{-1/2}t}{2}\right] d\omega \\ &\quad + \left[ \frac{3\sqrt{\pi}}{c+1} \sum_{l=0}^\infty \frac{(5/2)_l}{(3/2)_l l!} \right] \int_0^\infty \omega^{l+(1/2)} \exp\left[\frac{cb\omega^{-1/2}t}{2}\right] d\omega. \end{aligned} \quad (31)$$

To solve the integrals in equation (32), we assume that  $-R = (cb\omega^{(-1/2)t})/(2)$ ; then, equation (32) becomes

$$\begin{aligned} M_Z(t) &= -\left[ \frac{2c}{(c+1)} \sum_{l=0}^\infty \frac{(2)_l}{(3/2)_l l!} - \frac{2}{c+1} \sum_{l=0}^\infty \frac{(2)_l}{(1/2)_l l!} \right] \\ &\quad \times \frac{(bct)^{2l}\Gamma(-2l+2)}{2^{2l+1}} + \left[ \frac{-\sqrt{\pi}c}{2(c+1)} \sum_{l=0}^\infty \frac{(3/2)_l}{(1/2)_l l!} \right] \\ &\quad \times (-2) \left(\frac{bct}{2}\right)^{2l+1} \Gamma(-2l+1) + \left[ \frac{3\sqrt{\pi}}{c+1} \sum_{l=0}^\infty \frac{(5/2)_l}{(3/2)_l l!} \right] \\ &\quad \times (-2) \left(\frac{bct}{2}\right)^{2l+3} \Gamma(-2l+3). \end{aligned} \quad (32)$$

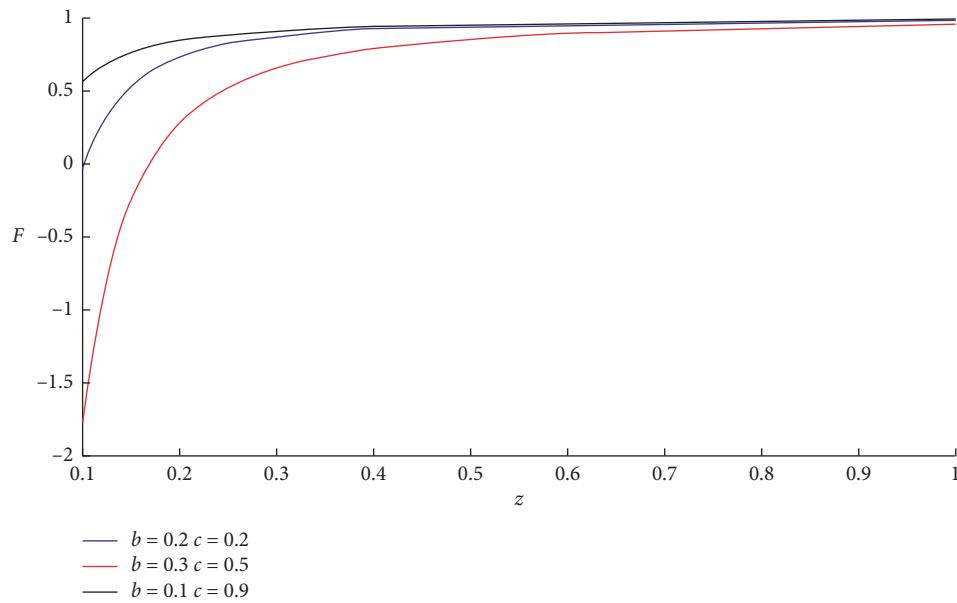


FIGURE 1: The graph of CDF for equation (18); we take three cases of the values of the parameters: first case  $b = c = 0.2$ , second case  $b = 0.3$  and  $c = 0.5$ , and final case  $b = 0.1$  and  $c = 0.9$ .

