Advanced Studies in Pure Mathematics 11, 1987
Commutative Algebra and Combinatorics
pp. 93–109

Distributive Lattices, Affine Semigroup Rings and Algebras with Straightening Laws

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Summary. A lattice L is called integral over a field k if there exists a homogeneous ASL (algebra with straightening laws) domain on L over k. By virtue of fundamental structure theorem of Birkhoff, we can prove that every finite distributive lattice is integral.

Introduction

What properties of a finite poset (partially ordered set) H guarantee the existence of an ASL domain on H over a field k? This is an interesting question lying between commutative algebra and combinatorics.

Investigate many concrete examples of ASL which appeared in classical invariant theory, and we may well hope that every ASL which is an integral domain should have some good properties. If we will try to analyze some ring-theoretical properties of ASL domains, then the above question should arise of necessity.

For example, it is a starting point of recent works [12] and [13], in which the final goal is to classify all the three dimensional homogeneous Gorenstein ASL domains, to determine all the posets on which there exist three dimensional homogeneous Gorenstein ASL domains.

The main purpose of this paper is first to construct an ASL domain $\mathscr{R}_k[D]$ (see § 2) on any finite distributive lattice D over a field k, secondly to calculate the canonical module of $\mathscr{R}_k[D]$ explicitly and give a combinatorial interpretation to the number of minimal generators of this module, and thirdly to determine what kind of distributive lattices are Gorenstein.

This article is divided into four sections. In Section 1, we recall some fundamental definitions and terminologies on commutative algebra and combinatorics. In addition, we shall remark that the tensor product $\mathscr{R}_1 \otimes_k \mathscr{R}_2$ of two ASL's $\mathscr{R}_1$ and $\mathscr{R}_2$ over a field k is again an ASL in a natural way. This result will be used in Section 4. Moreover, for each finite poset H, we will associate a positive integer $t(H)$ and show that $t(H)=1$ if and only if H is pure. This number $t(H)$ will play an essential role in Section 3.

Received October 28, 1985.

In Section 2, on any finite distributive lattice D we shall construct an ASL domain $\mathscr{R}_k[D]$ over a field k. This construction depends in an essential way on a classical fundamental structure theorem (cf. Birkhoff [2, p. 59]) of finite distributive lattices. Using Birkhoff's theorem the k-algebra $\mathscr{R}_k[D]$ will be expressed as an affine semigroup ring. We shall also investigate some ring-theoretical properties of $\mathscr{R}_k[D]$ and show that $\mathscr{R}_k[D]$ is normal, rational and Cohen-Macaulay.

By the way, Stanlay [18] obtained the explicit expression of the canonical module of a normal affine semigroup ring. In Section 3, using Stanley's result, we shall calculate the canonical module of $\mathscr{R}_k[D]$ explicitly and, to determine when $\mathscr{R}_k[D]$ is Gorenstein, we shall compute the number of minimal generators of this module.

In final Section 4, we will try to classify all finite distributive lattices from a viewpoint of some Gorenstein properties for a poset proposed in [20]. We will see a remarkable gap between Gorenstein posets and weakly Gorenstein posets.

The author wishes to express his hearty thanks to Professor Kei-ichi Watanabe for valuable stimulative suggestions and continuous discussions on these topics.

§ 1. Preliminaries from commutative algebra and combinatorics

We here summarize basic definitions and results on commutative algebra and combinatorics.

a) All posets (partially ordered set) to be considered are finite.

The *length* of a chain (totally ordered set) X is $\sharp(X)-1$, where $\sharp(X)$ is the cardinality of X as a set.

The *rank* of a poset H, denoted by rank(H), is the supremum of lengths of chains contained in H.

The *height* (resp. *depth*) of an element $\alpha \in H$ is the supremum of lengths of chains descending (resp. ascending) from α, and written $\text{height}_H(\alpha)$ (resp. $\text{depth}_H(\alpha)$).

A *poset ideal* in a poset H is a subset I such that $\alpha \in I$, $\beta \in H$ and $\beta \leq \alpha$ together imply $\beta \in I$.

A poset is called *pure* if its all maximal chains have the same length.

A *clutter* is a poset in which no two elements are comparable.

We refer to Birkhoff [2, Chapter I] for the basic definitions and notation in lattice theory.

b) Let H be a finite set and N the set of non-negative integers. We denote by N^H the set of maps from H to N. A *monomial* $\mathscr{M}$ on H is an element of N^H. The *support* of a monomial $\mathscr{M}$ is the set $\text{Supp}(\mathscr{M})=\{x \in H; \mathscr{M}(x)\neq 0\}$. A monomial $\mathscr{M}$ is called *standard* if $\text{Supp}(\mathscr{M})$ is a chain.

If $\mathscr{R}$ is a commutative ring and an injection $\varphi: H \hookrightarrow \mathscr{R}$ is given, then to each monomial $\mathscr{M}$ on H we may associate

$$\varphi(\mathscr{M}) := \prod_{x \in H} \varphi(x)^{\mathscr{M}(x)} \in \mathscr{R}.$$

We will usually identify H with $\varphi(H)$ and $\varphi(\mathscr{M}) \in \mathscr{R}$ is also called a monomial if there is no confusion. It will be clear from the context whether an abstract monomial or an element of $\mathscr{R}$ is intended.

Now let k be a field and $\mathscr{R}$ a commutative k-algebra. Suppose that H is a finite poset with an injection $\varphi: H \hookrightarrow \mathscr{R}$. Then we call $\mathscr{R}$ an *algebra with straightening laws* on H over k if the following conditions are satisfied:

(ASL-1) The set of standard monomials is a basis of the algebra $\mathscr{R}$ as a vector space over k.

(ASL-2) If α and β in H are incomparable (written as $\alpha \not\sim \beta$) and if

$$(*) \qquad \alpha\beta = \sum_i r_i \gamma_{i1}\gamma_{i2} \cdots \gamma_{ip_i},$$

where $0 \neq r_i \in k$ and $\gamma_{i1} \leq \gamma_{i2} \leq \cdots$, is the unique expression for $\alpha\beta \in \mathscr{R}$ as a linear combination of distinct standard monomials guaranteed by (ASL-1), then $\gamma_{i1} \leq \alpha, \beta$ for every i.

Note that the right-hand side of the relation in (ASL-2) is allowed to be the empty sum $(=0)$, but that, though 1 is a standard monomial, no $\gamma_{i1}\gamma_{i2}\cdots\gamma_{ip_i}$ can be 1. The relations (*) are called the *straightening relations* for $\mathscr{R}$.

We denote by $[\alpha\beta]$ the set of standard monomials which appear in the right-hand side of the relation for $\alpha\beta$ with $\alpha \not\sim \beta$. More generally, for a monomial $\alpha_1\alpha_2\cdots\alpha_p$, we denote by $[\alpha_1\alpha_2\cdots\alpha_p]$ the set of standard monomials which appear in the standard monomial expression of $\alpha_1\alpha_2\cdots\alpha_p$.

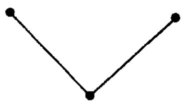
It is well known that the dimension of $\mathscr{R}$ as a k-algebra coincides with rank $(H)+1$ (see [4]).

c) An ASL $\mathscr{R}$ on a poset H is called *graded* if there is a grading $\mathscr{R} = \bigoplus_{n \geq 0} \mathscr{R}_n$ such that $\mathscr{R}_0 = k$ and each element of H is homogeneous of positive degree.

When $H \subset \mathscr{R}_1$, we say that $\mathscr{R}$ is *homogeneous.*

For a graded ASL $\mathscr{R}$ we denote by $\mathscr{P}_{\mathscr{R}}(\theta)$ the Poincaré series $\sum_{n \geq 0} (\dim_k \mathscr{R}_n)\theta^n$.

d) Let $\mathscr{R}$ (resp. $\mathscr{R}'$) be an ASL on a poset H (resp. H') over a field k. The tensor product $\mathscr{R} \otimes_k \mathscr{R}'$ will turn out to be an ASL in the following way. We make $H \oplus H' := H \cup H'$ (disjoint union) a poset by preserving the order of H and H', and by setting $\alpha < \alpha'$ for all $\alpha \in H$ and $\alpha' \in H'$, for example,



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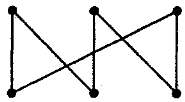

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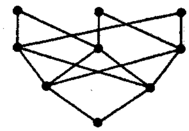


Figure 1 Figure 2 Figure 3

We inject $H \oplus H'$ into $\mathscr{R} \otimes_k \mathscr{R}'$ by sending $\alpha \in H$ (resp. $\alpha' \in H'$) to $\alpha \otimes 1$ (resp. $1 \otimes \alpha$). Now it is easily checked that $\mathscr{R} \otimes_k \mathscr{R}'$ is an ASL on $H \oplus H'$ over k.

e) Given a poset H, we will write $\bar{H}$ for the poset obtained by adjoining a new element, written $-\infty$, to H such that $-\infty < \alpha$ for all $\alpha \in H$. We use the covension that $-\infty$ is never an element of H.

A map $\nu: \bar{H} \to N$ is called *order reversing* if $x \leq y$ in $\bar{H}$ implies $\nu(x) \geq \nu(y)$. Moreover, a map $\nu: \bar{H} \to N$ is called *strictly order reversing* if (i) $\nu(x) \neq 0$ for all $x \in \bar{H}$ and (ii) $x < y$ in $\bar{H}$ implies $\nu(x) > \nu(y)$. We denote by $\mathscr{T}(H)$ the set of strictly order reversing maps from $\bar{H}$ to N.

Now we shall define a partial order in $\mathscr{T}(H)$. Let $\nu, \nu' \in \mathscr{T}(H)$. We will write $\nu \geq \nu'$ if the following conditions are satisfied:

(i) $\nu(x) \geq \nu'(x)$ for all $x \in \bar{H}$.

(ii) The map $\nu - \nu'$ defined by $(\nu - \nu')(x) := \nu(x) - \nu'(x)$ is order reversing map from $\bar{H}$ to N.

We denote by $\mathscr{T}_0(H)$ the set of minimal elements of $\mathscr{T}(H)$ with respect to the above partial order. Define $t(H) = \sharp(\mathscr{T}_0(H))$, and we have

Lemma. $t(H) = 1$ *if and only if H is pure.*

Proof. Define $d \in \mathscr{T}_0(H)$ by

$$d(x) = 1 + \mathrm{depth}_{\bar{H}}(x) \qquad (x \in \bar{H}).$$

First, we shall show that "if" part. If H is pure, then the above map d is a unique minimal element in $\mathscr{T}(H)$. In fact, for any map $\nu \in \mathscr{T}(H)$, we have $\nu(x) \geq d(x)$ for all $x \in \bar{H}$ by the definition of depth. Moreover, if $x < y$ in $\bar{H}$ and

$$x = z_0 < z_1 < z_2 < \cdots < z_n = y$$

is one of the unrefinable chains combining x with y, then, since H is pure, $d(z_i) + 1 = d(z_{i-1})$, hence

$$\begin{aligned} \nu(z_i) - d(z_i) &\leq \{\nu(z_{i-1}) - 1\} - d(z_i) \\ &= \{\nu(z_{i-1}) - 1\} - \{d(z_{i-1}) - 1\} \\ &= \nu(z_{i-1}) - d(z_{i-1}). \end{aligned}$$

So, we have

$$\nu(y)-d(y)\leq \nu(x)-d(x),$$

and $\nu\geq d$ in $\mathscr{T}(H)$.