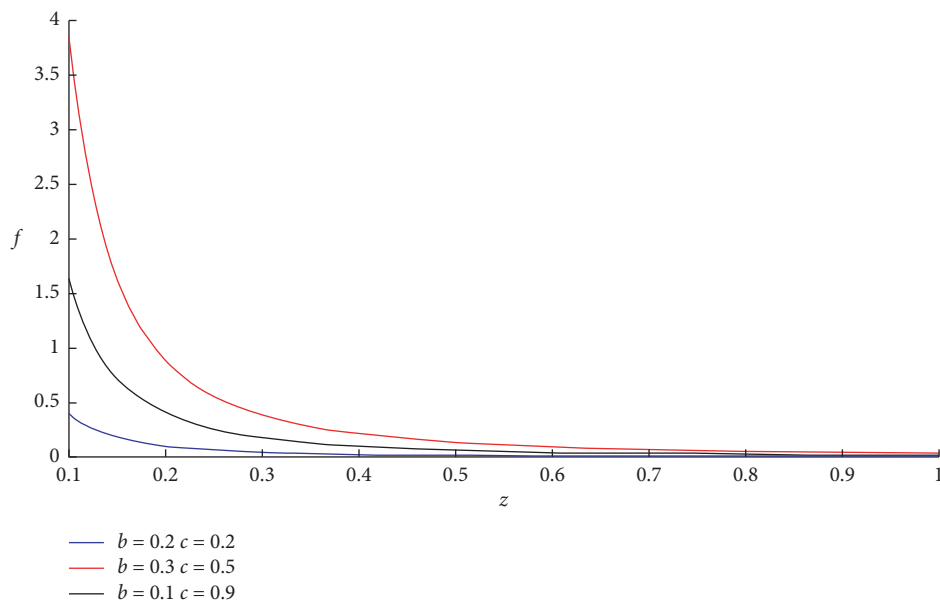


FIGURE 2: The graph of PDF for equation (23); we take three cases of the values of the parameters: first case  $b = c = 0.2$ , second case  $b = 0.3$  and  $c = 0.5$ , and final case  $b = 0.1$  and  $c = 0.9$ .

where  $\Gamma(-2l+2)$ ,  $\Gamma(-2l+1)$ , and  $\Gamma(-2l+3)$  are gamma functions of the negative integer numbers.

using some special functions. Let  $X \sim W(2, b)$  and  $Y \sim L(c)$ ; then, the CDF of  $Z = XY$  can be computed by

### 3. Distribution of the Product $Z = XY$

In this section, we derive CDF, PDF, MGF, plots of the CDF and PDF, m-moment, mean, and variance of  $Z = XY$  by

$$\begin{aligned}
 F(z) &= \text{Pro}\left(X \leq \frac{y}{z}\right) = \int_0^\infty F_X\left(\frac{y}{z}\right) f(y) dy \\
 &= \int_0^\infty \left[1 - \exp\left[-\frac{y^2}{b^2 z^2}\right]\right] \frac{c^2}{c+1} (y+1) \exp[-cy] dy \\
 &= 1 - \frac{c^2}{c+1} \frac{bz}{2} \Psi\left(\frac{1}{2}, \frac{1}{2}; \frac{b^2 c^2 z^2}{4}\right) - \frac{c^2}{c+1} \frac{b^2 c^2}{4} \Psi\left(1, \frac{1}{2}; \frac{b^2 c^2 z^2}{4}\right).
 \end{aligned} \tag{33}$$

Equation (33) is CDF of  $Z = XY$  by the confluent hypergeometric function. By using equation (8), equation (33) can be written as the following:

$$\begin{aligned}
 F(z) &= 1 - \frac{bzc^2}{2(c+1)} \left[ \sqrt{\pi} \left[ {}_1Y_1\left(\frac{1}{2}; \frac{1}{2}; \theta\right) \right] - 2\theta^{1/2} \left[ {}_1Y_1\left(\frac{1}{2}; \frac{1}{2}; \theta\right) \right] \right] \\
 &\quad - \frac{b^2 c^4}{4(c+1)} \left[ 2 \left[ {}_1Y_1\left(1; \frac{1}{2}; \theta\right) \right] - 2\sqrt{\pi} \theta^{1/2} \left[ {}_1Y_1\left(\frac{3}{2}; \frac{3}{2}; \theta\right) \right] \right].
 \end{aligned} \tag{34}$$