Next we shall show "only if" part. Suppose that H is not pure. Then there exist two elements α, $\beta \in \bar{H}$ such that (i) α covers β, namely, $\beta<\alpha$ and no element of $\bar{H}$ is properly between α and β, and (ii) $\mathrm{depth}_{\bar{H}}(\beta)>\mathrm{depth}_{\bar{H}}(\alpha)+1$. (Note that H does not necessarily have this property, for example,

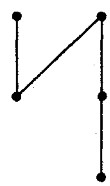


Figure 4

It is essential that $\bar{H}$ has a unique minimal element $-\infty$.) Define $\tilde{d} \in \mathscr{T}(H)$ by

$$\tilde{d}(x)=\begin{cases} d(x) & \text{if } x=\beta \text{ or } x\not<\alpha \text{ or } x>\alpha \\ 1+d(x) & \text{if } x\leq\alpha \text{ and } x\neq\beta. \end{cases}$$

Then $\tilde{d}(x)\geq d(x)$ for all $x \in \bar{H}$, however, though $\beta<\alpha$, we have $(\tilde{d}-d)(\beta)=0$ and $(\tilde{d}-d)(\alpha)=1$. Hence d and $\tilde{d}$ are imcomparable, so $\mathscr{T}_0(H)$ must contain an element other than d. Thus $t(H)\geq 2$. Q.E.D.

Example. Let H be the following poset

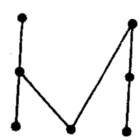


Figure 5

Then $\mathscr{T}_0(H)$ consists of

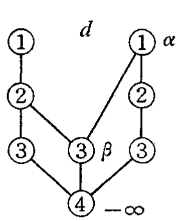


Figure 6

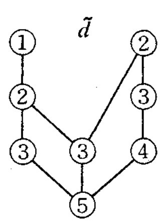


Figure 7

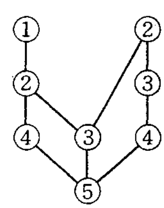


Figure 8

so $t(H)=3$.

§ 2. Construction of $\mathcal{R}_k[D]$

a) To begin with, we shall construct an ASL domain $\mathcal{R}_k[D]$ on any distributive lattice D over a field k.

Recall that a classical fundamental structure theorem of Birkhoff [2, p. 59] states that there exists a unique poset P such that $D=J(P)$, where $J(P)$ is the set of poset ideals of P, ordered by inclusion.

Let $P=\{p_1, p_2, \cdots, p_n\}$, where $n=\text{rank}(D)$. We will embed D into the polynomial ring $k[T, X_1, X_2, \cdots, X_n]$ by the injective map

$$\begin{array}{cccc} \varphi\colon D(=J(P)) & \longrightarrow & k[T, X_1, X_2, \cdots, X_n] \\ \cup & & \cup \\ I & \longmapsto & T \prod_{p_i \in I} X_i, \end{array}$$

where I is a poset ideal of P. We denote by $\mathcal{R}_k[D]$ the subring of $k[T, X_1, X_2, \cdots, X_n]$ which is generated by $\{\varphi(I)\}_{I \in D}$, namely,

$$\mathcal{R}_k[D]=k[\{\varphi(I)\}_{I \in D}].$$

Note that

$$\varphi(I)\varphi(J)=\varphi(I\cap J)\varphi(I\cup J) \tag{2.1}$$

for all $I, J \in D$.

Now we shall show that $\mathcal{R}_k[D]$ is an ASL on D over k. To prove the linear independence of the set of standard monomials, we have only to see that if $\mathcal{M}$ and $\mathcal{N}$ are distinct standard monomials on D then $\varphi(\mathcal{M})\neq\varphi(\mathcal{N})$ in $\mathcal{R}_k[D]$, because $\varphi(\mathcal{M})$ and $\varphi(\mathcal{N})$ are monomials (with coefficients 1) in $k[T, X_1, X_2, \cdots, X_n]$ in usual sence.

For a standard monomial $\mathcal{M}$ on D we express

$$\varphi(\mathcal{M})=T^{\xi(\mathcal{M})} \prod_{i=1}^{n} X_i^{\xi_i(\mathcal{M})}.$$

Then a poset ideal $I\ (\neq\phi)$ of P is contained in $\text{Supp}(\mathcal{M})$ if and only if $\xi_i(\mathcal{M})>\xi_j(\mathcal{M})$ for all i, j such that $p_i \in I$, $p_j \notin I$. In fact, the "only if" part is almost obvious. To see the "if" part, assume that $I \notin \text{Supp}(\mathcal{M})$. If $\{I\} \cup \text{Supp}(\mathcal{M})$ is a chain of D, then $\xi_i(\mathcal{M})=\xi_j(\mathcal{M})$ for all i, j such that $p_i \in I$, $p_j \notin I$ and

$$\{I-\{p_i\}, I, I\cup\{p_j\}\} \cup \text{Supp}(\mathcal{M})$$

is a chain of D. If $\{I\} \cup \mathrm{Supp}(\mathscr{M})$ is not a chain of D, then there exists $J \in \mathrm{Supp}(\mathscr{M})$ with $I \not\subset J$ and $J \not\subset I$. We have $\xi_i(\mathscr{M}) < \xi_j(\mathscr{M})$ for all i, j such that $p_i \in I$, $p_i \notin J$, $p_j \notin I$, $p_j \in J$.

This fact means that $\varphi(\mathscr{M}) \neq \varphi(\mathscr{N})$ if $\mathrm{Supp}(\mathscr{M}) \neq \mathrm{Supp}(\mathscr{N})$. While in the case of $\mathrm{Supp}(\mathscr{M}) = \mathrm{Supp}(\mathscr{N})$, it is easy to see that $\mathscr{M} \neq \mathscr{N}$ implies $\varphi(\mathscr{M}) \neq \varphi(\mathscr{N})$, because in the vector space Q^{n+1}, where Q is the field of rational numbers, the vectors $e_0 = (1, 0, 0, \cdots, 0)$, $e_1 = (1, 1, 0, \cdots, 0)$, $\cdots$, $e_n = (1, 1, 1, \cdots, 1)$ are linearly independent.

Thanks to (2.1), $\mathscr{R}_k[D]$ satisfies the axiom (ASL-2). Hence by virtue of [4, Proposition 1.1], $\mathscr{R}_k[D]$ is an ASL on D over k and we have

$$\mathscr{R}_k[D] \simeq k[X_\alpha; \alpha \in D]/I_D, \tag{2.2}$$

where $k[X_\alpha; \alpha \in D]$ is the polynomial ring in $\sharp(D)$-variables over k and

$$I_D = (X_\alpha X_\beta - X_{\alpha\wedge\beta} X_{\alpha\vee\beta}; \alpha \not\sim \beta). \tag{2.3}$$

Example. Let $D = J(P)$ be the distributive lattice of Figure 10.

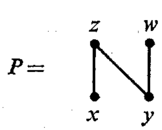


Figure 9

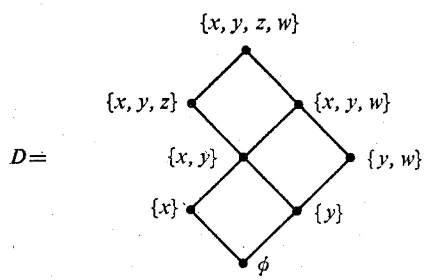


Figure 10

Then $\mathscr{R}_k[D]$ is a subring of $k[t, x, y, z, w]$ which is generated by all monomials written in Figure 11.

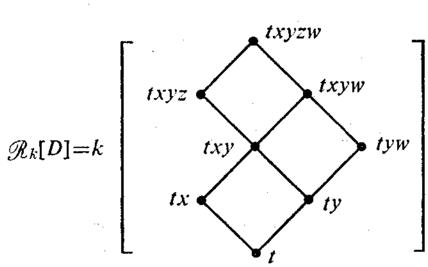


Figure 11

b) Next we shall investigate some ring-theoretical properties of $\mathscr{R}_k[D]$.

Normality. A monomial (in usual sense) $T^{\xi}\prod_{i=1}^{n}X_i^{\xi_i}$ in $k[T, X_1, X_2, \cdots, X_n]$ is contained in $\mathscr{R}_k[D]$ if and only if, for some order preserving bijection

$$\sigma: P \longrightarrow [n]=\{1, 2, \cdots, n\}, \tag{2.4}$$

we have

$$\xi \geq \xi_{i(1)} \geq \cdots \geq \xi_{i(n)}, \tag{2.5}$$

where $p_{i(j)}=\sigma^{-1}(j)$.

Hence $\mathscr{R}_k[D]$ is normal by the Hochster's criterion [14, p. 320]. Note that some combinatorial aspects of above maps (2.4) are investigated in Stanley [17]. Also, refer to Garsia [6] and [23].

Rationality. The k-algebra $\mathscr{R}_k[D]$ is rational over k, that is, the quotient field of $\mathscr{R}_k[D]$ is a purely transcendental extension of the base field k. In fact, $\mathscr{R}_k[D]$ contains $TX_{i(1)}\cdots X_{i(s)}$ for all $s=0, 1, \cdots, n$, where $p_{i(j)}=\sigma^{-1}(j)$ and σ is one of the above order preserving bijections. Thus the quotient field of $\mathscr{R}_k[D]$ is just $k(T, X_1, \cdots, X_n)$.

Cohen-Macaulayness. It is well known that distributive lattices are Cohen-Macaulay over an arbitrary field (see, for example, [1] or [3]). Combining this result with the fundamental theorem on ASL (cf. [4, Corollary 7.2]), we can conclude that $\mathscr{R}_k[D]$ is Cohen-Macaulay.

On the other hand, apart from the above argument, the Cohen-Macaulayness of $\mathscr{R}_k[D]$ is also deduced from the main result of Hochster [14], which says that every normal subring (of a polynomial ring over a field) generated by a finite number of monomials is always Cohen-Macaulay.

c) We summarize the discussions in **a)** and **b)** in the following

Theorem. *Let k be a field and L a finite lattice. We denote by $\mathscr{R}_k[L]$ the k-algebra*

$$k[X_\alpha; \alpha \in L]/(X_\alpha X_\beta - X_{\alpha\wedge\beta}X_{\alpha\vee\beta}; \alpha \not\sim \beta).$$

Then the following conditions are equivalent:

(i) *$\mathscr{R}_k[L]$ is an ASL on L over k.*

(ii) *$\mathscr{R}_k[L]$ is an integral domain.*

(iii) *L is a distributive lattice.*

Moreover, if these conditions are satisfied, then $\mathscr{R}_k[L]$ is Cohen-Macaulay, normal and rational.

Proof. We have only to show that (i) implies (iii) and that (ii) implies (iii).