Equation (34) is CDF of  $Z$  by generalized hypergeometric function, where  $\theta = (b^2 c^2 z^2)/4$ . The PDF of  $Z = XY$  can be considered:

$$\begin{aligned}
 f(z) &= \int_0^\infty \frac{1}{x} f_Y\left(\frac{z}{x}\right) f_X(x) dx \\
 &= \frac{2c^2}{b^2(c+1)} \int_0^\infty \left(1 + \frac{z}{x}\right) \exp\left[-\frac{x^2}{b^2}\right] \exp\left[-\frac{cz}{x}\right] dx \\
 &= \frac{2c^2}{b^2(c+1)} \int_0^\infty \left(1 + \frac{z}{x}\right) \exp\left[-\frac{x^2}{b^2}\right] \left[ \sum_{l=0}^\infty \frac{(-1)^l (cz)^l x^{-l}}{l!} \right] dx \\
 &= \frac{2c^2}{b^2(c+1)} \sum_{l=0}^\infty \frac{(-1)^l (cz)^l}{l!} \left[ \int_0^\infty x^{-l} \exp\left[-\frac{x^2}{b^2}\right] dx \right] \left[ 1 + z \int_0^\infty x^{-l-1} \exp\left[-\frac{x^2}{b^2}\right] dx \right] \\
 &= \frac{2c^2}{b^2(c+1)} \sum_{l=0}^\infty \frac{(-1)^l (cz)^l}{l!} \left[ \frac{b^{-l+1}}{2} \Gamma\left(\frac{-l+1}{2}\right) \right] \left[ 1 + z \frac{b^{-l}}{2} \Gamma\left(\frac{-l}{2}\right) \right],
 \end{aligned} \tag{35}$$

where  $R = (x^2)/(b^2)$ . So,

$$f(z) = \frac{2c^2}{b^2(c+1)} \sum_{l=0}^\infty \frac{(-1)^l (cz)^l}{l!} \left[ \frac{b^{-l+1}}{2} \Gamma\left(\frac{-l+1}{2}\right) \right] \left[ 1 + z \frac{b^{-l}}{2} \Gamma\left(\frac{-l}{2}\right) \right]. \tag{36}$$

Figure 3 is the plot of CDF in equation (33). Equation (36) is PDF of  $Z$  by gamma function of the negative integer number.