First, assume that $\mathscr{R}_k[L]$ is an ASL on L over k. Let α, β and γ be any elements of L. We will calculate the standard monomial expression of $\alpha^2\beta\gamma$ in two ways. Namely,

$$\begin{aligned}\alpha^2\beta\gamma &= (\alpha\beta)(\alpha\gamma)\\ &= \{(\alpha\wedge\beta)(\alpha\vee\beta)\}\{(\alpha\wedge\gamma)(\alpha\vee\gamma)\}\\ &= \{(\alpha\wedge\beta)(\alpha\wedge\gamma)\}\{(\alpha\vee\beta)(\alpha\vee\gamma)\}\\ &= (\alpha\wedge\beta\wedge\gamma)\{(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)\}\{(\alpha\vee\beta)\wedge(\alpha\vee\gamma)\}(\alpha\vee\beta\vee\gamma)\end{aligned}$$

$$\begin{aligned}\alpha^2\beta\gamma &= \alpha^2(\beta\gamma)\\ &= \alpha^2(\beta\wedge\gamma)(\beta\vee\gamma)\\ &= \{\alpha(\beta\wedge\gamma)\}\{\alpha(\beta\vee\gamma)\}\\ &= (\alpha\wedge\beta\wedge\gamma)\{\alpha\wedge(\beta\vee\gamma)\}\{\alpha\vee(\beta\wedge\gamma)\}(\alpha\vee\beta\vee\gamma).\end{aligned}$$

So, thanks to (ASL-1), we have

$$\begin{aligned}\alpha\wedge(\beta\vee\gamma) &= (\alpha\wedge\beta)\vee(\alpha\wedge\gamma)\\ \alpha\vee(\beta\wedge\gamma) &= (\alpha\vee\beta)\wedge(\alpha\vee\gamma),\end{aligned}$$

for all $\alpha, \beta, \gamma \in L$. Here L is distributive.

Secondly, assume that L is not distributive. Then L must contain one of the following two lattices

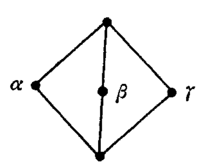


Figure 12

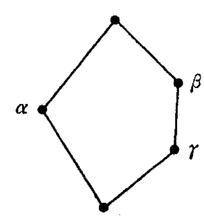


Figure 13

as a sublattice (cf. [2, Chapter II]). In both cases, we have

$$\alpha\beta = (\alpha\wedge\beta)(\alpha\vee\beta) = (\alpha\wedge\gamma)(\alpha\vee\gamma) = \alpha\gamma.$$

So, $\mathscr{R}_k[L]$ cannot be an integral domain. Q.E.D.

d) A finite lattice L is called *integral* over a field k if there exists a homogeneous ASL domain on L over k. The above theorem in **c)** states that every finite distributive lattice is integral over an arbitrary field k. We will propose two reasonable conjectures concerning integral lattices.

Conjecture. 1) Every modular lattice is integral over some field.
2) Every integral lattice is Cohen-Macaulay.

But the situation is still very obscure.

Example. 1) The following lattice L_1 is a non-modular integral lattice, while L_2 is a non-integral Cohen-Macaulay (in fact, Gorenstein) lattice (cf. [12]).

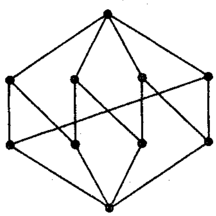
$L_1=$ 

Figure 14

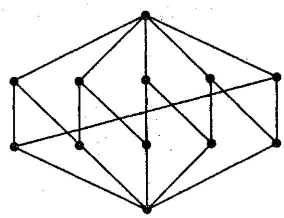
$L_2=$ 

Figure 15

2) Let n be an arbitrary positive integer, k an infinite field and L a modular lattice

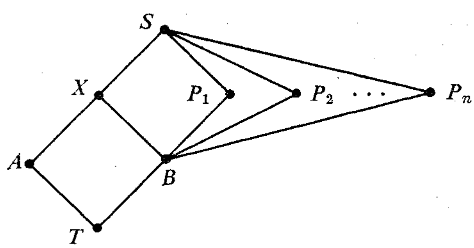
$L=$ 

Figure 16

Then we can construct a homogeneous ASL domain $\mathscr{R}$ on L over k by means of

$$T=xy^2,\quad A=x^2y,\quad B=yz^2,\quad X=xz^2,\quad S=w^3,$$
$$P_i=y^2z^2/(x-p_iy)\quad (i=1, 2, \cdots, n),$$

where $0\neq p_i \in k$, $p_i\neq p_j$ if $i\neq j$, and x, y, z, w are indeterminates over k. The straightening relations of $\mathscr{R}$ are

$$AB=TX,$$
$$P_iA=T(B+p_iP_i),\quad P_iX=B(B+p_iP_i),$$
$$P_iP_j=B(P_i-P_j)/(p_i-p_j)\quad (i\neq j).$$

Thanks to a criterion [13, Lemma 10], it can be checked that $\mathscr{R}$ is, in fact, an ASL on L over k. Also, by [13, Proposition 3], we see that $\mathscr{R}$ is normal.

Note that if k is a finite field and n is sufficiently large then L is not integral over k.

e) Now we shall consider the argument discussed in **c)** in a more general situation.

Let $\mathscr{R}$ be a homogeneous ASL (not necessarily a domain) on a lattice L over a field k. Assume that $\mathscr{R}$ satisfies the following condition:

($\sharp$) For each incomparable pair α and β, $[\alpha\beta]\neq\phi$ (see § 1, **b)** for the definition of $[\alpha\beta]$) and in the straightening relation $\alpha\beta=\sum r_i\gamma_{i1}\gamma_{i2}$ $(0\neq r_i \in k)$ we have $\gamma_{i1}\leq\alpha\wedge\beta$, $\alpha\vee\beta\leq\gamma_{i2}$.