The MGF of  $Z$  can be computed as

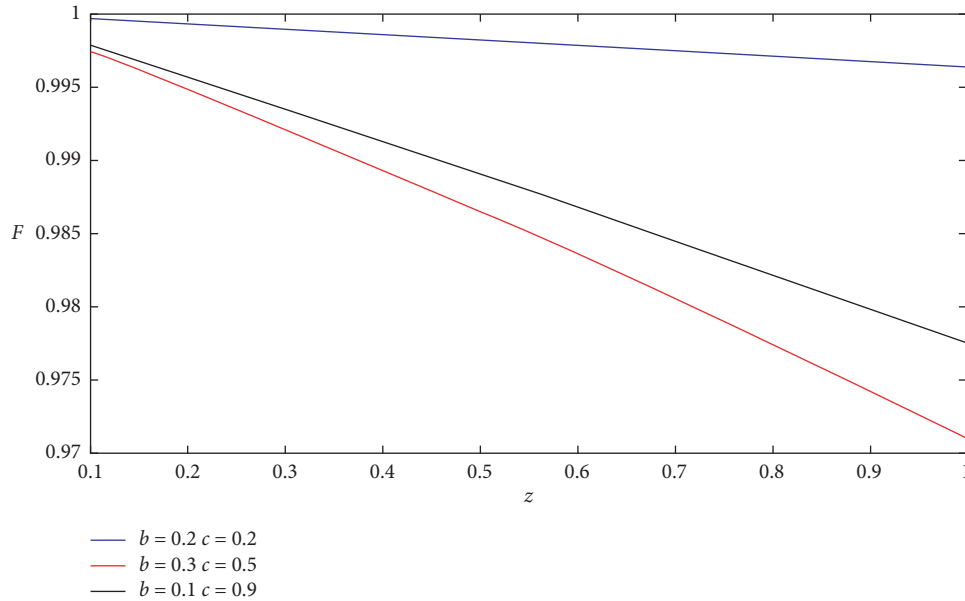


FIGURE 3: The graph of CDF for equation (33); we take three cases of the values of the parameters: first case  $b = c = 0.2$ , second case  $b = 0.3$  and  $c = 0.5$ , and final case  $b = 0.1$  and  $c = 0.9$ .

$$\begin{aligned}
 M_Z(t) &= \frac{2c^2}{b^2(c+1)} \sum_{l=0}^{\infty} \frac{(-1)^l (c)^l}{l!} \left[ \frac{b^{-l+1}}{2} \Gamma\left(\frac{-l+1}{2}\right) \int_0^{\infty} z^l \exp[tz] \right. \\
 &\quad \left. + \frac{b^{-l}}{2} \Gamma\left(\frac{-l}{2}\right) \int_0^{\infty} z^{l+1} \exp[tz] \right] \\
 &= \frac{2c^2}{b^2(c+1)} \sum_{l=0}^{\infty} \frac{(-1)^l (c)^l}{l!} \left[ \frac{b^{-l+1}}{2} \Gamma\left(\frac{-l+1}{2}\right) \left[ \frac{(-1)^l \Gamma(l+1)}{t^{l+1}} \right] \right. \\
 &\quad \left. + \frac{b^{-l}}{2} \Gamma\left(\frac{-l}{2}\right) \left[ \frac{(-1)^l \Gamma(l+2)}{t^{l+2}} \right] \right],
 \end{aligned}
 \tag{37}$$

where  $-R = tz$ . Therefore,

$$\begin{aligned}
 M_Z(t) &= \frac{2c^2}{b^2(c+1)} \sum_{l=0}^{\infty} \frac{(-1)^l (c)^l}{l!} \left[ \frac{b^{-l+1}}{2} \Gamma\left(\frac{-l+1}{2}\right) \left[ \frac{(-1)^l \Gamma(l+1)}{t^{l+2}} \right] \right. \\
 &\quad \left. + \left[ \frac{b^{-l}}{2} \Gamma\left(\frac{-l}{2}\right) \left[ \frac{(-1)^l \Gamma(l+2)}{t^{l+2}} \right] \right] \right].
 \end{aligned}
 \tag{38}$$

Equation (38) is MGF of  $Z$ .

### 4. Conclusion

In this paper, we discussed distribution of the ratio independent Weibull and Lindley random variables. In this approach, we derived the CDF, PDF, and MGF of the ratio independent Weibull and Lindley random variables. Also, the plots of CDF and PDF are drawn. The m-moment, mean, and variance are calculated. The CDF and PDF are derived in two formulas with respect to each one of them, first formula by confluent hypergeometric function and another formula by generalized hypergeometric function.

We studied distribution of the product independent Weibull and Lindley random variables. In this case, the CDF, PDF, and MGF are derived from the product independent Weibull and Lindley random variables. The plots of CDF and PDF are drawn. The CDF is derived in two formulas: first formula by using confluent hypergeometric function and another formula by using generalized hypergeometric function. However, PDF is derived by using gamma function of negative integer numbers.

### Data Availability

The data used to support the findings of this study are included within the article.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### Acknowledgments

The authors thank every person who helped them to complete this paper.

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