If this condition ($\sharp$) is satisfied, then L must be distributive. To prove this, suppose that α, β and γ are any elements of L with $\alpha\wedge\gamma=\beta\wedge\gamma$, $\alpha\vee\gamma=\beta\vee\gamma$ and that $\alpha\gamma=\sum s_i\delta_{i1}\delta_{i2}$ $(0\neq s_i \in k)$, $\beta\gamma=\sum t_j\varepsilon_{j1}\varepsilon_{j2}$ $(0\neq t_j \in k)$ are the standard monomial expressions of $\alpha\gamma$ and $\beta\gamma$. Note that, for example, if α and γ are comparable then we may consider $\alpha\gamma=(\alpha\wedge\gamma)(\alpha\vee\gamma)$. Then we have

$$\sum s_i\delta_{i1}\beta\delta_{i2}=\sum t_j\varepsilon_{j1}\alpha\varepsilon_{j2} \quad (=\alpha\beta\gamma).$$

Since all monomials $\delta_{i1}\beta\delta_{i2}$, $\varepsilon_{j1}\alpha\varepsilon_{j2}$ are standard, α must coincide with β by (ASL-1). Hence L is distributive.

Some interesting examples of ASL which appeared in classical invariant theory, such as coordinate rings of Grassmann varieties (cf. [4, III, 11]), satisfy the condition ($\sharp$). So analyzing above examples assures us that the following question is worthy to investigate.

Question. Assume that $\mathscr{R}$ is a homogeneous ASL domain on a distributive lattice D over a field k and that $\mathscr{R}$ satisfies the condition ($\sharp$). Then is $\mathscr{R}$ normal?

Note that there exists a family of three dimensional homogeneous Gorenstein ASL domains which are not normal, see [12, Example g)].

§ 3. When is $\mathscr{R}_k[D]$ Gorenstein?

In this section we shall calculate the canonical module of $\mathscr{R}_k[D]$ and determine when $\mathscr{R}_k[D]$ is Gorenstein.

a) Let k be a field, $\mathscr{S}\subset N^r$ $(r>0)$ an affine semigroup, that is, $\mathscr{S}$ is a finitely generated additive semigroup with identity, and $k[T]=k[T_1, T_2, \cdots, T_r]$ the polynomial ring in r-variables over k. We denote by $k[\mathscr{S}]$

the affine semigroup ring

$$k[T^w; w \in \mathscr{S}] \quad (\subset k[T])$$

of $\mathscr{S}$ over k, where $T^w = T_1^{w_1} T_2^{w_2} \cdots T_r^{w_r}$ if $w = (w_1, w_2, \cdots, w_r)$.

We say that $\mathscr{S}$ is a normal affine semigroup if $k[\mathscr{S}]$ is normal. Note that this condition does not depend on the field k, see Hochster [14, Proposition 1]. As was used in Section 2, **b**), Hochster's criterion [14, p. 320] says that $\mathscr{S}$ is normal if and only if the following condition is satisfied: If n is a positive integer and if $\alpha, \beta, \gamma \in \mathscr{S}$ satisfy $n\alpha = n\beta + \gamma$, then $\gamma = n\gamma'$ for some $\gamma' \in \mathscr{S}$.

If $\mathscr{S}$ is normal, define

$$(3.1) \qquad K_{\mathscr{S}} = \{\alpha \in \mathscr{S};\ \text{for all } \beta \in \mathscr{S} \text{ there is an integer } n > 0 \text{ and an element } \gamma \in \mathscr{S} \text{ such that } n\alpha = \beta + \gamma\}.$$

By virtue of Stanley [18, p. 82] if $\mathscr{S}$ is normal and $\mathscr{R} = k[\mathscr{S}]$ then the canonical module $K_{\mathscr{R}}$ of $\mathscr{R}$ coincides with $k[K_{\mathscr{S}}]$.

b) Now we shall compute the canonical module of $\mathscr{R}_k[D]$.

Let $D = J(P)$, $P = \{p_1, p_2, \cdots, p_n\}$, be a distributive lattice and $\mathscr{S}(D) \subset N^{n+1}$ ($n = \text{rank}\,(D)$) an affine semigroup

$$(3.2) \qquad \mathscr{S}(D) = \{(w_0, w_1, \cdots, w_n) \in N^{n+1};\ w_0 \geq w_i \text{ for all } i > 0 \text{ and } w_i \geq w_j \text{ if } p_i \leq p_j \text{ in } P\}.$$

Then $\mathscr{S}(D)$ is normal and $\mathscr{R}_k[D]$ is naturally isomorphic to $k[\mathscr{S}(D)]$. By the definition (3.1), it is easily seen that

$$(3.3) \qquad K_{\mathscr{S}(D)} = \{(w_0, w_1, \cdots, w_n) \in \mathscr{S}(D);\ w_0 > w_i > 0 \text{ for all } i > 0 \text{ and } w_i > w_j \text{ if } p_i < p_j \text{ in } P\}.$$

c) A subset U of an affine semigroup $\mathscr{S}$ is called a $\mathscr{S}$-ideal if $\mathscr{S} + U \subset U$. In this case $k[U]$ is a (ring-theoretical) ideal of an affine semigroup ring $k[\mathscr{S}]$. A $\mathscr{S}$-ideal U is always finitely generated in the sense that U can be expressed as $\bigcup_{i=1}^{q} (u_i + \mathscr{S})$ for some $u_1, u_2, \cdots, u_q \in U$ and the minimal generating system for U is uniquely determined. We denote by $\mu(U)$ the number of minimal generators.

If $\mathscr{S}$ is a normal affine semigroup then $K_{\mathscr{S}}$ is a $\mathscr{S}$-ideal. We are interested in $\mu(K_{\mathscr{S}})$. In our case of $\mathscr{R}_k[D]$, $D = J(P)$, $\mu(K_{\mathscr{S}(D)})$ coincides with the combinatorial number $t(P)$ treated in Section 1, **e**). Namely,

Theorem. $\mu(K_{\mathscr{S}(D)}) = t(P)$.

Proof. Let $P = \{p_1, p_2, \cdots, p_n\}$ and $\bar{P} = P \cup \{p_0\}$ ($p_0 = -\infty$). For each $\nu \in \mathscr{T}(P)$, we define

$$w^{(\nu)}=(\nu(p_0), \nu(p_1), \cdots, \nu(p_n)) \in K_{\mathscr{S}(D)}.$$

This definition gives a one-to-one correspondence between $\mathscr{T}(P)$ and $K_{\mathscr{S}(D)}$.

We shall show that $w^{(\nu)}$ is contained in the minimal generating system for $K_{\mathscr{S}(D)}$ if and only if $\nu \in \mathscr{T}_0(P)$. In fact, $w^{(\nu')}$ is not a member of the minimal generating system for $K_{\mathscr{S}(D)}$ if and only if

$$w^{(\nu')}=w^{(\nu)}+w \tag{3.4}$$

for some $w^{(\nu)} \in K_{\mathscr{S}(D)}$ and some $w=(w_0, w_1, \cdots, w_n) \in \mathscr{S}(D)$ with $w \neq (0, 0, \cdots, 0)$. By (3.2), $w_i \geq w_j$ provided $p_i \leq p_j$. Hence (3.4) implies $\nu' > \nu$ in $\mathscr{T}(P)$ (see § 1, **e)** for the definition of the partial order of $\mathscr{T}(P)$), so $\nu' \notin \mathscr{T}_0(P)$, and conversely. Q.E.D.

d) Now, $\mathscr{R}_k[D]$ is Gorenstein if and only if $\mu(K_{\mathscr{S}(D)})=1$. Thus, combining the above theorem in **c)** with the Lemma in Section 1, **e)**, we have a main result in this section.

Corollary. *$\mathscr{R}_k[D]$, $D=J(P)$, is Gorenstein if and only if P is pure.*

e) By the way, in [18] Stanley has obtained the necessary and sufficient condition for a Cohen-Macaulay graded domain $\mathscr{R}=\bigoplus_{n\geq 0}\mathscr{R}_n$ to be a Gorenstein ring in terms of the Poincaré series of $\mathscr{R}$. On the other hand, the Poincaré series of an graded ASL $\mathscr{R}=\bigoplus_{n\geq 0}\mathscr{R}_n$ on a poset H is completely determined by $\deg(\alpha)$, $\alpha \in H$, because its free basis over k is the set of standard monomials.

Hence, thanks to Corollary in **d)**, we immediately get the following result of Stanley and Buchweitz, which is contained in [4, II, 9)].

Corollary. *Let $D=J(P)$ be a distributive lattice and $\mathscr{R}=\bigoplus_{n\geq 0}\mathscr{R}_n$ a graded ASL domain on D over a field k. Assume that*

$$\deg(\alpha)+\deg(\beta)=\deg(\alpha\wedge\beta)+\deg(\alpha\vee\beta)$$

for all α, $\beta \in D$. Then $\mathscr{R}$ is Gorenstein if and only if P is pure.

Using "β-invariant" defined in Stanley [17], it is possible to calculate the Poincaré series of a homogeneous ASL domain $\mathscr{R}=\bigoplus_{n\geq 0}\mathscr{R}_n$ on a distributive lattice $D=J(P)$. Refer to a survay [3] for more details.

§ 4. Classification of distributive lattices from a viewpoint of Gorenstein properties for a poset

In this final section we shall classify all finite distributive lattices from a viewpoint of some "Gorenstein" properties for a poset proposed in Watanabe [20, § 4].

a) Let us first state some definitions from [20].

A *weighted poset* (H, ω) is a couple of a poset H and a map ω from H to $N-\{0\}$, called a *weight* on H.

We will call a graded ASL $\mathscr{R}=\bigoplus_{n\geq 0}\mathscr{R}_n$ on H over a field k a graded ASL on (H, ω) if $\alpha \in \mathscr{R}_{\omega(\alpha)}$ for every $\alpha \in H$.

A Poincaré series $\mathscr{P}_{(H,\omega)}(\theta)$ of a weighted poset (H, ω) is defined to be the Poincaré series $\mathscr{P}_{\mathscr{R}}(\theta)$ of any graded ASL $\mathscr{R}$ on (H, ω).

Definition. Let (H, ω) be a weighted poset. We say that

(i) (H, ω) is *weakly Gorenstein* over a field k if H is Cohen-Macaulay over k and there exists a graded ASL $\mathscr{R}$ on (H, ω) over k which is a Gorenstein ring.

(ii) (H, ω) is *numerically Gorenstein* over k if H is Cohen-Macaulay over k and the Ponicaré series $\mathscr{P}_{(H,\omega)}(\theta)$ satisfies

$$\mathscr{P}_{(H,\omega)}(1/\theta)=(-1)^d\theta^\lambda\mathscr{P}_{(H,\omega)}(\theta)$$

for some integer λ and $d=1+\operatorname{rank}(H)$.

Consult [20] for further informations.

b) In the following we fix the base field k and omit the word "over k".

By [18], if (H, ω) is weakly Gorenstein, then (H, ω) is numerically Gorenstein. Also, if (H, ω) is numerically Gorenstein and if there exists a graded ASL domain $\mathscr{R}$ on (H, ω), then (H, ω) is weakly Gorenstein.

Example. Let D be a distributive lattice

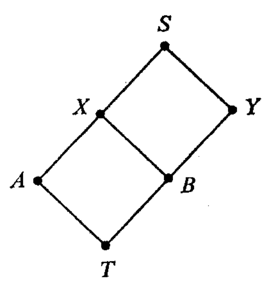
$D=$ 

Figure 17

Let ω_i $(i=1, 2)$ be weights on D defined by

$$\omega_1(\alpha)=\begin{cases}2 & \text{if } \alpha=A, B\\ 1 & \text{if } \alpha\neq A, B,\end{cases} \qquad \omega_2(\alpha)=\begin{cases}2 & \text{if } \alpha=X, Y\\ 1 & \text{if } \alpha\neq X, Y.\end{cases}$$

Then we have

$$\mathscr{P}_{(D,\omega_i)}(\theta)=(1+\theta+\theta^2)/(1-\theta)^3(1-\theta^2)$$

for $i=1, 2$. So, both (D, ω_1) and (D, ω_2) are numerically Gorenstein.

On the other hand, (D, ω_1) is weakly Gorenstein, however (D, ω_2) is not weakly Gorenstein. In fact, let $\mathscr{R}_i$ $(i=1, 2)$ be a graded ASL on (D, ω_i) over a field k and suppose that we have

$$XY=rB \quad (\text{resp. } XY=Bf(B, X, Y))$$

in $\mathscr{R}_1/(T, S)$ (resp. $\mathscr{R}_2/(T, S)$), where $r \in k$ and $f=f(x, y, z)$, $\deg(f)=3$, is a homogeneous polynomial in 3-variables x, y, z with $\deg(x)=1$, $\deg(y)=\deg(z)=2$. Now, $T, A+B, X+Y, S$ are a regular sequence on $\mathscr{R}_i$ and we have

$$\mathscr{R}_1/(T, A+B, X+Y, S)\simeq k[\xi, \eta]/(\eta^2, \xi\eta, \xi^2-r\eta)$$
$$(\text{resp. } \mathscr{R}_2/(T, A+B, X+Y, S)\simeq k[\xi, \eta]/(\xi^2, \xi\eta, \eta^2-\xi f(-\xi, \eta, -\eta))$$
$$\simeq k[\xi, \eta]/(\xi^2, \xi\eta, \eta^2)),$$

where ξ and η are indeterminates over k with $\deg(\xi)=1$, $\deg(\eta)=2$. Hence, $\mathscr{R}_1$ is Gorenstein if and only if $r\neq 0$, and $\mathscr{R}_2$ cannot be Gorenstein.

Note that there exists a graded ASL domain on (D, ω_1). For example, let x, y, t, s be indeterminates over k and put $T=t$, $A=t^2x/y$, $B=xy$, $X=x$, $Y=y$, $S=s$. Then we get a graded ASL domain $\mathscr{R}$ on (D, ω_1) whose straightening relations are $AB=T^2X^2$, $AY=T^2X$, $XY=B$.

c) Now we will state our final result in this paper.

Theorem. *Let $D=J(P)$ be a distributive lattice.*

1) *D is Gorenstein if and only if P is of the form*

(4.1) $$P=C_0\oplus C_1\oplus\cdots\oplus C_s,$$

*where each C_i is a clutter and $s=$*rank(P).

2) *Assume that a weight ω on D satisfies*

$$\omega(\alpha)+\omega(\beta)=\omega(\alpha\wedge\beta)+\omega(\alpha\vee\beta)$$

for all $\alpha, \beta\in D$. Then the following conditions are equivalent:

(i) *(D, ω) is weakly Gorenstein.*

(ii) *(D, ω) is numerically Gorenstein.*

(iii) *P is pure.*

Proof. 1) To begin with, suppose that the poset P is of the form (4.1). Let $B_i=J(C_i)$ be a Boolean lattice and β_i its unique minimal element. We denote by $\mathscr{R}_i$ (resp. $\mathscr{R}'_i$) the Stanley-Reisner ring on B_i (resp. $B_i-\{\beta_i\}$). Since Boolean lattices are Gorenstein (cf. [3, P. 615]), $\mathscr{R}_i$ is a Gorenstein ring, hence $\mathscr{R}'_i$ is also Gorenstein. Now, the Stanley-Reisner ring $\mathscr{R}$ on $D=J(P)$ is of the form

$$\mathscr{R}=\mathscr{R}_1\otimes_k\mathscr{R}'_2\otimes_k\cdots\otimes_k\mathscr{R}'_{s-1}\otimes_k\mathscr{R}'_s,$$

hence $\mathscr{R}$ is Gorenstein by [21].

Conversely, assume that the poset P is not of the form (4.1). Then, there exist two elements α, $\beta\in P$ with $\alpha\not>\beta$, and $\text{height}_P(\alpha)=1+\text{height}_P(\beta)$. We will choose $\gamma\in P$ with $\text{height}_P(\gamma)=\text{height}_P(\beta)$, and $\gamma<\alpha$. Define six poset ideals of P as follows.

$$\begin{aligned}
&I_1=\{x\in P;\ \text{height}_P(x)\leq\text{height}_P(\beta),\ x\neq\beta,\gamma\}\\
&I_2=I_1\cup\{\beta\}\\
&I_3=I_1\cup\{\gamma\}\\
&I_4=I_1\cup\{\beta,\gamma\}\\
&I_5=I_1\cup\{\alpha,\gamma\}\\
&I_6=I_1\cup\{\alpha,\beta,\gamma\}.
\end{aligned}$$

Then, in D, the open interval $\{\alpha\in D;\ I_1<\alpha<I_6\}$ is of the form

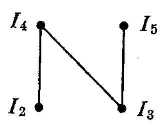


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Figure 18

hence D is not Gorenstein by [15, (5.5), (5.6)].

2) This result follows from Section 3, **d)** and **e)**. Q.E.D.

A somewhat surprising generalization of the above result 2) can be found in [23].

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